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CURVED ELEMENTS IN A MIXED FINITE ELEMENT METHOD CLOSE TO THE EQUILIBRIUM MODEL

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INTRODUCTION

Recently we have derived a new type of mixed finite element method [1], [2] which gives approximate solutions to the Dirichlet boundary problems for one elliptic equation and for the system of plane elasticity. The Galerkin approximations are vector-functions, converging to the à priori chosen components of co-gradient (or of the stress tensor) and to the solution itself.

The present paper extends the method of [1] to domains with smooth boundary, using the curved elements along the boundary. Some L_2-error estimates are presented, similar to those of [1]. Although only one equation is considered here, the same approach is applicable to the elliptic systems such as in [2].

In Section 1 we recall the main features of mixed finite element model of [1]. Section 2 shows the application of two types of curved elements, namely (i) the elements, describing the boundary segments exactly, analyzed by Zlámal [3] and (ii) the elements, interpolating the boundary segments, the theory of which was given by Ciarlet and Raviart [4]. In Section 3 we apply in particular the quadratic interpolation of the boundary.

1. VARIATIONAL FORMULATION AND A MIXED FINITE ELEMENT MODEL

First let us present notation, used throughout the paper. Consider a bounded domain $\Omega \subset E_n$ with a Lipschitz boundary Γ (cf. [5] for the definition of a Lipschitz boundary). By $\mathscr{L}(X, Y)$ we denote the space of linear, bounded mappings from X into Y.

$W^{k,2}(\Omega)$, $k \geqq 0$, integer, will denote the Sobolev space of functions, the generalized derivatives of which up to the order k are elements of $L_2(\Omega)$ (square-integrable). Using $(.;.)$ for the scalar product in $L_2(\Omega)$, we introduce the norm in $W^{k,2}(\Omega)$ by

$$\|v\|_{k,\Omega} = \Big[\sum_{|\alpha|\leq k} (D^\alpha v;\, D^\alpha v)\Big]^{1/2}$$

and the system of seminorms by

$$|v|_{j,\Omega} = \Big[\sum_{|\alpha| = j} (D^\alpha v;\, D^\alpha v)\Big]^{1/2} \quad (j \leq k)$$

where

$$D^\alpha v(\mathbf{x}) = \frac{\partial^{|\alpha|} v(\mathbf{x})}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}, \quad \alpha = (\alpha_1, \dots, \alpha_n), \quad |\alpha| = \sum_{i=1}^{n} \alpha_i$$

and α_i are non-negative integers.

In case $k = 0$ we set $W^{0,2}(\Omega) = L_2(\Omega)$ and write simply $\|v\|_{0,\Omega} = \|v\|_\Omega$. Moreover, the subscript Ω is omitted if any misunderstanding is not possible.

$W_0^{1,2}(\Omega)$ denotes the subspace of $W^{1,2}(\Omega)$ of functions, the traces of which vanish on Γ.

A repeated Latin index implies always summation over the range $1, 2, \dots, n$, unless exceptions are stated explicitely.

In the present Section we give a brief summary of some results of [1], which will be used later.

Let us consider the following problem

$$-a_{ij} \frac{\partial^2 u}{\partial x_i\, \partial x_j} = f \quad \text{in} \quad \Omega\,, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Gamma\,,$$

where $f \in L_2(\Omega)$, $a_{ij} = \text{const.}$ form a symmetric positive definite matrix $n \times n$. The weak solution of (1.1) is defined as a function of $W_0^{1,2}(\Omega)$, which minimizes the quadratic functional $\mathscr{L}(u) = \frac{1}{2}A(u, u) - (f; u)$ on the space $W_0^{1,2}(\Omega)$, where

$$A(u, v) = \left(a_{ij} \frac{\partial u}{\partial x_i};\ \frac{\partial v}{\partial x_j}\right).$$

Let us introduce

$$\mathscr{H} = [W^{1,2}(\Omega)]^n$$

and the bilinear form $B(\lambda, \mu)$ on $\mathscr{H} \times \mathscr{H}$ as follows:

$$B(\lambda, \mu) = (b_{ij}\lambda_i;\, \mu_j) - \gamma^{-1}(\operatorname{div} \lambda - \alpha_j \lambda_j;\, \operatorname{div} \mu - \alpha_j \mu_j) \tag{1.2}$$

where b_{ij} are elements of a matrix inverse to $[a_{ij}]$, $\alpha = (\alpha_1, \dots, \alpha_n)$ is a constant non-zero vector, $\gamma = a_{ij}\alpha_i\alpha_j$, $\operatorname{div} \lambda = \partial\lambda_j/\partial x_j \equiv \lambda_{j,j}$. In [1] we have shown the connection between the solution of (1.1) and the solution of the following problem:

to find $\lambda^0 \in \mathscr{H}$, such that

$$B(\lambda^0, \mu) = \gamma^{-1}(f; \operatorname{div} \mu - \alpha_j \mu_j) \quad \forall \mu \in \mathscr{H} . \tag{1.3}$$

The connection is given by the

Theorem 1. *Let the weak solution of* (1.1) *u belong to* $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ *for any* $f \in L_2(\Omega)$. *Then* (1.3) *has precisely one solution* $\lambda^0 \in \mathscr{H}$ *and it holds :*

$$\lambda_i^0 = a_{ij}\left(\frac{\partial u}{\partial x_j} + \alpha_j u\right), \quad i = 1, 2, \ldots, n \tag{1.4}$$

$$u = \gamma^{-1}(-f - \operatorname{div} \lambda^0 + \alpha_j \lambda_j^0) \tag{1.4'}$$

$$\sum_{i=1}^{n} \|\lambda_0^i\| + \|\operatorname{div} \lambda^0\| \leqq c\|f\| \tag{1.5}$$

where c is a constant independent of f.

Remark 1.1. The relation (1.3) is necessary and sufficient to the fact, that the stationary value of the functional

$$\mathscr{S}(\lambda) = (b_{ij}\lambda_i; \lambda_j) - \gamma^{-1}\|\operatorname{div} \lambda - \alpha_j \lambda_j + f\|^2$$

is attained at $\lambda = \lambda^0$.

Remark 1.2. The regularity assumptions of Theorem 1 are satisfied e.g. if $\Gamma \in C^\infty$ or if $\Omega \subset E_2$ is a convex polygon and $a_{ij} = \delta_{ij}$.

On the basis of (1.3) the Galerkin approximations can be defined [1]. Here we present only a survey of results, which will be needed in what follows. For simplicity let us restrict ourselves only to the model problem

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \subset E_n \\ u &= 0 \quad \text{on} \quad \Gamma . \end{aligned} \tag{1.6}$$

A more complex problem, namely that for the system of equations of linear plane elasticity, has been analyzed in [2].

Let us set

$$\alpha_1 = \alpha_2 = \ldots = \alpha_{n-1} = 0, \quad \alpha_n = \alpha \tag{1.7}$$

and define

$$\alpha = \alpha_0 h^{-1-\varepsilon} \tag{1.8}$$

where $h \in (0, 1\rangle$, $\alpha_0 > 0$, $\varepsilon > 0$.

Introducing

$$\bar{\lambda}_n = \alpha^{-1}\lambda_n, \quad \bar{\mu}_n = \alpha^{-1}\mu_n$$

we obtain from (1.4)

$$\lambda_i^0 = \frac{\partial u}{\partial x_i}, \quad i = 1, \ldots, n-1; \quad \bar{\lambda}_n^0 = \frac{h^{1+\varepsilon}}{\alpha_0} \frac{\partial u}{\partial x_n} + u .$$

Substituting for λ_n, μ_n, we may write

$$B(\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n) = \bar{B}(\lambda_1, \ldots, \bar{\lambda}_n; \mu_1, \ldots, \bar{\mu}_n),$$

$$\bar{B}(\lambda_1, \ldots, \bar{\lambda}_n; \mu_1, \ldots, \bar{\mu}_n) = \sum_{j=1}^{n-1} (\lambda_j; \mu_j) + \alpha^2(\bar{\lambda}_n; \bar{\mu}_n) - \tag{1.9}$$

$$- \frac{1}{\alpha^2} \Big(\sum_{j=1}^{n-1} \lambda_{j,j} + \alpha\bar{\lambda}_{n,n} - \alpha^2\bar{\lambda}_n; \sum_{j=1}^{n-1} \mu_{j,j} + \alpha\bar{\mu}_{n,n} - \alpha^2\bar{\mu}_n\Big) =$$

$$= \sum_{i,j=1}^{n-1} \mathscr{A}_{ij}(\lambda_i, \mu_j) + \sum_{j=1}^{n-1} \mathscr{A}_{nj}(\bar{\lambda}_n, \mu_j) + \sum_{j=1}^{n-1} \mathscr{A}_{jn}(\lambda_j, \bar{\mu}_n) + \mathscr{A}_{nn}(\bar{\lambda}_n, \bar{\mu}_n)$$

where

$$\mathscr{A}_{ij}(\lambda_i, \mu_j) = \delta_{ij}(\lambda_i; \mu_j) - \alpha^{-2}(\lambda_{i,i}; \mu_{j,j}) \qquad i, j = 1, \ldots, n-1 \text{ (no sum)} \tag{1.10}$$

$$\mathscr{A}_{nj}(\bar{\lambda}_n, \mu_j) = (-\alpha^{-1}\bar{\lambda}_{n,n} + \bar{\lambda}_n; \mu_{j,j}) \qquad j = 1, \ldots, n-1$$

$$\mathscr{A}_{jn}(\lambda_j, \bar{\mu}_n) = (\lambda_{j,j}; -\alpha^{-1}\bar{\mu}_{n,n} + \bar{\mu}_n) \qquad j = 1, \ldots, n-1$$

$$\mathscr{A}_{nn}(\bar{\lambda}_n, \bar{\mu}_n) = -(\bar{\lambda}_{n,n}; \bar{\mu}_{n,n}) + \alpha[(\bar{\lambda}_{n,n}; \bar{\mu}_n) + (\bar{\lambda}_n; \bar{\mu}_{n,n})]$$

Moreover, let us introduce another bilinear form

$$\tilde{B}(\lambda_1, \ldots, \bar{\lambda}_n; \mu_1, \ldots, \bar{\mu}_n) = \bar{B}(\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, -\bar{\mu}_n) = \tag{1.11}$$

$$= \sum_{i,j}^{n-1} \mathscr{A}_{ij}(\lambda_i, \mu_j) + \sum_{j=1}^{n-1} \mathscr{A}_{nj}(\bar{\lambda}_n, \mu_j) - \sum_{j=1}^{n-1} \mathscr{A}_{jn}(\lambda_j, \bar{\mu}_n) - \mathscr{A}_{nn}(\bar{\lambda}_n, \bar{\mu}_n) .$$

It is readily seen that the problem to find $\bar{\lambda}^0 = (\lambda_1^0, \ldots, \bar{\lambda}_n^0) \in \mathscr{H}$ such that

$$\tilde{B}(\lambda_1^0, \ldots, \bar{\lambda}_n^0; \mu_1, \ldots, \bar{\mu}_n) = \alpha^{-2}\Big(f; \sum_{j=1}^{n-1} \mu_{j,j} - \alpha\bar{\mu}_{n,n} + \alpha^2\bar{\mu}_n\Big) \ \forall \bar{\mu} \in \mathscr{H} \tag{1.12}$$

is equivalent to (1.3).

In order to define Galerkin approximations, we introduce two families of finite-dimensional subspaces V_h, V_{h_n}, $0 < h \leqq 1$, $0 < h_n \leqq 1$, which satisfy the following assumptions:

(i) (Conformity)

$$V_h \subset W^{1,2}(\Omega), \quad V_{h_n} \subset W_0^{1,2}(\Omega),$$

(ii) (Approximability) $\exists \varkappa \geqq 2$, $\forall v \in W^{\varkappa,2}(\Omega)$ $\exists \chi \in V_h$:

$$\|v - \chi\|_j \leqq ch^{\varkappa - j}\|v\|_\varkappa ;$$

$$\exists \varkappa_n \geqq 2, \ \forall w \in W^{\varkappa_n,2}(\Omega) \cap W_0^{1,2}(\Omega) \ \exists \psi \in V_{h_n}$$

$$\|w - \psi\|_j \leqq ch^{\varkappa_n - j}\|v\|_{\varkappa_n} \quad (j = 0, 1) .$$

(iii) (Inverse inequality) A constant C exists, independent of χ and h, such that for sufficiently small h and any $\chi \in V_h$

$$\|\chi\|_1 \leqq ch^{-1}\|\chi\| .$$

Denote

$$V(h, h_n) = (V_h)^{n-1} \times V_{h_n} \subset (W^{1,2}(\Omega))^n .$$

We say that an element $\bar{\lambda}^h \in V(h, h_n)$ is a *Galerkin approximation* to the solution $\bar{\lambda}^0 = (\lambda_1^0, \ldots, \bar{\lambda}_n^0)$ of the problem (1.12) if

$$(1.13) \qquad \tilde{B}(\bar{\lambda}^h, \bar{\mu}) = \alpha^{-2}(f; \sum_{j=1}^{n-1} \mu_{j,j} - \alpha\bar{\mu}_{n,n} + \alpha^2\bar{\mu}_n) \quad \forall \bar{\mu} \in V(h, h_n)$$

In [1] the following rate of convergence is proved for the Galerkin approximations:

Theorem 2. *Let the solution u of the problem (1.6) belong to $W^{q,2}(\Omega) \cap W_0^{1,2}(\Omega)$, where $q \geqq \max(\varkappa + 1, \varkappa_n)$ and to $W^{2,2}(\Omega)$ for any $f \in L_2(\Omega)$. Then for sufficiently small h the Galerkin approximations are defined uniquely by (1.13) and it holds*

$$(1.14) \qquad \sum_{j=1}^{n-1} \left\| \frac{\partial u}{\partial x_j} - \lambda_j^h \right\| + \left\| \frac{\partial u}{\partial x_n} - \frac{\partial \bar{\lambda}^h}{\partial x_n} \right\| + \|u - \bar{\lambda}_n^h\| \leqq$$
$$\leqq c[h^{\varkappa-1} + h^{\varepsilon} + h^{-1}h_n^{\varkappa_n} + h_n^{\varkappa_n - 1}] \|u\|_q .$$

2. SPACES OF CURVED FINITE ELEMENTS

In the error estimate (1.14) the essential role is played by the assumptions (i), (ii), (iii), which are imposed on the spaces V_h, V_{h_n}. In [1] we have dealt with the most simple application of Theorem 2, namely with the case that a plane domain Ω is a convex polygon and V_h, V_{h_n} are spaces of triangular elements.

It is the aim of the present paper to study the case $\Omega \subset E_n$, $\Gamma \in C^{k+1}$, where $k \geqq 1$ is an integer. Here we shall consider also curved elements along the boundary. The section will be divided into two parts. In part **A** we sketch the technique developed in [3] for the case that the curved side of the boundary elements (in E_2) coincides exactly with the corresponding arc of the boundary Γ, while in part **B** we shall employ the elements, the curved side of which approximate Γ only (see [4]).

As the proof of (iii) goes through analogously in the both parts **A** and **B**, first we prove an auxiliary lemma. To this end we introduce a new equivalent norm in $W^{k,2}(\Omega)$. Let $u \in W^{k,2}(\Omega)$. By means of $D^j u(\mathbf{x}) \in \mathscr{L}((E_n)^j, E_1)$, $(j \leqq k)$ we denote j-th derivative of u at the point $\mathbf{x}$.

Define

$$[D^j u(\mathbf{x})] = \sup_{\zeta_i \neq 0} \frac{|D^j u(\mathbf{x})(\zeta_1, \ldots, \zeta_j)|}{\|\zeta_1\| \cdot \|\zeta_2\| \cdots \|\zeta_j\|}, \quad \zeta_i \in E_n$$

and set

(2.1) $$[[u]]_{k,\Omega} = \left(\sum_{j=0}^{k} [u]_{j,\Omega}^2\right)^{1/2},$$

where

$$[u]_{j,\Omega} = \left(\int_\Omega [D^j u(\mathbf{x})]^2 \, d\mathbf{x}\right)^{1/2}.$$

It is readily verified that such constants $c_1, c_2 > 0$ exist, independent of u, that

(2.2) $$c_1 |u|_{j,\Omega} \leqq [u]_{j,\Omega} \leqq c_2 |u|_{j,\Omega} \quad (0 \leqq j \leqq k)$$

Let $\hat{K} \subset E_n$ be a fixed domain with Lipschitz boundary, F a C^1-diffeomorphism between $\hat{K}$ and $K = F(\hat{K})$ and $\hat{P} \subset W^{1,2}(\hat{K})$ a finite-dimensional space of functions defined on $\hat{K}$. If we denote by P_j the space of polynomials of the degree at most j, then it holds usually $P_m \subset \hat{P} \subset P_n$ ($m \leqq n$ integers) in practice.

Let us set

(2.3) $$P = \{p;\ \exists \hat{p} \in \hat{P},\ p(\mathbf{x}) = \hat{p}(F^{-1}(\mathbf{x})), \text{ where } \mathbf{x} = F(\hat{\mathbf{x}}),\ \hat{\mathbf{x}} \in \hat{K}\}.$$

Then $\dim P = \dim \hat{P}$ and $P \subset W^{1,2}(K)$ (cf. [5], chpt. 2, § 3).

Lemma 3. *For $p \in P$ it holds*

(2.4) $$\|p\|_{1,K} \leqq c \left\{1 + \sup_{\mathbf{x}\in K} [DF^{-1}(\mathbf{x})]^2 \frac{\sup_{\hat{\mathbf{x}}\in\hat{K}} |J_F(\hat{\mathbf{x}})|}{\inf_{\hat{\mathbf{x}}\in\hat{K}} |J_F(\hat{\mathbf{x}})|}\right\}^{1/2} \|p\|_K$$

where c does not depend on p and K.

Proof. $p \in P \Rightarrow p = \hat{p}(F^{-1}(\mathbf{x})),\ \hat{p} \in \hat{P};\ Dp(\mathbf{x})\,\zeta = D\hat{p}(F^{-1}(\mathbf{x}))\, DF^{-1}(\mathbf{x})\,\zeta\ \ \zeta \in E_n$; $[Dp(\mathbf{x})] \leqq [D\hat{p}(\hat{\mathbf{x}})]\,[DF^{-1}(\mathbf{x})]$, $(\hat{\mathbf{x}} = F^{-1}(\mathbf{x}))$

(2.5) $$[p]_{1,K} = \left(\int_K [Dp(\mathbf{x})]^2 \, d\mathbf{x}\right)^{1/2} \leqq \sup_{\mathbf{x}\in K} [DF^{-1}(\mathbf{x})] \sup_{\hat{\mathbf{x}}\in\hat{K}} |J_F(\hat{\mathbf{x}})|^{1/2} [\hat{p}]_{1,\hat{K}}$$

where $J_F(\mathbf{x})$ is the value of the Jacobian of the mapping F at $\mathbf{x}$. As $\hat{P}$ is a finite-dimensional space and D a linear operator, we obtain

(2.6) $$[\hat{p}]_{1,\hat{K}} \leqq \hat{c}[\hat{p}]_{0,\hat{K}} = \hat{c}\left(\int_{\hat{K}} |\hat{p}(\hat{\mathbf{x}})|^2 \, d\hat{\mathbf{x}}\right)^{1/2} \leqq \hat{c} \frac{[p]_{0,K}}{\inf_{\hat{\mathbf{x}}\in\hat{K}} |J_F(\hat{\mathbf{x}})|^{1/2}}.$$

The assertion of the lemma follows from (2.2), (2.5) and (2.6).

A. Let us restrict ourselves to the domains $\Omega \subset E_2$ with the boundary $\Gamma \in C^{k+1}$ ($k \geqq 1$, k integer). Let $Q_1, Q_2 \in \Gamma$, $Q_3 \in \Omega$. By the curved element K we call a closed set bounded with the straight-line segments Q_1Q_3, Q_2Q_3 and the boundary arc $\widehat{Q_1Q_2}$. Let h be the maximal side $(0 < h \leqq 1)$ and ϑ the minimal angle of the triangle $Q_1Q_2Q_3$. Assume that such $\vartheta_0 > 0$ exists independent of h, such that $\vartheta \geqq \vartheta_0$ if $h \to 0$.

Let $\hat{K}$ be the triangle with the following vertices: $\hat{P}_1 = [0, 0]$, $\hat{P}_2 = [1, 0]$, $\hat{P}_3 = [0, 1]$. If we know the parametric representation of $\widehat{Q_1Q_2}$, we may construct the mapping $F(\xi, \eta) = [x_1(\xi, \eta); x_2(\xi, \eta)]$ which is a C^k-diffeomorphism $\hat{K}$ onto K for h sufficiently small (cf. [3]).

Moreover, for the Jacobian J_F of the mapping F and the inverse mapping F^{-1} we have the following estimates

$$c_1 h^2 \leqq |J_F(\xi, \eta)| \leqq c_2 h^2 \quad \forall [\xi, \eta] \in \hat{K}\,, \tag{2.7}$$

where c_1, c_2 are positive constants independent of K and h;

$$[DF^{-1}(\mathbf{x})] \leqq ch^{-1} \quad \forall \mathbf{x} = [x_1, x_2] \in K\,. \tag{2.8}$$

According to [3], $\hat{P} = P_{2m-1}$, $(m = 1, 2, \ldots)$, the polynomials being determined uniquely by the following conditions

$$\begin{aligned} &D^i \hat{p}(\hat{P}_j) \quad |i| \leqq m - 1,\ j = 1, 2, 3 \\ &D^i \hat{p}(\hat{P}_0) \quad |i| \leqq m - 2,\ \hat{P}_0 \text{ is the centre of gravity} \end{aligned} \tag{2.9}$$

of $\hat{K}$ and for $m = 1$ the conditions at $\hat{P}_0$ are not included. The derivatives in (2.9) are taken in the sense of § 1.

We can easily derive a local variant of the inverse inequalities.

Lemma 4. *For $\forall p \in P$ (defined by means of (2.3)) and sufficiently small h it holds*

$$\|p\|_{1,K} \leqq ch^{-1} \|p\|_K \tag{2.10}$$

where c is independent of h, p and the element K.

Proof. Follows immediately from (2.4), (2.7) and (2.8).

Next let us construct the "triangulation" $\mathcal{T}_h$ of Ω, i.e. let us represent $\bar{\Omega}$ as the sum of a finite number both of curved elements along the boundary Γ and of internal triangular elements, satisfying the usual requirements imposed on their mutual position. Each "triangulation" $\mathcal{T}_h$ will be characterized by the two following parameters: h is the maximal side and ϑ the minimal angle of all "triangles" $K_i \in \mathcal{T}_h$. Assume that $\mathcal{T}_h$ is regular, i.e. a constant $\vartheta_0 > 0$ exists, independent of h such that $\vartheta \geqq \vartheta_0$ if $h \to 0$. Note that each straight element-triangle can be analyzed by means of the isoparametric technique as an affine image of $\hat{K}$. The local inverse inequality (2.10) holds for straight elements (see e.g. [6]) as well as the approximability (see e.g. [3], [4]).

Define

$$\text{(2.11)} \qquad V_h = \{v \in C(\bar{\Omega}) \cap W^{1,2}(\Omega),\ v|_{K_i} \in P,\ \forall K_i \in \mathcal{T}_h\}$$

$$V_{h_n} = \{v \in V_h;\ v|_\Gamma = 0\}.$$

Remark 2.1. We can consider more general cases, when the basic space $\hat{P}$ used for V_h is different from that used for V_{h_2}.

The properties of approximability of spaces V_h, V_{h_2} are given in the following

Theorem 3. (cf. [3]) *Let $\mathcal{T}_h$ be a regular triangulation of Ω, $\Gamma \in C^{k+1}$, $u \in W^{k,2}(\Omega)$ $k \geqq m + 1$, $m \geqq 1$. Then there exists a function $\chi \in V_h$, such that:*

$$\text{(2.12)} \qquad \|u - \chi\|_{j,\Omega} \leqq ch^{\varkappa - j}\|u\|_{\varkappa,\Omega}\,, \quad j = 0, 1\,; \quad \varkappa = \min(2m, k)\,,$$

where c does not depend on u, h.

An analogous theorem is true for $u \in W^{k,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and its approximation in the space V_{h_2}.

Theorem 4. *It holds*

$$\text{(2.13)} \qquad \|v\|_{1,\Omega} \leqq ch^{-1}\|v\|_\Omega \quad \forall v \in V_h \ (v \in V_{h_2})$$

with a constant c independent of v, h.

Proof follows immediately from (2.10) and the local inverse inequalities for straight elements.

By virtue of the choice of V_h, V_{h_2}, Theorems 3 and 4, the conditions (i), (ii), (iii) are satisfied and Theorem 2 can be applied to the error estimates of Galerkin approximations.

B. Let us consider curved elements, which do not describe the boundary exactly but only approximately. A detailed analysis of such elements can be found in [4]. Here we present only their construction briefly and restrict ourselves to the case of Lagrange interpolation.

Let $\hat{\Sigma} = \{\hat{a}_i\}_{i=1}^N$ be N different points of E_n, $\hat{K} = \operatorname{conv}(\hat{\Sigma})$ the closed convex hull and $\hat{P}$ a finite-dimensional space of functions defined on $\hat{K}$. Let $\Sigma = \{a_i\}_{i=1}^N$ be another array of N points of E_n, and suppose that there exists a mapping F from E_n into E_n such that

$$\text{(2.14)} \qquad \left| \begin{array}{l} F \text{ is a simple mapping of } \hat{K} \text{ onto } F(\hat{K})\,, \\ F = (F_1, F_2, \ldots, F_n),\ F_i \in \hat{P}\ (i \leqq n),\ F(\hat{a}_i) = a_i,\ i = 1, \ldots, N\,. \end{array} \right.$$

The set $K = F(\hat{K})$ together with P defined by (2.3) will be called a *curved element.* While the mapping F of part **A**, describing arcs of Γ, could be relatively complicated, in the present case its components are from $\hat{P}$, consequently polynomials in practice.

As the properties of approximability are concerned, there hold results similar to those for "straight" elements, provided that the curved elements do not differ "too much" from the straight ones. Let us show this on the example of simplicial elements.

Let $\hat{\Sigma} = \{\hat{a}_i\}_{i=1}^N$ be N different points of E_n, $N \geqq n + 1$ where

$$\hat{a}_i = (0, \ldots, 1, 0, \ldots, 0) \quad 1 \leqq i \leqq n\,,$$
$$\hat{a}_{n+1} = (0, \ldots, 0)$$

and $\hat{K} = \operatorname{conv}(\hat{\Sigma})$ is a non-degenerate n-simplex in E_n with the vertices $\hat{a}_1, \ldots, \hat{a}_{n+1}$. Let (Σ_h), $(0 < h \leqq 1)$ be a family of sets, each of them consisting of N different points, i.e. $\Sigma_h = \{a_i\}_{i=1}^N$, $N \geqq n + 1$ (the subscripts h will be omitted for simplicity) and such that $a_1, \ldots, a_{n+1}$ are vertices of a n-simplex $\tilde{K}_h = \tilde{F}_h(\hat{K})$, where $\tilde{F}_h$ is an affine regular mapping. Suppose that for each $h \in (0, 1\rangle$ there exists a mapping F_h of the form (2.14) and let $K_h = F_h(\hat{K})$ be the family of curved elements. Denote h the diameter of $\tilde{K}_h$, ϱ_h the diameter of a hypersphere inscribed in $\tilde{K}_h$ and assume that there exists $\alpha_0 > 0$, independent of h, such that

(2.15)
$$\frac{\varrho_h}{h} \geqq \alpha_0\,.$$

Let $\hat{\Pi}_{\hat{K}}\hat{u} \in \hat{P}$ be the Lagrange interpolate of the function $\hat{u}$ on $\hat{\Sigma}$ and define $\Pi_K u(\mathbf{x}) = \hat{\Pi}_{\hat{K}}\hat{u}(F^{-1}(\mathbf{x})) \in P$, where $u = \hat{u}(F^{-1}(\mathbf{x}))$. Then the following result is true:

Theorem 5. *Let the following conditions be satisfied*:

α) $P_k \subset \hat{P} \subset C^{k+1}(\hat{K})$, $k \geqq 1$, k integer

β) *the family* (K_h) *is regular in the sense of* (2.15)

γ) $\forall i : n + 2 \leqq i \leqq N$, $\|a_i - \tilde{a}_i\| = 0(h^2)$, *where* $\tilde{a}_i = \tilde{F}_h(a_i)$

δ) *for every integer* j, $2 \leqq j \leqq k + 1$ *there exists a constant* $\hat{c}_j$, *independent of* h *and such that*

(2.16)
$$\sup_{\hat{\mathbf{x}} \in \hat{K}} [D^j F_h(\hat{\mathbf{x}})] \leqq \hat{c}_j h^j\,.$$

Then for every integer m, $0 \leqq m \leqq k + 1$, *there exists a constant* $\hat{c}$, *independent of* h, u *and such that*

(2.17)
$$|u - \Pi_K u|_{m,K_h} \leqq \hat{c} h^{k+1-m} \|u\|_{k+1,K_h}$$

holds for all $u \in W^{k+1,2}(K_h)$, $k + 1 > \frac{1}{2}n$.

For the proof we refer to [4].

If $J_h(\hat{\mathbf{x}})$ $(\tilde{J}_h(\hat{\mathbf{x}}))$ is the Jacobian of the mapping $F_h(\hat{\mathbf{x}})$ $(\tilde{F}_h(\hat{\mathbf{x}}))$, and if $\alpha) - \delta)$ are satisfied, then we have (cf. [4])

(2.18)
$$c_1|\tilde{J}_h(\hat{\mathbf{x}})| \leqq |J_h(\hat{\mathbf{x}})| \leqq c_2|\tilde{J}_h(\hat{\mathbf{x}})| \quad \forall \hat{\mathbf{x}} \in \hat{K}$$

$$(2.19) \qquad \sup_{x \in K} [D F_h^{-1}(\mathbf{x})] \leqq c_3 \sup_{x \in K} [D F_h^{-1}(\mathbf{x})] .$$

As $\tilde{F}_h(\hat{\mathbf{x}}) = \tilde{B}_h \hat{\mathbf{x}} + \mathbf{b}$, where $\tilde{B}_h$ is a regular matrix $n \times n$, $\tilde{J}_h(\hat{\mathbf{x}}) = \text{const.}$ and it follows that (cf. [4])

$$(2.19') \qquad \sup_{x \in K} [DF_h^{-1}(\mathbf{x})] = \|\tilde{B}_h^{-1}\| \leqq ch^{-1}$$

where c does not depend on h.

Using Lemma 3, (2.18) and (2.19) we obtain the local inverse inequality of the form (2.10) on K_h for the function of P.

The "triangulation" $\mathscr{T}_h$ of Ω will consists from the curved elements along the boundary and straight elements. The curved elements have only one curved side, namely that, which approximates Γ. In the interior of Ω we employ the straight elements, i.e. affine images of $\hat{K}$. Suppose that $\mathscr{T}_h$ is regular, i.e., such constant $\alpha_0 > 0$ exists that

$$\frac{\varrho_i}{h_i} \geqq \alpha_0 \ \forall K_i \in \mathscr{T}_h .$$

Let $\Omega_h = \bigcup_{K_i \in \mathscr{T}_h} K_i$ with the boundary Γ_h. In general $\Omega_h \neq \Omega$. Define the finite-dimensional spaces $V_h(\Omega_h)$, $V_{h_n}(\Omega_h)$:

$$V_h(\Omega_h) = \{v \in C(\bar{\Omega}_h) \cap W^{1,2}(\Omega_h),\ v|_{K_i} \in P,\ \forall K_i \in \mathscr{T}_h\},$$

$$(2.20) \qquad V_{h_n}(\Omega_h) = \{v \in V_h(\Omega_h),\ v|_{\Gamma_h} = 0\}, \quad V(h, h_n) = [V_h(\Omega_h)]^{n-1} \times V_{h_n}(\Omega_h) .$$

If the curved elements satisfy the conditions of Theorem 5, then (2.17) and the well-known properties of straight elements yield the approximability (ii) by spaces V_h, V_{h_n}. Likewise the inverse inequality for $V_h(\Omega_h)$ follows easily.

It is necessary to define newly the Galerkin approximations and the sense of their convergence. Let $\Omega^0 \supset \bar{\Omega}$ be a bounded domain. Then $\Omega^0 \supset \Omega_h$ for sufficiently small h. Let the right-hand side f be defined on Ω^0 and suppose that each component of the solution $\bar{\lambda}^0$ can be extended from $W^{1,2}(\Omega)$ into $W^{1,2}(\Omega^0)$. We say that $\bar{\lambda}^h \in V(h, h_n)$ is a Galerkin approximation of $\bar{\lambda}^0$, if

$$(2.21) \qquad \tilde{B}_{\Omega_h}(\bar{\lambda}^h; \mu) = \alpha^{-2}(f; \sum_{j=1}^{n-1} \mu_{j,j} - \alpha\mu_{n,n} + \alpha^2 \mu_n)_{\Omega_h} \overset{\text{def}}{\equiv} [f; \mu]_{\Omega_h} \ \forall \mu \in V(h, h_n) .$$

Here $\tilde{B}_{\Omega_h}$ denotes the bilinear form (1.11), where the corresponding scalar products are integrated on Ω_h.

We define for $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathscr{H}(\Omega_h) = [W^{1,2}(\Omega_h)]^n$ the following

$$\|\lambda\|_{h,h_n} = \sum_{i=1}^{n-1} \|\lambda_i\|_{\Omega_h} + \|\lambda_{n,n}\|_{\Omega_h} ,$$

$$\|\lambda\|_{h,\Omega_h} = \sum_{i=1}^{n-1} \|\lambda_i\|_{\Omega_h} + h^{-1}\|\lambda_n\|_{\Omega_h} + \sum_{j=1}^{n-1} \|\lambda_{j,j}\|_{\Omega_h} + \|\lambda_{n,n}\|_{\Omega_h} ,$$

$$\|\lambda\|_{*,\Omega_h} = \sum_{i=1}^{n-1} \|\lambda_i\|_{\Omega_h} + h \sum_{i=1}^{n=1} \|\lambda_{i,i}\|_{\Omega_h} + \|\lambda_{n,n}\|_{\Omega_h} .$$

Lemma 5. *For any $\lambda \in \mathscr{H}(\Omega_h)$ and $\mu \in V(h, h_n)$ it holds*

$$|\tilde{B}_{\Omega_h}(\lambda; \mu)| \leqq c\|\lambda\|_{h,\Omega_h} \cdot \|\mu\|_{*,\Omega_h} , \tag{2.22}$$

where c is independent of h and Ω_h.

Proof is the same as in [1].

The error estimate will be derived on the basis of

Theorem 6. *Let*

$$\|v\|_{1,\Omega_h} \leqq ch^{-1}\|v\|_{\Omega_h} \ \forall v \in V_h(\Omega_h) , \tag{2.23}$$

where c does not depend on h and Ω_h.

Then

$$\|\bar{\lambda}^0 - \bar{\lambda}^h\|_{h,h_n} \leqq c\left[\inf_{\mu \in V(h,h_n)} \|\bar{\lambda}^0 - \mu\|_{h,\Omega_h} + \sup_{\substack{\omega \neq 0 \\ \omega \in V(h,h_n)}} \frac{|\tilde{B}_{\Omega_h}(\bar{\lambda}^0 - \bar{\lambda}^h; \omega)|}{\|\omega\|_{h,h_n}}\right] . \tag{2.24}$$

Proof. Let us consider an arbitrary $\tilde{\lambda} \in V(h, h_n)$. Using (1.7), (1.8), (1.10), (1.11) and (1.23) we obtain

$$\tilde{B}_{\Omega_h}(\tilde{\lambda}; \tilde{\lambda}) \geqq c \sum_{i=1}^{n-1} \|\tilde{\lambda}_i\|^2_{\Omega_h} - (n-1)\, \alpha_0^{-2} h^{2+2\varepsilon} \sum_{i=1}^{n-1} \|\tilde{\lambda}_{i,i}\|^2_{\Omega_h} + \|\tilde{\lambda}_{n,n}\|^2_{\Omega_h} \geqq$$

$$\geqq (1 - ch^{2\varepsilon}) \sum_{i=1}^{n-1} \|\tilde{\lambda}_i\|^2_{\Omega_h} + \|\tilde{\lambda}_{n,n}\|^2_{\Omega_h} \geqq c\|\tilde{\lambda}\|^2_{h,h_n}$$

for sufficiently small h.

Let $\bar{\lambda}^h \in V(h, h_n)$ be the Galerkin approximation of $\bar{\lambda}^0$ and $\tilde{\mu} \in V(h, h_n)$ arbitrary. Then Lemma 5, (2.23) and inequalities

$$\|\lambda\|_{h,h_n} \leqq \|\lambda\|_{h,\Omega_h} \quad \forall \lambda \in \mathscr{H}(\Omega_h) ,$$

$$\|\mu\|_{*,\Omega_h} \leqq c\|\mu\|_{h,h_n} \quad \forall \mu \in V(h, h_n)$$

result in

$$c\|\bar{\lambda}^h - \tilde{\mu}\|^2_{h,h_n} \leqq \tilde{B}_{\Omega_h}(\bar{\lambda}^h - \tilde{\mu}; \bar{\lambda}^h - \tilde{\mu}) = \tag{2.25}$$

$$= \tilde{B}_{\Omega_h}(\bar{\lambda}^h - \bar{\lambda}^0; \bar{\lambda}^h - \tilde{\mu}) + \tilde{B}_{\Omega_h}(\bar{\lambda}^0 - \tilde{\mu}; \bar{\lambda}^h - \tilde{\mu}) .$$

Denoting $\omega = \bar{\lambda}^h - \tilde{\mu} \in V(h, h_n)$, we may write

$$c\|\bar{\lambda}^h - \tilde{\mu}\|_{h,h_n} \leqq \|\bar{\lambda}^0 - \tilde{\mu}\|_{h,\Omega_h} + \sup_{\substack{\omega \neq 0 \\ \omega \in V(h,h_n)}} \frac{|\tilde{B}_{\Omega_h}(\bar{\lambda}^h - \bar{\lambda}^0; \omega)|}{\|\omega\|_{h,h_n}}.$$

Finally the assertion (2.24) follows easily, using also the "triangle" inequality.

Remark 2.1. The occurence of the second term in the right hand side of (2.24) corresponds with the fact that the approximate solution is sought on the region $\Omega_h \neq \Omega$. While in part **A** we could choose the space V_{h_n} of trial functions, approximating the last component, different from V_h, by means of which the approximations of the first $(n - 1)$ components is defined, here the situation is different. In the present case the "triangulation" $\mathcal{T}_h$ and the space $\hat{P}$ are the same both for $V_h(\Omega_h)$ and $V_{h_n}(\Omega_h)$. A modification, however, is possible as will be shown in the next section in Remark 3.1. First we define the space $V_{h_n}(\Omega_{h_n})$ by choosing $\hat{P}$ and $\mathcal{T}_{h_n}$, then carry out the "triangulation" $\mathcal{T}''_{h_n}$ of Ω_{h_n} in a suitable fashion in the sense of part **A** and set

$$V_h(\Omega_{h_n}) = \{v \in C(\bar{\Omega}_{h_n}),\ v|_{K_i} \in P,\ \forall K_i \in \mathcal{T}''_{h_n}\}.$$

Here $\hat{P}$ is the same both for $V_{h_n}(\Omega_{h_n})$ and $V_h(\Omega_{h_n})$.

3. APPLICATION OF QUADRATIC INTERPOLATION

To illustrate the general theory of Lagrange interpolation of part **B** let us consider the following model problem

$$\text{(3.1)} \qquad \begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega \subset E_2\,, \\ u &= 0 \quad \text{on} \quad \Gamma \in C^\infty\,, \end{aligned}$$

where $f \in W^{m,2}(\Omega)$, $m \geqq 1$.

It is well-known (cf. [5]) that precisely one solution $u \in W^{m+2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ exists such that

$$\int_\Omega \operatorname{grad} u \,.\, \operatorname{grad} v \, dx\, dy = \int_\Omega f v \, dx\, dy \quad \forall v \in W_0^{1,2}(\Omega)$$

and

$$\|u\|_{q+2,\Omega} \leqq c\|f\|_{q,\Omega}\,, \quad m \geqq q \geqq 0\,.$$

The solution u (the right-hand side f) can be extended from $W^{m+2,2}(\Omega)$ into $W^{m+2,2}(E_2)$ (from $W^{m,2}(\Omega)$ into $W^{m,2}(E_2)$) in such a way, that (cf. [5], chpt. 2, § 3)

$$\text{(3.2)} \qquad \|\tilde{u}\|_{m+2,E_2} \leqq c\|u\|_{m+2,\Omega}\,; \quad \|\tilde{f}\|_{m,E_2} \leqq c\|f\|_{m,\Omega}$$

where $\tilde{\mu}$ and $\tilde{f}$ denotes the extension of u and f on E_2, respectively. Then also the extensions of the components λ_1^0, $\bar{\lambda}_2^0$ of the solution $\bar{\lambda}^0$ onto E_2 are defined through

$$\lambda_1^0 = \frac{\partial \tilde{\mu}}{\partial x_1}, \quad \bar{\lambda}_2^0 = \frac{h^{1+\varepsilon}}{\alpha_0} \frac{\partial \tilde{u}}{\partial x_2} + \tilde{u} . \tag{3.3}$$

Let $h > 0$. Choosing N points $Q_1, Q_2, \ldots, Q_N$ on Γ such that $\operatorname{dist}(Q_j, Q_{j+1}) \leqq h$ ($j = 1, \ldots, N$; $Q_{N+1} = Q_1$), we may construct a polygon Ω_h' determined by the vertices $Q_1, \ldots, Q_N$. Let $\mathscr{T}_h'$ be a regular triangulation of Ω_h' with the following properties:

(a) $\operatorname{diam}(K_i) \leqq h \ \forall K_i \in \mathscr{T}_h'$,

(b) every triangle has at most 2 vertices on Γ,

(c) the whole segment Q_jQ_{j+1}, ($j = 1, \ldots, N$) represents a side of a K_i.

The elements, having precisely two points on Γ, will be called *boundary elements*, the other *interior* ones. We shall modify the side Q_jQ_{j+1} of every boundary element, as follows: in the centre of Q_jQ_{j+1} we construct a perpendicular line and denote its intersection with Γ by $Q_{j+1/2}$. The boundary arc of Γ will be approximated by the parabola determined by the triple Q_j, $Q_{j+1/2}$, Q_{j+1}. The domain with this modified piecewise parabolic boundary will be denoted by Ω_h and the corresponding system of elements by $\mathscr{T}_h$. The construction can be described in terms of isoparametric technique. Let $\hat{K}$ be the basic triangle with the vertices $\hat{a}_1 = [0, 0]$, $\hat{a}_2 = [1, 0]$, $\hat{a}_3 = [0, 1]$ and the centres of sides $\hat{a}_4 = [\frac{1}{2}, 0]$, $\hat{a}_5 = [\frac{1}{2}, \frac{1}{2}]$, $\hat{a}_6 = [0, \frac{1}{2}]$. Every interior element is an affine image of $\hat{K}$. Every boundary element $K \in \mathscr{T}_h$ is an image of $\hat{K}$ for a mapping $F : E_2 \to E_2$, both components of which are polynomials of the degree at most two, and which is determined uniquely by the conditions $F(\hat{a}_i) = a_i$ ($i = 1, \ldots, 6$) (see the figure 1), where $a_5 = \frac{1}{2}(a_2 + a_3)$, $a_6 = \frac{1}{2}(a_1 + a_3)$ and a_4 is found like $Q_{j+1/2}$.

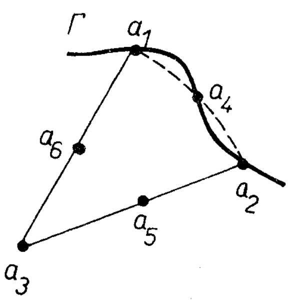

Fig. 1.

It is proved in [4], that in this case the mapping F is a C^1-diffeomorphism between $\hat{K}$ and K. Let the function $\hat{v}$ be defined on $\hat{K}$. By $\hat{\Pi}_{\hat{K}}\hat{v}$ we denote the quadratic Lagrange interpolate of $\hat{v}$ on $\hat{K}$, i.e.

(i) $\hat{\Pi}_{\hat{K}}\hat{v} \in \hat{P} = P_2$,

(ii) $\hat{\Pi}_{\hat{K}}\hat{v}(\hat{a}_i) = \hat{v}(\hat{a}_i)$, $i = 1, \ldots, 6$.

Let $K \in \mathscr{T}_h$, $K = F(\hat{K})$ and set $v(\boldsymbol{x}) = \hat{v}(F^{-1}(\boldsymbol{x}))$, $\boldsymbol{x} \in K$. The Lagrange interpolate $\Pi_K v$ of the function v on K will be defined through the relation $\Pi_K v(\boldsymbol{x}) = \hat{\Pi}_{\hat{K}}\hat{v}(F^{-1}(\boldsymbol{x}))$. For the error estimate we deduce from Theorem 5

$$(3.4) \qquad |v - \Pi_K v|_{j,K} \leqq ch^{k-j}\|v\|_{k,K}\,, \quad k = 2, 3\,; \quad j \leqq k\,; \quad \forall v \in W^{k,2}(K)\,.$$

Let P be the space defined in (2.3) with $\hat{P} = P_2$,

$$V_h = \{v \in C(\bar{\Omega}_h) \cap W^{1,2}(\Omega_h),\ v|_K \in P,\ \forall K \in \mathscr{T}_h\}\,,$$
$$V_{h_2} = \{v \in V_h,\ v|_{\Gamma_h} = 0\}$$

and define

$$r_h \in \mathscr{L}(W^{k,2}(\Omega_h);\, V_h) \cap \mathscr{L}(W^{k,2}(\Omega_h) \cap W_0^{1,2}(\Omega_h);\, V_{h_2})$$

by the relation

$$r_h v = \Pi_K v \quad \text{on} \quad K \in \mathscr{T}_h\,.$$

From (3.4) it follows

$$(3.5) \qquad \|v - r_h v\|_{j,\Omega_h} \leqq ch^{k-j}\|v\|_{k,\Omega_h}\,, \quad k = 2, 3\,; \quad j \leqq k\,; \quad \forall v \in W^{k,2}(\Omega_h)\,.$$

In the forthcoming proof we shall need two auxiliary lemmas.

Lemma 6. *Let $f \in W_0^{1,2}(\Omega)$ and set*

$$f_\eta = \begin{cases} f & \text{in} \quad \Omega^{2\eta} = \{x \in \Omega,\ \operatorname{dist}(x, \Gamma) < \eta\} \\ 0 & \text{in} \quad \Omega_{2\eta} = \{x \in \Omega,\ \operatorname{dist}(x, \Gamma) > 2\eta\} \end{cases}$$

Then

$$(3.6) \qquad \|f_\eta\|_\Omega \leqq c\eta\|f\|_{1,\Omega}$$

where c does not depend on η.

Proof. A particular case of a lemma of [7].

Next we shall extend also $\mu = (\mu_1, \mu_2) \in V(h, h_2)$ from Ω_h onto $\Omega \setminus \Omega_h$, likewise we extend the functions u and f out of Ω. As $\mu_2 \in V_{h_2} \subset W_0^{1,2}(\Omega_h)$, it suffices to extend μ_2 out of Ω_h by zero. For μ_1 the situation is more difficult. Let K be a boundary curved element, determined by the points $a_1, \ldots, a_6$, and consider an adjoint element K^*, which is determined by the points $a_1^*, \ldots, a_6^*$, where $a_1^* = a_1$, $a_2^* = a_2$, $a_4^* = a_4$ and the remaining points are symmetric to the corresponding points of K with respect to the straight line $a_1 a_2$ (see fig. 2).

If P is the space associated with the element $K = F(\hat{K})$ according to (2.3), let P^* denote the space associated with $K^* = F^*(\hat{K})$ (with the same $\hat{P} = P_2$).

Let $\varphi \in P$, $\varphi^* \in P^*$ be determined uniquely by the conditions $\varphi(a_i) = \alpha_i = \varphi^*(a_i^*)$ $(\alpha_i \in E_1,\ i = 1, \ldots, 6)$ and set

$$\Phi(\mathbf{x}) = \begin{cases} \varphi(\mathbf{x}) & \text{on } K, \\ \varphi^*(\mathbf{x}) & \text{on } K^*. \end{cases}$$

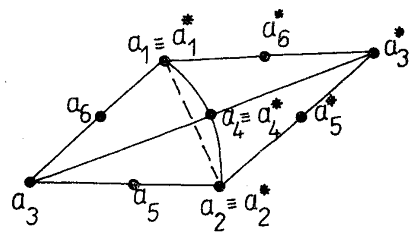

Fig. 2.

Then Φ defines an extension of φ from K on K^*. It is readily seen that $\Phi \in W^{1,2}(K \cup K^*)$. To this end it suffices to show that $\varphi|_{\widehat{a_1 a_2}} = \varphi^*|_{\widehat{a_1 a_2}}$. As both F and F^* is a C^1-diffeomorphism, it holds

$$(3.7) \qquad \varphi(\mathbf{x})|_{\widehat{a_1 a_2}} = \hat{\varphi}(\hat{\mathbf{x}}), \quad \varphi^*(\mathbf{x})|_{\widehat{a_1 a_2}} = \hat{\varphi}^*(\hat{\mathbf{x}}^*), \quad \hat{\mathbf{x}}, \hat{\mathbf{x}}^* \in \hat{a}_1 \hat{a}_2$$
$$\hat{\mathbf{x}} = F^{-1}(\mathbf{x}), \quad \hat{\mathbf{x}}^* = F^{*-1}(\mathbf{x}).$$

The assertion follows easily from (3.7) and from the fact, that both components of F, F^*, φ and φ^* are quadratic polynomials and $a_1^* = a_1$, $a_2^* = a_2$, $a_4^* = a_4$, therefore $\hat{\mathbf{x}} = \hat{\mathbf{x}}^*$ and $\hat{\varphi} = \hat{\varphi}^*$ on $\hat{a}_1 \hat{a}_2$.

Extension of μ_1 on $\Omega \setminus \Omega_h$ will consist of the extensions from every boundary element $K_i = F_i(\hat{K})$ onto its adjoint $K_i^* = F_i^*(\hat{K})$ $(i = 1, \ldots, N(h))$, described above. For sufficiently small h we have obviously

$$\Omega \subset \bigcup_{i=1}^{N(h)} K_i^* \cup \Omega_h .$$

The extension of μ_1 onto Ω will be denoted by $\tilde{\mu}_1$, the space of all extension by $\tilde{V}(h)$ and set

$$\tilde{V}(h, h_2) = \tilde{V}(h) \times \tilde{V}(h_2)$$

(recall that the functions of $\tilde{V}(h_2)$ are extended by zero out of Ω_h). We may write

$$\Omega \setminus \Omega_h = \bigcup_{i=1}^{N(h)} G_i ,$$

where $G_i \subset K_i^*$.

As $\operatorname{dist}(\Gamma, \Gamma_h) \leqq ch^3$, $\operatorname{dist}(Q_j, Q_{j+1}) \leqq h$ $(j = 1, \ldots, N;$ Γ_h is the boundary of $\Omega_h)$ and the conditions (2.19), (2.19′) hold, we obtain

$$(3.8) \qquad \|\tilde{\mu}_1\|^2_{\Omega\setminus\Omega_h} = \sum_{i=1}^{N(h)} \int_{G_i} |\tilde{\mu}_1(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leqq \sum_{i=1}^{N(h)} \max_{\mathbf{x}\in K_i^*} |\tilde{\mu}_1(\mathbf{x})|^2 \operatorname{mes}(G_i) \leqq$$
$$\leqq ch^4 \sum_{i=1}^{N(h)} \max_{\hat{\mathbf{x}}\in\hat{K}} |\hat{\mu}_1^{(i)}(\hat{\mathbf{x}})|^2 \leqq ch^4 \sum_{i=1}^{N(h)} \|\hat{\mu}_1^{(i)}\|^2_{\hat{K}} =$$
$$= ch^4 \sum_{i=1}^{N(h)} \int_{\hat{K}} |\mu_1^{(i)}(F_i(\hat{\mathbf{x}}))|^2 \,\mathrm{d}\hat{\mathbf{x}} \leqq ch^4 \sum_{i=1}^{N(h)} \sup_{\mathbf{x}\in K_i} |J_{F_i^{-1}}(\mathbf{x})| \int_{K_i} |\mu_1^{(i)}(\mathbf{x})|^2 \,\mathrm{d}\mathbf{x} \leqq$$
$$\leqq ch^3 \|\mu_1\|^2_{\Omega_h} \quad (\hat{\mu}_1^{(i)}(\mathbf{x}) = \tilde{\mu}_1(F_i^*(\hat{\mathbf{x}})),\ \mu_1^{(i)}(\mathbf{x}) = \hat{\mu}_1^{(i)}(F_i^{-1}(\mathbf{x}))),$$

using also Fubini's theorem and the equivalence of the norms in $C(\hat{K})$ and $L_2(\hat{K})$ on finite-dimensional subspaces. Similarly we may write

$$(3.9) \qquad \|\tilde{\mu}_{1,1}\|^2_{\Omega\setminus\Omega_h} = \sum_{i=1}^{N(h)} \|\tilde{\mu}_{1,1}\|^2_{G_i} \leqq c \sum_{i=1}^{N(h)} \int_{G_i} [D\,\tilde{\mu}_1(\mathbf{x})]^2 \,\mathrm{d}\mathbf{x} \leqq$$
$$\leqq c \sum_{i=1}^{N(h)} \max_{\mathbf{x}\in K_i^*} [D\,\tilde{\mu}_1(\mathbf{x})]^2 \operatorname{mes}(G_i) \leqq$$
$$\leqq ch^4 \sum_{i=1}^{N(h)} \sup_{\mathbf{x}\in K_i^*} [D\,F^{*-1}(\mathbf{x})] \max_{\hat{\mathbf{x}}\in\hat{K}} [D\,\hat{\mu}_1^{(i)}(\hat{\mathbf{x}})]^2 \leqq ch^2 \sum_{i=1}^{N(h)} \|\hat{\mu}_1^{(i)}\|^2_{\hat{K}}$$

making use of (2.19) and of the fact that D is bounded as a linear operator on a finite-dimensional space. The remaining estimates are the same as in the previous case. We obtain

$$(3.10) \qquad \|\tilde{\mu}_{1,1}\|^2_{\Omega\setminus\Omega_h} \leqq ch\|\mu_1\|^2_{\Omega_h}.$$

Lemma 7. *For any $\tilde{\mu}_1 \in \tilde{V}(h)$ it holds*

$$(3.11) \qquad \|\tilde{\mu}_1\|_{\Delta(\Omega,\Omega_h)} \leqq ch^{3/2}\|\mu_1\|_{\Omega_h}$$
$$(3.12) \qquad \|\tilde{\mu}_{1,1}\|_{\Delta(\Omega,\Omega_h)} \leqq ch^{1/2}\|\mu_1\|_{\Omega_h}$$

where $\Delta(\Omega, \Omega_h) = (\Omega \setminus \Omega_h) \cup (\Omega_h \setminus \Omega)$ and c is independent of h.

Proof.

$$\|\tilde{\mu}_1\|^2_{\Delta(\Omega,\Omega_h)} = \|\tilde{\mu}_1\|^2_{\Omega\setminus\Omega_h} + \|\tilde{\mu}_1\|^2_{\Omega_h\setminus\Omega}.$$

The first term has been estimated in (3.8), the second one can be treated similarly. (3.12) follows from the estimate (3.10) by an analogous way.

Using the same approach, we can show that

$$(3.13) \qquad \|\omega_2\|_{\Omega_h\setminus\Omega} \leqq ch^{3/2}\|\omega_2\|_{\Omega_h} \ \forall \omega_2 \in V(h_2) \subset W_0^{1,2}(\Omega_h)$$
$$(3.14) \qquad \|\omega_{2,2}\|_{\Omega_h\setminus\Omega} \leqq ch^{1/2}\|\omega_2\|_{\Omega_h}.$$

The estimate (3.13) can be improved on the basis of Lemma 6 and the inverse inequality (cf. (3.20′) below).

The main result of the section is the following

Theorem 7. *Let the extended solution u of (3.1) belong to $W^{p,2}(\tilde{\Omega})$, the solution $\bar{\lambda}^0 = (\lambda_1^0, \bar{\lambda}_2^0)$ of (1.12) be extended on $\tilde{\Omega}$ and $\bar{\lambda}^h \in V(h, h_2)$ denote its Galerkin approximation.*

Then

$$\|\bar{\lambda}^0 - \bar{\lambda}^h\|_{h,h_2} \leqq ch^\delta \|u\|_{p,\Omega} + ch^\gamma [\|u\|_{3,\Omega} + \|f\|_\Omega],$$

where

$$(3.15)\qquad \begin{vmatrix} \delta = \min[1, \varepsilon] & \textit{for} & p = 3, \\ \delta = \min[2, \varepsilon] & \textit{for} & p = 4, \\ \gamma = \min[2, \frac{3}{2} + \varepsilon]. & & \end{vmatrix}$$

Proof. Using the definition of $\|\cdot\|_{h,\Omega_h}$, (3.2), (3.3) and the properties of approximability by the space $V(h, h_2)$, (cf. (3.5)), we show easily that

$$(3.16)\qquad \inf_{\omega \in V(h,h_2)} \|\bar{\lambda}^0 - \omega\|_{h,\Omega_h} \leqq ch^\delta \|u\|_{p,\Omega}$$

where δ is defined in (3.15). Now

$$\sup_{\substack{\omega \neq 0 \\ \omega \in V(h,h_2)}} \frac{|\tilde{B}_{\Omega_h}(\bar{\lambda}^0 - \bar{\lambda}^h; \omega)|}{\|\omega\|_{h,h_2}} = \sup_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} \in \tilde{V}(h,h_2)}} \frac{|\tilde{B}_{\Omega_h}(\bar{\lambda}^0 - \bar{\lambda}^h; \tilde{\omega})|}{\|\tilde{\omega}\|_{h,h_2}}.$$

As $\Omega_h = \Omega \cup (\Omega_h \setminus \Omega) \setminus (\Omega \setminus \Omega_h)$, we may write on the basis of (2.21)

$$\begin{aligned} \tilde{B}_{\Omega_h}(\bar{\lambda}^0 - \bar{\lambda}^h; \tilde{\omega}) &= \tilde{B}_{\Omega_h}(\bar{\lambda}^0; \tilde{\omega}) - [\tilde{f}; \tilde{\omega}]_{\Omega_h} = \\ &= -\tilde{B}_{\Omega\setminus\Omega_h}(\bar{\lambda}^0; \tilde{\omega}) + \tilde{B}_\Omega(\bar{\lambda}^0; \tilde{\omega}) + \\ &+ \tilde{B}_{\Omega_h\setminus\Omega}(\bar{\lambda}^0; \tilde{\omega}) + [\tilde{f}; \tilde{\omega}]_{\Omega\setminus\Omega_h} - [\tilde{f}; \tilde{\omega}]_\Omega - [\tilde{f}; \tilde{\omega}]_{\Omega_h\setminus\Omega}. \end{aligned}$$

Inserting

$$\tilde{B}_\Omega(\bar{\lambda}^0; \tilde{\omega}) = [\tilde{f}; \tilde{\omega}]_\Omega$$

we obtain finally

$$(3.16')\qquad \begin{aligned} \sup_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} \in \tilde{V}(h,h_2)}} \frac{|\tilde{B}_{\Omega_h}(\bar{\lambda}^0 - \bar{\lambda}^h; \tilde{\omega})|}{\|\tilde{\omega}\|_{h,h_2}} &\leqq \sup_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} \in \tilde{V}(h,h_2)}} \frac{|\tilde{B}_{\Omega\setminus\Omega_h}(\bar{\lambda}^0; \tilde{\omega})|}{\|\tilde{\omega}\|_{h,h_2}} + \\ + \sup_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} \in V(h,h_2)}} \frac{|\tilde{B}_{\Omega_h\setminus\Omega}(\bar{\lambda}^0; \tilde{\omega})|}{\|\tilde{\omega}\|_{h,h_2}} &+ \sup_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} \in \tilde{V}(h,h_2)}} \frac{|[\tilde{f}; \tilde{\omega}]_{\Omega\setminus\Omega_h}|}{\|\tilde{\omega}\|_{h,h_2}} + \sup_{\substack{\tilde{\omega} \neq 0 \\ \tilde{\omega} \in \tilde{V}(h,h_2)}} \frac{|[\tilde{f}; \tilde{\omega}]_{\Omega_h\setminus\Omega}|}{\|\tilde{\omega}\|_{h,h_2}}. \end{aligned}$$

We show the estimates for the first and third term. According to (1.10) we may write

$$(3.17)\qquad \begin{aligned} \tilde{B}_{\Omega\setminus\Omega_h}(\bar{\lambda}^0; \tilde{\omega}) &= \mathscr{A}_{11,\Omega\setminus\Omega_h}(\lambda_1^0; \tilde{\omega}_1) + \mathscr{A}_{21,\Omega\setminus\Omega_h}(\bar{\lambda}_2^0; \tilde{\omega}_1) - \\ &- \mathscr{A}_{12,\Omega\setminus\Omega_h}(\lambda_1^0; \tilde{\omega}_2) - \mathscr{A}_{22,\Omega\setminus\Omega_h}(\bar{\lambda}_2^0; \tilde{\omega}_2). \end{aligned}$$

Let $\Omega^0 \supset \bar{\Omega}$. For sufficiently small h, $\Omega_h \subset \Omega^0$. We estimate the members of (3.17) by means of Lemma 7 and (3.2), as follows

$$(3.18)\quad \left|\mathscr{A}_{11,\Omega\setminus\Omega_h}(\lambda_1^0;\tilde{\omega}_1)\right| \leqq \|\lambda_1^0\|_{\Delta(\Omega,\Omega_h)} \|\tilde{\omega}_1\|_{\Delta(\Omega,\Omega_h)} + $$
$$+ ch^{2+2\varepsilon}\|\lambda_{1,1}^0\|_{\Delta(\Omega,\Omega_h)} \|\tilde{\omega}_{1,1}\|_{\Delta(\Omega,\Omega_h)} \leqq$$
$$\leqq ch^{3/2}\|\tilde{\omega}_1\|_{\Omega_h} \max_{x\in\bar{\Omega}^0} |\lambda_1^0(x)| \left(\operatorname{mes}(\Delta(\Omega,\Omega_h))\right)^{1/2} + ch^{5/2+2\varepsilon}\|u\|_{2,\Omega} \|\tilde{\omega}_1\|_{\Omega_h} \leqq$$
$$\leqq c(h^3 + h^{5/2+2\varepsilon}) \|u\|_{3,\Omega} \|\tilde{\omega}\|_{h,h_2}.$$

Here we make use of the estimates (cf. (3.8)) $\operatorname{mes}(\Delta(\Omega,\Omega_h)) \leqq ch^4 N(h)$, $N(h) \leqq ch^{-1}$ and of the imbeding of $W^{2,2}(\Omega^0)$ into $C(\bar{\Omega}^0)$. Similarly

$$(3.19)\quad \left|\mathscr{A}_{21,\Omega\setminus\Omega_h}(\bar{\lambda}_2^0;\tilde{\omega}_1)\right| \leqq ch^{1+\varepsilon}\|\bar{\lambda}_{2,2}^0\|_{\Delta(\Omega,\Omega_h)}\|\tilde{\omega}_{1,1}\|_{\Delta(\Omega,\Omega_h)} +$$
$$+ \|\bar{\lambda}_2^0\|_{\Delta(\Omega,\Omega_h)} \|\tilde{\omega}_{1,1}\|_{\Delta(\Omega,\Omega_h)} \leqq c(h^2 + h^{3/2+\varepsilon}) \|u\|_{3,\Omega} \|\tilde{\omega}\|_{h,h_2},$$

$$(3.20)\quad \left|\mathscr{A}_{12,\Omega\setminus\Omega_h}(\lambda_1^0;\tilde{\omega}_2)\right| \leqq ch^{1+\varepsilon}\|\lambda_{1,1}^0\|_{\Delta(\Omega,\Omega_h)} \|\tilde{\omega}_{2,2}\|_{\Delta(\Omega,\Omega_h)} +$$
$$+ \|\lambda_{1,1}^0\|_{\Delta(\Omega,\Omega_h)} \|\tilde{\omega}_2\|_{\Delta(\Omega,\Omega_h)} \leqq ch^{3/2+\varepsilon}\|u\|_{2,\Omega} \|\tilde{\omega}\|_{h,h_2} + \|\lambda_{1,1}^0\|_{\Delta(\Omega,\Omega_h)} \|\tilde{\omega}_2\|_{\Delta(\Omega,\Omega_h)}$$

by virtue of the Friedrichs inequality

$$\|\tilde{\omega}_2\|_{\Omega_h} = \|\tilde{\omega}_2\|_{\Omega^\circ} \leqq c\|\tilde{\omega}_{2,2}\|_{\Omega^\circ} = c\|\tilde{\omega}_{2,2}\|_{\Omega_h}$$

with c independent of h, Ω_h.

To estimate $\|\tilde{\omega}_2\|_{\Delta(\Omega,\Omega_h)} = \|\omega_2\|_{\Omega_h\setminus\Omega}$ we use Lemma 6. We know that $\operatorname{dist}(\Gamma,\Gamma_h) \leqq ch^3$. Inserting $\eta = ch^3$ and defining $(\omega_2)_\eta$ as in Lemma 6, we obtain from (3.6)

$$(3.20')\quad \|\omega_2\|_{\Omega_h\setminus\Omega} \leqq \|(\omega_2)_\eta\|_{\Omega_h} \leqq ch^3\|\omega_2\|_{1,\Omega_h} \leqq$$
$$\leqq ch^2\|\omega_2\|_{\Omega_h} \leqq ch^2\|\omega_{2,2}\|_{\Omega_h} \leqq ch^2\|\tilde{\omega}\|_{h,h_2}$$

using also the inverse inequality on $\tilde{\Omega}$ and the Friedrichs inequality for ω_2. Consequently, we have

$$\left|\mathscr{A}_{12,\Omega\setminus\Omega_h}(\lambda_1^0,\tilde{\omega}_2)\right| \leqq c(h^{3/2+\varepsilon} + h^2) \|u\|_{3,\Omega}\|\tilde{\omega}\|_{h,h_2}.$$

As $\mathscr{A}_{22,\Omega\setminus\Omega_h}(\bar{\lambda}_2^0;\tilde{\omega}_2) = 0$ follows from $\tilde{\omega}_2 = 0$ on $\Omega \setminus \Omega_h$, we estimate only $\mathscr{A}_{22,\Omega_h\setminus\Omega}$. Note that

$$\int_{\Omega_h\setminus\Omega} \frac{\partial}{\partial x_2}(u\tilde{\omega}_2)\,\mathrm{d}\mathbf{x} = 0$$

because $u = 0$ on Γ and $\tilde{\omega}_2 = 0$ on Γ_h. Thus using also (3.12), (3.14) we derive

$$\left|\mathscr{A}_{22,\Omega_h\setminus\Omega}(\bar{\lambda}_2^0;\tilde{\omega}_2)\right| \leqq \left|(\bar{\lambda}_{2,2}^0;\tilde{\omega}_{2,2})\right| + \alpha\left|\{(\bar{\lambda}_{2,2}^0;\tilde{\omega}_2) + (\bar{\lambda}_2^0;\tilde{\omega}_{2,2})\}\right| \leqq$$
$$\leqq ch^{3/2+\varepsilon}\|u\|_{2,\Omega} \|\omega_2\|_{\Omega_h} + c\operatorname{mes}(\Delta(\Omega,\Omega_h)) \max_{x\in\bar{\Omega}^\circ}\left|\frac{\partial u}{\partial x_2}(\mathbf{x})\right| \|\tilde{\omega}_{2,2}\|_{\Omega_h\setminus\Omega} +$$

$$+ ch^{-1-\varepsilon}\left|\left\{h^{1+\varepsilon}\left(\frac{\partial^2 u}{\partial x_2^2}\,;\,\tilde{\omega}_2\right) + \left(\frac{\partial u}{\partial x_2}\,;\,\tilde{\omega}_2\right) + (u;\tilde{\omega}_{2,2}) + h^{1+\varepsilon}\left(\frac{\partial u}{\partial x_2}\,;\,\tilde{\omega}_{2,2}\right)\right\}\right| \leqq$$

$$\leqq c(h^{3/2+\varepsilon} + h^2)\,\|u\|_{3,\Omega}\,\|\tilde{\omega}\|_{h,h_2} + ch^3\|u\|_{2,\Omega}\,\|\tilde{\omega}_2\|_{1,\Omega_h} +$$

$$+ c\max_{x\in\bar{\Omega}^\circ}\left|\frac{\partial u}{\partial x_2}(\mathbf{x})\right| \operatorname{mes}(\Delta(\Omega,\Omega_h))\,h^{1/2}\|\tilde{\omega}_2\|_{\Omega_h} \leqq c(h^{3/2+\varepsilon} + h^2)\,\|u\|_{3,\Omega}\,\|\tilde{\omega}\|_{h,h_2}\,.$$

Finally

$$(3.22)\qquad |[\tilde{f};\tilde{\omega}]_{\Omega\setminus\Omega_h}| \leqq ch^{2+2\varepsilon}\|\tilde{f}\|_{\Delta(\Omega,\Omega_h)}\,\|\tilde{\omega}_{1,1}\|_{\Delta(\Omega,\Omega_h)} +$$

$$+ ch^{1+\varepsilon}\|\tilde{f}\|_{\Delta(\Omega,\Omega_h)}\,\|\tilde{\omega}_{2,2}\|_{\Delta(\Omega,\Omega_h)} + \|\tilde{f}\|_{\Delta(\Omega,\Omega_h)}\,\|\tilde{\omega}_2\|_{\Delta(\Omega,\Omega_h)} \leqq$$

$$\leqq c(h^{3/2+\varepsilon} + h^3)\,\|f\|_\Omega\,\|\tilde{\omega}\|_{h,h_2}\,.$$

The assertion of Theorem 7 follows from (3.16), (3.16′) and the estimates of the form (3.17)–(3.22).

Theorem 8. *Let u be the extended solution of* (3.1), $u \in W^{p,2}(\Omega^0)$, $\bar{\lambda}^h = (\lambda_1^h, \bar{\lambda}_2^h) \in V(h, h_2)$ *the solution of* (2.21). *Then it holds*

$$(3.23)\qquad \left\|\frac{\partial u}{\partial x_1} - \lambda_1^h\right\|_{\Omega_h} + \left\|\frac{\partial u}{\partial x_2} - \frac{\partial \bar{\lambda}_2^h}{\partial x_2}\right\|_{\Omega_h} + \|u - \bar{\lambda}_2^h\|_{\Omega_h} \leqq$$

$$\leqq ch^\delta\|u\|_{p,\Omega} + ch^{\gamma_1}[\|u\|_{3,\Omega} + \|f\|_\Omega]$$

where δ is defined in (3.15) *and* $\gamma_1 = \min[1 + \varepsilon, 2]$.

Proof. We may write

$$\left\|\frac{\partial u}{\partial x_1} - \lambda_1^h\right\|_{\Omega_h} + \left\|\frac{\partial u}{\partial x_2} - \frac{\partial \bar{\lambda}_2^h}{\partial x_2}\right\|_{\Omega_h} \leqq \left\|\frac{\partial u}{\partial x_1} - \lambda_1^h\right\|_{\Omega_h} +$$

$$+ \left\|\frac{h^{1+\varepsilon}}{\alpha_0}\frac{\partial^2 u}{\partial x_2^2} + \frac{\partial u}{\partial x_2} - \frac{\partial \bar{\lambda}_2^h}{\partial x_2}\right\|_{\Omega_h} + \frac{h^{1+\varepsilon}}{\alpha_0}\|u\|_{2,\Omega} =$$

$$= \|\lambda_1^0 - \lambda_1^h\|_{\Omega_h} + \|\bar{\lambda}_{2,2}^0 - \bar{\lambda}_{2,2}^h\|_{\Omega_h} + \frac{h^{1+\varepsilon}}{\alpha_0}\|u\|_{2,\Omega} \leqq \|\bar{\lambda}^0 - \bar{\lambda}^h\|_{h,h_2} + \frac{h^{1+\varepsilon}}{\alpha_0}\|f\|_\Omega\,.$$

Using Theorem 7 and Friedrichs inequality we obtain (3.23).

Remark 3.1. From the practical point of view it is important, that a finer mesh can be used for λ_1 than for $\bar{\lambda}_2$. Let us illustrate the approach on the preceding example, thus showing also how to deal also with more general cases.

First we carry out the "triangulation" $\mathscr{T}_{h_2}$ of the region Ω and construct the space $V_{h_2}(\Omega_{h_2})$ with $\hat{P} = P_2$. A finer "triangulation" $\mathscr{T}_h$ of the domain Ω_{h_2} can be obtained by a triangulation of every element $K_i \in \mathscr{T}_{h_2}$. Here the "triangulation" of a curved element $K_i \in \mathcal{T}_{h_2}$ is to be comprehended in the sense of part **A** of the preceding Section, i.e. the curved boundary (which is now described by a quadratic function) is reproduced exactly.

We set

$$V_h(\Omega_{h_2}) = \{v \in C(\bar{\Omega}_{h_2}),\ v|_{K_i} \in P,\ \forall K_i \in \mathcal{T}'_h\}$$

where P is defined by means of $\hat{P} = P_2$ through (3.2).

The rate of convergence, analogous to (3.23), follows by the same line of thought as previously.

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Souhrn

KŘIVOČARÉ PRVKY VE SMÍŠENÉ METODĚ KONEČNÝCH PRVKŮ, BLÍZKÉ ROVNOVÁŽNÉMU MODELU

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V [1], [2] autoři odvodili nový typ smíšené metody konečných prvků, spočívající v tom, že Galerkinovy aproximace jsou vektorové funkce, jejichž $(n - 1)$ složek konverguje k vybraným složkám co-gradientu řešení a zbývající složka k řešení samotnému. Tato práce je pokračováním [1]. Studuje se možnost použití dvou typů křivočarých elementů, zavedených jednak podle [3] a jednak v [4]. V poslední části je postup ukázán na modelovém příkladu a je odvozen řád konvergence Galerkinovských aproximací.

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