Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 23 (1973), No. 1, 86–94

Persistent URL: http://dml.cz/dmlcz/101148

Czechoslovak Mathematical Journal, 23 (98) 1973, Praha

CURVES OF CONSTANT BREADTH

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(Received January 17, 1972)

We shall leave out the history of curves of constant breadth in the plane; an informative survey can be found in [3] and also in Z. NÁDENÍK'S work [8]. In 1914 M. FUJIWARA [4] made an attempt to generalize the results to closed curves in Euclidean space E_3. He requires the closed curve to have the following property: each normal hyperplane has with the curve only two common points, the distance of them being constant. W. BLASCHKE [2] completed Fujiwara's conclusions by proving the existence of such a curve. A fundamental turn was made by Z. Nádeník's works [8] and [9] which disclose a much closer connection with the plane case. Z. Nádeník dealt with closed curves in spaces of even dimensions whose spherical image of last normals is the hypercircle Γ, i.e., a curve with all curvatures constant. When considering a centrally symmetrical hypercircle Γ, Z. Nádeník defined opposite points of the curve (see p. 447) and then the breadth of the curve as the distance B of hyperosculating hyperplanes at the opposite points. For certain two types of the hypercircle Γ, the curve always lies on one side of its each hyperosculating hyperplane and the breadth in the direction of the vector of the last normal is always positive; these properties are completely analogous to those of oval curves in the plane. Conclusions reached by Z. Nádeník for curves of constant breadth B also show a very close connection with the simpler case in the plane. The generalization of Segre's theorem given by Z. Nádeník in [7] was of great importance for this study.

In this article we shall consider closed space curves $\mathscr{C}_s$ as defined in [11] (p. 75); all notations and starting assumptions – except the assumption (d) – will remain unchanged. However, we shall strenghthen the assumption concerning the class p of the curve $\mathscr{C}_s$ by requiring $p \geqq 2n - 1$. It was shown in the proof of Theorem V [11] that curves $\mathscr{C}_s$ can exist only in spaces of even dimensions; in the same paper also some relations important for our further study were established. Let us recall in particular that for a curve $\mathscr{C}_s$ the following relations hold (α being the arc on the spherical image $\mathscr{C}_s^*$ of tangent vectors or the curve $\mathscr{C}_s$):

$$(1) \qquad \boldsymbol{n}(\alpha + \tfrac{1}{2}a) = -\boldsymbol{n}(\alpha)\,, \quad \mathbf{e}_\nu(\alpha + \tfrac{1}{2}a) = -\mathbf{e}_\nu(\alpha) \quad (\nu = 1, \ldots, n - 2)\,,$$

$$(2) \qquad \varkappa_j(\alpha + \tfrac{1}{2}a) = \varkappa_j(\alpha) \quad (j = 2, \ldots, n - 1)\,.$$

The curve $\mathscr{C}_s$ can be represented parametrically by means of vectors from the moving Frenet's n-hedron

$$\mathbf{x}(\alpha) = h_0(\alpha)\,\mathbf{t}(\alpha) - h_1(\alpha)\,\mathbf{e}_1(\alpha) - \ldots - h_{n-2}(\alpha)\,\mathbf{e}_{n-2}(\alpha) - h(\alpha)\,\mathbf{n}(\alpha)\,. \tag{3}$$

The function $h(\alpha) = -\mathbf{x}(\alpha)\,.\,\mathbf{n}(\alpha)$ is called the supporting function of the curve $\mathscr{C}_s$ and the function $B(\alpha) = h(\alpha) + h(\alpha + \frac{1}{2}a)$ the breadth of the curve $\mathscr{C}_s$. Denoting further $B_0(\alpha) = h_0(\alpha) + h_0(\alpha + \frac{1}{2}a)$, $B_\nu(\alpha) = h_\nu(\alpha) + h_\nu(\alpha + \frac{1}{2}a)$ $(\nu = 1, \ldots, n-2)$, then it follows from (1) and (3) that the position of the radius-vectors at any pair of opposite points of the curve $\mathscr{C}_s$ is given by the relation

$$\begin{aligned} \mathbf{x}(\alpha) - \mathbf{x}(\alpha + \tfrac{1}{2}a) = B_0(\alpha)\,\mathbf{t}(\alpha) - B_1(\alpha)\,\mathbf{e}_1(\alpha) - \ldots \\ \ldots - B_{n-2}(\alpha)\,\mathbf{e}_{n-2}(\alpha) - B(\alpha)\,\mathbf{n}(\alpha)\,. \end{aligned} \tag{4}$$

When α is taken for the parameter, Frenet's formulae read as follows:

$$\begin{aligned} \mathbf{t}'(\alpha) &= \mathbf{e}_1(\alpha)\,, \\ \mathbf{e}_1'(\alpha) &= -\mathbf{t}(\alpha) + \varkappa_2(\alpha)\,\mathbf{e}_2(\alpha)\,, \\ \mathbf{e}_2'(\alpha) &= -\varkappa_2(\alpha)\,\mathbf{e}_1(\alpha) + \varkappa_3(\alpha)\,\mathbf{e}_3(\alpha)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ \mathbf{e}_{n-2}'(\alpha) &= -\varkappa_{n-2}(\alpha)\,\mathbf{e}_{n-3}(\alpha) + \varkappa_{n-1}(\alpha)\,\mathbf{n}(\alpha)\,, \\ \mathbf{n}'(\alpha) &= -\varkappa_{n-1}(\alpha)\,\mathbf{e}_{n-2}(\alpha)\,. \end{aligned} \tag{5}$$

Differentiating (4) with respect to α, modifying the righthand side by means of (5) and comparing both sides, we obtain the following system of conditions:

$$\begin{aligned} r(\alpha) + r(\alpha + \tfrac{1}{2}a) - B_0'(\alpha) &= B_1(\alpha)\,, \\ B_1'(\alpha) &= B_0(\alpha) + \varkappa_2(\alpha)\,B_2(\alpha)\,, \\ B_2'(\alpha) &= -\varkappa_2(\alpha)\,B_1(\alpha) + \varkappa_3(\alpha)\,B_3(\alpha)\,, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ B_{n-2}'(\alpha) &= -\varkappa_{n-2}(\alpha)\,B_{n-3}(\alpha) + \varkappa_{n-1}(\alpha)\,B(\alpha)\,, \\ B'(\alpha) &= -\varkappa_{n-1}(\alpha)\,B_{n-2}(\alpha)\,. \end{aligned} \tag{6}$$

Eliminating B_0 and B_ν $(\nu = 1, \ldots, n-2)$ from the system (6) we obtain – assuming the radius of flexion r to be known – a certain linear differential equation of the n-th order

$$G(B(\alpha)) = r(\alpha) + r(\alpha + \tfrac{1}{2}a)\,. \tag{7}$$

The solutions of the corresponding homogeneous equation

$$G(B(\alpha)) = 0 \tag{8}$$

are n coordinates of the vector $\boldsymbol{n}$. This can be directly seen from the comparison of (5) and (6) for $r(\alpha) + r(\alpha + \frac{1}{2}a) = 0$. Wronskian $W(\alpha)$ of these solutions differs from zero for each $\alpha \in \langle 0, a)$, as the equality

$$W(\alpha) = [\boldsymbol{n}(\alpha), \boldsymbol{n}'(\alpha), \ldots, \boldsymbol{n}^{(n-1)}(\alpha)] = \prod_{j=2}^{n-2} \varkappa_j^j(\alpha) \tag{9}$$

holds; (9) follows from (5) and its derivation will be sketched later (see No 1). Therefore the coordinates of the vector $\boldsymbol{n}$ form a fundamental system of the equation (8).

I.

Analogously to Z. Nádeník in [8], we shall establish first of all the necessary and sufficient conditions for a curve $\mathscr{C}_s$ to be a curve of constant breadth. Some direct consequences will be given here, too.

Theorem 1 (see No 1): *Let $A(\alpha)$ be the coefficient of $B(\alpha)$ in the equation (7). A curve $\mathscr{C}_s$ is a curve of constant breadth B if and only if the relation*

$$A(\alpha)\, B = r(\alpha) + r(\alpha + \tfrac{1}{2}a) \tag{10}$$

is valid. If the curve $\mathscr{C}_s$ has a constant sum of radii of flexion at the opposite points, then it is a curve of constant breadth B if and only if the coefficient $A(\alpha) = A$ is constant as well.

The following theorem characterizes curves $\mathscr{C}_s$ of constant breadth geometrically:

Theorem II (see No 2): *A curve $\mathscr{C}_s$ is of constant breadth B if and only if the vector $\mathbf{e}_{n-2}(\alpha)$ for each $\alpha \in \langle 0, \frac{1}{2}a)$ is perpendicular to the connecting line of the opposite points.*

Theorem III (see No 3): *On a curve $\mathscr{C}_s$ of the length L and constant breadth B the following relations hold:*

$$L = B \int_0^{a/2} A(\alpha)\, \mathrm{d}\alpha\,, \tag{11}$$

$$r(\alpha) + r(\alpha + \tfrac{1}{2}a) = \frac{A(\alpha)}{\int_0^{a/2} A(\alpha)\, \mathrm{d}\alpha} L. \tag{12}$$

To a curve $\mathscr{C}_s$ of constant breadth, the analogue of Barbier's theorem (see [1]) also applies.

Theorem IV (see No 4): *All curves $\mathscr{C}_s$ of the same constant breadth and with the same spherical image $\mathscr{C}_s^*$ of tangents also have the same length.*

E. MEISSNER [5] (p. 327) proved that the centre of gravity of an oval curve of constant breadth coincides with its centre of gravity of curvature. The notion of the centre of gravity of curvature was introduced by J. STEINER [10]; it stands for the point $\mathbf{Z} = 1/a \int_{\mathscr{C}} \mathbf{x}(s)\, k_1(s)\, \mathrm{d}s$ (s is the arc on $\mathscr{C}$). Also the curves studied by Z. Nádeník in [8] have the same property: the centre of gravity of the curve coincides with Steiner's centre of gravity of curvature (see [8], p. 544). Meissner's theorem evidently holds for curves which have the centre, regardless of the fact whether they are curves of constant breadth. Relation (10) is not fully analogous to the corresponding relation on an oval curve of constant breadth; the fact that the sum of radii of flexion at opposite points is constant (as it follows from Theorem I) does not mean that any curve $\mathscr{C}_s$ is also of constant breadth and vice versa. The analogue of Meissner's theorem holds when expressed in the following form:

Theorem V (see No 5): *On a curve $\mathscr{C}_s$ with constant sum of radii of flexion at opposite points the centre of gravity coincides with Steiner's centre of gravity of curvature if and only if*

$$\int_{\mathscr{C}_s*} r(\alpha) \{\mathbf{x}(\alpha) - \mathbf{x}(\alpha + \tfrac{1}{2}a)\}\, \mathrm{d}\alpha = \mathbf{0}\,.$$

II.

If between curves $\mathscr{K}$, ${}^1\mathscr{K}$ with parametric equations $\mathbf{z} = \mathbf{z}(\alpha)$, ${}^1\mathbf{z} = {}^1\mathbf{z}({}^1\alpha)$ there exists such a one-to-one mapping given by the relation ${}^1\alpha = f(\alpha)$ that the couple of moving n-hedrons at the corresponding points is invariant with respect to the group of Euclidean motions, then, following E. ČECH and Z. Nádeník (see [6], pp. 57–58), we shall call these curves Bertrand's curves.

In this section we shall describe the properties of some special curves $\mathscr{C}_s$ of constant breadth which with respect to themselves are Betrand's curves. In order to simplify the formulations it is suitable to denote by $\hat{E}_0(\alpha)$, $\hat{E}_{2i}(\alpha)$ the hyperplanes passing through the point $\mathbf{x}(\alpha)$ of the curve $\mathscr{C}_s$ and perpendicular to the vectors $\mathbf{t}(\alpha)$, $\mathbf{e}_{2i}(\alpha)$ ($i = 1, \ldots, \frac{1}{2}(n - 2)$) for each $\alpha \in \langle 0, a)$.

Theorem VI (see No 6): *On a curve $\mathscr{C}_s$ the hyperplanes $\hat{E}_0(\alpha)$, $\hat{E}_{2i}(\alpha)$ coincide at each couple of opposite points if and only if $\mathscr{C}_s$ is a curve of constant breadth with constant ratios of curvatures $k_{2i}(\alpha)/k_{2i+1}(\alpha)$ ($i = 1, \ldots, \frac{1}{2}(n - 2)$). These curves are Bertrand's curves, have a constant sum of radii of flexion at opposite points and the vectors $\mathbf{t}(\alpha)$, $\mathbf{e}_{2i}(\alpha)$ are always perpendicular to the connecting line of the opposite points.*

The curves $\mathscr{C}_s$ from Theorem VI are such that none of functions B_0, B, B_ν ($\nu = 1, \ldots, n - 2$) changes its sign. At the same time they are the only curves $\mathscr{C}_s$ having this property; this can be easily verified by means of (6).

Theorem VII (see No 7): *On a curve $\mathscr{C}_s$ of constant breadth B with constant ratios of curvatures $k_{2i}(\alpha)/k_{2i+1}(\alpha)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ the position of radius-vectors at opposite points can be expressed by the relation*

$$(13) \qquad \mathbf{x}(\alpha + \tfrac{1}{2}a) - \mathbf{x}(\alpha) = \Lambda B\, \mathbf{e}_1(\alpha) + \sum_{\mu=2}^{(n-2)/2} \Lambda B \prod_{j=1}^{\mu-1} \frac{\varkappa_{2j}(\alpha)}{\varkappa_{2j+1}(\alpha)}\, \mathbf{e}_{2\mu-1}(\alpha) + B\, \mathbf{n}(\alpha)\,,$$

where Λ is the constant coefficient from (10) of the following form:

$$(14) \qquad \Lambda = \prod_{i=1}^{(n-2)/2} \frac{\varkappa_{2i+1}(\alpha)}{\varkappa_{2i}(\alpha)}\,.$$

As we have already mentioned in the introduction to Theorem V, the constant sum of radii of flexion at opposite points does not guarantee that a curve $\mathscr{C}_s$ is of constant breadth. Even the constant coefficient Λ itself does not imply that $\mathscr{C}_s$ is a curve of constant breadth (see No 8). Nevertheless, the following theorem holds:

Theorem VIII (see No 8): a) *In case that a curve $\mathscr{C}_s$ has – in addition to the constant ratios of curvatures $k_{2i}(\alpha)/k_{2i+1}(\alpha)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ – also a constant sum of radii of flexion at opposite points, then $\mathscr{C}_s$ is Bertrand's curve of constant breadth.*

b) *A curve $\mathscr{C}_s$ with a constant flexion and constant ratios of curvatures $k_{2i}(\alpha)$: : $k_{2i+1}(\alpha)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ is Bertrand's spherical curve of constant breadth.*

Let us now pass to the proofs of the above theorems.

1. First of all the identity (9) is to be proved. If we substitute the corresponding expression for $\mathbf{n}'(\alpha)$ from (5) into the determinant $[\mathbf{n}, \mathbf{n}', \ldots, \mathbf{n}^{(n-1)}]$, then the vectors $\mathbf{n}(\alpha)$ and $\mathbf{e}_{n-2}(\alpha)$ will be found in the first two columns. Differentiating the other columns, arranging then by means of (5) and omitting the summands which would contribute zero determinants, we obtain the determinant $[\mathbf{n}, -\varkappa_{n-1}\mathbf{e}_{n-2}, \varkappa_{n-1}\varkappa_{n-2}\mathbf{e}_{n-3}, -\varkappa_{n-1}\varkappa_{n-2}\varkappa_{n-3}\mathbf{e}_{n-4}, \ldots, \varkappa_{n-1}\varkappa_{n-2}\ldots\varkappa_2\mathbf{e}_1, -\varkappa_{n-1}\varkappa_{n-2}\ldots\varkappa_2\mathbf{t}]$, from which (9) follows directly.

Now the proof of Theorem I: If $B(\alpha) = B = \text{const}$, then (7) has the form (10) where $\Lambda(\alpha)$ is formed only by the ratios of curvatures $\varkappa_j(\alpha)$ $(j = 2, \ldots, n-1)$ and their derivatives. Conversely, if (10) holds, then one solution of the non-homogeneous equation (7) is $B(\alpha) = B$; the general solution of the homogeneous equation (8) is $B(\alpha) = \mathbf{v} \cdot \mathbf{n}(\alpha)$, where $\mathbf{v}$ is an arbitrary fixed vector. Thus the general solution of the equation (7) has the form $B(\alpha) = B + \mathbf{v} \cdot \mathbf{n}(\alpha)$. As the breadth $B(\alpha)$ is a periodic function with the period $\frac{1}{2}a$, it necessarily follows from (1) that also $B(\alpha) = B - \mathbf{v} \cdot \mathbf{n}(\alpha)$. Therefore $\mathbf{v} = \mathbf{0}$ and $B(\alpha) = B$. The latter part of the theorem is a direct consequence of the former.

2. The validity of this assertion is evident from the last formula (6) and from (4).

3. On any curve $\mathscr{C}_s$ of constant breadth B (10) holds. Then we have for the length L of the curve $\mathscr{C}_s$:

$$L = \int_0^{a/2} \{r(\alpha) + r(\alpha + \tfrac{1}{2}a)\}\, d\alpha = B \int_0^{a/2} \Lambda(\alpha)\, d\alpha\,.$$

Comparing (10) and (11) we obtain (12).

4. (11) holds for the length of the considered curve $\mathscr{C}_s$; the coefficient $\Lambda(\alpha)$ is formed only by the ratios of curvatures $\varkappa_j(\alpha)$ $(j = 2, \ldots, n-1)$ and their derivatives. These ratios of curvatures are uniquely determined by the spherical image $\mathscr{C}_s^*$.

5. Let $R = r(\alpha) + r(\alpha + \frac{1}{2}a) = \text{const}$. For the centre of gravity $\mathbf{Y}$ of the curve $\mathscr{C}_s$ the following equation holds (s is the arc on $\mathscr{C}_s$):

$$\begin{aligned}\mathbf{Y} &= \frac{1}{L}\int_{\mathscr{C}_s} \mathbf{x}(s)\, ds = \frac{1}{L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, r(\alpha)\, d\alpha = \\ &= \frac{1}{2L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, r(\alpha)\, d\alpha + \frac{1}{2L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha + \tfrac{1}{2}a)\, r(\alpha + \tfrac{1}{2}a)\, d\alpha = \\ &= \frac{1}{2L}\int_{\mathscr{C}_s*} r(\alpha)\, \{\mathbf{x}(\alpha) - \mathbf{x}(\alpha + \tfrac{1}{2}a)\}\, d\alpha + \frac{R}{2L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha + \tfrac{1}{2}a)\, d\alpha\,.\end{aligned}$$

According to the assumption, the first integral equals zero. As $L = \frac{1}{2}aR$, the second one equals $1/a \int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, d\alpha = 1/a \int_{\mathscr{C}_s} \mathbf{x}(s)\, r^{-1}(s)\, ds$, which is Steiner's centre of gravity of the curvature of the curve $\mathscr{C}_s$.

If both centres of gravity coincide, i.e., $1/a \int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, d\alpha = 1/L \int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, r(\alpha)\, d\alpha$ and if we substitute for a the corresponding expression from the formula for the length L, then

$$\frac{R}{2L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, d\alpha = \frac{1}{2L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha)\, r(\alpha)\, d\alpha + \frac{1}{2L}\int_{\mathscr{C}_s*} \mathbf{x}(\alpha + \tfrac{1}{2}a)\, r(\alpha + \tfrac{1}{2}a)\, d\alpha\,.$$

Writing R under the integral sign and replacing R by the corresponding sum we obtain an equality which is just another form of the equality required by the theorem; the required equality is always fulfilled on oval curves and on curves dealt with by Z. Nádeník in [8]. It should be mentioned that the proof proceeds similarly to that of Z. Nádeník in [8] (pp. 547–548).

6. When the hyperplanes $\hat{E}_0(\alpha)$ and $\hat{E}_{2i}(\alpha)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ on a curve $\mathscr{C}_s$ coincide at each couple of opposite points, then the functions B_0, B_{2i} identically equal zero. Differentiation of (4) yields the following equation:

$$-\{r(\alpha)+r(\alpha+\tfrac{1}{2}a)\}\,\mathbf{t}(\alpha)=$$

$$=\sum_{i=1}^{(n-2)/2} B'_{2i-1}\mathbf{e}_{2i-1}(\alpha)+B'(\alpha)\,\mathbf{n}(\alpha)+\sum_{i=1}^{(n-2)/2} B_{2i-1}(\alpha)\,\mathbf{e}'_{2i-1}(\alpha)+B(\alpha)\,\mathbf{n}'(\alpha)\,.$$

Let us compare now both sides of this equation and replace $\mathbf{e}'_{2i}(\alpha)$ and $\mathbf{n}'(\alpha)$ by the corresponding expressions from (5). We then see that the functions B_{2i-1} and B are constant and, further, that also the ratios of curvatures k_{2i}/k_{2i+1} and the sum of radii of flexion at opposite points are constant. The equation (4) has therefore the form

$$\mathbf{x}(\alpha+\tfrac{1}{2}a)-\mathbf{x}(\alpha)=\sum_{i=1}^{(n-2)/2} B_{2i-1}\,\mathbf{e}_{2i-1}(\alpha)+B\,\mathbf{n}(\alpha)\,, \tag{6,1}$$

B_{2i-1} and B being positive constants.

Conversely: If B is constant, then it follows from the last equation (6) that $B_{n-2}(\alpha)=0$. The recurrent procedure with constant ratios of curvatures k_{2i}/k_{2i+1} $(i=1,\ldots,\frac{1}{2}(n-2))$ leads to the conclusion that the functions B_0, B_{2i} identically equal zero and the functions B_{2i-1} are necessarily constant; hence also $r(\alpha)+$ $+\,r(\alpha+\frac{1}{2}a)=B_1$ is constant.

From (6,1) and (1) it may be seen that these curves $\mathscr{C}_s$ are Bertrand's curves.
Also the assertion that the vectors $\mathbf{t}(\alpha)$, $\mathbf{e}_{2i}(\alpha)$ are perpendicular to the connecting line of the opposite points is a consequence of (6,1).

7. First the following recurrent relation will be proved: Let ν be an even index $(\nu=4,\ldots,n-2)$. If on the considered curve $\mathscr{C}_s$ the equality

$$B_{n-\nu+1}(\alpha)=\frac{\varkappa_{n-1}(\alpha)\,\varkappa_{n-3}(\alpha)\ldots\varkappa_{n-\nu+3}(\alpha)}{\varkappa_{n-2}(\alpha)\,\varkappa_{n-4}(\alpha)\ldots\varkappa_{n-\nu+2}(\alpha)}\,B \tag{7,1}$$

holds for some ν, then also the equality

$$B_{n-\nu-1}(\alpha)=\frac{\varkappa_{n-1}(\alpha)\,\varkappa_{n-3}(\alpha)\ldots\varkappa_{n-\nu+1}(\alpha)}{\varkappa_{n-2}(\alpha)\,\varkappa_{n-4}(\alpha)\ldots\varkappa_{n-\nu}(\alpha)}\,B \tag{7,2}$$

is valid. To prove it (6) will be used. Each equation of the system (6) with an even index on the left hand side is, for the considered ν, given by the relation

$$B'_{n-\nu}(\alpha)=-\varkappa_{n-\nu}(\alpha)\,B_{n-\nu-1}(\alpha)+\varkappa_{n-\nu+1}(\alpha)\,B_{n-\nu+1}(\alpha)$$

Substituting (7,1) into this equation we obtain (7,2), since $B_{n-\nu}(\alpha)=0$ according to Theorem VI. From the last equation (6) it follows that the condition (7,1) holds for $v = 4$. Thus the recurrence is confirmed and consequently (13) also holds according to (6,1). Further from (7,2) and from the first equation (6) we obtain for $v = n - 2$

$$(7,3) \qquad r(\alpha) + r(\alpha + \tfrac{1}{2}a) = B_1 = \prod_{i=1}^{(n-2)/2} \frac{\varkappa_{2i+1}(\alpha)}{\varkappa_{2i}(\alpha)} B .$$

(14) follows from (10) and (7,3).

8a) The first step will be to prove that, in consequence of the fact that the ratios of curvatures $k_{2i}(\alpha)/k_{2i+1}(\alpha)$ $(i = 1, \ldots, \frac{1}{2}(n-2))$ on the curve $\mathscr{C}_s$ are constant, the coefficient $A(\alpha)$ at $B(\alpha)$ in the equation (7) has the form (14); therefore it is constant. The first equation of the system (6) is

$$(8,1) \qquad B_0'(\alpha) + B_1(\alpha) = r(\alpha) + r(\alpha + \tfrac{1}{2}a) .$$

From the last equation (6) we get $B_{n-2}(\alpha) = - B'(\alpha)/\varkappa_{n-1}(\alpha)$, and from the last but one, $B_{n-3}(\alpha) = (\bullet) + (\varkappa_{n-1}(\alpha)/\varkappa_{n-2}(\alpha))\, B(\alpha)$. The expression $(\bullet)$ contains only the derivatives of $B(\alpha)$; we are only interested in the coefficient at $B(\alpha)$ in the equation (7). Recurrently we find out that the functions $B_0(\alpha)$ and $B_{2i}(\alpha)$ are expressed only by derivatives of the function $B(\alpha)$, while the function $B(\alpha)$ itself appears only in expressions of the functions $B_{2i-1}(\alpha)$, namely,

$$(8,2) \qquad B_{2i-1}(\alpha) = [\bullet] + \frac{\varkappa_{n-1}(\alpha)\, \varkappa_{n-3}(\alpha) \ldots \varkappa_{2i+1}(\alpha)}{\varkappa_{n-2}(\alpha)\, \varkappa_{n-4}(\alpha) \ldots \varkappa_{2i}(\alpha)} B(\alpha) .$$

The expression $[\bullet]$ includes only the derivatives of $B(\alpha)$; the fact that A has the form (14) follows from (8,1) and (8,2). From Theorem I it follows that the curve $\mathscr{C}_s$ is of constant breadth B and from Theorem VI that it is Bertrand's curve.

b) The mutual position of the radius-vectors at opposite points on these curves $\mathscr{C}_s$ is given by the equation (13). When the curve $\mathscr{C}_s$ has a constant flexion, it is centrally symmetrical. If the origin of the coordinate system is chosen at the centre of symmetry it follows from (13) that this curve is spherical.

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