CURVES ON 2-MANIFOLDS AND ISOTOPIES

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In [2, 3] R. Baer proved that two homotopic simple closed curves, on an orientable closed 2-manifold M of genus greater than one, are isotopic. He applied this theorem to show that a homeomorphism of M, which is homotopic to the identity, is isotopic to the identity. There have been investigations of the same theorems by W. Brodel [4] and W. Mangler [8].

This paper is a detailed and exhaustive investigation into the theorems of Baer. We shall assume throughout that M is a triangulated connected 2-manifold. But we shall not require M to be compact, nor to be without boundary. The paper is as self-contained as possible. In particular it does not depend on the work cited above.

An essentially new feature is that we treat the case where a basepoint is held fixed. In Theorem 6.3 it is proved that if $h \simeq 1 : M, * \to M, *$, where M is a closed 2-manifold and h is a homeomorphism, then h is isotopic to the identity, keeping the basepoint fixed. This can be regarded as a first step towards proving the conjecture that the space of all such homeomorphisms is contractible, provided M is not a 2-sphere or a projective plane.

In the main part of the paper, we deal with the case where the maps are piecewise linear. In the Appendix we show how to deduce the corresponding topological theorems, by proving that topological imbeddings or homeomorphisms can be approximated by piecewise linear imbeddings or homeomorphisms. In § 5 we prove a number of results for combinatorial manifolds of arbitrary dimension, all of which are more or less already known, but appear not to be in print.

In preparing this paper I have been helped by conversations with a number of people, in particular W. Browder, E. C. Zeeman, H. Zieschang and M. W. Hirsch. Dr. Zieschang pointed out a mistake in my first proof of Theorem 4.1, collaborated with me in finding the crucial counterexample, published in [5], and has proved a number of the results in this paper indepedently. Professor Hirsch told me how to prove the results in the Appendix and discussed with me at great length some early attempts to prove Theorem 4.1.

§ 1. Definitions and some standard theorems

1.1. Let N be an n-dimensional manifold and P a submanifold of dimension $n-1$. If there is an imbedding $h:P\times[-1,1]\to N$, which is the identity on $P\times 0$, and such that

$$h^{-1}\partial N=\partial P\times[-1,1],$$

then we say that P is *two sided* in N.

1.2. Let P be a two sided submanifold of N. If $N-P$ is connected, we form a covering $\tilde{N}$, called the *cyclic P-covering of N* as follows.

When we cut N along P, we obtain a connected manifold N_0 with two copies of P in its boundary. We denote these copies by P_0' and P_0''. Let $p_0:N_0\to N$ be the map which identifies P_0' and P_0''. For each integer r, let N_r be a copy of N_0, and let P_r' correspond to P_0' and P_r'' to P_0''. Let $p_r:N_r\to N$ be a copy of p_0. The covering N is obtained by taking $\bigcup_{-\infty<r<\infty}N_r$ and identifying P_r'' to P_{r+1}' for each r.

The covering map $p:\tilde{N}\to N$ is the map induced by using p_r on N_r for each r. Such a covering is illustrated in Diagram 1.

We have the exact sequence

$$0\to\pi_1\tilde{N}\xrightarrow{p_*}\pi_1N\to Z\to 0, \qquad 1.3$$

where Z is the cyclic infinite group of coverings translations of N.

1.4. We say that the imbeddings $f_0, f_1:X\to Y$ are isotopic, if there is a level-preserving imbedding $F:X\times I\to Y\times I$ which agrees with f_0 on $X\times 0$ and with f_1 on $X\times 1$. We say that f_0 and f_1 are *ambient isotopic* if there is a level-preserving homeomorphism

$$G: Y\times I\to Y\times I,$$

which is the identity on $Y\times 0$ and $g: Y\to Y$ on $Y\times 1$, and such that $f_1=gf_0$. The maps F and G are called *isotopies*.

The following results will be used repeatedly.

1.5. Proposition. *Let $p:X'\to X$ be an r-sheeted covering, where r is finite and X is a finite complex. Then the Euler characteristics satisfy $\chi(X')=r\chi(X)$.*

Proof. For each simplex in X there are r simplexes lying over it in X'.

1.6. Lemma. *Let $p:M'\to M$ be a covering of the 2-manifold M, and let M' be non-compact. Let $D\subset M$ be a 2-dimensional disk, with $p|\partial D$ 1-1. Then $p|D$* is 1-1.

Proof. By lifting D to the universal cover of M', we see that we may assume without loss of generality that M' is the universal cover of M. Let $x, y\in D$, with $px=py$. We shall assume $x\neq y$ and deduce a contradiction.

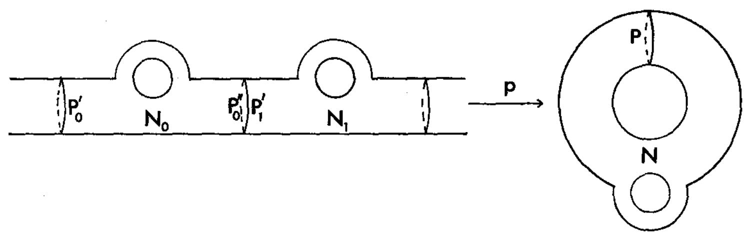

Diagram 1.

Let $\tau: M' \to M'$ be a covering translation such that $\tau x = y$. $D \cup \tau D$ is therefore compact and connected. Since $\partial D \cap \tau\partial D = \emptyset$, we easily check that $D \cup \tau D$ is a submanifold of M'. The boundary of this submanifold is either ∂D or $\tau\partial D$. (It cannot be $\partial D \cup \tau\partial D$, since ∂D is contractible in $D \cup \tau D$.) So we have either $\tau D \subset D$ or $D \subset \tau D$. Without loss of generality let $\tau D \subset \operatorname{int} D$. Then $\tau^k D \subset \operatorname{int} \tau^{k-1} D$ for all k. But $\operatorname{int} D \cap p^{-1} p \partial D$ contains only a finite number of components, so this is a contradiction.

1.7. Theorem. *If a simple closed curve $S \subset M$ is homotopic to zero, it bounds a disk.*

Proof. If M is a two-sphere, then this is the Schoenflies Theorem [9, p. 169].

If M is a projective plane, then S^2 is its universal cover. The inverse image of S consists of two disjoint circles S' and S'' in S^2, which bound disjoint disks D' and D''. The covering translation interchanges D' and D''. Therefore each disk is mapped homeomorphically into M, which proves the theorem when M is a projective plane.

Now suppose M is not a two-sphere or a projective plane. By glueing on $\partial M \times [0, 1)$ to M (identifying $\partial M \times 0$ with ∂M), we can assume M has no boundary. By 1.6 we only need to prove the theorem when M is simply connected.

By the Appendix we may assume S is a subcomplex of some triangulation of M. S is homologous to zero, mod 2. Let C be the chain such that $\partial C = S$. The union of the 2-simplexes in C is a compact submanifold N of M and $\partial N = S$.

N cannot be a Möbius band, for then we could construct a double covering of M by taking two copies of $M - N$ and glueing them to the double cover of N. So N must be a disk. For otherwise there would be a non-separating two-sided simple closed curve T in N, and we could construct a covering of M by taking the cyclic T-covering (see 1.2).

1.8. Corollary. *A non-compact simply connected 2-manifold M without boundary is a plane.*

Proof. Let $M_1, M_2, \ldots$ be a sequence of compact submanifolds of M with the following properties:

1) $M_i \subset \operatorname{int} M_{i+1}$.
2) Each component of $M - \operatorname{int} M_i$ is non-compact.
3) $\bigcup_i M_i = M$.

Applying 1.7 to ∂M_i, we see that for each i, M_i is a disk. Therefore $M_{i+1} - \operatorname{int} M_i$ is homeomorphic to $S^1 \times I$. We can now construct a homeomorphism of M with the plane, by mapping M_i onto the circle with radius i, and extending this map by induction.

§ 2. Two-sided imbeddings of curves without basepoint

2.1. THEOREM. *If f_0 and f_1 are homotopic (piecewise linear) imbeddings of S^1 in $\operatorname{int} M$, such that $f_0 S^1$ is two-sided and does not bound a disk, then f_0 is (piecewise linear) ambient isotopic to f_1, by an isotopy which is fixed outside a compact subset of $\operatorname{int} M$.*

Remarks. 1) If M is orientable and $f_0 S^1$ bounds a disk, then the theorem is false. For we can define $f_1 S^1$ to bound a disk with the opposite orientation. We have $f_0 \simeq f_1$, but since an ambient isotopy preserves orientations, f_0 is not isotpic to f_1. If M is non-orientable the theorem remains true even if $f_0 S^1$ bounds a disk.

2) If $f_0 S^1$ or $f_1 S^1$ meets ∂M we can push it off ∂M using an isotopy (not ambient). So this type of imbedding can be included if we delete the word "ambient" from the theorem.

Before tackling Theorem 2.1, we prove some lemmas.

2.2. LEMMA. *Let X be a compact subset of $\operatorname{int} M$ and let G be a finitely generated subgroup of $\pi_1(M, *)$, where the basepoint is in $\operatorname{int} M$. Then there is a compact connected 2-dimensional submanifold N of $\operatorname{int} M$ with $X \subset \operatorname{int} N$, such that $\pi_1(N, *) \to \pi_1(M, *)$ is one-one and contains G in its image. Each component of $\operatorname{int} M - \operatorname{int} N$ can be assumed non-compact and to have only one boundary component in common with N.*

Proof. We can represent the generators of G by a finite number of simple closed curves in $\operatorname{int} M$. We take a finite connected subcomplex of $\operatorname{int} M$, which contains X and these curves, and then take a regular neighbourhood in $\operatorname{int} M$. We enlarge this submanifold N so that each component of $\operatorname{int} M - \operatorname{int} N$ is non-compact. We further enlarge N so that every component of $\operatorname{int} M - \operatorname{int} N$ has only one boundary component in common with N. The result now follows from van Kampen's Theorem.

2.3. LEMMA. *Let α and β be elements of $\pi_1 M$, where M is not a projective plane. If $\alpha \neq e$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$, then M is a Klein bottle.*

Proof. Let $\tilde{M}$ be the covering space of M corresponding to the subgroup of $\pi_1 M$ generated by α and β. If $\tilde{M}$ is not closed, then by 2.2, $\pi_1 \tilde{M}$ is free. Since $\beta^{-1}\alpha\beta = \alpha^{-1}$, we

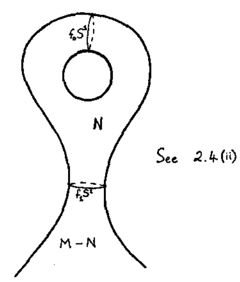

Diagram 2.

obtain the same word when we cyclically reduce α as when we cyclically reduce α^{-1}. This can be seen to lead to a contradiction and we deduce that $\tilde{M}$ is closed. Since $H_1\tilde{M}$ has at most two generators, $\tilde{M}$ is a sphere, a projective plane, a torus or Klein bottle. Therefore $\tilde{M}$ is a Klein bottle, which covers M a finite number of times. Hence M has Euler characteristic zero and is non-orientable. It follows that M is a Klein bottle.

There is a one-one correspondence between conjugacy classes in $\pi_1(M, *)$ and homotopy classes of maps $S^1 \to M$.

2.4. Lemma. *Let $f_0, f_1 : S^1 \to M$ be disjoint (piecewise linear) imbeddings and let f_0S^1 be two-sided. Suppose that the conjugacy class represented by f_1 is not $\{e\}$ and is contained in the normal subgroup generated by $\{f_0\}$. Then either:*

(i) *There is a cylinder $S^1 \times I$ in M, whose boundaries are f_0S^1 and f_1S^1; or*

(ii) *There is a submanifold N of M, which is homeomorphic to a Klein bottle with a disk removed or to a torus with a disk removed. $\partial N = f_1S^1$ and f_0S^1 is a non-separating curve on N. (See Diagram 2.)*

If f_0 and f_1 represent the same conjugacy class then (ii) *cannot occur, and the (piecewise linear) imbedding $F : S^1 \times I \to M$ of* (i) *can be chosen so that $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$.*

Proof. Let $h : S^1 \times [-1, 1] \to \operatorname{int} M - f_1S^1$ be an imbedding such that $h(x, 0) = f_0(x)$. We take $M - h(S^1 \times (-1, 1))$ and identify $h(S^1 \times -1)$ to a point P_{-1} and $h(S^1 \times 1)$ to a point P_1. If f_0S^1 separates M, this gives us two 2-dimensional manifolds, M_{-1} and M_1 with $P_{-1} \in M_{-1}$ and $P_1 \in M_1$ (see Diagram 3). If f_0S^1 does not separate M, we obtain a single connected 2-manifold, which we call M_1, and we put $M_{-1} = \emptyset$. Let

$$p : M - h(S^1 \times (-1, 1)) \to M_1 \cup M_{-1}$$

be the identification. Without loss of generality let $f_1S^1 \subset p^{-1}M_1$.

We define $F : M \to M_1$, agreeing with p on $p^{-1}M_1$. If f_0S^1 separates M, we define $F(M - p^{-1}M_1) = P_1$ (see Diagram 3). If f_0S^0 does not separate M, we define $Fh(x, t) = w(t)$

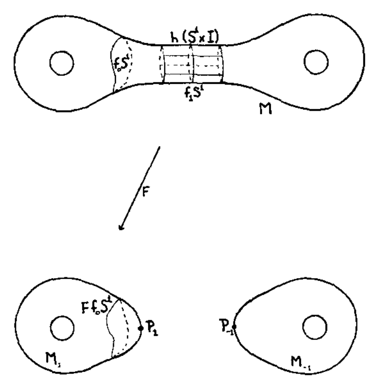

Diagram 3.

where w is a path in M_1 joining P_{-1} to P_1 (see Diagram 4). Then Ff_0 represents the trivial conjugacy class in $\pi_1 M_1$. It follows from the hypothesis that $Ff_1 \simeq 0 : S^1 \to M_1$.

Therefore $Ff_1 S^1$ bounds a disk in M_1. Case (i) arises when the disk contains only one of the points P_1 and P_{-1}. Case (ii) arises when the disk contains both points. If f_0 and f_1 represent the same conjugacy class, they are homotopic and case (ii) cannot occur, for in case (ii) $f_1 S^1$ bounds mod 2 in M, while f_0 does not bound mod 2. (To see that f_0 does not bound, map M onto a torus or Klein bottle by pinching $M - \operatorname{int} N$ to a point.) If the cylinder of (i) does not give the required map F, it gives us a homotopy between f_0 and $f_1 g$, where g is a rotation of S^1 about a diameter. (We are using 5.3 when $n = 1$.) We deduce that $f_0 \simeq f_0 g$. We choose a fixed point of g as the basepoint in S^1, and let its image under f_0 be the basepoint in M. Let f_0 represent $\alpha \in \pi_1(M, *)$. The homotopy $f_0 \simeq f_0 g$ gives us an element β such that $\beta^{-1}\alpha\beta = \alpha^{-1}$. It follows from the hypotheses that M is not the projective plane. By 2.3, M is a Klein bottle.

Now $f_0 S^1$ does not separate, for if it did, α would be in the centre of $\pi_1 M$. Therefore the complement of the cylinder given by (i) is another cylinder. This second cylinder is the one which satisfies the lemma.

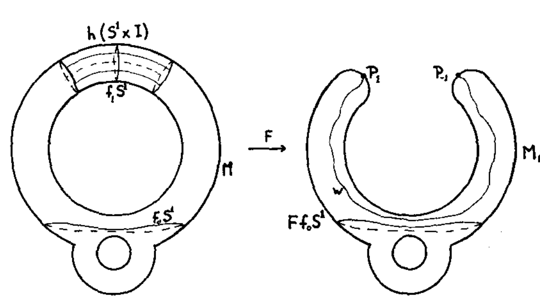

Diagram 4.

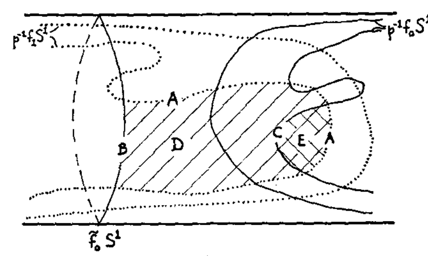

Diagram 5.

Proof of 2.1. Without loss of generality f_0 and f_1 can be assumed piecewise linear (see the Appendix). By an ambient isotopy of f_1, we can assume that $f_0S^1 \cap f_1S^1$ consists of a finite number of points, and that at each such point f_0S^1 and f_1S^1 actually cross.

2.5. LEMMA. *Suppose that $f_0S^1 \cap f_1S \neq \emptyset$. Then there is a disk in M, whose boundary consists of an arc in f_0S^1 and an arc in f_1S^1.*

This lemma shows that there is an ambient isotopy of f_1 which makes f_0S^1 and f_1S^1 disjoint. Applying the last part of Lemma 2.4 then completes the proof of Theorem 2.1.

Proof of Lemma 2.5. Let $p:\tilde{M} \to M$ be a covering with cyclic fundamental group generated by $\tilde{f}_0:S^1 \to M$, a lifting of f_0. Therefore $\tilde{M}$ is orientable. It follows from Lemma 2.2 that int $\tilde{M}$ is homeomorphic to the open cylinder $S^1 \times R$. Let $\tilde{f}_1 \simeq \tilde{f}_1$ be the lifting of $f_1 \simeq f_0$. Both $\tilde{f}_0S^1$ and $\tilde{f}_1S^1$ separate $\tilde{M}$, for otherwise they would be homotopic to zero.

If $f_0S^1 \cap f_1S^1 \neq \emptyset$, there must be an arc A in $p^{-1}f_1S^1$ such that $A \cap \tilde{f}_0S^1 = \partial A$. (To see this, we examine separately the cases $\tilde{f}_1S^1 \cap \tilde{f}_0S^1 = \emptyset$ and $\tilde{f}_1S^1 \cap \tilde{f}_0S^1 \neq \emptyset$. Remember that the components of $p^{-1}f_1S^1$ are not necessarily circles.) Then A together with an arc B in $\tilde{f}_0S^1$ bounds a disk D in $\tilde{M}$. All the components of $D \cap p^{-1}f_1S^1$ are arcs with their endpoints on B. Without loss of generality we may replace A by an innermost such arc. Then $\operatorname{int} D \cap p^{-1}f_1S^1 = \emptyset$. Similarly we find an arc C in $p^{-1}f_0S^1$, so that a subarc of A together with C bounds a disk $E \subset D$ and int E does not meet $p^{-1}f_0S^1$ or $p^{-1}f_1S^1$. (See Diagram 5.) Now $f_0S^1 \cap f_1S^1$ contains more than one point, since it contains $p\partial A$. Therefore $p|\partial E$ is one-one and so by 1.6 $p|E$ is one-one. This completes the proof of Lemma 2.5 and of Theorem 2.1.

§ 3. Arcs and one-sided imbeddings

3.1. THEOREM. *Let A and B be two (piecewise linear) arcs in M such that $\partial M \cap A = \partial A = \partial B = \partial M \cap B$, and which are homotopic keeping the endpoints fixed. Then they are (piecewise linearly) ambient isotopic by an isotopy which is fixed on ∂M and outside a compact subset of M.*

Proof. By the Appendix, we can assume the arcs are piecewise linear. Then we assume that they meet in only a finite number of points, and that at each intercsetion point in $\operatorname{int} M$, the arcs actually cross.

Theorem 3.1 follows from the next lemma.

3.2.. LEMMA. *There is a disk in M whose boundary consists of an arc in A and an arc in B.*

Proof. We look at the universal cover of M and notice that each component of the inverse image of A separates, and similarly for B. The lemma follows by a similar argument to that of 2.5.

3.3. THEOREM. *Let f_0 and f_1 be homotopic (piecewise linear) imbeddings of S_1 in* $\operatorname{int} M$. *If $f_0 S^1$ is one-sided, there is a (piecewise linear) ambient isotopy of f_0 to f_1, which is fixed outside a compact subset of* $\operatorname{int} M$.

Proof. By the Appendix, we can assume f_0 and f_1 are piecewise linear. A regular neighbourhood of $f_0 S^1$ is a Möbius band. Since f_0 does not lift to the orientable double cover of M, neither does f_1. Hence a neighbourhood of $f_1 S^1$ is a Möbius band.

Let P be an abstract Möbius band, with central curve S^1. We have imbeddings $h_0, h_1: P \to \operatorname{int} M$, which agree with f_0, f_1 respectively on the central curve.

Let $F: S^1 \times I \to M$ be the homotopy between f_0 and f_1. If we cut $S^1 \times I$ along $x \times I$, we obtain a square. We glue two such squares together and obtain a homotopy between the loop traversing $f_0 S^1$ twice and the loop traversing $f_1 S^1$ twice. Therefore $h_0 | \partial P \simeq h_1 | \partial P$.

If M is a projective plane, then Remark 1) after the statement of Theorem 2.1 shows that there is an ambient isotopy throwing $h_0 | \partial P$ onto $h_1 | \partial P$. If M is not a projective plane, this follows from Theorem 2.1 itself. If M is not a Klein bottle, the isotopy obviously throws $h_0 P$ onto $h_1 P$. If M is a Klein bottle, this is still true but not obvious. It follows since $f_0 S^1$ and $f_1 S^1$ are homologous mod 2.

To complete the proof of Theorem 3.3, we need only the following theorem.

3.4. THEOREM. *If P is a Möbius band and F is a (piecewise linear) homeomorphism of P onto itself which is the identity on ∂P, then F is (piecewise linearly) isotopic to the identity by an isotopy which is fixed on ∂M.*

Proof. Let A be an arc such that $A \cap \partial P = \partial A$ and such that A does not separate P. Let N be the universal cover of P and let a lifting B of A start at X and end at Y (see Diagram 6). Let α be a generator for the group of covering translations. Without loss of generality Y lies between $\alpha^{2r-1} X$ and $\alpha^{2r+1} X$ ($r \geqslant 0$). We claim that $r = 0$. For if $r > 0$, then B

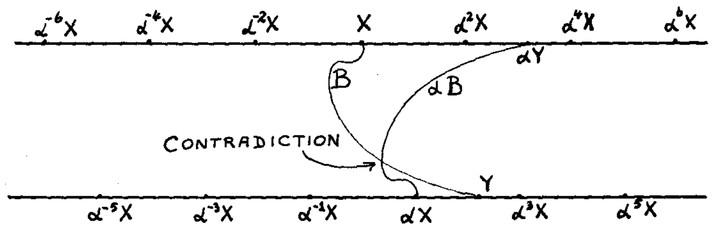

Diagram 6.

separates αX from $\alpha^{2r}X$, and hence B separates αX from αY. But αB joins αX to αY in the complement of B, so this is a contradicition.

It follows that any arc satisfying the same conditions as A, and with the same endpoints, is homotopic to A without moving the endpoints. It follows from Theorem 3.1 that the two arcs are isotopic, and so we can assume that F is the identity on A (see 5.2). We cut P along A and apply Alexander's Theorem (see 5.2) to complete the proof.

§ 4. Curves with basepoint

We wish to prove the following theorem.

4.1. THEOREM. *Let f_0, f_1 be (piecewise linear) imbeddings of S^1 in* int M, *such that $f_0 \simeq f_1 : S^1, * \to M, *$. If $f_0 S^1$ does not bound a disk or Möbius band in M, then there is a (piecewise linear) ambient isotopy between f_0 and f_1 keeping the basepoint fixed and which is fixed outside a compact subset of* int M.

Remarks. 1) If $f_0 S^1$ bounds a disk, we obtain a counterexample by letting $f_1 S^1$ bound a disk with the opposite orientation. (Recall that an isotopy keeping the basepoint fixed will preserve orientations in a neighbourhood of the basepoint.)

2) In the case $f_0 S^1$ bounds a Möbius band, a counterexample is presented in [5].

In order to prove Theorem 4.1, we need some information about the fundamental group of a 2-dimensional manifold.

4.2. THEOREM. *Let S be a simple closed curve in M, which does not bound a Möbius band or a disk. Let $\delta \in \pi_1(M, *)$ be represented by a single circuit of S and let $\delta = \beta^k$ where $k \geq 0$. Then $\delta = \beta$.*

Proof. We attach $\partial M \times [0, 1)$ to M by identifying $\partial M \times 0$ with ∂M. In this way we can assume without loss of generality that $\partial M = \emptyset$. The theorem is obviously true for a projective plane, so we assume that M is some other 2-manifold.

The following will be used in the proof. There are three distinct types of simple closed curves in a Möbius band: those which bound disks, those which bound Möbius bands, and those which are homotopic to the central curve of the Möbius band.

First we suppose that β generates $\pi_1 M$. We apply 2.2 to obtain a compact submanifold N of M with $S \subset \operatorname{int} M$ and $\pi_1(N, *) \to \pi_1(M, *)$ an isomorphism. It follows that N is a cylinder or Möbius band. If N is a cylinder, then δ generates $\pi_1 N$ and so $\delta = \beta$. If M is a Möbius band, then S must be homotopic to the central curve, δ generates $\pi_1 N$ and again $\delta = \beta$.

If β does not generate $\pi_1 M$, we assume $\delta \neq \beta$, and deduce a contradiction. We take the covering space $\tilde{M}$, whose fundamental group is generated by β. S lifts to $T \subset \tilde{M}$. Now T cannot bound a disk in $\tilde{M}$, for then we would have $T \simeq 0$ and so $S \simeq 0$ and then by 1.7 S would bound a disk in M. It follows from the preceding paragraph that T bounds a Möbius band in $\tilde{M}$. Then T is orientation preserving. Therefore so is S and so is every conjugate of δ. Therefore every component of the inverse image of S, which lies in the Möbius band, must bound a Möbius band. We look at the innermost such Möbius band P. The image of P in M is a compact 2-manifold Q with boundary S, which is covered by P. By 1.5 the Euler characteristic of Q is zero and it is non-orientable. Therefore Q is a Möbius band, which contradicts our hypothesis.

4.3. LEMMA. *$\pi_1 M$ has no elements of finite order unless M is a projective plane. If α and β are elements of $\pi_1 M$ which commute and do not lie in the same cyclic subgroup, then M is a torus or a Klein bottle, and both α and β are orientation preserving.*

Proof. Suppose $\alpha \in \pi_1 M$ has finite order. Let $\tilde{M}$ be a covering space of M whose fundametal group is cyclic of prime order p. Then $H_1(\tilde{M}; Z) = Z_p$ and so, by the Universal Coefficient Theorem, $H_2(\tilde{M}; Z_p)$ is non-zero. Therefore $\tilde{M}$ is a closed manifold with Euler characteristic 1. It follows from 1.5 that M is a closed manifold with Euler characteristic one. So M is a projective plane.

Let α and β commute, and suppose they do not lie in a cyclic subgroup. Then they generate a free abelian subgroup of rank two. Let $\tilde{M}$ be the covering space with this fundamental group. Then $\tilde{M}$ is homotopy equivalent to a torus. Therefore it is closed orientable and has Euler characteristic zero. It follows that α and β preserve orientation, and that M has Euler characteristic zero. This proves the lemma.

Proof of 4.1. Without loss of generality, f_0 and f_1 can be assumed piecewise linear.

4.1.1. We first assume M is not a torus and that if M is a Klein bottle, then $f_0 S^1$ is one-sided.

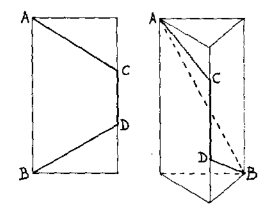

Diagram 7.

By Theorems 2.1 and 3.3, there is a piecewise linear isotopy $h_t:M \to M$, which is constant outside a compact subset of int M, and such that $h_0=1$ and $h_1 f_0 = f_1$. Then $\{h_t * \mid 0 \leqslant t \leqslant 1\}$ is a piecewise linear path in M, which represents an element β in $\pi_1(M, *)$. Let $\{f_0\}=\{f_1\}=\alpha \in \pi_1(M, *)$. We have $\beta^{-1}\alpha\beta=\alpha$ from the homotopy $h_t f_0$. By 4.3, β and α are in the same cyclic subgroup. By 4.2 and the hypotheses of 4.1, α generates this subgroup. Hence β is a power of α. Applying an isotopy which rotates $f_0 S^1$ within itself, we see that without loss of generality we can assume $\beta = e$.

We triangulate M so that $\{h_t *\}$ is a simplicial loop. Since it is homotopic to zero, the loop can be changed to a constant path by a sequence of moves of the following form. If x, y, z, ar three vertices lying in the same simplex, and the loop has xyz occurring as three consecutive vertices, we can replace this part of the loop by xz. Conversely we can replace xz by xyz.

We already have an isotopy $H: M \times I \to M \times I$ which moves the basepoint. In Diagram 7, we show the product of a 1-simplex of M with part of I, and the product of a 2-simplex of M with part of I. We put $F(AB)=ACDB$ or $F(ACDB)=AB$. Using such maps F, we change H to an isotopy which preserves basepoints.

4.1.2. We now assume that M is a torus or a Klein bottle, and $f_0 S^1$ is two-sided but does not bound a Möbius band.

$f_0 S^1$ does not separate M. Let $\tilde{M}$ be the cyclic $f_0 S^1$-covering of M (see 1.2). Let $\tilde{f}_0$ be a lifting of f_0. There is a homeomorphism $\tilde{M} = f_0 S^1 \times R$, such that $\tilde{f}_0 S^1$ corresponds to $f_0 S^1 \times 0$ and every covering translation α satisfies $p_2 \alpha(z,x) = x + n$, for some integer n depending only on α.

By an isotopy of M keeping the basepoint fixed, we can assume that $f_1 S^1$ meets $f_0 S^1$ in a finite number of points, and that where they meet they actually cross, except possibly at the basepoint. We assume that there are intersection points other than the basepoint and we will show how to reduce their number, so that finally $f_0 S^1 \cap f_1 S^1 = *$.

Let $\tilde{f}_0 *$ be the basepoint of $\tilde{M}$, and let $\tilde{f}_1 : S^1, * \to \tilde{M}, *$ be the lifting of f_1. Without loss of generality, we suppose that for every neighbourhood U of the basepoint in S^1,

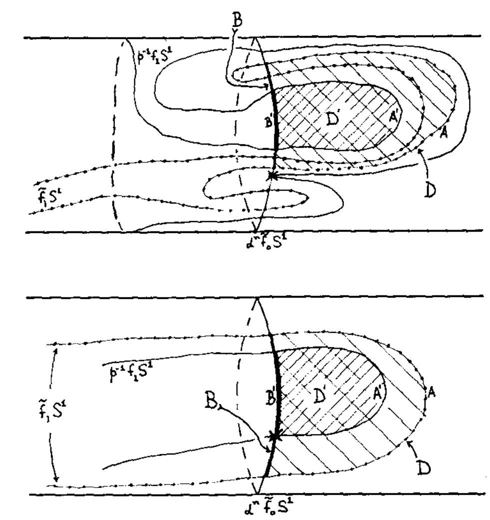

Diagram 8.

$p_2\tilde{f}_1 U$ contains points which are strictly positive. Let α be the covering translation such that $p_2\alpha(z,x) = x+1$, so that α generates the group of covering translations. Let $n \geqslant 0$ be the largest integer such that $\tilde{f}_1 S^1 \cap \alpha^n \tilde{f}_0 S^1 \neq \emptyset$. Let A be an arc in $\tilde{f}_1 S^1$ whose endpoints are in $\alpha^n \tilde{f}_0 S^1$ and such that $p_2 A \subset \{x;\, x \geqslant n\}$. Then A and an arc B in $\alpha^n \tilde{f}_0 S^1$ bound a disk D in $\tilde{M}$. Let $A' \subset p^{-1} f_1 S^1$ be an innermost arc in D, so that A' together with a subarc B' of B bounds a disk D' in $\tilde{M}$. (See Diagram 8.) We have

$$\operatorname{int} D' \cap p^{-1} f_1 S^1 = \operatorname{int} D' \cap p^{-1} f_0 S^1 = \emptyset.$$

Therefore $\alpha^n \tilde{f}^1 S^1 \cap \operatorname{int} D' = \emptyset$ and so $\alpha^n *$ is not in the interior of B'. (The last statement is the subtle part of this proof, and does not go through except in the special circumstances of 4.1.2.) By 1.6, pD' is a disk in M, bounded by the arcs pA' and pB'. The basepoint in M is either disjoint from pD', or it is one of the common endpoints of pA' and pB'. We can therefore perform an isotpy keeping the basepoint fixed, and reducing the number of intersection points by either two or one.

Finally we have $f_0 S^1 \cap f_1 S^1 = *$. Also $f_0 S^1 \cup f_1 S^1$ bounds the interior of a disk (whose closure is a disk with two points on the boundary identified to the basepoint). We perform an isotopy which moves f_1 across this disk to f_0.

§ 5. Some special results

5.1. THEOREM. *Let N be a closed, combinatorial manifold of any dimensional and let*

$$h: N \times [0, \infty) \to N \times [0, \infty)$$

be a piecewise linear homeomorphism which is the identity on $N \times 0$. Then h is piecewise linearly isotopic to the identity by an isotopy which is fixed on $N \times 0$.

Proof. By [6, Theorem 4], we can assume h is the identity on $N \times [0, 1]$.

We shall define a piecewise linear homeomorphism

$$\Phi : [0, \infty) \times [0, \infty) \to [0, \infty) \times [0, \infty).$$

We triangulate the domain by taking as 1-skeleton all lines through integer points which are parallel to the co-ordinate axes or have slope 1. If m and n are integers, we put

$$\begin{aligned} \Phi(n, m) &= (n-m, m) && \text{if } 0 \leqslant m \leqslant n-1 \\ &= (2^{n-m-1}, m) && \text{if } 0 \leqslant n-1 \leqslant m \\ &= (0, m) && \text{if } n = 0. \end{aligned}$$

We extend Φ linearly to each 2-simplex.

We now define a piecewise linear homeomorphism

$$\Psi : N \times [0, \infty) \times [0, \infty) \to N \times [0, \infty) \times [0, \infty).$$

We put
$$\Psi(x, s, t) = (1 \times \Phi^{-1})\{h(x, p_1\Phi(s, t)), t\}.$$

We have $\Psi(x, s, 0) = (h(x, s), 0)$, $\Psi(x, 0, t) = (x, 0, t)$ and if $t \geqslant s-1$, $\Psi(x, s, t) = (x, s, t)$. Let $\sigma : [0, 1) \to [0, \infty)$ be a piecewise linear homeomorphism. We define

$$H : N \times [0, \infty) \times [0, 1] \to N \times [0, \infty) \times [0, 1]$$

by

$$H(x, s, t) = (1 \times 1 \times \sigma^{-1})\Psi(x, s, \sigma t) \quad \text{if} \quad t < 1 \quad \text{and} \quad H(x, s, 1) = (x, s, 1).$$

H is a piecewise linear isotopy of h to the identity which is fixed on $N \times 0$.

We next prove a version of Alexander's Theorem [1].

5.2. Theorem. *If $h : D, * \to D, *$ is a (piecewise linear) homeomorphism of an n-disk, which is the identity on the boundary sphere, then there is a (piecewise linear) isotopy of h to the identity, which is fixed on ∂D and on the basepoint.*

Proof. If the basepoint is in ∂D, we choose x to be any point in $\operatorname{int} D$. If the basepoint is not in ∂D, we choose x to be the basepoint. Let $\phi : D \to \Delta^n$ be a piecewise linear equivalence of D with an n-simplex. We triangulate D so that x is a vertex and so that ϕh is linear on every simplex of D. (The latter condition is of course only imposed if h is piecewise linear.) We triangulate $D \times I$ by taking the triangulation of D on $D \times 1$, the product triangulation

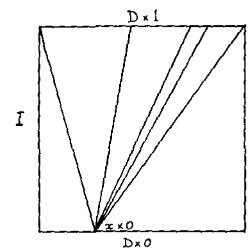

Diagram 9.

of $\partial D \times I$ and then regarding $D \times I$ as the cone on $(D \times 1) \cup (\partial D \times I)$ with $x \times 0$ as the conepoint. (See Diagram 9.) We define a level preserving homeomorphism $H: D \times I \to D \times I$ as follows. $H | \partial D \times I$ is the identity; $H | D \times 1$ is h; $H(x \times 0) = (x \times 0)$; and

$$(\phi \times 1) H: D \times I \to \Delta^n \times I$$

is linear on each rectilinear arc in $D \times I$, such that one end of the arc is $x \times 0$ and the other lies on $(D \times 1) \cup (\partial D \times I)$.

5.3. THEOREM. *Let $h: S^n, * \to S^n, *$ be an orientation preserving piecewise linear homeomorphism. Then there is a piecewise linear isotopy of h to the identity, which is fixed on the basepoint.*

Proof. The result is proved by induction on n. Let D be a piecewise linear disk in S^n, with the basepoint in its interior. Then by an isotopy we can assume $hD \subset \operatorname{int} D$. Since $D - \operatorname{int} hD$ is piecewise linearly equivalent to $S^{n-1} \times I$ [10], there is an isotopy changing h so that $hD = D$. By the theorem in dimension $(n-1)$, we can assume that $h | \partial D$ is the identity. By Alexander's Theorem (see Theorem 5.2) we have our result.

5.4. THEOREM. *Let $h: R^n, * \to R^n, *$ be an orientation preserving piecewise linear homeomorphism of Euclidean n-space. Then there is a piecewise linear isotopy, which is fixed on the basepoint, of h to the identity.*

Proof. As above we can assume h is the identity on a disk D containing the basepoint in its interior. The result follows from Theorem 5.1.

5.5. THEOREM. *There are two (piecewise linear) isotopy classes of (piecewise linear) homeomorphisms $P^2, * \to P^2, *$. If the basepoint is allowed to move, there is only one isotopy class.*

Proof. Given a homeomorphism $h: P^2, * \to P^2, *$, we can, without loss of generality, assume it is piecewise linear (see the Appendix). A basepoint preserving homeomorphism can be put into one of two classes, depending on whether it preserves or reverses the orientation of a neighbourhood of the basepoint. Suppoxe it preserves the orientation. Then we can assume it is the identity on a disk containing the basepoint in its interior. The first part of 5.5 now follows from Theorem 3.4.

If we rotate S^2 through an angle π, keeping the north and south poles fixed, then we induce an isotopy of P^2. If we take the basepoint of P^2 to be the image of a point in the equator of S^2, we have an isotopy of the identity on P^2 to a homeomorphism $P^2, \ast \to P^2, \ast$ which reverses orientations near the basepoint. This proves the second part of 5.5.

5.6. THEOREM. *Let $h: S^1 \times I, \ast \to S^1 \times I, \ast$ be a (piecewise linear) orientation preserving homeomorphism. Then there is a (piecewise linear) isotopy of h to the identity, which preserves the basepoint*

Proof. $h(S^1 \times 0) = S^1 \times 0$ and $h(S^1 \times 1) = S^1 \times 1$. Applying Theorem 4.1 to the circle $S^1 \times t$ which contains the basepoint, we see that without loss of generality, we can assume the basepoint is in $S^1 \times 1$. By 5.3 we can assume that $h \mid S^1 \times 0 \cup S^1 \times 1$ is the identity.

Let the basepoint be $x \times 1$. By rotating $S^1 \times 0$ through a multiple of 2π and keeping $S^1 \times 1$ fixed, we can assume that $h(x \times I)$ is homotopic to $x \times I$ with the endpoints fixed. By Theorem 3.1, we can assume that h is the identity on $S^1 \times 0 \cup S^1 \times 1 \cup x \times I$. To complete the proof, we cut the cylinder along $x \times I$ and apply Alexander's Theorem (see 5.2).

5.7. THEOREM. *Let $h: S^1 \times (-\infty, \infty), \ast \to S^1 \times (-\infty, \infty), \ast$ be a (piecewise linear) orientation preserving homeomorphism. Then there is a (piecewise linear) isotopy of h to the identity which preserves the basepoint.*

Proof. By Theorem 4.1, we can assume that h is the identity on the circle $S^1 \times t$ containing the basepoint. The result follows from Theorem 5.1.

5.8. THEOREM. *Let M be a (closed or open) Möbius band and let h be a (peicewise linear) homeomorphism of M, which is homotopic to the identity. Then h is (piecewise linearly) isotopic to the identity.*

Proof. First let M be a closed Möbius band. Then, since ∂M represents twice the generator of $H_1(M; Z)$, we have $h \mid \partial M \simeq 1: \partial M \to \partial M$. By Theorem 5.3 there is an isotopy changing $h \mid \partial M$ to the identity. We can then apply Theorem 3.4.

Now let M be an open Möbius band and let P be a closed Möbius band in M. Then

$hP \subset Q$ where Q is a closed Möbius band concentric with P. Now $Q - \operatorname{int} P = S^1 \times I$, so we can change h by an isotopy so that $hP = Q$, and then by a further isotopy so that $hP = P$. The result already proved for P, together with Theorem 5.1. gives the desired conclusion.

§ 6. Homeomorphisms

In this section we shall prove that (under appropriate hypotheses), if a homeomorphism of a 2-manifold is homotopic to the identity, it is isotopic to the identity. In the course of the proofs, we need to change various homeomorphisms and homotopies. We show how to make these changes in the next two lemmas.

6.1. LEMMA. *Let M be a 2-manifold other than P^2 or S^2. Let C be a piecewise linear circle in $\operatorname{int} M$ or in ∂M. Let $H: M \times I \to M$ be a piecewise linear homotopy of the identity to a piecewise linear homeomorphism h such that $hC = C$.* If $H(C \times I) \not\subset C$, *we assume that C does not bound a disk or a Möbius band. If M is a Klein bottle or torus, we assume that H keeps a basepoint in C fixed. Let U be a regular neighbourhood of C.*

Then there exists a piecewise linear ambient isotopy $\Phi: M \times I \to M \times I$ of the identity to a homeomorphism ϕ, and a homotopy $H': M \times I \to M$ such that:

a. *Φ is the identity outside $U \times I$ and $\Phi(C \times I) = C \times I$.*

b. *H' is a homotopy of the identity to ϕh. H' is the identity on $C \times t$ for each $t \in I$ and $H'(U \times I) \subset H(U \times I)$.*

c. *$H' = H$ on $\operatorname{Cl}(M - U) \times I$.*

d. *If H keeps a basepoint fixed, so do H' and Φ.*

Proof. If the basepoint is in C, we choose $x \in C$ to be the basepoint. Otherwise we choose $x \in C$ arbitrarily and without loss of generality we assume that the basepoint (if there is one) is not in U. One can check that there is no loss of generality in assuming $hx = x$. Let N be a regular neighbourhood of $H(C \times I)$.

During the homotopy H, x moves through a loop in $H(C \times I)$ which represents $\gamma \in \pi_1(N, x)$. Let C represent $\alpha \in \pi_1(N, x)$. Then $\gamma^{-1}\alpha\gamma = \alpha^{\varepsilon}$, where $\varepsilon = \pm 1$. Now $\gamma = e$ if M is a torus or a Klein bottle, because then x is the basepoint which does not move during the homotopy H. If M is not a torus or a Klein bottle, then by 2.3 $\varepsilon = 1$ and by 4.3, α and γ are in the same cyclic subgroup of $\pi_1(N, x)$. By 4.2 or by $H(C \times I) \subset C$, $\gamma = \alpha^n$ for some integer n. It follows that there is no loss of generality in assuming

$$\gamma = e \in \pi_1(N, x) \approx \pi_1(H(C \times I), x),$$

and if $H(C \times I) \subset C$ that $\gamma = e \in \pi_1(C, x)$. By 5.3 we can assume $h | C$ is the identity.

We construct a map $K: M \times I \times I \to M$ as follows;

for all $y \in M$, $K(y, s, 0) = H(y, s)$, $H(y, 0, t) = y$, $H(y, 1, t) = hy$;
for all $y \in C$, $K(y, s, 1) = y$;
for all $y \in \mathrm{Cl}(M - U)$, $K(y, s, t) = H(y, s)$;
if there is a basepoint preserved by H, then $K(*, s, t) = *$.

We extend K to $x \times I \times I$. (If $x = *$, this has already been done.) We extend to a map from $x \times I \times I$ into $H(C \times I)$. We now define K on $C \times I \times I$. Since $M \neq P^2, S^2$, we have $N \neq P^2, S^2$ and so $\pi_2 H(C \times I) = \pi_2 N = 0$. So we can extend to a map from $C \times I \times I$ into $H(C \times I)$. Using the homotopy extension theorem, we extend to a map from $U \times I \times I$ into $H(U \times I)$.

$H'(y, s) = K(y, s, 1)$ satisfies the lemma.

6.2. Lemma. *Let M be a 2-manifold other than P^2 or S^2. Let A be a piecewise linear arc, with* $\mathrm{int} A \subset \mathrm{int} M$. *Let $H: M \times I \to M$ be a piecewise linear homotopy of the identity to a piecewise linear homeomorphism h such that $hA = A$. Suppose L is a subcomplex of M, such that $A \cap L = \partial A$ and H is a constant homotopy on L. Let U be a regular neighbourhood of A.*

Then there exists a piecewise linear ambient isotopy $\Phi: M \times I \to M \times I$ of the identity to a homeomorphism ϕ, and a homotopy $H': M \times I \to M$ such that:

a. *Φ is the identity on $L \times I$ and outside $U \times I$. $\Phi(A \times I) = A \times I$.*

b. *H' is a homotopy of the identity to ϕh. H' is the identity on $A \times t$ for each $t \in I$, $H'(U \times I) \subset H(U \times I)$ and H' is a constant homotopy on L.*

c. *$H' = H$ on* $\mathrm{Cl}(M - U) \times I$.

d. *If H keeps a basepoint fixed, so do H' and Φ.*

Proof. The proof is similar to that of 6.1, but is considerably less complicated.

6.3. Theorem. *Let $h \simeq 1: M, \partial M \to M, \partial M$ be a proper (piecewise linear) homotopy H. Then there is a (piecewise linear) isotopy of h to the identity. If the homotopy preserves a basepoint, then so does the isotopy.*

Proof. We first prove the theorem when H is piecewise linear.

By 5.3 and 5.4 we may assume that $h | \partial M$ is the identity. By 6.1 we may assume that the homotopy is constant on each compact component of ∂M. 6.1 b and 6.1 c show that the new homotopy is also proper. A non-compact component of ∂M is piecewise linearly homeomorphic to the real line R. We change the homotopy to the constant homotopy on each non-compact component of ∂M by using the linear structure on R. We extend this constant homotopy to a proper homotopy defined on the whole of M, by taking the same homotopy as before outside a neighbourhood of the non-compact components of M, and using the homotopy extension theorem for homotopies of maps $M \times I \to M$ to define the homotopy on the neighbourhood.

So we have reduced the problem to considering a proper homotopy $H: M \times I \to M$ of the identity to h, such that $H(y, t) = y$ if $y \in \partial M$. We continue the proof assuming that H keeps a basepoint in $\operatorname{int} M$ fixed. We construct by induction compact, connected, piecewise linear submanifolds M_i $(i = 1, 2, \ldots)$ of M with the following properties:

(1) If M_1 is a disk then $M_1 \cap \partial M \neq \emptyset$. (By 5.4 the theorem we are trying to prove is true if M is a plane. So we are assuming that M is not a plane.)

(2) The inverse image of the basepoint under H is contained in $\mathring{M}_1 \times I$. (We distinguish between $\mathring{X} = M - \mathrm{Cl}(M - X)$ and $\operatorname{int} X$ which is the subset of X consisting of those points with a neighbourhood in X homeomorphic to the Euclidean plane.)

(3) If M_1 is not a disk or a 2-sphere, we choose an essential simple closed curve S in $\operatorname{int} M_1$ containing the basepoint and not bounding a Möbius band. By 4.1 we can find an ambient isotopy of M, fixed outside a compact subset X of $\operatorname{int} M$, which changes $h|S$ to the identity on S. We insist that $X \subset \operatorname{int} M_2$.

(4) $H^{-1}H(M_i \times I) \subset M_{i+1} \times I$.

(5) Each component of $M - \mathring{M}_i$ is non-compact.

(6) $\bigcup M_i = M$.

Since $H | M \times 0$ is the identity, (3) implies that

$$H^{-1}M_i \subset \mathring{M}_{i+1} \times I \quad \text{and} \quad H(M_i \times I) \subset \mathring{M}_{i+1}.$$

Therefore $H(\partial M_i \times I) \subset \mathring{M}_{i+1} - M_{i-1}$ for each i. We apply 2.1 and 3.1 to each component of $\partial M_{2i} - \partial M$ and we find an isotopy of $\mathring{M}_{2i+1} - M_{2i-1}$, which is fixed outside a compact subset of $\mathring{M}_{2i+1} - M_{2i-1}$ and also on $\partial(\mathring{M}_{2i+1} - M_{2i-1})$, and which changes h to the identity on ∂M_{2i}. We do all these isotopies simultaneously $(i \geqslant 1)$. Then M_1, and hence the basepoint, is not moved by the isotopy. By 6.1 and 6.2 we may without loss of generality assume that the homotopy is constant on ∂M_{2i} for each $i \geqslant 1$.

If M_2 is a disk or a 2-sphere, we may assume that the homotopy $h \simeq 1$ is constant on M_2 by 5.2 and 5.3. If M_2 is not a disk or a 2-sphere, then by condition (3), satisfied by M_2, we can assume $h|S$ is the identity. By 6.1 we can assume the homotopy $h \simeq 1$ is constant on S.

Let $A_1, \ldots, A_{n(1)}$ be disjoint piecewise linear arcs in M_2 with $A_j \cap \partial M_2 \subset \partial A_j \subset S \cup \partial M_2$ and such that cutting M along S and along the arcs, cuts M_2 into disks. By induction on

$i\,(i \geqslant 2)$, we construct disjoint piecewise linear arcs $A_{n(i-1)+1}, \ldots, A_{n(i)}$ in $N_i = M_{2i} - \mathring{M}_{2i-2}$, such that $A_j \cap \partial N_i = \partial A_j$ $(n(i-1) < j \leqslant n(i))$, and such that cutting along the arcs cuts N_i into disks.

Suppose that the homotopy is constant on $A_1 \cup \ldots \cup A_{j-1}$. Without loss of generality, we may assume hA_j meets A_j in a finite number of points. Examining the universal cover of M, we see that each component of the inverse image of S, A_j or $\partial N_i - \partial M$ is an arc or a homeomorphic copy of the real line. Therefore each component of the inverse image of $N_i - S - \bigcup_{k=1}^{j-1} A_k$ is simply connected. Lifting hA_j and A_j to such a component, we see that hA_j is homotopic to A_j in the manifold obtained from N_i by cutting along S and $A_1, \ldots, A_{j-1}$ $(n(i-1) < j < n(i))$, the homotopy keeping the endpoints fixed. By 3.1, there is an ambient isotopy of $N_i - \bigcup_{k=1}^{j-1} A_k$, fixed outside a compact subset, on the boundary and on S, and taking hA_j to A_j. So without loss of generality, we may assume that $h|A_j$ is the identity and by 6.2 that the homotopy is constant on A_j.

Finally, by Alexander's Theorem (see 5.2), we find an isotopy of h to the identity.

If H is not piecewise linear, we change h by an isotopy (keeping the basepoint fixed) to a piecewise linear homoemorphism (see Appendix). We then approximate the homotopy by a piecewise linear homotopy which keeps ∂M within ∂M [14].

If H is does not preserve a basepoint, we pick an arbitrary point $*$ in $\operatorname{int} M$. During the homotopy $*$ moves along a path α from $*$ to $h*$. We perform an isotopy which moves $h*$ to $*$, along a path homotopic to the inverse of α. Changing h by this isotopy, we have $h* = *$ and during the homotopy $*$ moves through a contractible loop. It is now easy to replace H by a homotopy which keeps $*$ fixed, by using the homotopy extension theorem for homotopies of maps $M \times I \to M$. (The only thing to check is that the new homotopy is a proper map.)

6.4. Theorem. *Let M be a 2-manifold such that every component of ∂M is compact. Let h be a (piecewise linear) homeomorphism of M onto itself. If M is a plane, a closed cylinder, or an open cylinder, let h preserve orientations.*

a) *If h is homotopic to the identity, it is (piecewise linearly) isotopic to the identity.*

b) *Let $h \simeq 1 : M, * \to M, *$, and if M is an open or closed Möbius band, let h preserve orientations near the basepoint. Then there is a (piecewise linear) isotopy of h to the identity, keeping the basepoint fixed.*

Proof. We deal first with the special cases of a plane, open and closed cylinders and open and closed Möbius bands. Because of the results of § 5, we have only to prove the following lemma.

6.5. LEMMA. *Let M be an open or closed Möbius band. Then Theorem* 6.4b *is true.*

Proof. We start with the case where M is compact. If the basepoint is in ∂M then the result follows as in the proof of 5.8. Otherwise we take an orientation reversing curve S in $\operatorname{int} M$, which contains the basepoint. By 4.1 we may assume $h|S$ is the identity, and by 6.1 that the homotopy $h \simeq 1$ is constant on S. As in 5.8 we can assume that $h|\partial M$ is the identity.

We take an arc A in M such that $A \cap \partial M = \partial A$, $A \cap S = *$, and such that A does not separate M. S separates A into two subarcs A_1 and A_2. The universal cover U of M is a strip with two boundary components. We choose a basepoint in U and lift $h: M, * \to M, *$ to a homeomorphism $k: U, * \to U, *$. Since h preserves orientations near the basepoint, k sends each component of ∂U into itself. It follows that we can change h by an isotopy keeping the basepoint fixed, so that $k|\partial U$ becomes the identity. Then by examining the universal cover we see that A_1 is homotopic to hA_1, keeping the endpoints fixed, in the cylinder obtained from M by cutting along S. It follows from 3.1 that there is an isotopy on M fixed on ∂M and on S, of $h|A_1$ to the identity. The isotopy required in the lemma is obtained by applying Alexander's Theorem (see 5.2) to the disk obtained by cutting M along S and along A_1.

Now let M be an open Möbius band. Let P be a closed Möbius band in M with the basepoint in $\operatorname{int} P$. As in 5.8, we can assume that $hP = P$. By the result for closed Möbius bands, we can assume $h|P$ is the identity and the required result follows from 5.1 since $M - \operatorname{int} P = S^1 \times [0, \infty)$.

We now prove 6.4, assuming that M is not a plane, an open or closed cylinder, or an open or closed Möbius band. By 6.3 we can assume that $\partial M \neq \emptyset$ or that M is not compact.

If C is a component of ∂M, then $hC = C$ by 2.4. By 6.1 there is no loss of generality in supposing that the homotopy $h \simeq 1$ is constant on ∂M.

We first deal with the case where the homotopy preserves a basepoint in $\operatorname{int} M$. We choose an essential simple closed curve S in $\operatorname{int} M$ containing the basepoint and not bounding a Möbius band. By 4.1, we may assume that $h|S$ is the identity. By 6.1 we may assume that the homotopy $h \simeq 1$ is constant on S.

We construct compact, connected, piecewise linear submanifolds M_i $(i = 1, 2, \ldots)$ of M with the following properties.

(1) If a component of ∂M meets M_i, then the component is contained in $\mathring{M}_i$. $M_i \subset \mathring{M}_{i+1}$. $\bigcup M_i = M$.

(2) $S \subset \operatorname{int} M_1$. If $\partial M \neq \emptyset$, then $M_1 \cap \partial M \neq \emptyset$.

(3) M_1 is not a cylinder or a Möbius band.

(4) Each component of $M - \mathring{M}_i$ is non-compact.

(5) If P is a component of $M_{i+1}-\mathring{M}_i$ which is homeomorphic to $S^1\times I$, then the component of $M-\mathring{M}_i$ containing P is homeomorphic to $S^1\times[0,\infty)$.

Suppose that we have constructed an isotopy which is fixed on ∂M and on the basepoint and which changes $h|M_i$ to the identity. We shall construct an isotopy which is fixed on ∂M and on M_i and changes $h|M_{i+1}$ to the identity. Glueing all these isotopies together, we obtain a piecewise linear level preserving homeomorphism

$$M\times[0,\infty)\to M\times[0,\infty)$$

which is constant on M_i when $t\geqslant i$. Using a piecewise linear homeomorphism of $[0,\infty)$ with $[0,1)$, this gives a level preserving piecewise linear homeomorphism

$$M\times[0,1)\to M\times[0,1).$$

We can extend this to a piecewise linear, level preserving homeomorphism

$$M\times[0,1]\to M\times[0,1]$$

by using the identity on $M\times 1$. This will complete the proof.

We now perform the induction step. We suppose $h|M_i\cup\partial M$ is the identity and the homotopy $h\simeq 1$ is constant on $M_i\cup\partial M$. Using 5.1, we can change h to the identity on each component of $M-\mathring{M}_i$ which is homeomorphic to $S^1\times[0,\infty)$.

Let C be any component of $\partial M_{i+1}-\partial M_i$ not lying in such a component of $M-M_i$. We now examine the covering space of M with cyclic fundamental group, generated by C. Each component of the inverse image of $\partial M_i\cup\partial M_{i+1}$ except for a single component of the inverse image of C, is homeomorphic to the real line. (By 2.3, 4.3 and 4.2, no conjugate of the element represented by C is in the cyclic subgroup of the fundamental group generated by C.)

Let components of ∂M_{i+1} be $C_1, C_2, \ldots, C_n$. Suppose that the homotopy $h\simeq 1$ is constant on $C_1\cup\ldots\cup C_{j-1}$. Putting $C_j=C$ in the preceding paragraph, we see that C_j is homotopic to hC_j in $M-M_i-\bigcup_{k=1}^{j-1}C_k$. By 2.1 there is an isotopy changing $h|C_j$ to the identity, and the isotopy is fixed outside a compact subset of $M-M_i-\bigcup_{k=1}^{j-1}C_k$. By 6.1 we can assume that the homotopy $h\simeq 1$ is constant on C_j. So after n steps, the homotopy is constant on $M_i\cup\partial M_{i+1}$.

In order to make the homotopy constant on M_{i+1} we cut $M_{i+1}-\mathring{M}_i$ into disks with arcs and proceed as in the proof of 6.3. This proves the theorem when there is a basepoint in $\operatorname{int} M$.

To prove the theorem when there is no basepoint in $\operatorname{int} M$, we apply the argument in the last paragraph of the proof of 6.3. This gives us the isotopy we want.

§ 7. Applications to fibre bundles

In [11] and [12], 3-dimensional manifolds which are fibre bundles over a circle, with fibre a 2-manifold, are discussed.

THEOREM. *Let $E \to S^1$ be a fibre bundle, with fibre M, where M is a connected 2-manifold with each boundary component compact.*

a) *If M is not a plane, an open or closed cylinder, then the fibre bundle is determined up to equivalence* [13], *by the homeomorphism $\pi_1 E \to \pi_1 S^1$.*

b) *If M is a plane, an open or closed cylinder, the fibre bundle is determined up to equivalence by whether E is orientable and by the homeomorphism $\pi_1 E \to \pi_1 S^1$.*

Proof. We take as co-ordinate neighbourhoods two overlapping open intervals U and V in S^1. Then $U \cap V = P \cup Q$ where P and Q are disjoint open intervals. Let the co-ordinate transformation be $g: P \to G$ and $h: Q \to G$, where G is the group of the bundle. Let U' and V' be smaller open intervals in U and V respectively, such that $U' \cup V' = S^1$, and let $U' \cap V' = P' \cup Q'$, where $P' \subset P$ and $Q' \subset Q$. We extend $g|P'$ to $g': U' \to G$ by making $g|U' - P'$ constant. We extend $h|Q'$ to $h': V' \to G$ by making $h|V' - Q'$ constant. Using g' and h' to alter the co-ordinate transformations as in [13, p. 12], we find that we have a fibre bundle where the co-ordinate transformations are constant. We can then assume that the co-ordinate transformation $P' \to G$ maps to the identity and the co-ordinate transformation $Q' \to G$ maps to a homeomorphism $f: M \to M$. If M is the projective plane, f is isotopic to 1. If M is not a 2-sphere, a plane, an open or closed cylinder, then the isotopy class of f is determined by its homotopy class. Since the higher homotopy groups of M are zero, f is determined up to isotopy by the outer automorphism f_* of $\pi_1 M$. We have an exact sequence

$$0 \to \pi_1 M \to \pi_1 E \to \pi_1 S^1 \to 0.$$

Let $\alpha \in \pi_1 E$ map onto a chosen generator of $\pi_1 S^1$. Conjugation by α induces f_* on $\pi_1 M$. Therefore the outer automorphism is determined by the homeomorphism $\pi_1 E \to \pi_1 S^1$.

If M is a 2-sphere, a plane, an open or closed cylinder, then the isotopy class of f is fixed by its orientation class and the map $f_*: \pi_1 M \to \pi_1 M$ of the fundamental group.

The result follows in each case, because the isotopy of f to some standard map F gives a level preserving homeomorphism $M \times Q' \to M \times Q'$ where we have f at each level near one end of Q' and F at each level near the other end of Q'.

Appendix

We now refine some results of Baer [1, 2], which show how to approximate a homeomorphism of a 2-manifold by a piecewise linear homeomorphism. The method used here was suggested by M. W. Hirsch.

A1. Theorem. *Let $f: S^1, s \to \operatorname{int} M, m$ be an imbedding. Then there is an ambient isotopy of M, which is fixed on m and outside a compact subset of* $\operatorname{int} M$, *and which changes f to a piecewise linear imbedding.*

Proof. Let D be a small piecewise linear disk in $\operatorname{int} M$, which is a neighbourhood of m. Then $f^{-1}\operatorname{int} D$ consists of a disjoint union of open intervals. Let the closure of the open interval containing m be I_0 and let its endpoints be x and y. We draw piecewise linear arcs X and Y in D, so that $X \cap Y = m$ and $X \cap \partial D = fx$, $Y \cap \partial D = fy$ (see Diagram 10). There is a homeomorphism

$$fI_0, fx, fy, m \to X \cup Y, fx, fy, m.$$

According to the Schoenflies Theorem [9, p. 169] we can extend this to a homeomorphism of D onto itself which is the identity on ∂D. By 5.2, there is an isotopy of M which is fixed on m and on $M - D$, and which changes $f \mid I$ to a piecewise linear map.

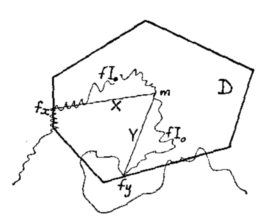

Diagram 10.

Let $D_1, \ldots, D_k$ be disks in $\operatorname{int} M - M$, such that $f(S^1 - \operatorname{int} I_0)$ is covered by $\operatorname{int} D_1, \ldots, \operatorname{int} D_k$. $f^{-1}\operatorname{int} D_i$ is a countable union of open intervals. So $S^1 - \operatorname{int} I_0$ is covered by a finite number of such open intervals $\operatorname{int} I_1, \ldots, \operatorname{int} I_n$, with closures $I_1, \ldots, I_n$. For convenience we put $I_0 = I_{n+1}$. Without loss of generality, we assume $I_r \cap I_s \neq \emptyset$ if and only if $|r-s| \leqslant 1$. For each i, we choose a subarc Ji of $\operatorname{int} I_i$, in such a manner that $\operatorname{int} J_0 \cup \ldots \cup \operatorname{int} J_n$ covers S^1.

For each r we shall try to find a homeomorphism φ_r of M, which is isotopic to the identity by an isotopy which is fixed on m and outside a compact subset of $\operatorname{int} M$, and such that $\varphi_r f$ is piecewise linear on $J_0 \cup \ldots \cup J_r$. We have already defined φ_0. Suppose that we have defined $\varphi_0, \ldots, \varphi_r$. If $r < n$, we try to define φ_{r+1}.

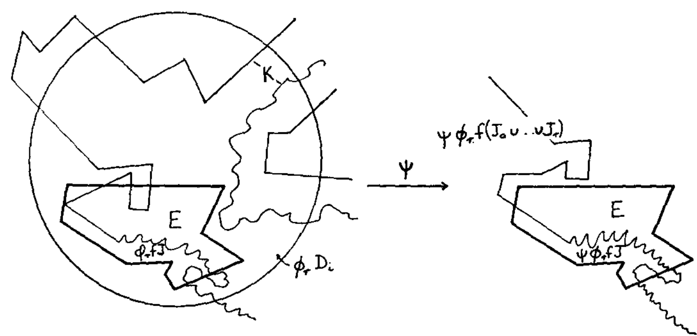

Diagram 11.

Let $\operatorname{int} I_{r+1}$ be a component of $f^{-1}\operatorname{int} D_i$. Now $\varphi_r \operatorname{int} D_i$ inherits the piecewise linear structure of M and is piecewise linearly homeomorphic to a plane.

$$K = \varphi_r f(S^1 - \operatorname{int} I_{r+1}) \cap \varphi_r \operatorname{int} D_i$$

is a closed subspace of $\varphi_r \operatorname{int} D_i$. We choose a compact piecewise linear submanifold E of $\varphi_r \operatorname{int} D_i$, which contains $\varphi_r f J_{r+1}$ in its interior, and which is disjoint from K. Without loss of generality, we can assume E is a disk.

Let $J_{r+1} \subset J \subset I_{r+1}$ where J is the closure of a component of $f^{-1}\varphi_r^{-1}\operatorname{int} E$.

There is a piecewise linear ambient isotopy of M, which is fixed on $M - \operatorname{int}\varphi_r D_i$ and J, of the identity to a homeomorphism ψ, such that $E \cap \psi\varphi_r f(J_0 \cup \ldots \cup J_r)$ is an arc in E, meeting ∂E only at an endpoint. (If $r = n$, we have two arcs instead of one.) Now we apply Alexander's Theorem to E as in the first paragraph of this proof to obtain an isotopy which is fixed outside E, of the identity to a homeomorphism η, such that $\eta\psi\varphi_r f | J$ is piecewise linear. We put $\varphi_{r+1} = \eta\psi\varphi_r$.

A2. Theorem. *Let $F: I \to M$ be an imbedding with $f^{-1}\partial M = \partial I$. Then there is an ambient isotopy of M, which is fixed on ∂M and outside a compact subset of M.*

Proof. We first make f piecewise linear near ∂I, by a method similar to that used in the first paragraph of the proof of A1. After this the proof is identical with that of A1.

A3. Lemma. *Let $g, h: X \to Y$ be ambient isotopic maps and let $H: X \times I \to Y \times I$ be such that $H(x, 0) = (hx, 0)$. Then there is an ambient isotopy of H to a map $G: X \times I \to Y \times I$, such that $G(x, 0) = (gx, 0)$. The isotopy is fixed on $Y \times 1$ and is the given isotopy on $Y \times 0$.*

Proof. Let $\Phi: Y \times I \to Y \times I$ be the ambient isotopy with $\Phi_0 = 1_Y$ and $\Phi_1 h = g$. We define $\Psi: Y \times I \times I \to Y \times I \times I$ by

$$\begin{aligned} \Psi(y, s, t) &= (y, s, t) && \text{if} \quad t \leqslant 2s \\ &= (\Phi_{t-2s} y, s, t) && \text{if} \quad t \geqslant 2s. \end{aligned}$$

We put $(G(x, s), 1) = \Psi(H(x, s), 1)$.

A4. Theorem. *Let $h: M, * \to M, *$ be a homeomorphism. There is an ambient isotopy which is fixed on the basepoint and which changes h to a piecewise linear homeomorphism.*

Proof. $h|\partial M$ is a homeomorphism of ∂M onto itself. This homeomorphism is isotopic to a piecewise linear homeomorphism. By A3 we can extend this to an isotopy of M, using a collar neighbourhood of ∂M. So we can assume $h|\partial M$ is piecewise linear.

Let $M = \bigcup M_i$, where $* \in \partial M_1$, each M_i is a compact piecewise linear submanifold of M and $M_i \subset \mathring{M}_{i+1}$. By Theorem A1, we can assume that h is piecewise linear on each circle in $M_i \cap \mathrm{Cl}(M - M_i)$. By Theorem A2 we can assume h is piecewise linear on each arc in $M_i \cap \mathrm{Cl}(M - M_i)$.

This reduces the problem to the case where M is a compact manifold with boundary and $h|\partial M$ is piecewise linear. We cut M into disks with piecewise linear arcs. We can assume h is piecewise linear on each arc by Theorem A2. Finally we make h piecewise linear on each disk by using Alexander's Theorem (see 5.2).

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