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CURVES WITH MAXIMALLY COMPUTED CLIFFORD INDEX

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Abstract
We say that a curve X of genus g has maximally computed Clifford index if the Clifford index c of X is, for $c > 2$, computed by a linear series of the maximum possible degree $d < g$; then $d = 2c + 3$ resp. $d = 2c + 4$ for odd resp. even c. For odd c such curves have been studied in [6]. In this paper we analyze if/how far analoguous results hold for such curves of even Clifford index c.

1. Introduction

Let X denote a smooth irreducible projective curve defined over the complex numbers, and let $g \geq 4$ resp. $c \geq 0$ denote its genus resp. its Clifford index. We say that a (complete and base point free) linear series g^r_d on X, or a divisor in it, computes c if $d < g$, $r > 0$ and $d - 2r = c$. It is well known ([5, Thm. C]) that in this case we have $d \leq 2c + 4$ if X is neither hyper- nor bi-elliptic (which certainly holds for $c > 2$). For $c > 2$ we say that the Clifford index c of X is *maximally computed* if X has a g^r_d computing c of the maximal possible degree, i.e. $d = 2c + 3$ resp. $d = 2c + 4$ if c is odd resp. even. Such curves exist for every $c > 2$ ([5, 3.3]) and examples are constructed on K3 surfaces.

Let X be such a curve. Then we have $g = d + 1$ ([5, 3.2.5]).

For odd c we also know: X has gonality $c + 3$ and infinitely many pencils g^1_{c+3} ([5, 3.2.2 and 2.3]), and by [6], 3.6 and 3.7 the g^r_d is the only series on X computing c (in particular, it is half-canonical, i.e. $|2g^r_d|$ is the canonical series of X, and very ample); moreover, the g^r_d is even normally generated.

For even c our knowledge on X is less complete ([5], [10]) mainly because a basic Diophantine equation ([6, sections 1 and 2]) valid for X in the case of odd c is not available if X has even Clifford index. One knows, for even c:

- X has gonality $c + 2$,
- for every pencil $|D|$ of degree $c + 2$ on X there is a pencil $|D'|$ of degree $c + 2$ on X such that $g^r_d = |D + D'|$ ([5, 3.2.3 and 3.2.4]),
- X has no base point free pencil of degree $c + 3$ ([5, 3.2.1]),
- X has no series computing c of degree e with $3(c + 2)/2 < e < 2(c + 2) = d$ ([13, Cor. 1]); note that this implies that our g^r_d must be very ample.

In [5, 3.3.2] the following ”recognition theorem” is proved: On any k-gonal curve ($k \geq 3$) having only finitely many base point free pencils of degree k and $k + 1$, a linear series g^r_d,

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$r \geq 2$, computing its Clifford index c computes c maximally and is the only linear series computing c which is not a pencil. (Note that c is even, then, and the g^r_d is half-canonical.) Moreover, it follows that the g^r_d is even normally generated: Since there are (by assumption) only finitely many g^1_{c+2} the curve embedded into $\mathbb{P}^r$ by the g^r_d lies on only finitely many quadrics of rank ≤ 4 which implies (cf. [2, III, ex. D-1 and V, ex. C-7]) that it is quadratically normal, and to see that it is n-normal for all other integers $n \geq 1$ we can use Green's results on Koszul-cohomology, as is done in [6, proof of Theorem 3.6].

However, there are curves whose (even) Clifford index c is computed maximally which have infinitely many pencils of degree $c+2$; this will be shown in the next section where we discuss the case $c = 4$ in greater detail. So the recognition theorem does not always answer the

Question. Is, on our X, the g^r_d the only linear series computing c which is not a pencil?

In this paper we deal with this Question. For $c \equiv 0 \bmod 4$ we prove in Section 3 that every effective divisor of X computing c is contained in a divisor of the g^r_d; in particular, the g^r_d is then the only linear series on X computing c maximally. And for $c = 4$, $c = 6$ and $c = 8$ we answer our Question in the affirmative. Finally, for X lying, via the g^r_d, on a K3 surface of degree $2r - 2$ we check if the divisor theory of the surface may be helpful to provide a negative answer.

Notation. The basic reference is [2]. For any curve X, Div(X) denotes its group of divisors and the symbol $\sim$ means the linear equivalence of divisors. For $D, E \in \mathrm{Div}(X)$ we write $D \leq E$ (and say that D is contained in E) if $E - D$ is effective, i.e. $E - D \geq 0$, and for linear series g^r_d, g^s_e on X the notation $g^r_d \subset g^s_e$ means that every divisor in g^r_d is contained in a divisor of g^s_e (equivalently, $|g^s_e - g^r_d| \neq \emptyset$). We sometimes identify a complete g^r_d on X with the point in the variety $W^r_d = W^r_d(X)$ corresponding to it via the Abel-Jacobi map. (Specifically, for a canonical divisor K_X of X the canonical series $|K_X|$ likewise is the only point in W^{g-1}_{2g-2}, for $g > 0$.)

2. Clifford index $c = 4$

For $c = 4$ we construct a curve whose Clifford index c is maximally computed and satisfies $\dim(W^1_6) > 0$.

Example. Let E denote a smooth elliptic curve and $S \to E$ be a ruled surface with invariant $e \geq 0$. Using the notations of [9, V, 2] we can find a smooth elliptic curve H in the numerical equivalence class of $C_0 + e \cdot f$ ($C_0^2 = -e$, f a fibre); we have $h^0(H) = e + 1$, and $-C_0 - H$ is a canonical divisor of S ([8, 3.3]). Observe that $H^2 = e$ and $C_0 \cdot H = 0$. For $e > 0$ we consider the divisor $D := 3H$ of S; then $|D|$ is base point free and so a general member X in $|D|$ is a smooth curve, by Bertini's theorem. Writing $X = X_1 + X_2$ with effective divisors X_1, X_2 of S, we thus must have $X_1 \cdot X_2 = 0$. If $X_1 \equiv \alpha C_0 + \beta f$ (here $\equiv$ denotes numerical equivalence) we have $X_2 \equiv (3 - \alpha)C_0 + (3e - \beta)f$ with integers $\alpha, \beta \geq 0$, $\alpha \leq 3$, $\beta \leq 3e$ ([8, 3.1]), and $X_1 \cdot X_2 = 0$ implies the relation $(2\beta - e\alpha)(2\alpha - 3) = 3e\alpha$ leading to $X_1 \equiv 0$ for $\alpha = 0$ resp. $X_2 \equiv 0$ for $\alpha = 3$ and $\beta = -e < 0$ for $\alpha = 1$ resp. $\beta = 4e > 3e$ for $\alpha = 2$. Thus it follows that X is irreducible, and its genus is, by adjunction, $g = 3e + 1$.

Now, let $e = 4$ and view the base curve E as an elliptic normal curve in $\mathbb{P}^{e-1} = \mathbb{P}^3$ (of degree $e = 4$); let S_0 denote the cone over E in $\mathbb{P}^4$. Blowing up the vertex of the elliptic cone S_0 we obtain a ruled surface $S \to E$ of invariant $e = 4$ as above ([9, V, 2.11.4]), and the blow down $S \to S_0 \subset \mathbb{P}^4$ is defined by $|H|$. Our curve $X \subset S$ from above blows down to a curve $X' \subset S_0$ of degree $X \cdot H = 3H^2 = 3e = 12 = g - 1$ and X' is smooth since it misses the vertex of S_0. Since $h^0(S, H - X) = h^0(S, -2H) = 0$ the linear series $|H|$ of S cuts out on X a (maybe, incomplete) linear series of degree 12 and dimension $h^0(H) - 1 = e = 4$. Hence X has Clifford index $c \leq 12 - 2 \cdot 4 = 4$. To see that $c = 4$ we recall that on a curve of genus 13 its Clifford index can be computed by pencils; so we have to show that the gonality k of X is 6. Since the natural map $\pi : X \subset S \to E$ has degree $X \cdot f = 3$ our curve X is a triple covering of an elliptic curve; in particular, X has infinitely many g^1_6. If $k < 6$ we obtain, according to Castelnuovo's genus formula for curves with independent morphisms ([2, VIII, ex. C-1]), that $g \leq (k - 1)(3 - 1) + 3g(E) = 2k + 1 \leq 11$, a contradiction. So $k = 6$, $c = 4$, and the series $|H|_X|$ is a complete (and very ample) g^4_{12} on X thus computing $c = 4$ maximally. Since $W^1_6(X)$ contains (at least) the one-dimensional irreducible component $\pi^* W^1_2(E)$ we clearly have $\dim(W^1_6(X)) > 0$.

Proposition 2.1. *Let X be a curve whose Clifford index $c = 4$ is computed maximally. Assume that $\dim(W^1_6) > 0$. Then X admits a triple covering $\pi : X \to E$ over an elliptic curve E, $\pi^*(W^1_2(E))$ is the only infinite irreducible component of W^1_6, and this component is singular with finitely many singularities. Furthermore, X has only one series g^4_{12} (computing c maximally), and the variety W^4_{12} is not reduced.*

Proof. By de Franchis' theorem, on any k-gonal curve X with an infinite set S of g^1_k either infinitely many g^1_k in S are compounded of the same irrational involution or there are only finitely many compounded g^1_k in S. For $k = 6$, in the latter case such a curve is a smooth plane septic ($g = 15$) or we have $g \leq 11$ ([4]), and in the first case infinitely many g^1_6 in S are induced by a covering $\rho : X \to Y$ over a non-hyperelliptic curve Y of genus 3 or by a triple covering $\pi : X \to E$ over an elliptic curve E. Now, let X be a curve whose Clifford index $c = 4$ is computed maximally and admitting infinitely many g^1_6. Since $g = 13$ we then are in the first case from above.

Assume that $\rho : X \to Y$ is a double covering of X over a curve Y of genus 3. Then Y is a smooth plane quartic, and every g^1_6 on X is induced by ρ since otherwise we would have $g \leq (6-1)(2-1)+2g(Y) = 11$, by Castelnuovo's genus formula for curves with independent morphisms. Hence we have $W^1_6(X) = \rho^*(W^1_3(Y)) = \rho^* K_Y - \rho^*(W_1(Y))$. Since we know that there are pencils g^1_6, h^1_6 on X such that $g^4_{12} = |g^1_6 + h^1_6|$ we thus have pencils L_1, L_2 of degree 3 on Y such that $g^4_{12} = |\rho^*(L_1) + \rho^*(L_2)|$. But (cf. [12, p. 1797])

$$h^0(X, \rho^*(L_1 + L_2)) = h^0(Y, L_1 + L_2) + h^0(Y, (L_1 + L_2) - D) = 4 + h^0(Y, L_1 + L_2 - D)$$

for a divisor D of Y such that $2D$ is linearly equivalent to the branch divisor B of ρ (i.e. B is made up by the points of Y over which ρ ramifies). So $2\deg(D) = \deg(B) = 2g-2-2(2g(Y)-2) = 16$, i.e. $\deg(D) = 8 > 6 = \deg(L_1 + L_2)$ which implies that $h^0(Y, L_1 + L_2 - D) = 0$. Thus we obtain $h^0(X, \rho^*(L_1 + L_2)) = 4$ which contradicts $|\rho^*(L_1 + L_2)| = g^4_{12}$.

So X admits a triple covering $\pi : X \to E$ over an elliptic curve E. Our very ample g^4_{12}

embeds X as a curve of degree 12 in $\mathbb{P}^4$. Assume that there is another series on X computing c maximally, i.e. a $h^4_{12} \neq g^4_{12}$. Then $|h^4_{12} - g^4_{12}| = \emptyset$, and, according to a refinement of the base point free pencil trick ([2, III, ex. B-6]) we have: $\dim(|h^4_{12} + g^4_{12}|) \geq 2 \cdot 4 - \dim(|h^4_{12} - g^4_{12}|) + 4 - 1 = 12 = g - 1$ whence $h^4_{12} = |K_X - g^4_{12}|$ and so $|2g^4_{12}| \neq |K_X|$. Thus it follows that $\dim(|2g^4_{12}|) = g - 2 = 11 = 3 \cdot 4 - 1$, and so a result of Castelnuovo ([2, p. 120]) implies that X lies on a non-degenerate surface S of minimal degree in $\mathbb{P}^4$, i.e. on a cubic rational normal scroll. But this is impossible: By Segre's formula for curves on a rational normal scroll whose ruling consists of n-secant lines for the curve, we obtain $13 = g = (n-1)(\deg(X)-1-(n/2)\deg(S)) = (n-1)(12-1-(n/2)\cdot 3)$ which cannot hold. Consequently, we see that a $h^4_{12} \neq g^4_{12}$ cannot exist on X, i.e. W^4_{12} is a point, and this point is not a smooth point of W^4_{12} since the tangent space to W^4_{12} at it has positive dimension ([2, IV, ex. A-2]; observe that the unique g^4_{12} on X is half-canonical).

$W^1_6(X)$ has the irreducible component $\pi^*(W^1_2(E))$. The argument in the beginning of this proof shows that a further infinite irreducible component of $W^1_6(X)$ gives rise to a second triple covering $\pi'^* : X \to E'$ over an elliptic curve E'; but applying Castelnuovo's genus bound for curves admitting independent morphisms to the pair (π, π') of coverings we get the contradiction $g \leq (3-1)(3-1) + 3g(E) + 3g(E') = 10$.

For simplicity we identify our g^4_{12} on X with the point ℓ of $W^4_{12}(X)$ corresponding to it. Then the irreducible component $\ell - \pi^*(W^1_2(E))$ of $W^1_6(X)$ coincides with $\pi^*(W^1_2(E))$. Hence there are four points $p_1, ..., p_4 \in E$ such that $\ell = |\pi^*(p_1 + ... + p_4)|$. Since, on E, $p_1 + ... + p_4 \sim 2q_1 + 2q_2$ for two points $q_1, q_2 \in E$ there exists a $g^1_6 = |\pi^*(q_1 + q_2)|$ on X such that $|2g^1_6| = \ell$, and since X has only finitely many 2-torsion points X has only a finite number of such g^1_6. Recall that the embedding series ℓ is the only g^4_{12} on X. Hence $|2g^1_6| = \ell$ is equivalent with $\dim|2g^1_6| \geq 4$, and it follows ([2, IV, 4.2]) that the g^1_6 in $\pi^*(W^1_2(E))$ satisfying $|2g^1_6| = \ell$ correspond to the singularities of the component $\pi^*(W^1_2(E))$ of $W^1_6(X)$. □

Though $\dim(W^1_6) > 0$ is possible, on every curve X whose Clifford index $c = 4$ is computed maximally only the unique g^4_{12} and the pencils of degree 6 compute c. To see this, recall that X has no series computing c of degree d with $3(c+2)/2 < d < 2(c+2)$, i.e. no g^3_{10}. A g^2_8 on X (computing c) cannot be simple since we know that $W^1_7 = W^1_6 + W_1$ which implies that $|g^2_8 - P|$ has a base point, for every point $P \in X$. So a g^2_8 on X is compounded thus inducing a double covering $\rho : X \to Y$ over a smooth plane quartic, i.e. over a non-hyperelliptic curve of genus 3. But in the proof of the Proposition we observed already that this is impossible.

Finally, we just note that one can show that the curve X of Proposition 2.1 is as in the example. (In fact, viewing X as being embedded by the g^4_{12} it lies in the intersection of two irreducible quadrics in $\mathbb{P}^4$, i.e. on a surface of degree 4 which turns out to be an elliptic cone.)

3. The main result

The following general result is elementary but useful, for our purposes.

Lemma 3.1. *On any curve Y of genus g and Clifford index c let D, E be effective divisors computing c. Then the greatest common divisor (D, E) of D and E has Clifford index* $\mathrm{cliff}((D, E)) \leq c$*, and if* $\dim|(D, E)| > 0$ *then (D, E) and one of the divisors $D + E - (D, E)$*

(*the "least common multiple" of* D *and* E) *resp. its dual* $K_Y - (D + E - (D, E))$ *compute* c.

Proof. Recall that, for a divisor Δ of Y, we have $\text{cliff}(\Delta) = \deg(\Delta) - 2h^0(\Delta) + 2$, $\text{cliff}(K_X - \Delta) = \text{cliff}(\Delta)$, and that the Clifford index c of Y is the minimum of all $\text{cliff}(\Delta)$ such that $h^0(\Delta) \geq 2$ and $h^1(\Delta) \geq 2$ holds.

It is easy to prove the inequality (cf. [14, 2.21])

$$\text{cliff}(D) + \text{cliff}(E) \geq \text{cliff}((D, E)) + \text{cliff}(D + E - (D, E)).$$

Since $\text{cliff}(D) = c = \text{cliff}(E)$ the first claim of the Lemma follows from this inequality provided that $\text{cliff}(D + E - (D, E)) \geq c$. So assume that $\text{cliff}(D + E - (D, E)) < c$. Since $h^0(D + E - (D, E)) \geq h^0(D) \geq 2$ we then must have $h^1(D + E - (D, E)) \leq 1$, and so we obtain $c > \text{cliff}(D + E - (D, E)) = \text{cliff}(K_Y - (D + E - (D, E))) = 2g - 2 - (\deg(D)+\deg(E)-\deg((D, E)))-2h^1(D+E-(D, E))+2 \geq \deg((D, E))$ (recall that $\deg(D) < g$ and $\deg(E) < g$). But $\deg((D, E)) < c$ implies that $h^0((D, E)) = 1$ whence it follows that $\text{cliff}((D, E)) = \deg((D, E)) < c$.

Assume that $h^0((D, E)) \geq 2$. We then have $\text{cliff}((D, E)) \geq c$, and by the (just proved) first claim of the Lemma we see that (D, E) computes c. Hence the inequality at the beginning of this proof shows that $\text{cliff}(D + E - (D, E)) \leq c$. Since $h^0(D + E - (D, E)) \geq 2$ it follows that $|D + E - (D, E)|$ or its dual series computes c (depending on which of these two series has degree $< g$) provided that $h^1(D + E - (D, E)) \geq 2$, too. But for $h^1(D + E - (D, E)) \leq 1$ we obtain $c \geq \text{cliff}(K_Y - (D + E - (D, E))) \geq 2g - 2 - (\deg(D)+\deg(E)-\deg((D, E))) \geq \deg((D, E))$ whence $h^0((D, E)) \leq 1$, a contradiction. □

From now on we use the following notation: X always denotes a curve of genus g whose Clifford index c is even and computed maximally. We set $d_0 := g - 1 = 2c + 4$, $r_0 := (d_0 - c)/2 = (c + 4)/2$, and $g^{r_0}_{d_0}$ is an arbitrary but fixed series on X (computing c maximally). Finally, I denotes the set of effective divisors D of X computing c such that $\deg(D) > c + 2$. (Clearly, $I \neq \emptyset$ since it contains the $g^{r_0}_{d_0}$.)

Theorem 3.2. *Assume that there is a divisor* $D \in I$ *which is not contained in a divisor of the* $g^{r_0}_{d_0}$. *Then* $c \equiv 2 \bmod 4$, D *computes* c *maximally and* $W^{r_0}_{d_0}$ *is infinite.*

Proof. For a divisor $D \in I$ let $d := \deg(D)$, and $r := \dim(|D|) = (d - c)/2 \geq 2$. Using a notation of [5], for any integer $e \geq r - 1$ the set

$$V^{r-2}_e(|D|) := \{E \in \text{Div}(X) : E \geq 0, \deg(E) = e \text{ and } \dim|D - E| \geq 1\}$$

is the variety of e-secant $(r - 2)$-plane divisors of X; if $V^{r-2}_e(|D|) \neq \emptyset$ every irreducible component Z of it has dimension $\dim(Z) \geq 2(r-1)-e$. By [5, 1.2] we know that $V^{r-2}_{2r-3}(|D|) \neq \emptyset$, and for $E \in V^{r-2}_{2r-3}(|D|)$ we have $|D-E| \in W^1_{c+3} = W^1_{c+2}+W_1$. Hence for every $E \in V^{r-2}_{2r-3}(|D|)$ there is exactly one point $P_E \in X$ such that $E + P_E \in V^{r-2}_{2r-2}(|D|)$. So the assignment $E \mapsto E+P_E$ defines a surjection $V^{r-2}_{2r-3}(|D|) \to V^{r-2}_{2r-2}(|D|)$ with finite fibres whence $\dim V^{r-2}_{2r-2}(|D|) = \dim V^{r-2}_{2r-3}(|D|) \geq 2(r-1)-(2r-3) = 1$. Let $i : V^{r-2}_{2r-2}(|D|) \to W^1_{c+2}$ be the natural map defined by $F \mapsto |D - F|$ for $F \in V^{r-2}_{2r-2}(|D|)$.

For any pencil L in the image of i there is a divisor $F \in V^{r-2}_{2r-2}(|D|)$ resp. a pencil L' of degree $c + 2$ on X such that $|D| = |L + F|$ resp. $g^{r_0}_{d_0} = |L + L'|$, and for any point P in the support of F we can find a divisor $E' \in L'$ containing P. Hence for any $E \in L$ the greatest common divisor $G := (E + F, E + E')$ of $E + F \in |D|$ and $E + E' \in g^{r_0}_{d_0}$ contains the divisor $E + P$. So $\deg(G) > \deg(E) = c + 2$, and by Lemma 3.1 we know that $\mathrm{cliff}(G) \leq c$. Since $\dim|G| \geq \dim|E| = 1$ we see that G computes c, i.e. $G \in I$.

Now assume that D is not contained in a divisor of the $g^{r_0}_{d_0}$. Then G is properly contained in $E + F \in |D|$, and so $\deg(G) < d$. Thus the divisor $H := (E + E') + (E + F) - G$ has degree $g - 1 + d - \deg(G) \geq g$, and, again by Lemma 3.1, $|K_X - H|$ is a linear series of degree at most $g - 2 = 2c + 3$ computing c which implies that $\deg(K_X - H) \leq 3(c + 2)/2$, i.e. we have $2(c + 2) - d + \deg(G) = \deg(K_X - H) \leq 3(c + 2)/2$. Hence $\deg(G) \leq d - (c + 2)/2$, and since $\deg(G) > c + 2$ we obtain $d > 3(c + 2)/2$. It follows that $d = 2c + 4 = g - 1$, i.e. $|D|$ is a $g^{(c+4)/2}_{2c+4}$ on X different from our chosen $g^{r_0}_{d_0}$.

CLAIM. Assume that X has a linear series computing c maximally which is different from our $g^{r_0}_{d_0}$. Then $W^{r_0}_{d_0}$ is infinite, and X has linear series of degree $3(c + 2)/2$ computing c.

To prove this claim let $h^{r_0}_{d_0}$ be a $g^{(c+4)/2}_{2c+4}$ on X different from our $g^{r_0}_{d_0}$. For any $L \in W^1_{c+2}$ there is a unique pair (L', L'') of different pencils L', L'' of degree $c+2$ on X such that $g^{r_0}_{d_0} = |L+L'|$ and $h^{r_0}_{d_0} = |L + L''|$. Let $L = |E|$.

Assume that L' and L'' are not compounded of the same involution. Then the General Position Theorem ([1, 4.1]) implies that there is a divisor $E' \in L'$ having with every divisor $E'' \in L''$ at most one point in common, and for every point P in the support of E' we can find a divisor $E'' \in L''$ containing P. With this choice we see, by Lemma 3.1, that $G := (E + E', E + E'') = E + (E', E'') = E + P$ is a divisor computing c which is impossible since $\deg(G) = c + 3$.

Hence the two pencils $L' = |g^{r_0}_{d_0} - L|$, $L'' = |h^{r_0}_{d_0} - L|$ are compounded of the same (irrational) involution. Then there is a covering $\pi : X \to Y$ of maximum possible degree n such that L', L'' are induced from pencils of degree $(c + 2)/n$ on the curve Y (in particular, n divides $c + 2$). For this pair (L', L'') specified by $L = |E|$ we can choose, for any point $P \in X$, unique divisors $E'_P \in L'$, $E''_P \in L''$ having the point P in common. Then the greatest common divisor (E'_P, E''_P) of E'_P and E''_P is the divisor $\pi^*(\pi(P))$ of degree n of X. (Clearly, $\dim|(E'_P, E''_P)| = 0$. Choosing $E'_Q \in L'$, $E''_Q \in L''$ having another point $Q \in X$ in common we either have $(E'_Q, E''_Q) = (E'_P, E''_P)$ - which happens only in the case $\pi(Q) = \pi(P)$ - or that (E'_Q, E''_Q) and (E'_P, E''_P) have no point in common.) The divisor $G_P := (E + E'_P, E + E''_P) = E + (E'_P, E''_P)$ has degree $\deg(G_P) = c + 2 + n = ((\lambda + 1)/\lambda)(c + 2)$ if $2 \leq \lambda := (c + 2)/n$, and according to Lemma 3.1 it computes c. We will show that $\lambda = 2$, i.e. $\deg(G_P) = 3(c + 2)/2$; then Y is an elliptic curve.

For $m \geq 2$ points $P_1, ..., P_m$ of X such that (E'_{P_i}, E''_{P_i}) and (E'_{P_j}, E''_{P_j}) have disjoint support for $1 \leq i < j \leq m$ we set $G_{P_1,...,P_m} := E + (E'_{P_1}, E''_{P_1}) + ... + (E'_{P_m}, E''_{P_m})$. Then $(G_{P_1,...,P_{m-1}}, G_{P_m}) = E$ computes c, and we have $G_{P_1,...,P_m} = G_{P_1,...,P_{m-1}} + G_{P_m} - E = G_{P_1,...,P_{m-1}} + G_{P_m} - (G_{P_1,...,P_{m-1}}, G_{P_m})$. Inductively applying Lemma 3.1 we see that $G_{P_1,...,P_m}$ computes c as long as $\deg(G_{P_1,...,P_m}) = c + 2 + mn = c + 2 + m(c + 2)/\lambda = (1 + (m/\lambda))(c + 2)$ is strictly smaller than g, i.e. for $m \leq \lambda$. If $\lambda \geq 3$ we choose $m = \lambda - 1$ and obtain that $G_{P_1,...,P_{\lambda-1}}$ is a divisor computing c of degree strictly between $3(c + 2)/2$ and $2(c + 2)$; this is not possible. Hence we have $\lambda = 2$. Then we choose $m = \lambda$ whence $\deg(G_{P_1,P_2}) = 2c + 4 = d_0$. Since, for $Q \in X$, we have $G_{P_1,P_2} \sim G_{P_1,Q}$ iff $(E'_{P_2}, E''_{P_2}) = (E'_Q, E''_Q)$ (i.e. $\pi(P_2) = \pi(Q)$) we see that - fixing P_1 but varying P_2 - we obtain this way infinitely many linear series on X which compute c maximally. This proves the claim.

Finally, we observe that $3(c+2)/2 = \deg(G_P) \equiv c \equiv 0 \bmod 2$ implies that $c \equiv 2 \bmod 4$. □

Corollary 3.3. *In the case* $c \equiv 0 \bmod 4$ *the* $g^{r_0}_{d_0}$ *is the only linear series on* X *computing* c *maximally.*

Remark. Let $V^n_e(g^{r_0}_{d_0}) := \{E \in \mathrm{Div}(X) : E \geq 0, \deg(E) = e \text{ and } \dim(|g^{r_0}_{d_0} - E|) \geq r_0 - 1 - n\}$; here $n \in \mathbb{Z}$ with $n \leq e-1$ and $n \leq r_0 - 1$. Choose an integer r such that $1 < r < r_0$ and set $d = c + 2r$ (note that $d_0 - d = 2(r_0 - r)$). The upshot of the Theorem, then, is that $V^{r_0-1-r}_{2(r_0-r)}(g^{r_0}_{d_0}) \cong W^r_d$ (via $E \mapsto |g^{r_0}_{d_0} - E|$). For $r = 1$ (i.e. $d = c+2$) this bijection is wrong since $V^{r_0-2}_{2r_0-2}(g^{r_0}_{d_0})$ is the set of all effective divisors of degree $2r_0 - 2 = c + 2$ of X which move in a non-trivial linear series, i.e. $V^{r_0-2}_{2r_0-2}(g^{r_0}_{d_0}) = \{0 \leq E \in \mathrm{Div}(X) : |E| = g^1_{c+2}\}$; so $V^{r_0-2}_{2r_0-2}(g^{r_0}_{d_0})$ is a $\mathbb{P}^1$-bundle over W^1_{c+2}.
The Theorem thus relates the question if $W^r_d \neq \emptyset$ $(1 < r < r_0)$ to the existence of a $2(r_0 - r)$-secant $(r_0 - 1 - r)$-plane for the curve X viewed as imbedded into $\mathbb{P}^{r_0}$ by the $g^{r_0}_{d_0}$. And for $2(c+2) > d > 3(c+2)/2$ (i.e. for $0 < r_0 - r < (c+2)/4$) we know that there is no such plane.

Corollary 3.4. *Assume that there exists a divisor* $D \in I$ *of degree* $d < g - 1$. *Then* W^1_{c+2} *contains a one-dimensional irreducible component* W *such that for every pencil* $L \in W$ *we have* $\dim|D - L| = 0$*, and the unique divisor in* $|D - L|$ *is contained in a divisor of the pencil* $|g^{r_0}_{d_0} - L|$ *of degree* $c + 2$.

Proof. We use the notation from the proof of the Theorem. Let $r := \dim(|D|)$ and $i|_Z : Z \to W^1_{c+2}$ be the natural map from an irreducible component Z of $V^{r-2}_{2r-2}(|D|)$ into W^1_{c+2}; recall that $\dim(Z) \geq 1$. Since there is no pencil of degree $2r - 2 = d - c - 2 < c + 2$ on X the map i is injective whence we have $\dim(i(Z)) \geq 1$. But since $\dim(W^1_{c+2}) \leq 1$ ([2, VII, ex. C-2]) it follows that $\dim(i(Z)) = 1 = \dim(Z)$. (In particular, $V^{r-2}_{2r-2}(|D|)$ is equi-dimensional of dimension 1.)

Let $W := i(Z)$. Then W is an infinite irreducible component of W^1_{c+2}, and for every $L \in W$ there is a divisor $F \in Z$ such that $|D| = |L + F|$. Since $\deg(F) = 2r-2 = d-(c+2) < c+2$ we have $|D - L| = \{F\}$, and, by the Theorem, F is contained in a divisor of the pencil $|g^{r_0}_{d_0} - L|$. □

Recall that $D \in I$, $\deg(D) < g - 1 = 2c + 4$ implies that $\deg(D) \leq 3(c+2)/2$, and for $c \equiv 0 \bmod 4$ we even have $d < 3(c+2)/2$ since $d \equiv c \equiv 0 \bmod 2$. We add the following observation.

Corollary 3.5. *In* Corollary 3.4*, if* $d < 3(c+2)/2$ *then* W^1_{c+2} *contains a one-dimensional irreducible component (namely* $g^{r_0}_{d_0} - W$*) such that no two different pencils in it are compounded of the same involution.*

Proof. In Corollary 3.4 we have $|g^{r_0}_{d_0} - D| \subset |g^{r_0}_{d_0} - L|$ for any $L \in W$. Setting $d = \deg(D)$ we clearly have $\deg(|g^{r_0}_{d_0} - D|) = d_0 - d$, and we know that $(c+2)/2 = 2(c+2) - 3(c+2)/2 \leq d_0 - d \leq (2c+4) - (c+4) = c$. In particular, $|g^{r_0}_{d_0} - D|$ consists of a single divisor $E \geq 0$.

Assume that two pencils $L' \neq L''$ in $g^{r_0}_{d_0} - W$ are compounded of the same involution thus giving rise to a covering $\pi : X \to Y$ of degree $n \geq 2$ such that L', L'' are induced from pencils of degree $(c+2)/n$ on the curve Y. We can choose divisors $E' \in L'$, $E'' \in L''$ whose greatest common divisor (E', E'') contains E. We may assume that $n = \deg((E', E''))$; then $n \geq \deg(E) \geq (c+2)/2$, and so we obtain $n = (c+2)/2 = \deg(E)$. Thus $d = 3(c+2)/2$; Y is an elliptic curve, then, and $g^{r_0}_{d_0} - W = \pi^*(W^1_2(Y))$. However, for $d < 3(c+2)/2$ this does not occur. □

We see that the divisor $D \in I$ in Corollary 3.5 endows X with a feature of its pencils of minimal degree which - observing that their Brill-Noether number is negative - is apparently only known to be shared by the smooth plane curves (of degree ≥ 6). Cf. Remark 3.8 in [6].

Corollary 3.6. *For integers d, r such that $c+2 \leq d \leq g-1$ and $d-2r = c$ we have dim$(W^r_d) \leq 1$.*

Proof. We have $\dim(W^1_{c+2}) \leq 1$ ([2, VII, ex. C-2]), and since $W^{r_0}_{d_0} \subset g^1_{c+2} + W^1_{c+2}$ for a fixed pencil g^1_{c+2} on X it follows that $\dim(W^{r_0}_{d_0}) \leq 1$. So we assume that $c+2 < d < d_0 = g-1$. Let K be an irreducible component of maximal dimension of W^r_d. Then $\bigcup_{g^r_d \in K} i(V^{r-2}_{2r-2}(g^r_d)) \subset W^1_{c+2}$ is a union of one-dimensional irreducible components $W_1, \ldots, W_n$ of W^1_{c+2}. If $K_j := \{g^r_d \in K | i(V^{r-2}_{2r-2}(g^r_d)) \supset W_j\}$ $(j = 1, \ldots, n)$ we thus have $K = K_1 \cup \ldots \cup K_n$. Fixing $L_j \in W_j$ we have, by Corollary 3.4, a map $\gamma_j : K_j \to \mathbb{P}^1$ which assigns to $g^r_d \in K_j$ that divisor of the pencil $|g^{r_0}_{d_0} - L_j|$ which contains the (unique) divisor $E = |g^{r_0}_{d_0} - g^r_d|$. Since E specifies g^r_d (and since the divisor $\gamma_j(g^r_d)$ of degree $c+2$ contains only a finite number of effective divisors of degree $d_0 - d \leq c$) the fibres of γ_j are finite. Choosing j such that $\dim(K_j) = \dim(K) = \dim(W^r_d)$ it follows that $\dim(W^r_d) \leq \dim(\mathbb{P}^1) = 1$. □

Corollary 3.7. *If the $g^{r_0}_{d_0}$ on X is not unique then every pencil of degree $c+2$ on X is induced by a pencil of degree 2 on a smooth elliptic curve (which is covered by X with $(c+2)/2$ sheets), and I consists of divisors of degree $3(c+2)/2$ and $2(c+2) = d_0$.*

Proof. Let $L \in W^1_{c+2}$. There are pencils $L', L'' \in W^1_{c+2}$ with $L'' \neq L$ such that $\dim(|L' + L|) = r_0 = \dim(|L' + L''|)$, and from the proof of the Claim in the proof of the Theorem we see that L and L'' are compounded of the same elliptic involution of order $(c+2)/2$. The remaining assertion follows from Corollary 3.5. □

Lemma 3.8. *X has no net computing c if $c > 8$.*

Proof. Assume that X has a net g^2_{c+4}. Then for every point $P \in X$ the pencil $g^2_{c+4}(-P)$ of degree $c+3$ has a base point since $W^1_{c+3} = W^1_{c+2} + W_1$. Hence the g^2_{c+4} is not simple. Then it induces a morphism $X \to Y$ of degree $m > 1$ upon an integral plane curve Y of degree $(c+4)/m$. If $m > 2$ or if Y has singularities the normalization of Y has a pencil of degree $d < (c+2)/m$ which induces a pencil of degree $md < c+2$ on X which cannot exist. Hence $m = 2$ and Y is a smooth plane curve of degree $(c+4)/2$. Then Y has genus $g(Y) = (1/2)((c+4)/2 - 1)((c+4)/2 - 2) = c(c+2)/8$, and by the Riemann-Hurwitz genus formula for coverings we obtain $2c + 5 = g \geq 2g(Y) - 1 = c(c+2)/4 - 1$, i.e. $(c-3)^2 \leq 33$ which implies $c \leq 8$. □

For $c = 6$ and $c = 8$ we don't know yet if X has no net computing c.

4. Clifford index $c = 6$ and $c = 8$

In this section we turn to the Question posed in the Introduction, for $c = 6$ and $c = 8$. In these cases the series computing c, besides those computing c maximally, are at most pencils, nets and webs. First, we reduce to pencils and nets, by the

Lemma 4.1. *Let $c = 6$ or $c = 8$. If X has a web computing c then it also has a net computing c.*

Proof. Assume that X has a g^3_{c+6}. Then this series is base point free and simple thus inducing a birational morphism onto an integral space curve X' of degree $c + 6$.

Let $D \in g^3_{c+6}$. The number ρ_2 of conditions imposed on quadrics in $\mathbb{P}^2$ by a general plane section of X' is at most $h^0(2D) - h^0(D)$, and from the proof of Corollary 1 in [13] we know that $h^0(2D) \geq 4 \cdot 3 - 2 = 10$, i.e. $|2D| = g^r_{2c+12}$ with $r \geq 9$. If $r \geq 10$ then X has a $g^{10}_{24} = |K_X - g^2_8|$ for $c = 6$ which is impossible resp. X has a $g^{10}_{28} = |K_X - g^2_{12}|$ for $c = 8$ in which case there is a net computing $c = 8$ on X. So we may assume that $r = 9$ whence $\rho_2 \leq 10 - 4 = 6 = 2\dim(|D|)$. By a lemma of Castelnuovo and Fano's extension of it ([3, 1.10 and 3.1]) this implies that X' lies on a surface S of degree at most 3 in $\mathbb{P}^3$. The proof of Corollary 1 in [13] shows that $X' \subset \mathbb{P}^3$ cannot lie on a quadric; so S is a cubic surface.

The projection $\pi : X' \to \mathbb{P}^2$ with center a smooth point of X' is birational onto its image Y since $c + 5$ is a prime number for $c = 6$ and $c = 8$. Hence Y is a plane curve of degree $c + 5$ which cannot be smooth. Since X has no base point free g^1_{c+3} all singular points of Y are triple points (points of multiplicity 3). Thus the fibre of π at a singular point of Y consists of 3 points of X'. Consequently, X' has a quadrisecant line through every smooth point. Clearly, then, all these lines must lie on the cubic S; since our g^3_{c+6} is complete this is only possible if S is an elliptic cone. The ruling of the cone makes X a 4-fold covering of an elliptic curve. In particular, X has infinitely many g^1_8 which is impossible for $c = 8$. For $c = 6$ we use Segre's formula for the arithmetic genus of a curve on an elliptic scroll whose ruling are n-secant lines for the curve,

$p_a(X') = (n-1)(\deg(X')-1-(1/2)n\deg(S))+n = 3(12-1-(1/2)\cdot 4\cdot 3)+4 = 19 > g = 17.$

So X' has at least one singular point; taking the projection $X' \to \mathbb{P}^2$ with center this point we obtain a net of degree $m \leq \deg(X') - 2 = c + 4 = 10$ on X. Since $c = 6$ we must have $m = 10$, and so we are done. □

Theorem 4.2. *For $c = 6$ and $c = 8$ the $g^{r_0}_{d_0}$ is the only non-pencil on X computing c.*

Proof. By Corollary 3.7, Lemma 4.1 for $c = 6$ resp. Corollary 3.3 for $c = 8$, the $g^{r_0}_{d_0}$ on X is unique (and so, in particular, half-canonical). By Lemma 4.1 it remains to show the non-existence of nets on X computing c. So assume there is a g^2_{c+4} on X. As in the proof of Lemma 3.8 we see that this net induces a double covering $\pi : X \to Y$ over a smooth plane curve Y of degree $(c + 4)/2$. Let σ ($\sigma^2 = id.$) denote the unique automorphism of X/Y.

By Theorem 3.2 there is an effective divisor D_c of X of degree $d_0 - (c + 4) = c$ such that $g^2_{c+4} = |g^{r_0}_{d_0} - D_c|$. Since the g^2_{c+4} is base point free the support of a general divisor $D' \in g^2_{c+4}$ consists of pairwise different points (is "separable") and is disjoint to the support of D_c. Since all divisors in our g^2_{c+4} are of the form $\pi^*(\delta)$ for a divisor δ in the unique net $g^2_{(c+4)/2}$

on Y the divisor D' (being separable) contains no ramification point of π and is σ-invariant (i.e. $\sigma D' = D'$).

Let $D_0 := D' + D_c$. Then $D_0 \in g^{r_0}_{d_0}$. Since the $g^{r_0}_{d_0}$ on X is unique we have $\sigma(g^{r_0}_{d_0}) = g^{r_0}_{d_0}$. In particular, $D' + D_c = D_0 \sim \sigma D_0 = \sigma D' + \sigma D_c = D' + \sigma D_c$, i.e. $\sigma D_c \sim D_c$. But $\dim|D_c| = 0$, and so it follows that $\sigma D_c = D_c$ and, then, $\sigma D_0 = D_0$.

Let $R_1, ..., R_n$ be the ramification points of π; then $R := R_1 + ... + R_n \in \mathrm{Div}(X)$ is the ramification divisor of π, and we have $n = 12$ for $c = 6$, $n = 4$ for $c = 8$. For a σ-invariant divisor $D = \sum_{i=1}^n k_i R_i + \sum_j l_j(P_j + \sigma(P_j)) \in \mathrm{Div}(X)$ with $P_j \neq R_i$ for all i, j we define a divisor $\pi_0 D$ of Y by $\pi_0 D := \sum_{i=1}^n [k_i/2]\pi(R_i) + \sum_j l_j \pi(P_j) \in \mathrm{Div}(Y)$, and we let $V_e(D) := \{f \in H^0(D) | f \circ \sigma = f\}$ resp. $V_o(D) := \{f \in H^0(D) | f \circ \sigma = -f\}$ be the even resp. odd part of $H^0(D)$. Then $\deg(\pi_0 D) \leq (1/2)\deg(D)$, and we have equality here iff $\pi^*(\pi_0 D) = D$. Furthermore, $V_e(D) \cong H^0(Y, \pi_0 D)$ (since $f \in V_e(D)$ has a pole of even order at every ramification point R_i of π), and $H^0(D) = V_e(D) \oplus V_o(D)$.

Let $V_e := V_e(D_0)$, $V_o := V_o(D_0)$. Since $H^0(Y, \pi_0 D') \cong V_e(D') \subset V_e$ we have $\dim(V_e) \geq h^0(\pi_0 D') = 3$. Furthermore, $\dim(V_e) = h^0(\pi_0 D_0)$ with $\deg(\pi_0 D_0) \leq d_0/2 = c+2 = 2\deg(Y)-2$. Since Y is a smooth plane curve it follows that $h^0(\pi_0 D_0) \leq 4$, and if $h^0(\pi_0 D_0) = 4$ holds then $\deg(\pi_0 D_0) = c+2$. So we see that $\dim(V_e) \leq 4$, and if $\dim(V_e) = 4$ then $\pi^*(\pi_0 D_0) = D_0$.

We first consider the case $\dim(V_e) = 3$, i.e. $V_e = V_e(D')$. Then $\dim(V_o) = h^0(D_0) - 3 = (((c+4)/2) + 1) - 3 = c/2$.

Let $D_c \leq R$ (i.e. $\pi_0 D_c = 0$); this is only possible for $c = 6$. By adjunction we have $K_X \sim \pi^*(K_Y) + R \sim \pi^*(2\delta) + R \sim 2D' + R$ for a divisor δ in the net $g^2_{(c+4)/2}$ on Y, and since $|D_0|$ is half-canonical we have $K_X \sim 2D_0 = 2D' + 2D_c$. Hence we have $2D_c \sim R$. For a suitable numbering of the ramification points $R_1, ..., R_{12}$ of π we thus have $2(R_1 + ... + R_6) \sim R_1 + ... + R_6 + R_7 + ... + R_{12}$, i.e. $R_1 + ... + R_6 \sim R_7 + ... + R_{12}$. But X has no g^1_6; hence it follows that $R_1 + ... + R_6 = R_7 + ... + R_{12}$ which is not true.

So we have $2R_i \leq D_c$ for some i or $P + \sigma(P) \leq D_c$ for a non-ramification point $P \in X$. Let $k_i \geq 2$ resp. $l \geq 1$ be the multiplicity of R_i resp. P in D_c; note that k_i is odd. Choose a basis $f_1, ..., f_{c/2}$ of V_o such that R_i resp. P is a pole of order k_i resp. l of these functions. Then there are $a_1, ..., a_{(c/2)-1} \in \mathbb{C}$ such that the functions $g_j := f_{c/2} - a_j f_j \in V_o$ $(j = 1, ..., (c/2) - 1)$ have a pole of order $k_i - 2$ at R_i resp. $l - 1$ at P (and $\sigma(P)$). Then the vector space $V_e \oplus \mathrm{span}(g_1, ..., g_{(c/2)-1})$ of dimension $\dim(V_e) + ((c/2) - 1) = (c/2) + 2$ gives rise to a linear series on X of dimension $(c/2) + 1$ and degree $\deg(D') + 2((c/2) - 1) = 2c + 2$. Since this series computes c we obtain a contradiction.

So we have $\dim(V_e) = 4$, i.e. $h^0(\pi_0 D_0) = 4$. Then $\pi^*(\pi_0 D_0) = D_0$ whence ([12, p. 1797])

$$((c+4)/2) + 1 = h^0(X, D_0) = h^0(X, \pi^*(\pi_0 D_0)) = h^0(Y, \pi_0 D_0) + h^0(Y, \pi_0 D_0 - E)$$

for a divisor E of Y such that $2E$ is linearly equivalent to the branch divisor $\pi_*(R)$ of π.

Thus we obtain $h^0(\pi_0 D_0 - E) = (c+6)/2 - 4 = (c-2)/2$, i.e. $h^0(\pi_0 D_0 - E) = 2$ for $c = 6$ and $h^0(\pi_0 D_0 - E) = 3$ for $c = 8$. But for $c = 6$ we have $\deg(E) = n/2 = 6$ and so $\deg(\pi_0 D_0 - E) = (1/2)\deg(D_0) - \deg(E) = (c+2) - 6 = 2$, i.e. $|\pi_0 D_0 - E|$ is a g^1_2 on Y which is impossible. Let $c = 8$. Then we have $\deg(E) = n/2 = 2$ whence $\deg(\pi_0 D_0 - E) = 8$, i.e. $|\pi_0 D_0 - E|$ is a g^2_8 on Y. Let δ be a divisor in the unique net g^2_6 on Y. Then there are points p_1, p_2, q_1, q_2 of Y such that $\pi_0 D_0 \sim 2\delta - p_1 - p_2$ and $\pi_0 D_0 - E \sim \delta + q_1 + q_2$. (In fact, it is well known that $W^2_8(Y) = W^2_6(Y) + W_2(Y) = |\delta| + W_2(Y)$ for a smooth plane sextic Y whence $W^3_{10}(Y) = |K_Y| - W^2_8(Y) = |3\delta| - (|\delta| + W_2(Y)) = |2\delta| - W_2(Y)$.) So we obtain $\delta + q_1 + q_2 \sim \pi_0 D_0 - E \sim 2\delta - p_1 - p_2 - E$, i.e. $\delta - E \sim p_1 + p_2 + q_1 + q_2$ which implies that $h^0(\delta - E) \geq 1$. But we have $3 = h^0(X, \pi^*(\delta)) = h^0(Y, \delta) + h^0(Y, \delta - E) = 3 + h^0(Y, \delta - E)$ which shows that $h^0(Y, \delta - E) = 0$, and this contradiction proves the Theorem. □

If a smooth curve in $\mathbb{P}^5$ on a cone over a 4-gonal canonical curve of genus 5 is cut out there by a quadric hypersurface it has maximally computed Clifford index 6 and infinitely many g^1_8; so Theorem 4.2 is, for $c = 6$, not merely a consequence of the recognition theorem stated in the Introduction.

5. X on a K3 surface

Viewing X as being embedded into $\mathbb{P}^{r_0}$ by our $g^{r_0}_{d_0}$ it possibly lies on a smooth projective K3 surface S of degree $2r_0 - 2$ in $\mathbb{P}^{r_0}$. (In fact, the examples of curves with maximally computed Clifford index have been constructed in this way, cf. [5, 3.2.6, 3.2.7].) If so, observing that $c < [(g-1)/2] = c + 2$ there exists an effective divisor D of S such that its restriction $D|_X$ to X computes c ([7]). Hence one may ask if it is possible to find an (unexpected) g^r_{c+2r} with $1 < r < r_0$ on $X \subset S$ with the aid of a suitable divisor of S. As a consequence of an interesting result of Knutsen for curves on a K3 surface ([11, 3.4]) we have the

Theorem 5.1. *Assume that X lies, as a curve of degree d_0, on a K3 surface S of degree $2r_0 - 2$ in $\mathbb{P}^{r_0}$. Then for every complete linear series $|D|$ of S without a base curve such that $D|_X$ computes c we have* $\deg(D|_X) = 2c + 4$ *or* $\deg(D|_X) = c + 2$.

Proof. Let H be a hyperplane section of S. We have $H^2 = \deg(S) = 2r_0 - 2 = c + 2$, $X^2 = 2g - 2 = 4c + 8$ and $H \cdot X = d_0 = 2c + 4$, i.e. $(H \cdot X)^2 = 4(c+2)^2 = H^2 X^2$ which implies, by the Hodge index theorem ([9, V, 1.9 and ex. 1.9]), that $X \sim ((H \cdot X)/H^2)H = 2H$. Since the canonical series of S is trivial we have $h^0(H - X) = h^0(-H) = h^2(H) = 0$ and $h^1(H - X) = h^1(X - H) = h^1(H) = 0$ ([15, 2.2]) whence by a standard exact sequence and by the Riemann-Roch theorem ([9, V, 1.6]) it follows that $h^0(X, H|_X) = h^0(H) = 2 + (1/2)H^2 = r_0 + 1$, i.e. $|H|_X| = g^{r_0}_{d_0}$.

Let D be an effective divisor of S such that $|D|$ has no base curve and $D|_X$ computes c. Then $D^2 \geq 0$, and since $\deg(D) = D \cdot H = (1/2) D \cdot X = (1/2)\deg(D|_X) < g - 1 = d_0 = \deg(X)$ we have $h^0(D - X) = 0$.

Assume that $h^1(D) = 0$. Then a standard exact sequence shows that $h^0(X, D|_X) = h^0(D) + h^1(D - X)$. Likewise, if X_0 is an arbitrary smooth irreducible curve in $|2H|$ we have $h^0(X_0, D|_{X_0}) = h^0(D) + h^1(D - X_0)$. Clearly, $D - X_0 \sim D - X$ implies that $h^1(D - X_0) = h^1(D - X)$ whence $h^0(X_0, D|_{X_0}) = h^0(X, D|_X)$. Since, by [7], X_0 has the same Clifford index c as X, we see that $D|_{X_0}$ computes the Clifford index of X_0.

Choose X_0 general in $|2H|$. Then X_0 has only finitely many pencils g^1_{c+2}, according to a theorem of Knutsen ([11, 3.4]), and since the Clifford index c of X_0 is maximally computed (by $H|_{X_0}$) there are no base point free g^1_{c+3} on X_0. Consequently, the recognition theorem (applied to X_0) shows that $D|_{X_0}$ computes c maximally or $|D|_{X_0}| = g^1_{c+2}$. Hence, for X, we have $h^0(X, D|_X) = r_0 + 1$ or (provided that $D^2 = 0$) $h^0(X, D|_X) = 2$.

Assume that $h^1(D) \neq 0$. Then $D \sim kE_0$ for an irreducible curve E_0 with $E_0^2 = 0$ and some integer $k \geq 2$ ([15, 2.6]). We have $k \deg(E_0|_X) = \deg(D|_X) \leq g - 1 = 2c + 4$, and since $h^0(X, E_0|_X) \geq h^0(E_0) \geq 2 + (1/2)E_0^2 = 2$ we have $\deg(E_0|_X) \geq c + 2$. Thus we obtain $k = 2$ and $\deg(D|_X) = 2c + 4$. □

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