Progress of Theoretical Physics, Vol. 64, No. 6, December 1980

Cusp Soliton of a New Integrable Nonlinear Evolution Equation

Miki WADATI, Yoshi H. ICHIKAWA* and Toru SHIMIZU**

Institute of Physics, University of Tokyo, Komaba, Tokyo 153
**Institute of Plasma Physics, Nagoya University, Nagoya 464*
***Institute of Physics, University of Tsukuba, Ibaraki 305*

(Received August 4, 1980)

Inverse scattering method for the already found integrable nonlinear evolution equation

$$q_t-2(1/\sqrt{1+q})_{xxx}=0$$

is presented. A new integrable nonlinear evolution equation

$$r_t+(1-r)^3 r_{xxx}=0,$$

which can be transformed into the above-mentioned equation, is shown to have a singular spiky soliton solution (cusp soliton).

§ 1. Introduction

Very recently, we have found a new series of integrable nonlinear evolution equations.[1] These nonlinear evolution equations have many interesting features mathematically and physically.

The nonlinear Schrödinger type equation,

$$iq_t+\left(\frac{q}{\sqrt{1+|q|^2}}\right)_{xx}=0\,,$$

was solved by the inverse scattering method.[2]

As for the *K-dv* type equation,

$$q_t+\left(\frac{q_x}{\sqrt{1+q^2}}\right)_{xx}=0\,,$$

it was applied to a physical problem concerning the determination of the shape of a one-dimensional droplet in a gravitational field.[3] Also, it has been shown that the equation which describes nonlinear transverse vibrations of elastic beams under tension is reduced to this *K-dv* type equation.[4]

In this paper, we shall study an integrable nonlinear evolution equation

$$q_t-2(1/\sqrt{1+q})_{xxx}=0 \tag{1·1}$$

under the boundary condition

$$q(x, t) \to 0 \quad \text{as} \quad |x| \to \infty . \tag{1·2}$$

The inverse scattering method is applied to Eq. (1·1) according to the same procedure as in Ref. 2).

The outline of the present paper is the following. In § 2, we shall introduce fundamental equations of the inverse scattering problem. Using the results, we shall derive the Gelfand-Levitan equation for our system in § 3. In § 4, we shall obtain one-soliton solution from the Gelfand-Levitan equation. The last section is devoted to discussion. There, we find that a new integrable nonlinear evolution equation,

$$r_t + (1-r)^3 r_{xxx} = 0 , \tag{1·3}$$

has a singular spiky soliton solution (cusp soliton). Equations (1·1) and (1·3) are related by a transformation

$$(1+q)^{-1/2} = 1 - r . \tag{1·4}$$

We shall also discuss the solutions derived by the inverse scattering method under the nonvanishing boundary condition.

§ 2. Scattering problem

We consider the following eigenvalue problem:

$$\psi_{xx} + \lambda^2 (1+q) \psi = 0 . \tag{2·1}$$

The time dependence of the eigenfunctions is chosen to be

$$\psi_t = 2\lambda^2 \left[\frac{2}{\sqrt{1+q}} \frac{\partial}{\partial x} - \left(\frac{1}{\sqrt{1+q}} \right)_x \right] \psi . \tag{2·2}$$

By assuming $\partial\lambda/\partial t = 0$, Eq. (1) arises as a compatibility condition of Eqs. (2·1) and (2·2).

We introduce the Jost functions by

$$\phi(\lambda, x) \to e^{-i\lambda x} \quad \text{as} \quad x \to -\infty , \tag{2·3a}$$

$$\psi(\lambda, x) \to e^{i\lambda x} \quad \text{as} \quad x \to \infty , \tag{2·3b}$$

and the scattering coefficients by

$$\phi(\lambda, x) = a(\lambda) \psi(-\lambda, x) + b(\lambda) \psi(\lambda, x) . \tag{2·4}$$

We investigate the analytic properties of $a(\lambda)$ and the Jost functions for large $|\lambda|$. From Eqs. (2·4) and (2·3), we have

$$\log a = \int_{-\infty}^{\infty} \sigma \, dx , \tag{2·5}$$

where

$$\sigma=\frac{\partial}{\partial x}\log\,(\phi e^{i\lambda x}). \tag{2·6}$$

Define

$$f=\phi e^{i\lambda x}. \tag{2·7}$$

Then, Eq. (2·1) becomes

$$f_{xx}-2i\lambda f_x+\lambda^2 qf=0. \tag{2·8}$$

Substitution of Eq. (2·6) with Eq. (2·7) into Eq. (2·8) yields

$$\sigma_x+\sigma^2-2i\lambda\sigma+\lambda^2 q=0. \tag{2·9}$$

We expand σ in power series of λ:

$$\sigma=\sum_{n=-1}^{\infty}\frac{\sigma_n}{(i\lambda)^n}. \tag{2·10}$$

Inserting this into Eq. (2·9) and equating the terms of the same powers of λ, we obtain

$$\sigma_{nx}+\sum_{l=-1}^{n+1}\sigma_l\sigma_{n-l}-2\sigma_{n+1}-\delta_{n,-2}q=0. \tag{2·11}$$

The first two conserved densities which vanish for $q=0$ are

$$\sigma_{-1}=1-\sqrt{1+q}, \tag{2·12a}$$

$$\sigma_0=-\frac{1}{4}\frac{\partial}{\partial x}\log\,(1+q). \tag{2·12b}$$

From Eqs. (2·5) and (2·10), we see that

$$\log a=i\lambda\varepsilon+O\left(\frac{1}{\lambda}\right), \tag{2·13}$$

where

$$\varepsilon=\int_{-\infty}^{\infty}\sigma_{-1}dx. \tag{2·14}$$

Using Eqs. (2·6)$\sim$(2·8), we have

$$\log\,(\phi e^{i\lambda x})=i\lambda\varepsilon_{-}-\frac{1}{4}\log\,(1+q)+O\left(\frac{1}{\lambda}\right), \tag{2·15}$$

where

$$\varepsilon_{-}=\int_{-\infty}^{x}\sigma_{-1}dx. \tag{2·16}$$

Similar analysis is possible for $\psi(\lambda,x)$. Summing up the results, as $|x|\to\infty$, we have

$$ae^{-i\lambda\varepsilon}=1+O\left(\frac{1}{\lambda}\right), \tag{2·17a}$$

$$\phi e^{i\lambda(x-\varepsilon_-)}=(1+q)^{-1/4}+O\left(\frac{1}{\lambda}\right), \tag{2·17b}$$

$$\psi e^{-i\lambda(x+\varepsilon_+)}=(1+q)^{-1/4}+O\left(\frac{1}{\lambda}\right), \tag{2·17c}$$

where

$$\varepsilon_+(x)=\int_x^\infty \sigma_{-1}dx\,. \tag{2·18}$$

§ 3. Gelfand-Levitan equation

In this section we shall consider the inverse problem for a system (2·1). We assume that q is on compact support. Then, $a(\lambda)e^{-i\lambda\varepsilon}$, $\phi e^{i\lambda(x-\varepsilon_-)}$ and $\psi e^{-i\lambda(x+\varepsilon_+)}$ are entire functions of λ.

From Eq. (2·4), we consider the integral:

$$\int_C \frac{d\lambda'}{\lambda'-\lambda}\frac{\phi(\lambda')e^{i\lambda'(x-\varepsilon_-)}}{a(\lambda')e^{-i\lambda'\varepsilon}}$$

$$=\int_C \frac{d\lambda'}{\lambda'-\lambda}\psi(-\lambda')e^{i\lambda'(x+\varepsilon_+)}+\int_C \frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi(\lambda')e^{i\lambda'(x+\varepsilon_+)}. \tag{3·1}$$

Here an integral path C is the contour in the complex λ plane, starting from $\lambda=-\infty+i0^+$, passing over all zeros of $a(\lambda)$, and ending at $\lambda=+\infty+i0^+$. Similarly, we define $\overline{C}$ to be the contour starting from $\lambda=-\infty+i0^-$, passing under all zeros of $a(-\lambda)$, and ending at $\lambda=+\infty+i0^-$. As the contour C becomes far away, then from Eqs. (2·17a) and (2·17b), we have

$$\text{l.h.s. of Eq. (3·1)}=-i\pi(1+q)^{-1/4}.$$

From Eq. (2·17c), similarly, we have

$$\begin{aligned}&\text{r.h.s. of Eq. (3·1)}\\&=-2i\pi\psi(-\lambda)e^{i\lambda(x+\varepsilon_+)}\\&\quad+\int_{\overline{C}} \frac{d\lambda'}{\lambda'-\lambda}\psi(-\lambda')e^{i\lambda'(x+\varepsilon_+)}+\int_C \frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi(\lambda')e^{i\lambda'(x+\varepsilon_+)}\\&=-2i\pi\psi(-\lambda)e^{i\lambda(x+\varepsilon_+)}+i\pi(1+q)^{-1/4}\\&\quad+\int_C \frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi(\lambda')e^{i\lambda'(x+\varepsilon_+)}.\end{aligned}$$

Therefore, we obtain

$$\psi(-\lambda)\,e^{i\lambda(x+\varepsilon_+)}=(1+q)^{-1/4}+\frac{1}{2\pi i}\int_C\frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\psi(\lambda')\,e^{i\lambda'(x+\varepsilon_+)}. \tag{3·2}$$

We introduce a kernel K by

$$\psi(\lambda,x)=e^{i\lambda(x+\varepsilon_+)}+i\lambda e^{i\lambda\varepsilon_+}\int_x^\infty K(x,s)\,e^{i\lambda s}ds\,. \tag{3·3}$$

The kernel K is assumed to satisfy

$$\lim_{s\to\infty}K(x,s)=0. \tag{3·4}$$

Substitution of Eq. (3·3) into Eq. (2·1) gives

$$1+q=[1-K(x,x)]^{-4}. \tag{3·5}$$

From Eqs. (3·2) and (3·3), we have

$$(1+q)^{-1/4}-1+\int_x^\infty i\lambda K(x,s)\,e^{i\lambda(x-s)}ds+\frac{1}{2\pi i}\int_C\frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}e^{2i\lambda'(x+\varepsilon_+)}$$
$$+\frac{1}{2\pi i}\int_C\frac{d\lambda'}{\lambda'-\lambda}\frac{b(\lambda')}{a(\lambda')}\int_x^\infty i\lambda' K(x,s)\,e^{i\lambda'(s+x+2\varepsilon_+)}ds=0\,. \tag{3·6}$$

Multiplying Eq. (3·6) by $(1/2\pi)\,e^{i\lambda(y-x)}/i\lambda$ and integrating with respect to λ from $-\infty$ to ∞, we arrive at the Gelfand-Levitan equation:

$$K(x,y)-F(x+y)-\int_x^\infty K(x,s)\,F'(s+y)\,ds=0 \tag{3·7}$$

for $x\leqq y$. Here $F(z)$ and $F'(z)$ are defined by

$$F(z)=\frac{1}{2\pi}\int_C\frac{b(\lambda)}{a(\lambda)}\frac{e^{i\lambda(z+2\varepsilon_+(x))}}{i\lambda}d\lambda\,, \tag{3·8a}$$

$$F'(z)=\frac{\partial F}{\partial z}=\frac{1}{2\pi}\int_C\frac{b(\lambda)}{a(\lambda)}e^{i\lambda(z+2\varepsilon_+(x))}d\lambda\,. \tag{3·8b}$$

The time-dependences of the scattering coefficients are determined from Eq. (2·2). The result is

$$a(\lambda,t)=a(\lambda,0)\,, \tag{3·9}$$

$$b(\lambda,t)=b(\lambda,0)\exp(8i\lambda^3t). \tag{3·10}$$

The zeros of $a(\lambda)$ in the upper half λ-plane are the bound state eigenvalues, which we shall designate by $\lambda_k\,(k=1,2,\cdots,N)$. When all the zeros of $a(\lambda)$ are simple, $F(z)$ can be expressed as

$$F(z,t)=\sum_{k=1}^{N}c_k(t)\frac{e^{i\lambda_k(z+2\varepsilon_+)}}{i\lambda_k}$$
$$+\frac{1}{2\pi}\int_{-\infty}^{\infty}\rho(\lambda,t)\frac{e^{i\lambda(z+2\varepsilon_+)}}{i\lambda}d\lambda\,, \tag{3·11}$$

where the time dependences of $c_k(t)$ and $\rho(\lambda,t)$ are

$$c_k(t)=c_k(0)\,e^{8i\lambda_k{}^3t}\,, \tag{3·12a}$$

$$\rho(\lambda,t)=\rho(\lambda,0)\,e^{8i\lambda^3t}\,. \tag{3·12b}$$

The set of Eqs. (3·5), (3·7), (3·11) and (3·12) determines a sought function $q(x,t)$. Given the scattering data $\{\rho(\lambda,0),\ \lambda$ real; $\lambda_k, c_k(0), k=1,2,\cdots,N\}$, we construct $F(z,t)$ by Eq. (3·11) with Eq. (3·12), then we solve Eqs. (3·7) for $K(x,y;t)$, and by Eq. (3·5) we can obtain $q(x,t)$. The function $\varepsilon_+(x)$ is determined from Eq. (2·18).

§ 4. One-soliton solution

We shall analyze a one-soliton solution. For the purpose, we restrict ourselves to the case that $a(\lambda)$ has only one simple zero in the upper half λ-plane and $\rho(\lambda,0)=0$ for real λ. Then Eqs. (3·8) become

$$F(z)=\frac{c_1(t)}{i\lambda_1}e^{i\lambda_1(z+2\varepsilon_+)}, \tag{4·1a}$$

$$F'(z)=c_1(t)\,e^{i\lambda_1(z+2\varepsilon_+)}. \tag{4·1b}$$

Substitution of Eqs. (4·1) into the Gelfand-Levitan equation (3·7) yields

$$K(x,y)=\frac{(c_1(t)/i\lambda_1)\,e^{i\lambda_1(x+y+2\varepsilon_+)}}{1+(c_1(t)/2i\lambda_1)\,e^{2i\lambda_1(x+\varepsilon_+)}}\,. \tag{4·2}$$

We put $i\lambda_1=\kappa(<0)$ and from Eq. (3·12a), we have

$$\frac{c_1(t)}{2i\lambda_1}=e^{-8\kappa^3t-2\kappa x_0}\,, \tag{4·3}$$

where the constant x_0 is defined by

$$\frac{c_1(0)}{2\kappa}=e^{-2\kappa x_0}\,. \tag{4·4}$$

Combining Eq. (4·2) with Eqs. (4·3) and (3·5), we obtain

$$1+q(x,t)=\tanh^{-4}[\kappa(x-x_0-4\kappa^2t+\varepsilon_+)]\,. \tag{4·5}$$

Differentiating Eq. (2·18) and using Eq. (4·5), we have

$$\frac{\partial}{\partial x}(x-x_0-4\kappa^2 t+\varepsilon_+(x))=\tanh^{-2}[\kappa(x-x_0-4\kappa^2 t+\varepsilon_+(x))]. \qquad (4\cdot 6)$$

From Eqs. (4·5), (4·6) and (2·18), we find that $\varepsilon_+(x)$ satisfies a relation

$$\varepsilon_+=\frac{1}{\kappa}\{1+\tanh[\kappa(x-x_0-4\kappa^2 t+\varepsilon_+)]\}. \qquad (4\cdot 7)$$

We observe that the one-soliton solution cannot be expressed in a closed form, however Eqs. (4·5) and (4·7) describe it completely.

§ 5. Discussion

First, we shall examine the one-soliton solution of Eq. (1·3). From Eqs. (1·4) and (4·5), we obtain

$$r(x,t)=\operatorname{sech}^2[\kappa(x-x_0-4\kappa^2 t+\varepsilon_+)]. \qquad (5\cdot 1)$$

Except for the presence of a function $\varepsilon_+(x)$, Eq. (5·1) is in a similar form to the one-soliton solution of the *K-dv* equation. Since ε_+ is given by Eq. (4·7), we can evaluate $r(x,t)$ numerically. The function $\varepsilon_+(u)$ ($u\equiv x-4\kappa^2 t$) and the one-soliton solution $r(u)$ for $\kappa=-1/2$ are plotted in Figs. 1 and 2, respectively. As shown in Fig. 2, the one-soliton solution of Eq. (1·3) has a singularity at the peak of the soliton. Therefore, we shall call it a singular spiky soliton (cusp soliton). The (regular) spiky envelope soliton has been first obtained in the study of circular polarized Alfven wave.[5]

Second, we consider the inverse scattering scheme under the nonvanishing boundary condition

$$q(x,t)\to q_0 \quad \text{as} \quad |x|\to\infty, \qquad (5\cdot 2)$$

instead of Eq. (1·2). By the same analysis as given in the previous sections, we obtain the following results:

$$1+q(x,t)=(1+q_0)\tanh^{-4}[\kappa(\sqrt{1+q_0}(x-x_0)-4\kappa^2 t+\varepsilon_+)], \qquad (5\cdot 3)$$

$$\varepsilon_+=\frac{1}{\kappa}\{1+\tanh[\kappa(\sqrt{1+q_0}(x-x_0)-4\kappa^2 t+\varepsilon_+)], \qquad (5\cdot 4)$$

$$r(x,t)=r_0\operatorname{sech}^2\left[\kappa\left(\frac{1}{r_0}(x-x_0)-4\kappa^2 t+\varepsilon_+\right)\right]+1-r_0, \qquad (5\cdot 5)$$

where

$$r_0=1/\sqrt{1+q_0}. \qquad (5\cdot 6)$$

We can obtain these results more directly. Equation (1·1) is invariant under the following scale transformations:

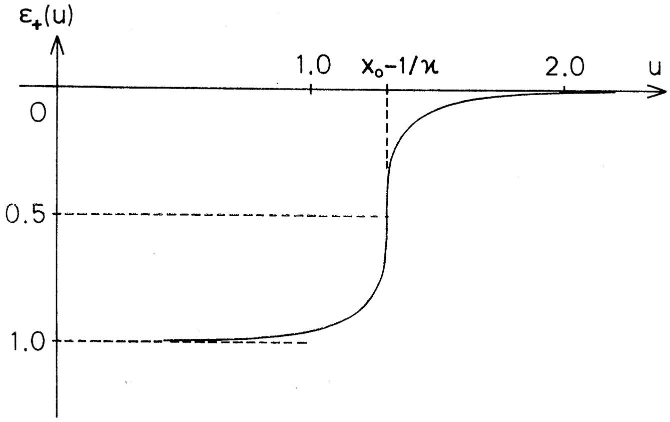

Fig. 1. The curve of $\varepsilon_+(u)$ for $\kappa=-1/2$.

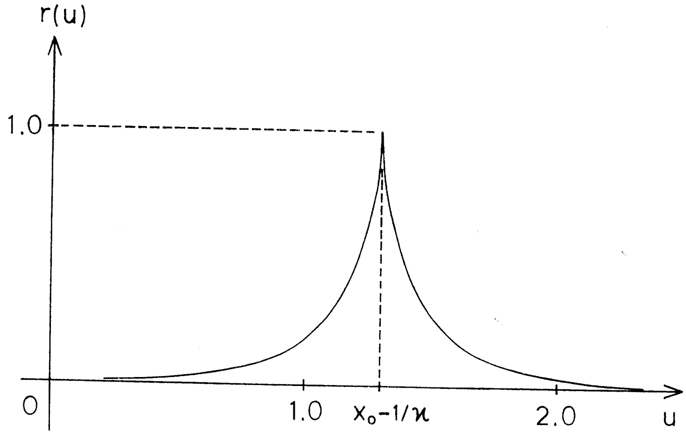

Fig. 2. The curve of the singular spiky soliton solution $r(u)$ for $\kappa=-1/2$ (cusp soliton).

$$x \to (1+q_0)^{1/2}x\,,$$
$$t \to t\,,$$
$$1+q \to (1+q_0)^{-1}(1+q)\,. \tag{5·7}$$

With this transformation, Eqs. (4·5) and (4·7) reduces to Eqs. (5·3) and (5·4), respectively. Corresponding to the transformation (5·7), Eq. (1·3) is invariant under a transformation

$$x \to r_0^{-1}x\,,$$
$$t \to t\,,$$
$$r^{-1}(r+r_0-1) \to r_0^{-1}(r+r_0-1)\,. \tag{5·8}$$

Therefore, the relation between Eqs. (5·1) and (5·5) is clear.

The application of Eqs. (1·1) and (1·3) to the physical system is under investigation.

References

1) M. Wadati, K. Konno and Y. H. Ichikawa, J. Phys. Soc. Japan **47** (1979), 1698.
2) T. Shimizu and M. Wadati, Prog. Theor. Phys. **63** (1980), 808.
3) T. Shimizu, Y. H. Ichikawa and M. Wadati, preprint.
4) Y. H. Ichicawa, K. Konno and M. Wadati, preprint.
5) Y. H. Ichikawa, K. Konno, M. Wadati and H. Sanuki, J. Phys. Soc. Japan **48** (1980), 279.