# CUSPIDAL PROJECTIONS OF SPACE CURVES

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## §1. Introduction

Let $C \subset \mathbb{P}^3_k$ be a smooth, algebraic curve. We say that C admits a cuspidal projection if there exists a point $v \in \mathbb{P}^3_k - C$ such that the linear projection $\pi: C \to \mathbb{P}^2_k$ from v satisfies (i) $\pi: C \to \pi(C)$ is birational, (ii) $\pi(C)$ has only cuspidal (unibranch) singularities.

D.Ferrand showed that a curve that admits a cuspidal projection is a set-theoretical complete intersection, if the base field k has positive characteristic [F1]. He therefore asked: which curves admit a cuspidal projection? What we present below grew out of an attempt to answer this question. The problem is viewed, and attacked, as a geometrical one, however, in the sense that the base field k is assumed to be of characteristic 0.

Suppose a cuspidal projection $\pi: C \to \mathbb{P}^2$ exists. Then clearly the centre of projection v has to lie on the tangent developable of C, and v has to be very singular on this surface. Namely, if C has degree d and genus g, then $\pi(C)$ has $\delta = \frac{1}{2}(d-1)(d-2) - g$ cusps (counted properly), because there are no other kinds of singularities (no self-crossings). For example, v could be a $\delta$-multiple point of the developable, or $\pi(C)$ could have cusps of higher order, arising from v lying on tangents to C at points of inflection or hyperosculation.

One expects a "general" curve $C \subset \mathbb{P}^3$ to be such that its tangent developable has no points of multiplicity greater than 3, that it has no points of inflection, and that cusps arising from projection along tangents at points of hyperosculation are not worse than double. Therefore, it is natural to believe that a general curve, with $\delta \geqslant 4$, does not admit a cuspidal projection. What we shall prove, is the following:

<u>Theorem 1</u>: Every curve with $\delta \leqslant 3$ admits a cuspidal projection.

<u>Theorem 2</u>: A general canonical curve of genus 4 does not admit a cuspidal projection.

Note that a canonical curve $C \subset \mathbb{P}^3$ of genus 4 is the complete intersection of a quadric and a cubic surface. Hence Theorem 2 indicates that there is no relation in general between the property of admitting a cuspidal projection and that of being a (set-theoretical) complete intersection. In fact, the curves in Ferrand's theorem are the set-theoretical intersection of a very special surface – namely the cone of the cuspidal projection – with some other surface, and, as he remarked later [F2], this can only happen in positive characteristic.

Theorem 1 is proved by examining each type of curve satisfying $\delta \leqslant 3$. Because of Castelnuovo's bound on the genus of a space curve, there are only three cases: the twisted cubic, and the elliptic and rational quartics. We study the possible configurations of the tangents to these curves; in particular, we use Telling's classification of rational quartics [T]. The proof also requires some general facts about the tangent developable and its singularities, and thus links up with classical enumerative geometry [Z] - the necessary material is gathered in §4. In the course of this proof, the various types of possible cuspidal projections are described.

Theorem 2 is proved by showing that the presence of certain phenomena, necessary for a cuspidal projection to exist, implies that the canonical curve does not have general moduli. We use the fact that these curves are complete intersections (to parametrize them), and that they lie on a quadric surface. However, the proof should illustrate what one would need to prove in order to generalize Theorem 2 to other curves.

I would like to thank Robin Hartshorne for suggesting looking at the canonical curves of genus 4 in this context.

## §2. Projection of branches

Fix a base field k, algebraically closed and of characteristic 0. Let $P = \mathbb{P}^3_k$ denote projective 3-space over k – in the coordinate free way we shall also write $P = \mathbb{P}(V)$, with V a 4-dimensional vector space over k. Let $C_0 \subset P$ be a reduced, closed curve, and let $h: C \to C_0$ denote its normalization. We shall assume that $C_0$ spans P, i.e., $C_0$ is not contained in a plane.

For each point $p \in C$ we can choose (affine) coordinates around $h(p) \in P$ such that the branch of $C_0$ determined by p has a (formal) parametrization at h(p) equal to

$$(+)\qquad \begin{aligned} x &= at^{l_1+1} + \dots \\ y &= bt^{l_2+2} + \dots \\ z &= ct^{l_3+3} + \dots \end{aligned}$$

with $abc \neq 0$ and $0 \leq l_1 \leq l_2 \leq l_3$. Even if h(p) is a singular point of $C_0$ we shall call the line $y = z = 0$ for the tangent to $C_0$ at p, and the plane $z = 0$ for the osculating plane to $C_0$ at p.

We call $k_i(p) = l_{i+1} - l_i$ the ith stationary index of the branch at the point (or of p, for short); thus $k_0(p)$ is the number of cusps of $C_0$ at p, $k_1(p)$ the number of flexes (points of inflection), and $k_2(p)$ the number of stalls (points of hyperosculation). A point $p \in C$ with $l_1 = l_2 = l_3 = 0$ is called regular – there are only a finite number of non-regular points. The triple $(l_1+1, l_2+2, l_3+3)$ is called the type of p. If C maps to a plane curve, the type of p with respect to this map will be a pair $(m_1+1, m_2+2)$.

Let $\pi: P - \{v\} \to \bar{P} \cong \mathbb{P}^2$ denote the projection from a point $v \in P$ onto a plane $\bar{P}$. The rational map $\pi|C_0 : C_0 \to \bar{P}$ is defined on $C_0$ if $v \notin C_0$, and is in any case defined on C; by abuse of notation, we shall call this morphism $\pi$ also.

Set $\bar{C} = \pi(C)$. Suppose $p \in C$ is of type $(l_1+1, l_2+2, l_3+3)$. The point $\pi(p)$ on the corresponding branch of $\bar{C}$ will be of type $(l_1+1, l_2+2)$, unless v is on the tangent to $C_0$ at p.

Suppose v is on the tangent to $C_0$ at p (but $v \neq h(p)$). Then the plane curve branch will have type $(l_2+2, l_3+3)$. This cusp will be ordinary, i.e., of type (2,3), if and only if p was regular on $C_0$.

The number $\delta(p)$ of (ordinary) double points of $\bar{\sigma}$ absorbed by the cusp $\pi(p)$ depends of course on the type, though it is not always determined by it. For example, if $(l_2+2, l_3+3) = 1$, then

$$\delta(p) = \tfrac{1}{2}(l_2+1)(l_3+2) .$$

In particular,

$$\delta(p) = \begin{cases} 1 & \text{if } p \text{ is a regular point} \\ 3 & \text{if } p \text{ is an ordinary flex (i.e., } l_1 = 0,\ l_2 = l_3 = 1 \text{).} \end{cases}$$

If p is an ordinary stall ($l_1 = l_2 = 0$, $l_3 = 1$), the type is no longer sufficient to determine $\delta(p)$. However, we have the following:

<u>Lemma 1</u>: If $p \in C$ has a parametrization

$$x = at + \dots$$
$$y = b_2t^2 + b_3t^3 + \dots$$
$$z = c_4t^4 + c_5t^5 + \dots$$

with $ab_2c_4 \neq 0$ and $b_2c_5 \neq b_3c_4$, then the projection of $C_o$ from a point on its tangent at p gives a cusp with $\delta(p) = 2$ ; in fact, the cusp is ramphoid of the 1st type.

<u>Remark</u>: Recall that a ramphoid cusp of the sth type is one which can be put in the form

$$x = t^2$$
$$y = at^4 + (\text{even powers of } t) + bt^{2s+3} + (\text{higher powers of } t),$$

with $ab \neq 0$. Such a cusp is equivalent to s+1 ordinary double points.

<u>Proof</u>: The cusp $\pi(p)$ is equivalent to one of the form

$$\bar{y} = y(b_2 + b_3t + \dots)^{-1} = t^2$$
$$\bar{z} = z(b_2 + b_3t + \dots)^{-1} = (c_4t^4 + \dots)(b_2^{-1} - b_2^{-2}b_3t + b_2^{-3}(b_3^2 - b_2b_4)t^2 + \dots$$
$$= b_2^{-1}c_4t^4 + b_2^{-2}(b_2c_5 - b_3c_4)t^5 + \dots \qquad .//$$

In order to determine how many other tangents a given tangent to $C_o$ meet, we need to know the type of singularity we get when we intersect the tangent developable of $C_o$ with a plane containing a tangent.

Set $T(p)$ = the tangent to $C_o$ at p, and let $X = \bigcup_{p \in C} T(p)$ denote the tangent developable of $C_o$. If $p \in C$ has a parametrization (+), then $h(p) \in X$ has a (formal) parametrization

$$x = at^{l_1+1} + \dots + s(a(l_1+1)t^{l_1} + \dots)$$
$$y = bt^{l_2+2} + \dots + s(b(l_2+2)t^{l_2+1} + \dots)$$
$$z = ct^{l_3+3} + \dots + s(c(l_3+3)t^{l_3+2} + \dots).$$

Since $T(p)$ is given by $y = z = 0$, a plane H containing $T(p)$ has an equation $\alpha y + \beta z = 0$, with $\alpha \neq 0$ if and only if H is not the osculating plane. Suppose it is not. Then $(H \cap X)_{red} = T(p) \cup D$, where the plane curve D has a singularity of type $(l_1+1, l_3+3)$ at $h(p)$. Moreover, $T(p)$ is the tangent to D at $h(p)$. Thus we have proved the following:

Lemma 2: The intersection number of D with $T(p)$ at $h(p)$ is given by

$$i(D,T(p);h(p)) = l_3 + 3 .$$

In §4 we shall return to a global study of the tangent developable X.

## §3. On the existence of cuspidal projections

Let $C \subset P$ be a smooth, irreducible curve of degree $r_0 = d$, genus $g$, and assume $C$ is not contained in a plane. Denote by $\delta$ the number of apparent double points of $C$, i.e., set

$$\delta = \tfrac{1}{2}(d-1)(d-2) - g .$$

Theorem 1: All curves $C \subset P$ with $\delta \leq 3$, admit a cuspidal projection.

Proof: Castelnuovo's bound on the genus of a space curve shows that $\delta \leq 3$ implies $d(d-2) \leq 12$, if $d$ is even, and $(d-1)^2 \leq 12$, if $d$ is odd. Therefore there are only three cases to consider:

1) $C \subset P$ is a twisted cubic, i.e., $\delta = 1$, $d = 3$, $g = 0$.

2) $C \subset P$ is an elliptic quartic, i.e., $\delta = 2$, $d = 4$, $g = 1$.

3) $C \subset P$ is a rational quartic, i.e., $\delta = 3$, $d = 4$, $g = 0$.

In each case we shall describe the possible cuspidal projections.

Case 1): The tangent developable $X$ of $C$ has no singularities outside its cuspidal edge $C$ (see §4). By projecting $C$ from any point on $X - C$, we obtain a plane cubic with one (ordinary) cusp. (Observe that, for degree reasons, any projection of $C$ is necessarily birational onto its image.)

Case 2): First of all, $C$ can have no flexes: Suppose $p \in C$ is a point of type $(1, l_2+2, l_3+3)$. The pencil $\{H_\lambda\}$ on $C$ cut out by planes containing the tangent $T(p)$ has a base point divisor equal to $(l_2+2)p$. By Riemann-Roch, then, we must have

$$2 = h^0(H-(l_2+2)p) = 4 - (l_2+2) + 1 - g = 2 - l_2$$

(since $l_2$ is equal to 0 or 1, $H-(l_2+2)p$ is non-special), hence $l_2 = 0$. Moreover, the stalls of $C$ are necessarily ordinary: since $d=4$, we have $l_3 \leq 1$. Thus $C$ has $k_2=16$ stalls (§4).

We shall show that $C$ admits two kinds of cuspidal projections - one gives a plane curve with one ramphoid cusp, the other a plane curve with two ordinary cusps.

Suppose $v \in P - C$ is a point such that the projection from $v$, $\pi : C \to \pi(C)$, is not birational. Then necessarily $\deg \pi = 2$ and $\deg \pi(C) = 2$, so $C$ is on a quadric cone with vertex $v$. Call this cone $K$. Now we know that $C$ is the base locus of a pencil of quadrics. If $C$ is on at least one smooth quadric, it will be on no more than 4 quadric cones, and $K$ must be one of these. If $C$ is not on a smooth quadric, there is a pencil of quadric cones containing $C$; we shall see below that this is impossible.

By the Riemann-Hurwitz formula, $\pi : C \to \pi(C)$ has 4 branch points, so $K$ contains the tangents to $C$ at 4 points. Let $p$ be one of these. The tangent plane $H$ to $K$ along $T(p)$ intersects $C$ only at $p$, hence with intersection number 4. Hence $H$ is a hyperosculating plane to $C$, so $p$ is a stall. Each of the 16 stall tangents thus intersects exactly 3 others, in the same point, and these points of intersection are the vertices of four quadric cones containing $C$. It follows that these four cones are the only quadric cones that contain $C$.

Let $p \in C$ be a stall, and $v \in T(p)$ any point different from $p$ and different from the vertex of the (unique) quadric cone containing $C$ and $T(p)$. Then the projection $\pi$ of $C$ from $v$ is birational onto its image, and $\pi(C)$ is a plane elliptic quartic with one ramphoid cusp (necessarily of the 1st type).

Consider now the nodal curve of $C$ (the "double curve" of the tangent developable $X$). Note that it contains no bitangents, since $C$ cannot have any. Assume first that the nodal curve is double (of multiplicity 2) on $X$. Then the projection of $C$ from any point on it, not on $C$ and different from the 4 vertices of the cones, is birational onto its image - the argument above shows that otherwise all tangents are stall tangents - and the projected curve will have 2 ordinary cusps. That the nodal curve is non empty, follows from the fact that it has degree (see §4)

$$b = \tfrac{1}{2}(r_1(r_1-1)-r_2-3r_0) = \tfrac{1}{2}(8\cdot 7 - 12 - 3\cdot 4) = 16.$$

Suppose the nodal curve had a component which was of multiplicity greater than 2 on X. Then the projection of C from any point on that component could not be birational onto its image, hence the component would consist of vertices of quadric cones containing C. As we have seen, this is impossible.

(Thus we have shown that the nodal curve is double; moreover, the four vertices of the cones are quadruple points on the nodal curve - this checks with the fact that this curve has a double point cycle of degree $3T = 48 = 4\cdot 4\cdot 3$ , see §4.)

Case 3): These curves have been classified by Telling [T]; she considered the various ways of projecting the rational normal quartic in $\mathbb{P}^4$ to $\mathbb{P}^3$. We shall distinguish between two cases: a) the general rational quartic, b) the equianharmonic rational quartic.

Observe first that we need not worry about birationality: Let $v \in P - C$ be a point, $\pi : C \to \pi(C)$ the projection from it. If $\pi$ is not birational, then necessarily $\deg \pi = 2$ , so C lies on a quadric cone - this is impossible since C is rational.

In case a), $C \subset P$ is a generic (or almost so) projection of the rational normal quartic in $\mathbb{P}^4$. It has only the kind of singularities that is predicted by dimension count - in particular the nodal curve is double. The tangent developable has T points of multiplicity 3, given by (§4)

$$T = \frac{1}{6}(r_1-4)((r_1-3)(r_1-2)-6g) = 4 \; .$$

Among these triple points are the $d(1,2) = 4$ points where a tangent meets the curve again - hence there are no points where 3 distinct tangents intersect (see also [T], p.55). It follows that such a C can not be projected onto a tricuspidal quartic.

For the number $k_1$ of flexes and $k_2$ of stalls, there are three possibilities (all flexes and stalls are ordinary, since $d = 4$).

Case $a_1$): $k_1 = 0$, $k_2 = 4$. (This is the most general C.)
The only possible type of cuspidal projection is obtained by projecting C from the (unique, see §4) point of intersection of a stall tangent with another tangent. This point is not on the curve: It is different from the stall ([T], pp.46-47). If it was another point on the curve, by projecting from it we would obtain a plane rational cubic with a ramphoid (double) cusp - this is impossible.

Thus C admits a projection onto a plane rational quartic with one ordinary cusp and one ramphoid cusp (necessarily of the 1st type).

Case $a_2$): $k_1 = 1$, $k_2 = 2$. This curve admits the same type of cuspidal projection as the one above, but also an additional one: Projecting from a point on the flex tangent, one obtains a plane quartic with one cusp, of type (3,4).

Case $a_3$): $k_1 = 2$ $k_2 = 0$. The projection from a point of one of the two flex tangents is the only type of possible cuspidal projection.

In case b), $C \subset P$ is the projection of the rational normal quartic from a general point on a certain quadric hypersurface in $\mathbb{P}^4$ (this quadric is the nucleus of the fundamental polarity, see [T], pp.8,65). In this case, the nodal curve is triple, there are no flexes, and hence 4 stalls (the nodal curve is a plane conic through these 4 points). The reason for the name of this curve, is that the 4 stalls form an equianharmonic set on the curve (which means their cross-ratios are equianharmonic).

Also, the d(1,2) = 4 (§4) points where a tangent meets the curve again (the socalled steinerian points of C), are just the stalls ([T], p.66). The only way of obtaining a cuspidal projection of the equianharmonic quartic, is to project it from a point on its nodal curve, not on C. The projected curve is a tricuspidal quartic (and all tricuspidal quartics are obtainable in this way). //

It seems natural to believe that a general curve of degree d and genus g, with $\delta \geqslant 4$, does <u>not</u> admit a cuspidal projection. In fact, this would follow if we could prove the following.

① A general curve $C \subset P$ of degree d and genus g has only such singularities that are predicted by a dimension count (in particular, C has no flexes and only ordinary stalls, the nodal curve is double, and there are no points of multiplicity greater than 3 on the tangent developable).

② For a general $C \subset P$ , the projection from a point on a stall tangent gives a ramphoid cusp of the 1st type.

If ① and ② hold, then, for a general $C \subset P$ , one can obtain only (the equivalent of) 3 or fewer cusps by projection - the types of projections are the ones described in the proof of Theorem 1.

If we want ① to be true (without being tautological), we must of course be careful about how to define "general". We have seen that for d=4 and g=0, ① is true when "general" means "most curves of degree 4 and genus 0". With a similar definition, ① fails for d=4 and g=1 (since these curves all have quadruple points on their tangent developable). So though there is a very natural definition of "general" in this case - namely "the intersection of two general quadrics" - it is not one that makes ① hold. Moreover, these general curves also have general moduli - hence it would not help to impose that condition. All one can expect is therefore that an ad hoc version of ① - sufficient for our purposes - would always be true. We shall prove this for the case d=6 and g=4. These are the canonical curves of genus 4, and they are the complete intersection of a quadric and a cubic surface.

Suppose a curve C lies on a quadric surface Q, and that C is of type (a,b) on Q. If a or b is greater than 3, then C has an infinity of quadri-secants - a phenomenon which is <u>not</u> "predicted by a dimension count". But then, curves that lie on a quadric are not usually (for big a or b) considered to be "general". It turns out, however, that the property in ② is easier to verify for curves that are the (complete) intersection of a quadric with another surface. We shall prove ② for the case $d=6$ and $g=4$, i.e., for canonical curves of genus 4. Note that a curve $C$ which is the intersection of a general quadric and a general cubic surface, has general moduli (i.e., $C$ is general as a curve of genus 4).

<u>Theorem 2</u>: A general canonical curve $C \subset P$ of genus 4 does not admit a cuspidal projection.

<u>Proof</u>: Since the embedding of $C$ is given by its canonical divisor, the non-regular points of $C \subset P$ are its Weierstrass points. A general curve of given genus has only normal Weierstrass points (e.g. [G-H], p.277), hence we may assume that $C$ has no flexes and only ordinary stalls. (If $p \in C$ has type $(1, l_2+2, l_3+3)$, then the gap sequence at $p$ is $(1,2,l_2+3,l_3+4)$.)

<u>Lemma 3</u>: Let $C \subset P$ be a general canonical curve, i.e., $C = Q \cap F$ is the intersection of a general quadric $Q$ and a general cubic surface $F$ in $P$. Let $p \in C$ be a stall and $v \in P - C$ a point on the tangent to $C$ at $p$. If $\pi : C \to \mathbb{P}^2$ denotes the projection of $C$ from $v$, then $\pi(p) \in \pi(C) \subset \mathbb{P}^2$ is a ramphoid cusp of the 1st type.

Assume this lemma holds. If $\pi : C \to \mathbb{P}^2$ is a cuspidal projection of a general $C \subset P$, there are only 4 possibilities: $\pi(C)$ has 6 ordinary cusps, or 4 ordinary and 1 ramphoid cusp, or 2 ordinary and 2 ramphoid cusps, or 3 ramphoid cusps. The next two lemmas imply that for a general $C$, none of these occur - hence the proof of the theorem will be completed by establishing Lemmas 3, 4, and 5.

<u>Lemma 4</u>: Suppose $C$ is the normalization of
(i) a plane sextic curve of genus 4, with 6 ordinary cusps, or

(ii) a plane sextic curve of genus 4, with 4 ordinary cusps and 1 ramphoid cusp.

Then C does not have general moduli.

Lemma 5: If $C \subset P$ is a general canonical curve of genus 4, then no two stall tangents intersect.

Proof of Lemma 3: Since all smooth quadrics are projectively equivalent, we shall fix one; call it Q. Then the curves C are parametrized by cubic surfaces: Let $A = k[F_{ijkl}]$ denote the ring of coefficients of cubic, homogeneous polynomials in 4 variables, and set $\mathcal{F} = \Sigma F_{ijkl} X_0^i X_1^j X_2^k X_3^l$ , the universal cubic. We have a (complete intersection) map

$$\Phi : \mathrm{Proj}(A[X_0, X_1, X_2, X_3]/(Q, \mathcal{F})) \longrightarrow \mathrm{Proj}(A) .$$

Let $U \subset \mathrm{Proj}(A)$ be the open subscheme such that, if $\mathcal{C} = \Phi^{-1}(U)$, $\Phi : \mathcal{C} \rightarrow U$ is smooth, of relative dimension 1. Thus we have a family of canonical curves of genus 4, containing the general ones.

The Weierstrass points on the fibres of $\Phi$ form an effective, relative divisor W on $\mathcal{C}$ over U. It can be defined as follows: There is a natural homomorphism ([P1], §6)

$$a^{g-1} : \Phi^* \Phi_* \Omega^1_{\mathcal{C}/U} \rightarrow \mathcal{P}^{g-1}_{\mathcal{C}/U}(\Omega^1_{\mathcal{C}/U}) .$$

Since $a^{g-1}$ is a map between locally free sheaves of rank g , we can take its determinant

$$\det a^{g-1} : \Phi^* \mathcal{M} \rightarrow \Lambda^g \mathcal{P}^{g-1}_{\mathcal{C}/U}(\Omega^1_{\mathcal{C}/U}) \cong (\Omega^1_{\mathcal{C}/U})^{\otimes g(g+1)} ,$$

where we have put $\mathcal{M} = \Lambda^g \Phi_* \Omega^1_{\mathcal{C}/U}$. The corresponding section

$$w_{\mathcal{C}/U} : \mathcal{O}_{\mathcal{C}} \rightarrow (\Omega^1_{\mathcal{C}/U})^{\otimes g(g+1)} \otimes \Phi^* \mathcal{M}^{-1}$$

defines the (relative) Weierstrass divisor

$$W = \mathrm{div}(w_{\mathcal{C}/U}) \in \mathrm{Div}^+(\mathcal{C}/U) .$$

Replacing U by a smaller open, we may assume that all Weierstrass points of the fibres $\mathcal{C}_u = \Phi^{-1}(u)$ are normal, i.e.,

$$\# W_u = (g-1)g(g+1) = 60$$

for all $u \in U$.

For $p \in \mathcal{C}$, set $u = \Phi(p)$ ; then $p \in \mathcal{C}_u = Q \cap \mathcal{F}'_u \subset \mathbb{P}^3$ . Set $p_0 = (1,0,0,0)$ and $Q_0$ the quadric defined by $X_0X_3 = X_1X_2$ . Define a scheme $\tilde{\mathcal{C}}$ by

$$\tilde{\mathcal{C}} = \{ (p,\alpha) \in \mathcal{C} \times PGL(3) \mid \alpha(p) = p_0 , \alpha(Q) = Q_0 \} .$$

Let $\varphi : \tilde{\mathcal{C}} \to \mathcal{C}$ denote the projection. Then $\varphi$ is smooth, of relative dimension 4: The fibres of $\varphi$ are

$$\varphi^{-1}(p) = \{ \beta \in PGL(3) \mid \beta(p_0) = p_0 , \beta(Q_0) = Q_0 \} .$$

Consider

$$G = \{ \beta \in PGL(3) \mid \beta(p_0) = p_0 \} ,$$

$$S = \{ \text{quadrics through } p_0 \} .$$

There is a surjective map $\gamma : G \to S$ , given by $\gamma(\beta) = \beta(Q_0)$, and $\gamma^{-1}(Q_0) = \varphi^{-1}(p)$ . Now the fibres of $\gamma$ have dimension 4, since $\dim G = 15 - 3 = 12$, and $\dim S = 9 - 1 = 8$.

We shall now define a rational map $\tilde{\mathcal{C}} \dashrightarrow A^5$ . Suppose $(p,\alpha) \in \tilde{\mathcal{C}}$ . Then $u = \Phi(p)$ corresponds to a homogeneous cubic polynomial

$$F = \sum F_{ijkl} X_0^i X_1^j X_2^k X_3^l ,$$

where the $F_{ijkl}$'s are defined up to multiplication by the same non-zero scalar. Note that $F_{3000} = 0$ , since $p \in \mathcal{C}_u = Q \cap F$. In affine coordinates $(x,y,z)$, with $X_0 \neq 0$, the equation of Q becomes $z = xy$ , and substituting this in the affine equation of F, we get a polynomial in two variables

$$h(x,y) = \sum h_{ij} x^i y^j ,$$

which, together with $z = xy$, determines $\mathcal{C}_u$.

Assume now that $h_{01} \neq 0$ (i.e., $F_{2010} \neq 0$). Then we normalize the $h_{ij}$'s so that $h_{01} = -1$. Expand y in terms of x:

$$y = a_1x + a_2x^2 + \dots + a_5x^5 + \dots ,$$

where

$$a_1 = h_{10}$$

$$a_2 = a_1^2h_{02} + a_1h_{11} + h_{20}$$

$$a_3 = a_2h_{11} + 2a_1a_2h_{02} + h_{30} + a_1h_{21} + a_1^2h_{12} + a_1^3h_{03}$$

$$a_4 = a_1^2h_{13} + a_1^2h_{22} + a_1h_{31} + 3a_1a_2h_{03} + 2a_1a_2h_{12} + a_2h_{21} + a_2^2h_{02} + a_3h_{11}$$

$$a_5 = a_4h_{11} + 2a_1a_4h_{02} + 2a_2a_3h_{02} + a_3h_{21} + a_2^2h_{12} + 2a_1a_3h_{12} + 3a_1^2a_3h_{03} + a_2h_{31} + 2a_1a_2h_{22} + 3a_1^2a_2h_{13} + a_1^2h_{32} + a_1^3h_{23}$$

etc.

Thus $p \in \mathcal{C}_u \subset P$ has a parametrization

$$(*) \qquad \begin{aligned} x &= x \\ y &= a_1x + a_2x^2 + \dots \\ z &= xy = a_1x^2 + a_2x^3 + \dots . \end{aligned}$$

Thus we have a map $\Psi : \tilde{\mathcal{C}} \dashrightarrow \mathbb{A}^5$, defined for all points $(p, \alpha)$ such that $h_{01} \neq 0$, by $\Psi(p,\alpha) = (a_1, a_2, a_3, a_4, a_5)$.

Suppose $(p,\alpha)$ is such that $h_{01} = 0$. Then, since p is a smooth point of $\mathcal{C}_u$, necessarily $h_{10} \neq 0$. We may then assume $h_{10} = -1$ to get a parametrization

$$\begin{aligned} y &= y \\ x &= h_{02}y^2 + (h_{02} + h_{03})y^3 + \dots \\ z &= h_{02}y^3 + (h_{02} + h_{03})y^4 + \dots . \end{aligned}$$

Since, by assumption, $\mathcal{C}_u$ has no flexes, we must have $h_{02} \neq 0$, so p is a regular point of $\mathcal{C}_u$. Hence $\Psi(p,\alpha)$ is defined whenever $p \in W$, so we get a morphism

$$\psi : \varphi^{-1}(W) \to \mathbb{A}^5 .$$

Suppose $\psi$ is defined at $(p,\alpha)$. The parametrization (*) is equivalent to

$$\begin{aligned} x &= x \\ z &= a_1x^2 + a_2x^3 + \dots \\ y &= (a_3 - a_1^{-1}a_2^2)x^3 + (a_4 - a_1^{-1}a_2a_3)x^4 + (a_5 - a_1^{-1}a_2a_4)x^5 + \dots \end{aligned}$$

If $p \in W$, then $a_3 = a_1^{-1}a_2^2$ holds, and - since any stall is ordinary - $a_4 \neq a_1^{-1}a_2a_3$. According to Lemma 1, the ramphoid cusp obtained by projecting the curve from a point on the stall tangent T(p), will be of the 1st type if

$$a_1(a_5 - a_1^{-1}a_2a_4) \neq a_2(a_4 - a_1^{-1}a_2a_3)$$

holds.

Set $V = \{a \in \mathbb{A}^5 \mid a_1 \neq 0,\ a_1a_4 \neq a_2a_3,\ a_2^2 = a_1a_3\}$.

Then $\Psi(\varphi^{-1}W) \subset V$. Moreover, we claim that $\psi : \varphi^{-1}W \to V$ is generically surjective: By construction, $\psi : \tilde{\mathcal{E}} \dashrightarrow \mathbb{A}^5$ factors through the map

$$\tilde{\varphi} : \tilde{\mathcal{E}} \to \tilde{U} = \{u \in U \mid p_0 \in \mathcal{F}_u\},$$

defined by $\tilde{\varphi}(p,\alpha) = \alpha(\mathcal{F}_{\Phi(p)})$.

Both this map and $U \dashrightarrow \mathbb{A}^5$ are generically surjective. Since $\psi^{-1}(\psi(\tilde{\mathcal{E}}) \cap V) = \varphi^{-1}W$, the claim follows. (If $\varphi^{-1}W$ has more than one irreducible component, $\psi$ is generically surjective on each of them, because of the homogeneous nature of the map $\psi$.)

Set

$$\begin{aligned} V_1 &= \{a \in V \mid a_1(a_1^2 - a_2a_4) = a_2(a_1a_4 - a_2a_3)\} \\ &= \{a \in V \mid a_1a_5 - a_2a_4 = a_2a_4 - a_3^2\}. \end{aligned}$$

Because of the independence of the defining equations,

$$\dim V_1 = \dim V - 1 ,$$

hence

$$\dim \psi^{-1}(V_1) < \dim \varphi^{-1}W$$

holds.

Since the property $(p,\alpha) \in \psi^{-1}(V_1)$ of the point $(p,\alpha)$ is independent of $\alpha$ , the above inequality implies

$$\dim \varphi(\psi^{-1}(V_1)) < \dim W .$$

Set $U_1 = \Phi(\varphi(\psi^{-1}V_1))$, and let $U_1$ denote its closure. The map $\Phi|W:W \to U$ is finite and onto, therefore

$$\dim U_1 < \dim U .$$

Hence we have found $U_0 = U - U_1$ , open and non-empty, with the property that any curve $\mathcal{C}_u$ , $u \in U_0$ , is such that its stalls satisfy the condition of the lemma. //

Proof of Lemma 4: Suppose $\bar{C} \subset \mathbb{P}^2$ is a plane, irreducible curve of degree d. Let N denote its normal bundle,

$$J = F^1(\Omega^1_{\bar{C}}) \subset \mathcal{O}_{\bar{C}}$$

its jacobian ideal, and $\pi: C \to \bar{C}$ its normalization. Set

$$\mathcal{L} = \pi^* N \otimes J\mathcal{O}_C .$$

Denote by r the dimension of the space of all plane curves of degree d with the same type (and number) of singularities as $\bar{C}$, or, r is the dimension of the space of locally trivial deformations of C. It is known ([A]; [Zar], VIII, §5), that if $\mathcal{L}$ is non-special, then

$$r = \dim H^0(C,\mathcal{L}) = \deg\mathcal{L} + 1 - g = d^2 - \deg(J\mathcal{O}_C)^{-1} + 1 - g .$$

This gives ([P2], 3.9)

$$r = 3d + 2g - 2 - \deg I^{-1} + 1 - g = 3d + g - 1 - \deg I^{-1} ,$$

where $I = F^0(\Omega^1_{C/\bar{C}})$ is the ramification ideal of $\pi: C \to \bar{C}$ .

The line bundle $\mathcal{L}$ is non-special if

$$\deg \mathcal{L} = 3d + 2g - 2 - \deg I^{-1} > 2g - 2 \quad ,$$

hence if $\deg I^{-1} < 3d$ .

Apply the above to $d = 6$ , $g = 4$. In case (i), we have $\deg I^{-1} = 6$ , and in case (ii), $\deg I^{-1} = 5$. In both cases, therefore, $\mathcal{L}$ is non-special. Thus we get $r = 15$ in case (i), and $r = 16$ in case (ii).

Modulo projective transformations, the dimension of the family is $r - 8$ . Hence plane sextics with (exactly) 6 ordinary cusps form a family of dimension 7, and plane sextics with 4 ordinary and 1 ramphoid cusp form a family of dimension 8. Since the moduli space of genus 4 curves has dimension $3g - 3 = 9$, a curve C which is the normalization of either of the above plane curves, cannot have general moduli. //

Remark: For plane sextics with the other two configurations of cusps - 2 ordinary and 2 ramphoid, or 3 ramphoid - we get families of dimensions 9 and 10, respectively. Hence the above method gives no conclusion in those cases.

Proof of Lemma 5: Suppose Q is a smooth quadric, and $C = Q \cap F$ a canonical curve. We observe that if p and q are distinct points of C lying on the same ruling L of Q, then the tangents T(p) and T(q) of C do not intersect. For, suppose they did. Since the tangent planes to Q at p and q intersect in L, this implies $T(p) = T(q) = L$. Hence L is a bitangent to C and has intersection multiplicity at least 4 with C, contrary to the fact that L intersects the cubic surface F, hence also C, in (the equivalent of) three points.

Let $p, q \in C$ be two points not on the same ruling of Q. After a projective transformation, we may assume

$$Q: X_0X_3 - X_1X_2 = 0$$

$$p = (1,0,0,0)$$

$$q = (0,0,0,1) \ .$$

The choice of coordinates gives us the coefficients $(F_{ijkl})$ of a cubic polynomial F (determined modulo Q and up to scalar multiplication). Since p, $q \in C = Q \cap F$, we have $F_{3000} = F_{0003} = 0$. The tangent planes to Q and to F at p and q are given by

$$T_Q(p):\ X_3 = 0 \qquad T_F(p):\ F_{2100}X_1+F_{2010}X_2+F_{2001}X_3 = 0$$

$$T_Q(q):\ X_0 = 0 \qquad T_F(q):\ F_{1002}X_0+F_{0102}X_1+F_{0012}X_2 = 0 .$$

The tangents to C at p and q are

$$T(p):\ X_3 = 0\ ,\ F_{2100}X_1 + F_{2010}X_2 = 0$$

$$T(q):\ X_0 = 0\ ,\ F_{0102}X_1 + F_{0012}X_2 = 0 .$$

The two tangents intersect if and only if

$$F_{2100}F_{0012} = F_{0102}F_{2010}$$

holds.

Assume $F_{2010} \neq 0$. The computations made in the proof of Lemma 3 show that the condition that p be a small, is that the following equality holds:

$$\begin{aligned} &F^4_{2010}F^2_{1200} - F^3_{2010}F_{2100}F_{1110}F_{1200} - F^3_{2010}F_{2100}F_{2001}F_{1200} \\ &- F^4_{2010}F_{2100}F_{0300} + F^3_{2010}F^2_{2100}F_{0210} + F^3_{2010}F^2_{2100}F_{1101} \\ &- F^2_{2010}F^3_{2100}F_{0120} - F^2_{2010}F^3_{2100}F_{1011} + F_{2010}F^4_{2100}F_{0030} \\ &+ F_{2010}F^3_{2100}F_{1020}F_{1110} + F_{2010}F^3_{2100}F_{1020}F_{2001} - F^4_{2100}F_{1020} = 0 . \end{aligned}$$

(This is the equation $a_2^2 - a_1a_3 = 0$, where $a_1 = h_{10} = -F_{2010}{}^{-1}F_{2100}$, $a_2 = \ldots$)

Note that this equation is invariant under the change $F_{ijkl} \rightsquigarrow F_{ikjl}$. Since at least one of $F_{2010}$ and $F_{2100}$ is non-zero, our assumption $F_{2010} \neq 0$ does not mean any loss of generality.

If we change $F_{ijkl}$ to $F_{ljki}$ in the above equation, we obtain the condition that the point q be a small. These conditions are seen, by inspection, to be independent, i.e., the fact that T(p) and T(q) intersect and p is a stall does not imply that q is a stall; or, if p and q are stalls, then by moving F (and keeping the stalls) we get a curve such that T(p) and T(q) don't intersect. Thus we may assume this to be true for any pair of stalls on a general canonical curve. In fact, similarly to the proof of Lemma 3, one can define

$$\mathcal{D} = \left\{ (p,q,\alpha) \in \mathcal{C} \times_U \mathcal{C} \times PGL(3) \;\middle|\; \begin{array}{l} p,\ q \text{ stalls},\ \alpha(p) = p_0, \\ \alpha(q) = q_0 = (0,0,0,1),\ \alpha(Q)=Q_0 \end{array} \right\}$$

and consider the map $\mathcal{D} \longrightarrow \{ u \in U \mid p_0, q_0 \in \mathcal{F}_u \}$ and argue as in that proof. //

<u>Remark</u>: By performing the same computations for a curve which is the intersection of two quadrics (an elliptic quartic curve), we find that, if T(p) and T(q) intersect and p is a stall, then necessarily q is a stall too. This is just as expected, since we have already seen (in the proof of Theorem 1, Case 2)) that for such curves, any stall tangent intersects three other stall tangents but no other tangents.

Theorem 2 says that general canonical curves of genus 4 do not admit a cuspidal projection. There exist, however, canonical curves that do. Recall the following:

Let $S \subset P$ be a smooth surface of degree d, and suppose $\pi : S \to \mathbb{P}^2$ is the projection of S from a general point v. The <u>curve of contact</u> $C \subset S$ of S with respect to $\pi$ can be defined as the ramification divisor $\Sigma^1(\pi)$ of $\pi$. Another description of C is that it is the intersection of S with the 1st polar $S_1$ of S with respect to v. The curve C has degree d(d-1) and genus $g = \frac{1}{2}d(d-1)(2d-5)+1$ ; moreover, the number of cusps of $\pi(C)$ (these are ordinary, since $\pi$ is a generic projection) is equal to the degree d(d-1)(d-2) of the ramification divisor $\Sigma^{1,1}(\pi) = \Sigma^1(\pi|C)$ of $\pi|C : C \to \pi(C)$ ([P3], §5).

Apply this to $d = 3$. Then $S: F = 0$ is a cubic surface, its 1st polar $S_1: Q = \sum v_i \frac{\partial F}{\partial X_i} = 0$ is a quadric, and the intersection of the cubic and quadric (not general as such) is a canonical curve C. The projected curve $\pi(C)$ has 6 ordinary cusps, hence no other singularities, and so $\pi: C \to \mathbb{P}^2$ is a cuspidal projection of C.

It is known that a smooth curve $D = S_1 \cap S_2$, where $S_i$ is a surface of degree $d_i$, is such that all its chords through a general point $v \in P - D$ lie on a cone of degree $(d_1-1)(d_2-1)$ (Valentiner, Noether - see [B], p.204). This is seen as follows: The projection from v, $\pi: D \to \mathbb{P}^2$, is birational onto its image $\bar{D} = \pi(D)$. The conductor ideal $\underline{C} = \underline{Hom}_{\bar{D}}(\pi_* \mathcal{O}_D, \mathcal{O}_{\bar{D}})$ of D in $\bar{D}$ satisfies

$$\pi_* \Omega^1_D = \omega_{\bar{D}} \otimes \underline{C} ,$$

where $\omega_{\bar{D}} = \mathcal{O}_{\bar{D}}(d_1 d_2 - 3)$ denotes the dualizing sheaf of $\bar{D}$. Since $\Omega^1_D = \mathcal{O}_D(d_1+d_2-4)$ holds, one sees that

$$H^0(\bar{D}, \underline{C} \otimes \mathcal{O}_{\bar{D}}(m)) \neq 0$$

if and only if $m \geq (d_1-1)(d_2-1)$. Therefore, the singularities of the plane curve $\bar{D}$ (as defined by the conductor) lie on a curve of degree $(d_1-1)(d_2-1)$.

The following converse is also true (Halphen - see [B], p.204; [G-P], p.32): Suppose a curve $D \subset P$ has degree $d_1 d_2$ and does not lie on a surface of degree $< \min(d_1, d_2)$. If the chords to D through a (general) point v lie on a cone of degree $(d_1-1)(d_2-1)$, then D is the complete intersection of two surfaces of degrees $d_1$ and $d_2$.

Therefore, a sextic curve $C \subset \mathbb{P}^2$ of genus 4, with 6 cusps, is the projection of a canonical curve $C \subset P$ if and only if the cusps lie on a conic. As Zariski observes ([Zar], p.223), there are 6-cuspidal sextics without this property - they also form a 15-dimensional family.

Remark: I do not know whether any 6-cuspidal sextic with all its cusps on a conic, can be obtained as the projection of the intersection of a cubic surface with its 1st polar.

Another question one can ask, is the following: Given d and g such that there exist space curves of degree d and genus g , does there exist one that admits a cuspidal projection?

Here are some cases where the answer is known to be yes:

1. $g = 0$ , all d .

The rational curves $\mathbb{P}^1_k \xrightarrow{\sim} C \subset \mathbb{P}^3_k$ , given by $(u,v) \rightsquigarrow (u^d, u^{d-1}v, uv^{d-1}, v^d)$ , were shown by Hartshorne (1964, [H]) to be set-theoretical complete intersections if char $k > 0$ . Ferrand [F1] observed that they admit a cuspidal projection: the projection of C from a point on one of the two inflectionary tangents give a plane curve with a monomial cusp of type (d-1,d); hence that plane curve can have no other singularities. (Ferrand went on to prove that any curve which admits a cuspidal projection is a set-theoretical complete intersection in positive characteristic ([F1], 2.3).)

2. $g = 1$ , $d = 4$ .

All curves admit a cuspidal projection (Theorem 1).

3. $g = 4$ , $d = 6$.

Examples were given above.

4. $g = 1$ or $2$ , $d = 5$ or $6$.

Suppose $\bar{C} \subset \mathbb{P}^2$ is a curve of degree d and genus g , with $\kappa = \frac{1}{2}(d-1)(d-2) - g$ ordinary cusps (hence no other singularities), and let $\pi : C \to \bar{C}$ denote its normalization. If $d > 2g$ , then $\mathcal{O}_C(1) = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)|C$ is very ample, hence it embeds C in $\mathbb{P}^N$ , where $N = d - g$. By factoring the given projection $\mathbb{P}^N \dashrightarrow \mathbb{P}^2$ generically via a $\mathbb{P}^3$, we obtain C embedded in $\mathbb{P}^3$ and such that the projection $\pi : C \to \bar{C} \subset \mathbb{P}^2$ is cuspidal.

Since there exist plane curves (because $d \leq 6$, see [Zar] , p.222) with

| | | |
|---|---|---|
| $g = 1$ | $d = 5$ | $\kappa = 5$ |
| | $d = 6$ | $\kappa = 9$ |
| $g = 2$ | $d = 5$ | $\kappa = 4$ |
| | $d = 6$ | $\kappa = 8$ , |

the above method applies to these cases.

(There also exist plane curves of genus 3 and degree 5 (resp.6), with 3 (resp. 7) cusps, but here $d > 2g$ no longer holds.)

## §4. The tangent developable and its singularities

Let $h: C \to C_0 \subset P$ be as in the beginning of §2. Recall that we can describe the tangent developable of $C_0$ in the following way [Pl]: Let $\mathcal{P}^m_C(1)$ denote the (m+1)-bundle of principal parts of order m of the line bundle $\mathcal{O}_C(1) = h^*\mathcal{O}_P(1)$. There are canonical maps

$$a^m: H^0(P, \mathcal{O}_P(1))_C = V_C \to \mathcal{P}^m_C(1).$$

The cokernel of $a^1$ is isomorphic to $\Omega^1_{C/P} \otimes \mathcal{O}_C(1)$. Hence, since C is a smooth curve, the image $\mathcal{P}^1 = \mathrm{Im}(a^1)$ is a rank 2 bundle on C. Setting $Y = \mathbb{P}(\mathcal{P}^1)$ we get a closed embedding $Y \hookrightarrow C \times P$. The tangent developable $X \subset P$ is then the image of Y under the projection onto the second factor. Note that the map $f: Y \to X$ is finite (this is true in arbitrary characteristic provided the curve $C_0$ is not strange) and birational, hence the singularities of X are resolved by normalization.

For m = 2,3 the homomorphisms $a^m$ are also generically surjective, since $C_0$ spans P. Set $\mathcal{P}^m = \mathrm{Im}(a^m)$. The bundle $\mathcal{P}^2$ represents the osculating planes of $C_0$, while $\mathcal{P}^3$ is isomorphic to $V_C$.

Let $r_0$ denote the degree of $C_0$ and g its (geometric) genus. The rank $r_1$ of $C_0$, defined as the number of tangents to $C_0$ meeting a given (general) line, is equal also to the degree of the tangent developable X. The class $r_2$ of $C_0$ is defined as the number of osculating planes containing a given (general point.

Set $k_i = \sum_{p \in C} k_i(p)$, i = 1,2,3. We shall use repeatedly the following formulas ([Pl], 3.2):

$$\begin{aligned} r_1 &= 2r_0 + 2g - 2 - k_0 \\ r_2 &= 3(r_0 + 2g - 2) - 2k_0 - k_1 \qquad (1) \\ k_2 &= 4(r_0 + 3g - 3) - 3k_0 - 2k_1 . \end{aligned}$$

The rank and class also have interpretations in terms of dual varieties. Recall that the dual variety $\check{Z} \subset \check{\mathbb{P}}^n$ of a variety $Z \subset \mathbb{P}^n$ is defined as the closure of the set of hyperplanes tangent to Z at smooth points. The dual variety $\check{C} \subset \check{P}$ of $C_0 \subset P$ is thus a ruled surface, in fact it is the tangent developable $X^*$ of the curve $C^* \subset \check{P}$, whose points are the osculating planes of $C_0$.

We shall call $C^*$ the dual curve of $C_o$. Since char $k = 0$, biduality holds: The dual variety of $\check{C}=X^*$ is the curve $C_o$ (which is also the dual curve of $C^*$), and the dual variety of $C^*$ is the tangent developable $X$ of $C_o$. One of the characterizations of a developable surface is just that it is a ruled surface whose tangent planes are constant along a generator, i.e., whose dual variety is a curve.

We have

$$r_1 = \text{degree of } X = \text{degree of } X^*$$

$$r_2 = \text{degree of } C^* = \text{rank of } X .$$

For the stationary indices $k_i^*$ of $C^*$ we have the duality

$$k_i^* = k_{2-i} , \quad i = 0,1,2.$$

Remarks: 1) For the duality between $C_o$ and $C^*$, see e.g. [Pl]. Moreover, let $G = \mathrm{Grass}_2(V)$ denote the Grassmann variety of lines in $P$ and $G^* = \mathrm{Grass}_2(V^\vee)$ the Grassmannian of lines in $\check{P}$. Then $X$ can be considered as a curve $\Gamma \subset G$; in fact, $C \to G$, defined by the quotient $V_C \to \mathcal{P}^1$, is the 1st associated map of $C_o$. Likewise, $X^*$ corresponds to a curve $\Gamma^* \subset G^*$, and the curves $\Gamma$ and $\Gamma^*$ are equal under the natural identification $G \cong G^*$([Pl], 5.2). The fact that the ruled surface $X$ (resp. $X^*$) is developable, is reflected on the curve $\Gamma$ (resp. $\Gamma^*$) by the fact that its tangents are all contained in $G$ (resp. $G^*$).

2) A proof, along the same lines, of the equality $\check{X} = C^*$ can be obtained using the functoriality of the bundles of principal parts ([Pl], §6): If $q:Y = \mathbb{P}(\mathcal{P}^1) \to C$ denotes the structure map, one shows that $q^* a^1 : q^* V_C \to q^* \mathcal{P}^2$ factors through $a_Y^1 : V_Y \to \mathcal{P}_Y^1(1)$, where $\mathcal{P}_Y^1(1)$ denotes the principal parts of order 1 of the line bundle $\mathcal{O}_Y(1) = f^*(\mathcal{O}_P(1)|_X)$. Since $a_Y^1$ is generically surjective, $\mathrm{Im}(a^1)$ and $q^*\mathcal{P}^2$ are isomorphic as quotients of $V_Y$, and hence the dual map of $X \subset P$ is defined on the normalization $Y$ of $X$ and has for image $C^* \subset \check{P}$.

3) Since the dual map of $X$ is defined on $Y$, it follows that the Nash transformation of $X$ is equal to the normalization $f:Y \to X$. Now $Y$ is smooth, hence the singular points of $X$ are such that the local Euler obstruction is equal to the multiplicity ([L-T], 6.1.4).

Let us examine the singularities of the tangent developable $X$ of $C_o$. Since $X$ has smooth normalization, it is of the type studied in [P3] - we shall use results and notations from there.

The surface $X$ has degree $\mu_0 = r_1$, rank $\mu_1 = r_2$, and class $\mu_2 = 0$. Its cuspidal edge consists of the curve $C_o$ and the tangents at flexes (taken with proper multiplicities) of $C_o$, hence has degree

$$c = r_0 + k_1.$$

This can (also) be seen as follows: The cuspidal edge is the image of the ramification divisor $R$ of $f:Y \to X$, hence it is defined by the ideal $F^0(\Omega^1_{Y/X}) = F^0(\mathrm{Coker}\ a^1_Y\colon V_Y \to \mathcal{P}^1_Y(1))$. A local study of $a^1_Y$ gives the support of $R$; moreover, it follows that the rational equivalence class of $R$ is given by

$$[R] = (c_1(\mathcal{P}^1_Y(1)) - c_1(q^*\mathcal{P}^2)) \frown [Y]$$

since $\mathrm{Im}(a^1_Y) = q^*\mathcal{P}^2$. Using the standard exact sequences

$$(2) \qquad 0 \to q^*\Omega^1_C \to \Omega^1_Y \to \Omega^1_{Y/C} \to 0$$

and

$$(3) \qquad 0 \to \Omega^1_{Y/C} \otimes \mathcal{O}_Y(1) \to q^*\mathcal{P}^1 \to \mathcal{O}_Y(1) \to 0,$$

one obtains

$$[R] = (q^*(c_1(\Omega^1_C) + c_1(\mathcal{P}^1) - c_1(\mathcal{P}^2)) + c_1(\mathcal{O}_Y(1))) \frown [Y].$$

This gives

$$c = \deg f_*[R] = 2g-2+r_1-r_2+r_1 = r_2-2r_1+r_0+k_1+2r_1-r_2 = r_0+k_1.$$

In addition to the cuspidal edge, the developable $X$ has a double (or higher multiple) curve, called the nodal curve of $C_o$. It consists of points that are on more than one tangent to $C_o$. Eventual bitangents to $C_o$ are part of the nodal curve.

Let $b$ denote the degree of the nodal curve. If the nodal curve is double and the flexes of $C_o$ are ordinary, we get from ([P3], Th.4)

$$2b = \mu_0(\mu_0 - 1) - \mu_1 - 3c = r_1(r_1-1) - r_2 - 3(r_0+k_1).$$

If the nodal curve consists of curves $D_j$, where $D_j$ is ordinary $j$-multiple, then the degrees $b_j$ of $D_j$ satisfy ([P3], §4)

$$\sum j(j-1)b_j = r_1(r_1-1) - r_2 - 3(r_0+k_1)$$

(still assuming the flexes to be ordinary).

The cuspidal edge and the nodal curve may themselves be singular, and also they have points of intersection - the points on X of higher multiplicity are found among these. In particular, if the nodal curve is double and the flexes ordinary, X will have a finite number of points of multiplicity $\geqslant 3$. In fact, one can define a triple point cycle $[M_3]$ on Y, with respect to $f:Y \to X$ ([K], Ch.V), and the number

$$T = \tfrac{1}{6}\deg f_*[M_3]$$

will be called the (total) number of triple points of X. Note that this number is well defined even if the nodal curve is not double and the flexes are not ordinary, since $[M_3]$ is defined as the double point cycle of the map from the double point scheme of f to Y (see [K], loc.cit.). As in ([P3], §5) we shall apply Kleiman's triple point formula ([K], V82) to compute T.

Proposition 1: The total number T of triple points of the tangent developable X of $C_0$ is given by

$$T = \frac{1}{6}(r_1-4)((r_1-3)(r_1-2) - 6g) .$$

Remarks: 1) Suppose $S \subset P$ is a ruled (non-developable) surface of degree m and genus g, with ordinary singularities, i.e., a double curve, a finite number of pinch points, and a finite number t of (ordinary) triple points. Then t is given by (e.g. [K], V84)

$$t = \frac{1}{6}(m-4)((m-3)(m-2) - 6g).$$

Now t is also equal to the number of tritangent planes to S: since the dual variety $\check{S} \subset \check{P}$ is a ruled surface which is also of degree m and genus g, the number $\check{t}$ of triple points of $\check{S}$ has the same expression as t, and the triple points of $\check{S}$ are just the tritangent planes of S.

Similarly, the tritangent planes to the developable X correspond to the triple points of the dual variety of $C_0$, which is the tangent developable $X^*$ of $C^* \subset \check{P}$. Since $C^*$ has the same rank and genus as $C_o$, the number $\check{T}$ of triple points of $X^*$ is equal to T.

2) The observations made in 1) show that the total number of triple points on a ruled surface (developable or not) is equal to its number of tritangent planes, and that this number is determined by its genus and its degree viewed as a curve $\Gamma$ in the Grassmann variety G of lines in P. Note that a triple point of the ruled surface corresponds to a Schubert cycle $\sigma_{2,0}$ (consisting of all lines through the point) on G which is trisecant to $\Gamma$. A tritangent plane corresponds to a Schubert cycle $\sigma_{1,1}$ (lines in the plane) which is trisecant to $\Gamma$. Now the $\sigma_{2,0}$-planes in G corresponds to the $\sigma_{1,1}$-planes in $G^*$; since $\Gamma = \Gamma^*$ under the identification $G = G^*$, the equality between the number of triple points and the number of tritangent planes is justified.

<u>Proof</u>: With notations as in ([P3], §5), let $\Gamma_c \subset Y$ denote the reduced ramification locus of $f:Y \to X$, and $\Gamma_b \subset Y$ the reduced inverse image via f of the nodal curve. Set $i = (\Gamma_b, \Gamma_c)$, and let $\mathcal{N} = f^*\Omega^1_P - \Omega^1_Y$ denote the virtual conormal bundle of $f:Y \to P$.

We shall apply Kleiman's triple point formula ([K], V.82) – as in ([P3], p.225) – to the map f. This we can do, because the double point scheme of f has the expected dimension, namely 1 ([K], p.387). Assume now that the nodal curve is double and the flexes of $C_o$ are ordinary. Then the formula yields

$$3T = \tfrac{1}{2} \deg f_*[M_3] = (\Gamma_b^{\,2}) + 4(\Gamma_c^{\,2}) + 4i - r_1(b+c) + c_2(\mathcal{N}).$$

<u>Lemma 6</u>: (i) $(\Gamma_c^{\,2}) = 2-2g+k_0+2k_1$

(ii) $(\Gamma_b^{\,2}) = (r_1-1)(r_1^{\,2}-7r_1-2k_0-4k_1)+9r_1+6k_0+12k_1$

(iii) $i = (r_1-4)(r_0+k_1)-k_0-2k_1$

(iv) $c_2(\mathcal{N}) = 2(r_1+2g-2)$.

<u>Proof of the lemma</u>: (i) Write $\Gamma_c = C \cup \bigcup_i F_i$, where $\{f(F_i)\}$ are the flex tangents to $C_o$, and $C \subset Y$ is the section of $q:Y \to C$ given by the quotient $\mathcal{P}^1 \to \mathcal{O}_C(1)$. If $\mathcal{N}_{C/Y}$ denotes the

conormal bundle of C in Y, then $(C^2) = -\deg \mathcal{N}_{C/Y}$. From the exact sequences ( 2 ), ( 3 ), and

$$0 \to \mathcal{N}_{C/Y} \to \Omega^1_Y|_C \to \Omega^1_C \to 0 ,$$

and the fact $q^*\Omega^1_C|_C = \Omega^1_C$, it follows that

$$\deg \mathcal{N}_{C/Y} = \deg \Omega^1_{Y/C} = \deg \mathcal{P}^1 - 2\deg \mathcal{O}_Y(1) = r_1 - 2r_0 = 2g-2-k_0 .$$

The formula follows from this, since we have $(C,F_i) = 1$ and $(F_i,F_j)=0$.

(ii) Set $H = c_1(f^*\mathcal{O}_X(1)) \cap [Y]$ and $\check{H} = c_1(\check{f}^*\mathcal{O}_{\check{P}}(1)) \cap [Y]$, where $\check{f}: Y \to \check{P}$ denotes the dual map of X (recall that $\check{f}(Y) = C^*$, the dual curve of $C_0$). Then we have ([P3], §5)

$$[\Gamma_b] = (r_1-1)H - 3[\Gamma_c] - \check{H} .$$

Since we have $(H^2) = \mu_0 = r_1$, $(H,\check{H}) = \mu_1 = r_2$, and $(\check{H}^2) = \mu_2 = 0$, this and (i) imply

$$(\Gamma_b^2) = r_1(r_1-1)^2 + 9(2k_1+k_0-2g+2) - 6(r_1-1)(r_0+k_1) - 2r_2(r_1-1) + 6r_2 .$$

The stated formula is a rewriting of this one, using the relations (1) between the $r_i$'s and $k_i$'s stated in the beginning of §4.

(iii) The expressions for $[\Gamma_b]$ and $[\Gamma_c]$ given above give

$$i = (\Gamma_b,\Gamma_c) = (r_1-1)(r_0+k_0) - 3(2-2g+k_0+2k_1) - \sigma ,$$

where $\sigma = (\Gamma_c,\check{H})$ is the class of immersion of $f(\Gamma_c)$ in X , i.e., $\sigma$ is the degree of $\check{f}(\Gamma_c)$ in $\check{P}$ . But $\check{f}(\Gamma_c) = C^*$, hence $\sigma = r_2$. The rest is again formal manipulations.

(iv) By the definition of $\mathcal{N}$ ,

$$\begin{aligned} c_2(\mathcal{N}) &= c_2(f^*\Omega^1_P - \Omega^1_Y) \\ &= c_2(f^*\Omega^1_P) - c_1(f^*\Omega^1_P)c_1(\Omega^1_Y) + c_1(\Omega^1_Y)^2 - c_2(\Omega^1_Y) \\ &= 6c_1(\mathcal{O}_Y(1))^2 + 4c_1(\mathcal{O}_Y(1))c_1(\Omega^1_Y) + c_1(\Omega^1_Y)^2 - c_2(\Omega^1_Y) . \end{aligned}$$

Now $q: Y \to C$ is a ruling, so $c_1(\Omega^1_Y)^2 = 2c_2(\Omega^1_Y)$ and

$$c_1(\Omega^1_Y) = c_1(q^*\Omega^1_C) + c_1(\Omega^1_{Y/C})$$
$$= c_1(q^*\Omega^1_C) + c_1(q^*\mathcal{P}^1) - 2c_1(\mathcal{O}_Y(1)) ,$$
$$c_1(\Omega^1_Y)^2 = -4(2g-2).$$

Hence,

$$c_2(\mathcal{N}) = 6r_1+4(2g-2+r_1-2r_1)-2(2g-2) = 2(r_1+2g-2) \ . \ //$$

To finish the proof of the proposition, use the lemma on the terms in the triple point formula to obtain

$$3T = \tfrac{1}{2}r_1((r_1-1)(r_1-2)-3r_0-3r_2+22-3k_1)-10r_0+2r_2+6k_0+2k_1-2(2g-2),$$

and then apply (1) to eliminate $r_0$, $r_2$, and $k_0$.

Though we assumed for simplicity the nodal curve to be double and the flexes to be ordinary, the same formula is valid in the general case: Replace $\Gamma_b$ by $\sum_j (j-1)\Gamma_{b_j}$ , if $f(\Gamma_{b_j})$ is j-multiple on X. If the flexes are not ordinary, each $F_i$ will count with a multiplicity $m_i$ in the double point cycle of f. Adjusting the expressions of the lemma accordingly, and plugging them into the triple point formula, one observes that the "bad" terms disappear, and one is left with exactly the same expression. //

Assume now that the nodal curve is double and the flexes ordinary. The points of multiplicity $\geq 3$ of X contains: the t points where three distinct tangents to $C_0$ meet, the $\gamma$ intersections (outside $C_0$) of the nodal curve with the flex tangents, the d(1,2) points where a tangent to $C_0$ cuts $C_0$ again, the $k_0$ cusps and the $k_1$ flexes of $C_0$.

Zeuthen gives the following formulas for t and d(1,2) [Z]:

$$t = \frac{1}{6} r_1(r_1^2-3r_1-9r_0-3r_2-9k_1) + \frac{1}{3} (39r_0-29r_1+21r_2+39k_1) ,$$

$$d(1,2) = r_1(r_0-6)+4r_0-6k_0-2k_1 \ .$$

(For a proof when $k_0 = k_1 = 0$, see [G-H], p.294. A formula for d(1,2) including the flexes is given by Le Barz [LB], also under

The assumption that $C_0$ is smooth.)

<u>Lemma 7</u> : $\gamma = (r_1-6)k_1$ .

<u>Proof</u>: Let H be a plane containing a tangent T(p) to $C_0$, and assume H is not osculating. Then $H \cap X = D \cup (l_2-l_1+1)T(p)$ , where the plane curve D has degree $r_1-(l_2-l_1+1)$ and a singularity of type $(l_1+1, l_3+3)$ at h(p). By lemma 2 , D and T(p) have intersection number $l_3+3$ at h(p). The number of intersections of T(p) and X outside h(p) is thus equal to

$$\gamma(p) = r_1-(l_2-l_1+1)-l_3-3 = r_1-4-l_3-l_2+l_1$$
$$= r_1-4-k_2(p)-2k_1(p)-k_0(p).$$

In particular, the tangent T(p) meets

$$\left.\begin{array}{l} r_1-4 \\ r_1-5 \\ \\ r_1-6 \end{array}\right\} \text{ other tangents if } \left\{\begin{array}{l} \text{p is a regular point} \\ \text{p is an ordinary cusp} \\ \quad \text{or an ordinary stall} \\ \text{p is an ordinary flex.} \end{array}\right.$$

//

<u>Remarks</u>: 1) With the above formulas, we obtain the following equality:

$$T = t + d(1,2) + \gamma + k_0 + 2k_1 .$$

Why the flexes, as opposed to the other triple points of X, appear with multiplicity 2, remains a mystery...

2) The stalls of $C_0$ are also points where the nodal curve intersects the cuspidal edge, but these points are singular on neither curve and are of multiplicity 2 on X (they are in fact "cuspidal" Whitney umbrellas).

In ([P1] ,5.1(ii)) it is shown that the dual (plane) curve of a generic plane projection of $C_0 \subset P$ , is equal to the corresponding plane section of the dual variety $X^* \subset \check{P}$ of $C_0$. Going through that proof, one verifies that the genericity assumption is in fact unnecessary. Therefore, the results on the nature of the singularities of plane sections of the developable (§2), could also have been obtained by duality from the knowledge of the singularities of plane projections. Note that if p is a point of type $(l_1+1, l_2+2, l_3+3)$, then the corresponding point on the dual curve $\check{C}$ is of type $(l_3-l_2+1, l_3-l_1+2, l_3+3)$ (see e.g.[B],p.184).

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