Cut Elimination in Coalgebraic Logics

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Abstract

We give two generic proofs for cut elimination in propositional modal logics, interpreted over coalgebras. We first investigate semantic coherence conditions between the axiomatisation of a particular logic and its coalgebraic semantics that guarantee that the cut-rule is admissible in the ensuing sequent calculus. We then independently isolate a purely syntactic property of the set of modal rules that guarantees cut elimination. Apart from the fact that cut elimination holds, our main result is that the syntactic and semantic assumptions are equivalent in case the logic is amenable to coalgebraic semantics. As applications we present a new proof of the (already known) interpolation property for coalition logic and newly establish the interpolation property for the conditional logics CK and $CK + ID$.

1 Introduction

Establishing the admissibility of the cut rule in a modal sequent calculus often allows establishing many other properties of the particular logic under scrutiny. Given that the sequent calculus enjoys the subformula property, the conservativity property is immediate: each formula is provable using only those deductive rules that mention exclusively operators that occur in the formula. As a consequence, completeness of the calculus at large immediately entails completeness of every subsystem that is obtained by removing a set of modal operators and the deduction rules in which they occur. Moreover, cut-free sequent systems admit backward proof search, as the logical complexity of a formula usually decreases when passing from the conclusion

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to the premise of a deductive rule to the premise. Given that contraction is admissible in the proof calculus, this yields – in presence of completeness – decidability and complexity bounds for the satisfiability problem associated with the logic under consideration [9, 2]. Finally, a cut-free system provides the necessary scaffolding to prove interpolation theorems by induction on cut-free proofs.

For normal modal logics, sequent calculi, often in the guise of tableau systems, have therefore – not surprisingly – received much attention in the literature [1, 5, 16]. In the context of non-normal logics, sequent calculi have been explored for regular and monotonic modal logics [6], for Pauly's coalition logic [7] and for a family of conditional logics [13]. All these logics are coalgebraic in nature: their standard semantics can be captured by interpreting them over coalgebras for an endofunctor on sets. This is the starting point of our investigation and we set out to derive sequent systems for logics with coalgebraic semantics and study their properties. Given a (complete) axiomatisation of a logic w.r.t. its coalgebraic semantics, we systematically derive a (complete) sequent calculus. In general, this calculus will only be complete if we include the cut rule. We show that cut free completeness, and therefore eliminability of cut, follows if the axiomatisation is *one-step cut-free complete*: every valid clause containing operators applied to propositional variables can be derived using a *single* modal deduction rule. The existence of a cut-free sequent calculus for coalgebraic logics is then exploited to establish conservativity, complexity and interpolation for modal logics in a coalgebraic framework. While conservativity and complexity of coalgebraic logics have already been established in [19] we believe that the results here offer additional conceptual insight. Regarding interpolation, we obtain a new proof of the (known) interpolation property for Coalition Logic [7] while interpolation for the conditional logics CK and $CK + ID$ [4] was left as future work in [13] and appears to be new.

On a technical level, we consider modal logics that are built from atomic propositions, propositional connectives and modal operators, that is, in contrast to earlier work (e.g. [10, 14, 18, 19]) we treat propositional variables as first-class citizens. This does not only provide a better alignment with standard texts in modal logic [4, 3] but is moreover a prerequisite for formulating the interpolation property.

As a consequence, we are lead to work with coalgebraic models, that is, coalgebras together with a valuation of the propositional variables, right from the start. Completeness and cut-free completeness is then proved via a terminal sequence argument, but over the extension of the signature functor to the slice category $\mathsf{Set}/\mathcal{P}(V)$ where V is the set of propositional variables.

This provides an alternative route to the shallow proof property of [19]. In this setting, we observe that one-step cut-free completeness corresponds to eliminability of cut. We then isolate purely syntactic conditions under which cut elimination holds. In essence, the set of modal rules has to be rich enough so that cuts between conclusions of modal rules can be absorbed into a single rule. If the rules are moreover strong enough to propagate contraction, we show that cut can be eliminated. This essentially amounts to completing the rule set so that cuts involving rule conclusions are in fact absorbed in the rule set, in strong analogy with Mints' comparison [12] between resolution and sequent proofs. It is interesting to note that the respective strengths of the syntactic and the semantic approach are identical: we show that the semantic coherence condition that guarantees admissibility of cut is equivalent to the syntactic requirement which is needed for cut elimination.

We summarise the coalgebraic semantics of modal logics in Section 2 and introduce modal sequent calculi in 3. Section 4 then establishes cut-free completeness and we discuss applications, in particular the interpolation property, in Section 6 before concluding with two open problems.

2 Coalgebraic and Logical Preliminaries

Given a category $\mathbb{C}$ and an endofunctor $F : \mathbb{C} \to \mathbb{C}$, an *$F$-coalgebra* is a pair (C, γ) where $C \in \mathbb{C}$ is an object of $\mathbb{C}$ and $\gamma : C \to FC$ is a morphism of $\mathbb{C}$. A *morphism* between F-coalgebras (C, γ) and (D, δ) is a morphism $m : C \to D \in \mathbb{C}$ such that $\delta \circ m = Fm \circ \gamma$. The category of F-coalgebras will be denoted by $\mathsf{Coalg}(F)$.

In the sequel, we will be concerned with F-coalgebras both on the category Set of sets and (total) functions and on the slice category $\mathsf{Set}/\mathcal{P}(V)$, for V a denumerable set of propositional variables that we keep fixed throughout the paper. Working with the slice category $\mathsf{Set}/\mathcal{P}(V)$ allows a convenient treatment of propositional variables. In particular, coalgebras on $\mathsf{Set}/\mathcal{P}(V)$ play the role of Kripke models, i.e. they come equipped with a valuation of propositional variables. Recall that an object of $\mathsf{Set}/\mathcal{P}(V)$ is a function $f : X \to \mathcal{P}(V)$ and a morphism $m : (X \xrightarrow{f} \mathcal{P}(V)) \to (Y \xrightarrow{g} \mathcal{P}(V))$ is a commuting triangle, that is, a function $m : X \to Y$ such that $g \circ m = f$. The projection functor mapping $(X \to \mathcal{P}(V)) \mapsto X$ is denoted by $U : \mathsf{Set}/\mathcal{P}(V) \to \mathsf{Set}$. For the remainder of the paper, we fix an endofunctor $T : \mathsf{Set} \to \mathsf{Set}$ and denote its extension to $\mathsf{Set}/\mathcal{P}(V)$ by $T/\mathcal{P}(V) : (\mathsf{Set}/\mathcal{P}(V)) \to (\mathsf{Set}/\mathcal{P}(V))$; the functor $T/\mathcal{P}(V)$ maps objects

$f : X \to \mathcal{P}(V)$ to the second projection mapping $TX \times \mathcal{P}(V) \to \mathcal{P}(V)$. Note that an object $M \in \mathsf{Coalg}(T/\mathcal{P}(V))$ is a commuting triangle necessarily of the form

$$\begin{tikzcd} C \arrow[rr, "\langle\gamma,\vartheta\rangle"] \arrow[dr, "\vartheta"'] & & TC \times \mathcal{P}(V) \arrow[dl, "\pi_2"] \\ & \mathcal{P}(V) & \end{tikzcd}$$

or equivalently a triple (C, γ, ϑ) where $(C, \gamma) \in \mathsf{Coalg}(T)$ and $\vartheta : C \to \mathcal{P}(V)$ is a co-valuation of the propositional variables. Passing from the co-valuation $\vartheta : C \to \mathcal{P}(V)$ to the valuation $\vartheta^\sharp : V \to \mathcal{P}(C)$ induced by the self-adjointness of the powerset functor, we can view $T/\mathcal{P}(V)$-coalgebras as T-coalgebras (C, γ) together with a valuation of propositional variables. $T/\mathcal{P}(V)$-coalgebras therefore play the role of *T-models* (T-coalgebras, which we see as frames, together with a valuation of propositional variables). In what follows, we will denote $T/\mathcal{P}(V)$-coalgebras as triples (C, γ, ϑ) as above and use $\mathsf{Mod}(T)$ to refer to the category $\mathsf{Coalg}(T/\mathcal{P}(V))$ of T-models. If $M = (C, \gamma, \vartheta)$ is a T-model, then we refer to $(C, \gamma) \in \mathsf{Coalg}(T)$ as the *underlying frame* of M.

On the syntactic side, we work with modal logics over an arbitrary modal similarity type (set of modal operators with associated arities) Λ. The set of Λ-*formulas* given by the grammar

$$\mathcal{F}(\Lambda) \ni A, B ::= p \mid A \wedge B \mid \neg A \mid \heartsuit(A_1, \dots, A_n)$$

where $p \in V$ and $\heartsuit \in \Lambda$ is n-ary. We use the standard definitions of the other propositional connectives, that is we put $A \vee B = \neg(\neg A \wedge \neg B)$, $A \to B = \neg A \vee B$, $\bot = p \wedge \neg p$ for some $p \in V$ and $\top = \neg\bot$. Note that, in contrast to the earlier treatments of coalgebraic modal logic [10, 14, 18, 19], the definition above includes propositional variables as first-class citizens. If S is a set (of formulas, or variables) then $\Lambda(S)$ denotes the set $\{\heartsuit(s_1, \dots, s_n) \mid \heartsuit \in \Lambda \text{ is } n\text{-ary}, s_1, \dots, s_n \in S\}$ of formulas comprising exactly one application of a modality to elements of S. We denote the set of propositional formulas over a set S by $\mathsf{Prop}(S)$.

To facilitate induction on the modal rank of a formula, we stratify the set $\mathcal{F}(\Lambda)$ by modal rank. That is, we put

$$\mathcal{F}_{-1}(\Lambda) = \emptyset \text{ and } \mathcal{F}_n(\Lambda) = \mathsf{Prop}(\Lambda(\mathcal{F}_{n-1}(\Lambda)) \cup V)$$

for $n \geq 0$. It is easy to see that $\mathcal{F}(\Lambda) = \bigcup_{n\in\omega} \mathcal{F}_n(\Lambda)$.

An S-*substitution* is a mapping $\sigma : V \to S$. We denote the result of simultaneously substituting $\sigma(p)$ for every $p \in V$ in a formula $A \in \mathcal{F}(\Lambda)$ by $A\sigma$. As usual, substitution associates to the right, i.e. $A\sigma\rho = (A\sigma)\rho$ for formulas $A \in \mathcal{F}(\Lambda)$ and substitutions $\sigma, \rho : V \to \mathcal{F}(\Lambda)$.

As in [14, 17], formulas of $\mathcal{F}(\Lambda)$ are interpreted over T-coalgebras provided that T extends to a Λ-*structure*, i.e. comes equipped with an assignment of predicate liftings (natural transformations)

$$[\![\heartsuit]\!] : 2^n \to 2 \circ T$$

to every n-ary modal operator $\heartsuit \in \Lambda$. Here $2 : \mathsf{Set} \to \mathsf{Set}$ is the contravariant powerset functor, and for any functor F, F^n denotes the n-fold product of F with itself, i.e. $F^n(X) = FX \times \cdots \times FX$. Explicitly, the naturality equation for $[\![\heartsuit]\!]$ translates into the requirement that $[\![\heartsuit]\!]$ commutes with inverse images, i.e.

$$[\![\heartsuit]\!]_X(f^{-1}[Z_1], \ldots, f^{-1}[Z_n]) = (Tf)^{-1}[[\![\heartsuit]\!]_Y(Z_1, \ldots, Z_n)]$$

for all maps $f : X \to Y$ and all subsets $Z_1, \ldots, Z_n \subseteq Y$. We usually leave the assignment of predicate liftings to modal operators implicit and simply use T to refer to the entire Λ-structure.

Given a Λ-structure T and $M = (C, \gamma, \vartheta) \in \mathsf{Mod}(T)$, the semantics of $A \in \mathcal{F}(\Lambda)$ is inductively given by

$$[\![\heartsuit(A_1, \ldots, A_n)]\!]_M = \gamma^{-1} \circ [\![\heartsuit]\!]_C([\![A_1]\!]_M, \ldots, [\![A_n]\!]_M)$$

and

$$[\![p]\!]_M = \{c \in C \mid p \in \vartheta(c)\}$$

for $p \in V$, together with the standard clauses for the propositional connectives.

If $M = (C, \gamma, \vartheta)$ is a T-model, semantic validity $[\![A]\!]_M = C$ is denoted by $M \models A$. We write $\mathsf{Mod}(T) \models A$ if $M \models A$ for all $M \in \mathsf{Mod}(T)$.

The completeness results that we establish later rely heavily on exploiting the semantic relation between formulas of $\mathsf{Prop}(V)$ (describing properties of states) and formulas of $\mathsf{Prop}(\Lambda(V))$ that describe properties of successors, in close analogy to coalgebra structures mapping states (elements of C) to successors in TC. The following notation is convenient for this purpose:

If $A \in \mathsf{Prop}(V)$, then every valuation $\tau : V \to \mathcal{P}(X)$ inductively defines a subset $[\![A]\!]_X^\tau \subseteq X$ by evaluation in the boolean algebra $\mathcal{P}(X)$ and we write $X, \tau \models A$ if $[\![A]\!]_X^\tau = X$.

For statements about successor states, i.e. formulas $A \in \mathsf{Prop}(\Lambda(V))$, we have that every valuations $\tau : V \to \mathcal{P}(X)$ induces a subset $[\![A]\!]^{\tau}_{TX} \subseteq TX$ given by inductively extending the assignment

$$[\![\heartsuit(p_1, \dots, p_n)]\!]^{\tau}_{TX} = [\![\heartsuit]\!]_V(\tau(p_1), \dots, \tau(p_n))$$

on atoms to the whole of $\mathsf{Prop}(\Lambda(V))$. We write $TX, \tau \models A$ if $[\![A]\!]^{\tau}_{TX} = TX$. Our techniques will be illustrated by the following two running examples:

Example 2.1 (Coalition Logic and Conditional Logic)**.**

(i) Coalition logic [15] allows reasoning about the coalitional power in games. We take $N = \{1, \dots, n\}$ to be a fixed set of *agents*, subsets of which are called *coalitions*. The similarity type Λ of coalition logic contains a unary modal operator $[C]$ for every coalition $C \subseteq N$. Informally, $[C]A$ expresses that coalition C has a collaborative strategy to force A. The coalgebraic semantics for coalition logic is based on the signature functor C defined by

$$\mathsf{C}X = \{(S_1, \dots, S_n, f) \mid \emptyset \neq S_i \in \mathsf{Set}, f : \textstyle\prod_{i \in N} S_i \to X\}.$$

(The fact that C is actually class-valued has no bearing on the further technical development.) The elements of $\mathsf{C}X$ are understood as *strategic games* with set X of states, i.e. tuples consisting of nonempty sets S_i of *strategies* for all agents i, and an *outcome function* $(\prod S_i) \to X$. A C-coalgebra is a *game frame* [15]. We denote the set $\prod_{i \in C} S_i$ by S_C, and for $\sigma_C \in S_C, \sigma_{\bar{C}} \in S_{\bar{C}}$, where $\bar{C} = N - C$, $(\sigma_C, \sigma_{\bar{C}})$ denotes the obvious element of $\prod_{i \in N} S_i$. A Λ-structure over C is defined by the predicate liftings

$$[\![[C]]\!]_X(A) = \{(S_1, \dots, S_n, f) \in \mathsf{C}X \mid \exists \sigma_C \in S_C. \forall \sigma_{\bar{C}} \in S_{\bar{C}}. f(\sigma_C, \sigma_{\bar{C}}) \in A\}.$$

(ii) The similarity type of the conditional logics *CK* and *CK* + *ID* contains the single binary modal operator $\Rightarrow$ that represents a non-monotonic conditional. The selection function semantics of *CK* is captured coalgebraically via the functor $\mathsf{CK}X = (2(X) \to \mathcal{P}(X))$ with $\to$ representing function space, and CK-coalgebras are standard conditional models [4]. We extend CK to a Λ-structure by virtue of the predicate lifting

$$[\![\Rightarrow]\!]_X(A, B) = \{f : 2X \to \mathcal{P}X \mid f(A) \subseteq B\}$$

which induces the standard semantics of *CK*. The conditional logic *CK* + *ID* additionally obeys the (rank-1) axiom $A \Rightarrow A$ and is interpreted over the functor $\mathsf{CK}_{\mathsf{Id}}X = \{f : 2(X) \to \mathcal{P}(X) \mid \forall A \subseteq X. f(A) \subseteq A\}$; note that $\mathsf{CK}_{\mathsf{Id}}$ is a subfunctor of CK. The functor $\mathsf{CK}_{\mathsf{Id}}$ extends to a Λ-structure by relativizing the interpretation of $\Rightarrow$ given above, i.e.

$$[\![\Rightarrow]\!]_X(A,B) = \{f \in \mathsf{CK}_{\mathsf{Id}}X \mid f(A) \subseteq B\}$$

for subsets $A, B \subseteq X$.

3 Sequent Systems for Coalgebraic Logics

Previous work on deduction in coalgebraic logics has focused on languages without propositional variables and deduction was formalised using Hilbert-style proof systems where propositional variables were simulated using nullary modalities. This contrasts with our treatment here where we treat propositional variables as first-class citizens in a Gentzen-style sequent calculus. If $S \subseteq \mathcal{F}(\Lambda)$ is a set of formulas, an *S-sequent* is a finite multiset of elements of $S \cup \{\neg A \mid A \in S\}$. We write $\mathcal{S}(S)$ for the set of S-sequents, and $\mathcal{S}$ for the set of $\mathcal{F}(\Lambda)$-sequents. As the logics we consider here are extensions of classical propositional logic, we work with single-sided sequent calculi and read sequents disjunctively. That is, a sequent corresponds to the disjunction of its elements, and we write $\check{\Gamma} = \bigvee \Gamma$ for the associated formula. We use the standard set-theoretic notation of union, intersection and subset also for multisets, respecting multiplicity. If $\Gamma \subseteq \mathcal{F}(\Lambda)$ is a multiset, we write $\mathrm{supp}(\Gamma)$ for its support, i.e. the set of elements of Γ, disregarding multiplicities. We identify a formula A with the singleton multiset $\{A\}$ whenever convenient and denote the multiset union of $\Gamma, \Delta \subseteq \mathcal{F}(\Lambda)$ by Γ, Δ. Combining both conventions, we write Γ, A for $\Gamma \cup \{A\}$.

Substitutions are applied pointwise to sequents: if σ is a substitution and Γ is a sequent, $\Gamma\sigma = \{A\sigma \mid A \in \Gamma\}$. In our terminology, a *sequent rule* is a tuple $(\Gamma_1, \dots, \Gamma_n, \Gamma_0)$ of sequents, usually written in the form

$$\frac{\Gamma_1 \quad \dots \quad \Gamma_n}{\Gamma_0} \quad \text{or} \quad \Gamma_1, \dots, \Gamma_n / \Gamma_0$$

where we silently identify sequent rules modulo reordering of the sequents in the premise.

Given a set $\mathbf{S}$ of sequent rules and a set $H \subseteq \mathcal{S}$ of additional hypotheses, the notion of deduction is standard: the set D of *$\mathbf{S} + H$-derivable sequents* is the least set that contains H and is closed under the rules in $\mathbf{S}$, i.e. it satisfies $\Gamma_0 \in D$ whenever $\Gamma_1, \dots, \Gamma_n / \Gamma_0 \in \mathbf{S}$ and $\Gamma_1, \dots, \Gamma_n \in D$. We write $\mathbf{S} + H \vdash \Gamma$ if Γ is an $\mathbf{S} + H$-derivable sequent, and $\mathbf{S} \vdash \Gamma$ in case $H = \emptyset$. A sequent rule $\Gamma_1, \dots, \Gamma_n/\Gamma_0$ is $\mathbf{S}$-*admissible* if $\mathbf{S} \vdash \Gamma_0$ whenever $\mathbf{S} \vdash \Gamma_i$ for all $i = 1, \dots, n$.

We use the following set $\mathbf{G}$ of sequent rules to account for the propositional part of our calculus

$$(Ax)\frac{}{\Gamma, A, \neg A} \qquad (\wedge)\frac{\Gamma, A \quad \Gamma, B}{\Gamma, A \wedge B} \qquad (\neg\wedge)\frac{\Gamma, \neg A, \neg B}{\Gamma, \neg(A \wedge B} \qquad (\neg\neg)\frac{\Gamma, A}{\Gamma, \neg\neg A}$$

where $A, B \in \mathcal{F}(\Lambda)$ and $\Gamma \in \mathcal{S}$ ranges over all $\mathcal{F}(\Lambda)$-sequents. We adopt the context-free version of the cut-rule and write $\mathbf{C}$ for the set containing of all instances

$$(\text{cut})\frac{\Gamma, A \quad \Delta, \neg A}{\Gamma, \Delta}$$

where $\Gamma, \Delta \in \mathcal{S}$ and $A \in \mathcal{F}(\Lambda)$. To facilitate arguments by induction on the modal rank of a formula, we write

$$\mathbf{S}_n = \{\frac{\Gamma_1 \dots \Gamma_k}{\Gamma_0} \in \mathbf{S} \mid \Gamma_i \in \mathcal{S}(\mathcal{F}_n(\Lambda)) \text{ for all } i = 0, \dots, k\}$$

for the set of rules in $\mathbf{S}$ whose premises and conclusions are restricted to sequents over S. In particular, this induces the sets $\mathbf{G}_n$ and $\mathbf{C}_n$, containing the propositional rules and instances of the cut rule, applied to formulas of modal rank at most n. The system $\mathbf{G}$ is a slight modification of the system **G3c** of [22] where only $A \in V$ is permitted in (Ax) and (Ax) as formulated here is admissible. Note that exchange rules are not needed as $\mathbf{G}$ is formulated in terms of multisets.

Note that $\mathbf{G}$ is complete w.r.t. propositional validity, i.e. $\mathbf{G} \vdash \Gamma$ iff $\check{\Gamma}$ is a propositional tautology. Our next task is to extend $\mathbf{G}$ with additional sequent rules to account for modal deduction. It has been shown in [17] that coalgebraic logics can always be completely axiomatised in *rank 1*, in particular, by a (possibly infinite) number of *one-step rules*, that is, rules whose premise is a purely propositional formula that have a purely modalised conclusion.

Definition 3.1. A *one-step rule* over a modal similarity type Λ is an $n+1$-tuple $(\Gamma_1, \dots, \Gamma_n, \Gamma_0)$, written as $\frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0}$ or $\Gamma_1 \dots \Gamma_n/\Gamma_0$, where $\Gamma_1, \dots, \Gamma_n \in \mathcal{S}(V)$ and $\emptyset \neq \Gamma_0 \in \mathcal{S}(\Lambda(V))$.

One-step rules describe the passage from statements about states (the premises) to a statement about successors (in the conclusion), analogously to the way in which the structure map $\gamma : C \to TC$ of a T-coalgebra (C, γ) provides us with a (structured) successor state for each world $c \in C$ of the model.

The definition above differs slightly from that given in [14, 17] in the sense that one-step rules in *op.cit.* are of the form ϕ/ψ where $\phi \in \mathsf{Prop}(V)$ is a purely propositional formula and ψ is a clause over atoms in $\Lambda(V)$. By passing from a propositional formula ϕ to its conjunctive normal form, every one-step rule in the sense of [14, 17] can accomodated in the above definition in a straightforward way.

Every set of one-step rules gives rise to a set of sequent rules by passing from a one-step rule to all its substitution instances, augmented with an additional weakening context.

Definition 3.2. Let $\mathbf{R}$ be a set of one-step rules. The set $\mathcal{S}(\mathbf{R})$ of sequent rules associated with $\mathbf{R}$ is the set of rules consisting of all instances of

$$\frac{\Gamma_1\sigma \quad \dots \Gamma_n\sigma}{\Gamma_0\sigma, \Delta}$$

where $\Gamma_1, \dots, \Gamma_n/\Gamma_0 \in \mathbf{R}$, $\sigma : V \to \mathcal{F}(\Lambda)$ is a substitution and $\Delta \in \mathcal{S}$ ranges over the set of $\mathcal{F}(\Lambda)$-sequents.

For our two running examples, the situation is as follows.

Example 3.3 (Coalition Logic and Conditional Logic)**.**

(i) In [19], Pauly's Coalition Logic [15] was axiomatised by the rules

$$\frac{\bigvee_{i=1}^{k} \neg a_i}{\bigvee_{i=1}^{k} \neg [C_i] a_i} \qquad \frac{\bigwedge_{i=1}^{k} a_i \to b \vee \bigvee_{j=1}^{l} c_j}{\bigwedge_{i=1}^{k} [C_i] a_i \to [D] b \vee \bigvee_{j=1}^{l} [N] c_j}$$

subject to the side condition that the C_i are pairwise disjoint; the second rule additionally requires that $C_i \subseteq D$ for all $i = 1, \dots, k$. These rules are one-step rules if we dissolve premise and conclusion into sequents, i.e. if we replace the formula $\bigwedge_{i=1,\dots,n} A_i \to \bigvee_{j=1,\dots,m} B_j$ by the sequent $\neg A_1, \dots, \neg A_n, B_1, \dots, B_m$.

The induced set $\mathcal{S}(\mathbf{R_C})$ of sequent rules is most economically presented if we abbreviate $\mathbf{A} = A_1, \dots, A_k$ for $A_1, \dots, A_k \subseteq V$ and $\mathbf{C} = (C_1, \dots, C_k)$ for $C_1, \dots, C_k \subseteq N$; in this case $[\mathbf{C}]\mathbf{A}$ represents the multiset $[C_1]A_1, \dots, [C_k]A_k$ of formulas. Using this notation, we obtain the following set of sequent rules, consisting of all instances of

$$(A)\frac{\neg \mathbf{A}}{\neg [\mathbf{C}]\mathbf{A}, \Gamma} \qquad (B)\frac{\neg \mathbf{A}, B, \mathbf{A}'}{\neg [\mathbf{C}]\mathbf{A},\ [D]B,\ [\mathbf{N}]\mathbf{A}', \Gamma}$$

where $\mathbf{N} = N, \dots, N$ and $\neg \Delta = \{\neg A \mid A \in \Delta\}$ for $\Delta \in \mathcal{S}$. Both rule schemas are subject to the side condition that the coalitions appearing in $\mathbf{C}$ are disjoint; rule (B) moreover requires that their union is a subset of D.

(ii) The axiomatisation of conditional logic in [4] contains the rules

$$\frac{\bigwedge_{i=1,\dots,n} b_i \to b_0}{\bigwedge_{i=1,\dots,n}(a \Rightarrow b_i) \to (a \Rightarrow b_0)} \qquad \frac{a \leftrightarrow a'}{(a \Rightarrow b) \to (a' \leftrightarrow b)}$$

that induce one-step rules $\mathbf{R}_{\mathbf{CK}_0}$ as above, with the logical equivalence in the right hand rule broken down into sequents $\neg a, a'$ and $\neg a', a$. Amalgamating both rules into one, we obtain the rule set $\mathbf{R}_{\mathbf{CK}}$ that consists of the one-step rules

$$\frac{\neg b_1, \dots, \neg b_n, b_0 \quad \neg a_0, a_1 \quad \dots \quad \neg a_0, a_n \quad a_1, \neg a_0 \quad \dots \quad a_n, \neg a_0}{\neg(a_1 \Rightarrow b_1), \dots, \neg(a_n \Rightarrow b_n), (a_0 \Rightarrow b_0)}$$

for every $n \in \omega$. As above, we abbreviate $\mathbf{B} = B_1, \dots, B_n$, $\mathbf{A} = A_1, \dots, A_n$ and $\mathbf{A} \Rightarrow \mathbf{B} = A_1 \Rightarrow B_1, \dots, A_n \Rightarrow B_n$. The associated sequent rules then take the form

$$(C)\frac{\neg \mathbf{B}, B_0 \quad \neg A_0, A_1 \quad \dots \quad \neg A_0, A_n \quad \neg A_1, A_0 \quad \dots \quad \neg A_n, A_0}{\neg(\mathbf{A} \Rightarrow \mathbf{B}),\ A_0 \Rightarrow B_0,\ \Delta}.$$

The set of one-step rules needed to axiomatise $CK + ID$ contains the additional rule

$$\frac{\neg a_0, a_1 \quad \neg a_1, a_0}{a_0 \Rightarrow a_1}$$

which induces the set

$$(ID)\frac{\neg A_0, A_1 \qquad \neg A_1, A_0}{A_0 \Rightarrow A_1, \Delta}$$

of sequent rules, where $A_1, A_2 \in \mathcal{F}(\Lambda)$ and $\Delta \in \mathcal{S}$. The rules (C) express that the second component obeys normality whereas the first behaves like the modal $\Box$ of neighbourhood frames and (ID) formalises an identity law. We integrate (C) and (ID) into the single rule set

$$(CI)\frac{\neg A_0, \neg \mathbf{B}, B_0 \quad \neg A_0, A_1 \quad \dots \quad \neg A_0, A_n \quad \neg A_1, A_0 \quad \dots \quad \neg A_n, A_0}{\neg(\mathbf{A} \Rightarrow \mathbf{B}),\ A_0 \Rightarrow B_0,\ \Delta}.$$

The rule set that extends $\mathbf{R}_{\mathbf{CK}}$ with all instances of (CI) is denoted by $\mathbf{R}_{\mathbf{CKId}}$.

It is the special format of one-step rules that facilitates inductive arguments over the modal rank of formulas. For the case of one-step rules, we have the following characterisation:

Lemma 3.4. *Let* $\mathbf{R}$ *be a set of one-step rules. Then*

$$\mathcal{S}(\mathbf{R})_n = \{\frac{\Gamma_1\sigma \dots \Gamma_n\sigma}{\Gamma_0\sigma, \Delta} \mid \Gamma_1 \dots, \Gamma_n/\Gamma_0 \in \mathbf{R}, \sigma : V \to \mathcal{F}_{n-1}(\Lambda), \Delta \in \mathcal{S}(\mathcal{F}_n(\Lambda)\}$$

where, for a set $\mathbf{S}$ *of sequent rules,* $\mathbf{S}_n$ *are those rules in* $\mathbf{S}$ *whose premises and conclusion are sequents over* $\mathcal{F}_n(\Lambda)$.

In the remainder of the paper, we will use sequent calculi that are induced by several different rule sets. In particular, we will consider sequent calculi with and without cut, and also calculi whose rules are restricted to formulas of fixed modal depth. This is reflected by the following convention:

Convention 3.5. If $\mathbf{S}_1, \dots, \mathbf{S}_n$ are sets of sequent rules and $H_1, \dots, H_k \subseteq \mathcal{S}$ is a set of additional hypotheses, we use the short form and write

$$\mathbf{S}_1 \dots \mathbf{S}_n + H_1 + \dots + H_m \vdash \Gamma$$

in case $(\mathbf{S}_1 \cup \dots \cup \mathbf{S}_n) + (H_1 \cup \dots + H_n) \vdash \Gamma$ for $\Gamma \in \mathcal{S}$. Moreover, if $\mathbf{R}$ is a set of one-step rules, we write $\mathbf{GR}$ for the the rule set $\mathbf{G} \cup \mathcal{S}(\mathbf{R})$. As a consequence, note that $\mathbf{GR}_n = \mathbf{G}_n \cup (\mathcal{S}(\mathbf{R}))_n$ for $n \in \omega$.

We start our analyis of the provability predicate $\mathbf{GR} \vdash$ by establishing that weakening and inversion are admissible in the relativised calculi $\mathbf{GR}_n$. This is most easily established using the following characterisation of $\mathbf{GR}_n$-provability: a sequent is $\mathbf{GR}_n$-provable iff it is $\mathbf{G}_n$-provable from the set of conclusions of $\mathcal{S}(\mathbf{R})_n$-rules whose premises are $\mathbf{GR}_{n-1}$-provable. That is, we have the following:

Lemma 3.6. *Let* $\mathbf{R}$ *be a set of one-step rules, and let* $n \in \omega$. *Then* $\mathbf{GR}_n \vdash \Gamma$ *iff*

$$\mathbf{G}_n + \{\Gamma_0\sigma, \Delta \mid \Gamma_1, \dots \Gamma_k/\Gamma_0 \in \mathbf{R}, \Delta \in \mathcal{S}(\mathcal{F}_n(\Lambda)), \\ \sigma : V \to \mathcal{F}_{n-1}(\Lambda), \forall_{1 \le i \le k} \mathbf{GR}_{n-1} \vdash \Gamma_i\sigma\} \vdash \Gamma$$

whenever $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$.

Proof. The case $n = 0$ is evident as $\mathcal{F}_{-1}(\Lambda) = \emptyset$. For $n > 0$, one proves the only-if direction by induction on the proof of Γ, noting that for $\Gamma \in \mathcal{S}(\mathcal{F}_{n-1}(\Lambda))$ we have $\mathbf{GR}_n \vdash \Gamma$ iff $\mathbf{GR}_{n-1} \vdash \Gamma$. □

One ingredient in the construction of sequent rules from one-step rules was the addition of a weakening context Γ to the conclusion of every substituted one-step rule. As a consequence, weakening is admissible:

Lemma 3.7 (Weakening lemma)**.** *Let* $\mathbf{R}$ *be a set of one-step rules. Then* $\mathbf{GR}_n \vdash \Gamma, A$ *whenever* $\mathbf{GR}_n \vdash \Gamma$ *and* $A \in \mathcal{F}_n(\Lambda)$*.*

Proof. By induction on the proof of $\mathbf{GR}_n \vdash \Gamma$ using Lemma 3.6. □

The same argument allows us to prove that inversion is admissible.

Lemma 3.8 (Inversion lemma)**.** *Let* $n \in \omega$*, and let* $\mathbf{R}$ *be a set of one-step rules. Then all instances of the inversion rules*

$$\frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_1} \qquad \frac{\Gamma, A_1, \wedge A_2}{\Gamma, A_2},$$

where $A_1, A_2 \in \mathcal{F}_n(\Lambda)$ *and* $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$*, are* $\mathbf{GR}_n$*-admissible.*

Proof. Standard induction on proofs in $\mathbf{GR}_n$ using Lemma 3.6. Note that inversion is automatic for formulas $\Gamma_0\sigma, \Delta$ where Γ_0 is the conclusion of a one-step rule, $\sigma : V \to \mathcal{F}_{n-1}(\Lambda)$ and $\Delta \in \mathcal{S}(\mathcal{F}_n(\Lambda))$, as the formulas in $\Gamma_0\sigma$ do not contain any top-level propositional connectives. □

Finally, we show that $\mathbf{GR}$-derivability is closed under uniform substitution. Again, this is carried out relative to the modal depth of formulas.

Lemma 3.9 (Substitution Lemma)**.** *Let* $\mathbf{GR}_n \vdash \Gamma$*, and let* $\sigma : V \to \mathcal{F}_k(\Lambda)$*. Then* $\mathbf{GR}_{n+k} \vdash \Gamma\sigma$*.*

Proof. Induction on the proof of $\mathbf{GR}_n \vdash \Gamma$. □

Lemma 3.7 and Lemma 3.8 entail the admissibility of weakening and inversion also in the calculus $\mathbf{GR}$. This is an easy consequence of the following observation which will be crucial in the following sections.

Proposition 3.10. *Let* $\mathbf{R}$ *be a set of one-step rules, and let* $\Gamma \in \mathcal{S}$*. Then* $\mathbf{GR} \vdash \Gamma$ *iff* $\mathbf{GR}_n \vdash \Gamma$ *for some* $n \in \omega$*. The corresponding statement holds for derivability in* $\mathbf{GRC}$*.*

Proof. As all rules in $\mathbf{GR}$ only have finitely many premises, any proof $\mathbf{GR}$ or $\mathbf{GRC}$ can be simulated in $\mathbf{GR}_n$ and $\mathbf{GRC}_n$, respectively, where n is large enough, i.e. such that all formulas occuring in the proof are elements of $\mathcal{F}_n(\Lambda)$. □

As a corollary, we have admissibility of weakening and contraction in the calculus $\mathbf{GR}$.

Corollary 3.11. *Let* $\mathbf{R}$ *be a set of one-step rules. Then all instances of weakening and inversion*

$$\frac{\Gamma}{\Gamma, A} \qquad \frac{\Gamma, \neg\neg A}{\Gamma, A} \qquad \frac{\Gamma, \neg(A_1 \wedge A_2)}{\Gamma, \neg A_1, \neg A_2} \qquad \frac{\Gamma, A_1 \wedge A_2}{\Gamma, A_1} \qquad \frac{\Gamma, A_1, \wedge A_2}{\Gamma, A_2}$$

where $\Gamma \in \mathcal{S}(\mathcal{F}(\Lambda))$ *and* $A, A_1, A_2 \in \mathcal{F}(\Lambda)$*, are* $\mathbf{GR}$*-admissible.*

We could have established the previous corollary directly, without the need to detour via the rank-n-derivability relation $\mathbf{GR}_n \vdash$. In fact, we never need to use the corollary above, but need to make crucial use of weakening and inversion in relativised form (Lemma 3.7 and Lemma 3.8).

This concludes our discussion of the basic properties of sequent systems induced by one-step rules. The next two sections are devoted to establish admissibility of cut and contraction, first semantically in the next section and then by a purely syntactic argument.

4 Soundness and Cut-Free Completeness

We now study the relationship between $\mathbf{GR}$-derivability and semantic validity. As in previous work, both soundness and completeness will be implied by one-step completeness of the rule set $\mathbf{R}$. However, we want to point out two subtle differences: (a) our proof deals with propositional variables directly and (b) it sheds light on the structure of proofs. In particular, we will see that a one-step complete rule set necessitates the use of cut to obtain completeness and eliminability of cut amounts to one-step cut-free completeness. We recall the definition of one-step soundness and one-step completeness, adapted to a sequent calculus setting from [14, 17]:

Definition 4.1. A set $\mathbf{R}$ of one-step rules is *one-step sound* w.r.t a Λ-structure T if, whenever $\Gamma_1, \dots, \Gamma_n/\Gamma_0 \in \mathbf{R}$, we have $TX, \tau \models \Gamma_0$ for each set X and each valuation $\tau : V \to \mathcal{P}(X)$ such that $X, \tau \models \Gamma_i$ for all $i = 1, \dots, n$. The set $\mathbf{R}$ is *one-step complete* if

$$\mathbf{GC}_1 + \{\Gamma_0\sigma \mid \frac{\Gamma_1 \dots \Gamma_n}{\Gamma_0} \in \mathbf{R}, \sigma : V \to \mathsf{Prop}(V), \forall_{1 \le i \le n}(X, \tau \models \Gamma_i\sigma)\} \vdash \Gamma$$

whenever $TX, \tau \models \Gamma$ for a set X, $\Gamma \in \mathcal{S}(\Lambda(V))$, and a $\mathcal{P}(X)$-valuation τ. Finally, $\mathbf{R}$ is *one-step cut-free complete* if, whenever $TX, \tau \models \Gamma$, we have

$$\Gamma_0\sigma \subseteq \Gamma$$

for some $\Gamma_1, \dots, \Gamma_n/\Gamma_0 \in \mathbf{R}$ and some substitution $\sigma : V \to V$ such that $X, \tau \models \Gamma_i\sigma$ for all $i = 1, \dots, n$.

It is an easy exercise to show that both **GR** and **GRC** are sound provided the rule set $\mathbf{R}$ is one-step sound. To align the coalgebraic semantics of $\mathcal{F}(\Lambda)$ with the system **GR**, we define the *interpretation* of a sequent Γ w.r.t. $M = (C, \gamma, \vartheta) \in \mathsf{Mod}(V)$ to be the semantics of the associated propositional formula, i.e. $[\![\Gamma]\!]_M = [\![\check{\Gamma}]\!]_M$, and accordingly $M \models \Gamma$ iff $M \models \check{\Gamma}$, $\mathsf{Mod}(T) \models \Gamma$ if $\mathsf{Mod}(T) \models \check{\Gamma}$.

Theorem 4.2 (Soundness)**.** *Let* $\mathbf{R}$ *be one-step sound for* T*. Then* $\mathsf{Mod}(T) \models \Gamma$ *if* $\mathbf{GRC} \vdash \Gamma$ *and, a fortiori,* $\mathsf{Mod}(T) \models \Gamma$ *if* $\mathbf{GR} \vdash \Gamma$*.*

Proof. We proceed by induction over the length of the derivation, where the only interesting cases are applications of rules $\Gamma_1, \ldots, \Gamma_n / \Gamma_0 \in \mathcal{S}(\mathbf{R})$. So suppose that $(C, \gamma, \vartheta) \in \mathsf{Mod}(T)$ and that Γ has been derived via an application of $\Gamma_1 \ldots \Gamma_n / \Gamma_0$. That is, we can find a one-step rule $\Gamma'_1 \ldots \Gamma'_n / \Gamma'_0 \in \mathbf{R}$ and a substitution $\sigma : V \to \mathcal{F}(\Lambda)$ such that $\Gamma_i = \Gamma_i \sigma$ for $i = 1, \ldots, n$ and $\Gamma_0 = \Gamma'_0 \sigma, \Delta$ for some $\Delta \in \mathcal{S}$. By the induction hypothesis, $[\![\Gamma_i \sigma]\!]_M = \top$ for all $i = 1, \ldots, n$. Consider the $\mathcal{P}(C)$-valuation $\tau(p) = [\![\sigma(p)]\!]_M$. We obtain $C, \tau \models \Gamma'_i$ for all $i = 1, \ldots, n$ in the one-step sense, and one-step soundness implies $TC, \tau \models \Gamma'_0 \sigma$. Consequently, $[\![\Gamma_0]\!]_M = [\![\Gamma'_0 \sigma, \Delta]\!]_M \supseteq [\![\Gamma'_0 \sigma]\!]_M = \top$ which concludes the proof. □

We now proceed to establish completeness and cut-free completeness directly by means of a semantic argument, and present a purely syntactic reconstruction in the following section. For the semantic approach, we prove completeness using a terminal sequence argument in the style of [14] which ties in very well with the proof of cut elimination in the next section. As we are dealing with models, i.e. coalgebras equipped with a valuation, we consider the terminal sequence of the endofunctor $T/\mathcal{P}(V)$ in the category $\mathsf{Set}/\mathcal{P}(V)$. We briefly recapitulate the terminal sequence construction, as used in [14], but phrased in a general categorical setting.

If $F : \mathbb{C} \to \mathbb{C}$ is an endofunctor on a category $\mathbb{C}$ with terminal object 1, the finitary part of the terminal sequence of F is the diagram consisting of

- the objects $F^n 1$ for $n \in \omega$ where F^n denotes n-fold application of F, and
- the morphisms $p^i_j : F^i 1 \to F^j 1$ defined by $p^{i+1}_i = F^i(! : F1 \to 1)$ and $p^{n+k}_n = p^{n+k}_{n+k-1} \circ \cdots \circ p^{k+1}_k$.

Every F-coalgebra (C, γ) gives rise to a canonical cone $(C, (\gamma_n)_{n \in \omega})$, where $\gamma_n : C \to F^n 1$, over the finitary part of the terminal sequence by stipulating that $\gamma_0 =! : C \to F^0 1 = 1$ where ! is the unique arrow given by finality of

$1 \in \mathbb{C}$, and $\gamma_{n+1} = F\gamma_n \circ \gamma$. We use the terminal sequence construction for the functor $F = T/\mathcal{P}(V)$, the terminal sequence of which is visualised in the following diagram.

$$\begin{tikzcd}
\overbrace{1 \times \mathcal{P}(V)}^{S_0} \arrow[d, "\pi_2"] & \overbrace{TS_0 \times \mathcal{P}(V)}^{S_1} \arrow[l, "p_0^1=!"'] \arrow[d, "\pi_2"] & \overbrace{TS_1 \times \mathcal{P}(V)}^{S_2} \arrow[l, "p_1^2=Tp_0^1\times\mathsf{id}"'] \arrow[d, "\pi_2"] & \cdots \arrow[l, "p_1^3=Tp_1^2\times\mathsf{id}"'] \\
\mathcal{P}(V) \arrow[r, equal] & \mathcal{P}(V) \arrow[r, equal] & \mathcal{P}(V) \arrow[r, equal] & \cdots
\end{tikzcd}$$

The key technique in the proof of completeness via a terminal sequence argument is to associate to every formula A of modal rank $\leq n$ an n-step semantics $[\![A]\!]_n$ over the n-th approximant $(T/\mathcal{P}(V))^n 1$ of the terminal sequence. In our case, we take a predicate over $(T/\mathcal{P}(V))^n 1$ to be a subset of $S_n = U((T/\mathcal{P}(V))^n 1)$. The formal definition is as follows:

Definition 4.3. The *n-step semantics* of $A \in \mathcal{F}_n(\Lambda) \subseteq S_n$ is inductively defined by $S_0 = \mathcal{P}(V)$ and

$$[\![p]\!]_0 = \{S \in \mathcal{P}(V) \mid p \in S\}$$

for $n = 0$ and $S_n = TS_{n-1} \times \mathcal{P}(V)$ together with

$$[\![p]\!]_n = \pi_2^{-1}(\{S \in \mathcal{P}(V) \mid p \in S\})$$

and

$$[\![\heartsuit(A_1, \dots, A_k)]\!]_n = \pi_1^{-1} \circ [\![M]\!]_{S_{n-1}}([\![A_1]\!]_{n-1}, \dots, [\![A_k]\!]_{n-1})$$

for $A_1, \dots, A_k \in \mathcal{F}_{n-1}(\Lambda)$ and $\heartsuit \in \Lambda$ an n-ary modality.

Note that $S_n = U((T/\mathcal{P}(V))^n 1)$. We can mediate between the n-step semantics and the semantics w.r.t $\mathsf{Mod}(T)$ as follows:

Lemma 4.4. *Let $A \in \mathcal{F}_n(\Lambda)$, let $M = (C, \gamma, \vartheta) \in \mathsf{Mod}(T)$, and let $(M, (\gamma_n)_{n\in\omega})$ be the canonical cone of M over the terminal sequence of $T/\mathcal{P}(V)$. Then $[\![A]\!]_M = (U\gamma_n)^{-1}([\![A]\!]_n)$ for all $A \in \mathcal{F}_n(\Lambda)$.*

Proof. By induction on n. For $n = 0$ we have $U\gamma_0 = \vartheta$ and $\vartheta^{-1}([\![p]\!]_0) = \vartheta^{-1}(\{S \subseteq V \mid p \in S\}) = \{c \in C \mid p \in \vartheta(c)\} = [\![p]\!]_M$. For $n > 0$, we obtain inductively $U\gamma_n = \langle TU\gamma_{n-1} \circ \gamma, \vartheta\rangle : C \to TS_{n-1} \times \mathcal{P}(V)$. This gives $(U\gamma_n)^{-1}([\![p]\!]_n) = (\pi_2 \circ \langle TU\gamma_n \circ \gamma, \vartheta\rangle)^{-1}(\{S \subseteq V \mid p \in S\}) = \vartheta^{-1}(\{S \subseteq$

$V \mid p \in S\}) = \{c \in C \mid p \in \vartheta(c)\} = [\![p]\!]_M$ as above. For modal formulas $\heartsuit(A_1, \dots, A_k)$ with $A_1, \dots, A_k \in \mathcal{F}_{n-1}(\Lambda)$ we obtain

$$\begin{aligned}
&(U\gamma_n)^{-1}([\![\heartsuit(A_1, \dots, A_k]\!]_n)) \\
=&\langle TU\gamma_{n-1} \circ \gamma, \vartheta\rangle^{-1} \circ \pi_1^{-1}([\![\heartsuit]\!]_{S_{n-1}}([\![A_1]\!]_{n-1}, \dots, [\![A_k]\!]_{n-1})) \\
=&\gamma^{-1} \circ (TU\gamma_{n-1})^{-1} \circ [\![\heartsuit]\!]_{S_{n-1}}([\![A_1]\!]_{n-1}, \dots, [\![A_k]\!]_{n-1}) \\
=&\gamma^{-1} \circ [\![\heartsuit]\!]_C \circ (U\gamma_{n-1})^{-1} \times \dots \times (U\gamma_{n-1})^{-1}([\![A_1]\!]_{n-1}, \dots, [\![A_k]\!]_{n-1}) \\
=&\gamma^{-1} \circ [\![\heartsuit]\!]([\![A_1]\!]_M, \dots, [\![A_k]\!]_M) \\
=&[\![\heartsuit(A_1, \dots, A_k)]\!]_M
\end{aligned}$$

using the induction hypothesis and naturality of $[\![\heartsuit]\!]$. □

We recall the following lemma, whose proof directly translates to a general categorical setting, from [14]:

Lemma 4.5. *Let $f^0 : 1 \to F1$ be a morphism of $\mathbb{C}$ and let $f^n = Ff^{n-1}$ inductively. Then $f_n^n = \mathsf{id}_{F^n 1}$ for all $n \in \omega$.*

This immediately implies that semantic validity of a sequent Γ, with $\Gamma \subseteq \mathcal{F}_n(\Lambda)$ is equivalent to validity w.r.t the n-step semantics.

Corollary 4.6. *Let $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$. Then $\mathsf{Mod}(T) \models \Gamma$ iff $[\![\Gamma]\!]_n = \top$.*

Proof. The 'if'-part is a consequence of Lemma 4.4 above. For the 'only if'-part assume that $\mathsf{Mod}(T) \models \Gamma$ and pick $f^0 : 1 \to (T/\mathcal{P}(V)1) \in \mathsf{Set}/\mathcal{P}(V)$ where 1 is a terminal object of $\mathsf{Set}/\mathcal{P}(V)$. Consider $M = (C, \gamma) \in \mathsf{Coalg}(T/\mathcal{P}(V))$ where $C = (T/\mathcal{P}(V))^n$ and $\gamma = (T/\mathcal{P}(V))^n(f^0)$. As $\mathsf{Mod}(T) \models \Gamma$ we have that $M \models \Gamma$ and Lemma 4.4 above implies that $[\![\Gamma]\!]_n = \top$. □

The proof of completeness (and later cut-free completeness) relies on the stratification of the provability predicate $\mathbf{GR}_n \vdash$ of $\mathbf{GR}$, indexed by modal rank. The following proposition is the key stepping stone in the completeness proof and relates validity in the n-step semantics to derivability in rank n.

Proposition 4.7. *Let $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda)$ be a sequent over $\mathcal{F}_n(\Lambda)$. Then $[\![\Gamma]\!]_n = \top$ implies that $\mathbf{GRC}_n \vdash \Gamma$ if $\mathbf{R}$ is one-step complete. If $\mathbf{R}$ is one-step cut-free complete, we have that $\mathbf{GR}_n \vdash \Gamma$.*

Proof. By induction on n. If $n = 0$ the statement follows from semantic completeness of $\mathbf{G}$. By the inversion lemma, it suffices to consider, for $n > 0$, the case

$$\Gamma = \neg\heartsuit_1 A_1, \dots, \neg\heartsuit_i A_i, \neg q_1, \dots, \neg q_j, \heartsuit'_1 A'_1, \dots, \heartsuit'_{i'} A'_i, q'_1, \dots, q'_{j'}$$

where $A_i, A'_{i'}$ are tuples of formulas in $\mathcal{F}_{n-1}(\Lambda)$ according to the arity of $\heartsuit_i$ and $\heartsuit'_{i'}$ and $q_j, q'_{j'} \in V$. By the definition of $[\![\cdot]\!]_n$ and elementary boolean algebra, we deduce that either

$$[\![\neg\heartsuit_1 A_1, \dots, \neg\heartsuit_i A_i, \heartsuit'_1 A'_1, \dots, \heartsuit'_{i'} A'_i]\!]_n = \top$$

or, alternatively,

$$[\![\neg q_1, \dots, \neg q_j, q'_1, \dots, q'_{j'} \dots, q'_{j'}]\!]_n = \top$$

holds. In the latter case, $\bigwedge_{k=1}^{j} q_j \to \bigvee_{k=1}^{j'} q'_k$ is a propositional tautology and the result follows as $\neg q_1, \dots, \neg q_j, q'_1, \dots, q'_{j'} \dots, q'_{j'}$ is neccessarily an axiom. So assume that the upper identity holds. This allows us to write $\Gamma = \Delta\tau$ where

$$\Delta = \neg\heartsuit_1 p_1, \dots, \neg\heartsuit_i p_i, \heartsuit'_1 p'_1, \dots, \heartsuit'_{i'} p'_{i'}$$

where p_i and $p'_{i'}$ are tuples of propositional variables according to the arity of $\heartsuit_i$ and $\heartsuit'_{i'}$, respectively and $\tau : V \to \mathcal{F}_{n-1}(\Lambda)$ is a substitution mapping the every component of p_i to the corresponding component of A_i, and similarly for p'_i. Write τ_{n-1} for the $\mathcal{P}(S_{n-1})$ valuation $p \mapsto [\![\tau(p)]\!]_{n-1}$. Then $TS_{n-1}, \tau_{n-1} \models \Delta$. We first assume that **R** is one-step complete and deal with one-step cut-free completeness later. By definition of one-step completeness, there exist $k \geq 0$ and one-step rules $\Gamma^l_1, \dots, \Gamma^l_{m_l}/\Gamma^l_0 \in \mathbf{R}$ together with substitutions $\sigma_l : V \to \mathsf{Prop}(V)$ for each $l = 1, \dots, k$ such that

- $\mathbf{GC} + \{\Gamma^l_0 \sigma_l \mid l = 1, \dots, k\} \vdash \Delta$
- $S, \tau_{n-1} \models \Gamma^l_m \sigma_l$ for all $l = 1, \dots, k$ and all $m = 1, \dots, m_l$.

Consequently, for $l = 1, \dots, k$ and $m = 1, \dots, m_l$ we have $[\![\Gamma^l_m \sigma_l \tau]\!]_{n-1} = \top$. By induction hypothesis, this implies that $\mathbf{GRC}_{n-1} \vdash \Gamma^l_m \sigma_l \tau$ for all $l = 1, \dots k$ and $m = 1, \dots, m_l$ whence $\mathbf{GRC}_n \vdash \Gamma^l_0 \sigma_l \tau$ for $1 \leq l \leq k$.

Combined with the fact that $\mathbf{GC}_n + \{\Gamma^l_0 \sigma_l \mid l = 1, \dots, k\} \vdash \Delta$ and the Substitution Lemma 3.9, we finally obtain $\mathbf{GRC}_n \vdash \Delta\tau = \Gamma$.

This finishes the proof in case **R** is one-step complete. We now assume that **R** is one-step cut-free complete. This allows us to assume $k = 1$ in the above, i.e. assuming that $TS_{n-1}, \tau_{n-1} \models \Delta$ we find a one-step rule $\Gamma_1, \dots, \Gamma_m/\Gamma_0 \in \mathbf{R}$ and $\sigma : V \to V$ so that $S_{n-1}, \tau \models \Gamma_l \sigma$ for $l = 1, \dots, m$ and $\Gamma_0 \sigma \subseteq \Delta$.

By induction hypothesis, we obtain $\mathbf{GR}_{n-1} \vdash \Gamma_l \sigma\tau$ for $l = 1, \dots, m$ as above. Since $\Gamma_0 \sigma \subseteq \Delta$ we can find $\Sigma \in \mathcal{S}$ such that $\Gamma_0 \sigma\tau, \Sigma = \Delta\tau$, which implies that there is a sequent rule $\Gamma_1 \sigma\tau \dots \Gamma_m \sigma\tau/\Delta\tau \in \mathcal{S}(\mathbf{R})$. Since $\mathbf{GR}_{n-1} \vdash \Gamma_l \sigma\tau$ for $l = 1, \dots, m$ we finally obtain $\mathbf{GR}_n \vdash \Delta\tau = \Gamma$. $\square$

Note that we needed the power of the cut rule at precisely one point in the above proof: To conclude $\Delta\tau$ from the sequent set $\Gamma_0^l\sigma\tau$ where $1 \le l \le k$ – the need for cut is eliminated if we use one-step cut-free complete rule sets. Completeness is now an easy corollary.

Corollary 4.8 (Completeness and cut free completeness)**.** *Let* $\mathbf{R}$ *be one-step complete for* T *and* $\mathsf{Mod}(T) \models \Gamma$ *for a sequent* $\Gamma \in \mathcal{S}(\mathcal{F}(\Lambda))$*. Then* $\mathbf{GRC} \vdash \Gamma$*. If moreover* $\mathbf{R}$ *is one-step cut-free complete, then* $\mathbf{GR} \vdash \Gamma$*.*

In particular, this gives us a semantic proof of cut elimination and admissibility of contraction.

Theorem 4.9. *Let* $\mathbf{R}$ *be one-step cut-free complete. Then all instances of the cut and contraction rules*

$$\frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta} \quad \textit{and} \quad \frac{\Gamma, A, A}{\Gamma, A}$$

where $\Gamma, \Delta \in \mathcal{S}$ *and* $A \in \mathcal{F}(\Lambda)$*, are admissible in* $\mathbf{GR}$.

One may argue that the above semantic proof yields a slightly weaker result than the syntactic proofs of Section 5, as we pre-suppose soundness and completeness w.r.t. a given Λ-structure. However, for every rank-1 logic we can always construct a Λ-structure for which the given rule set is one-step sound and one-step cut-free complete [21]. We conclude the section by re-visiting our two running examples.

Example 4.10. (i) It has been shown in [20] that, *mutatis mutandis*, the set of one-step rules $\mathbf{R_C}$ is one-step cut-free complete, and, as a consequence, cut is admissible in $\mathbf{GR_C}$.

(ii) We leave it to the reader to either show that $\mathbf{R_{CK_0}}$ is one-step complete or to infer one-step completeness of $\mathbf{R_{CK_0}}$ from one-step cut-free completeness of $\mathbf{R_{CK}}$ that we now set out to prove. Let $\Gamma = \{\neg(p_i \Rightarrow q_i) \mid i \in I\} \cup \{p'_j \Rightarrow q'_j \mid j \in J\}$, and let τ be a $\mathcal{P}(X)$-valuation such that $\mathsf{CK}X, \tau \models \Gamma$. We claim that there exists $j \in J$ such that

$$\bigcap_{i \in I_j} \tau(q_i) \subseteq \tau(q'_j), \tag{$*$}$$

where $I_j = \{i \in I \mid \tau(p_i) = \tau(p'_j)\}$. Assume, for a contradiction, that this is not the case. Then, for every $j \in J$, $\bigcap_{i \in I_j} \tau(q_i) \not\subseteq \tau(q'_j)$. Define the function $f : 2(X) \to \mathcal{P}(X)$ by

$$f(S) = \begin{cases} \bigcap_{i \in I_j} \tau(q_i) & S = \tau(p'_j) \\ \emptyset & \text{otherwise.} \end{cases}$$

(This is well-defined since $I_j = I_k$ whenever $\tau(p'_j) = \tau(p'_k)$.) Then $f(\tau(p_i)) \subseteq \tau(q_j)$ but for all $j \in J$ we have that $f(\tau(p'_j)) \not\subseteq q'_j$ by construction, contradicting $\mathsf{CK}X, \tau \models \Gamma$. Having thus proved the claim, we pick $j \in J$ satisfying $(*)$. We obtain $X, \tau \models \{\neg q_i \mid i \in I_0\}, q_j$. If $I_0 = \{i_1, \dots, i_k\}$, the claim follows as

$$\frac{\{\neg q_i \mid i \in I_0\}, q'_j \quad \neg p'_j, p_{i_1} \dots \neg p'_j, p_{i_k}, \neg p_{i_1}, p'_j \dots \neg p_{i_k}, p'_j}{\{\neg(p_i \Rightarrow q_i) \mid i \in I_0\}, p_j \Rightarrow q_j}$$

is a substitution instance of (C) whose premise is valid under τ.
This proof is easily modified to establish that also the rule set $\mathbf{R_{CKId}}$ is one-step cut-free complete for $\mathsf{CK}_{\mathsf{Id}}$: if Γ is as above, one proves that there exists $j \in J$ satisfying the weaker condition

$$\tau(p'_j) \cap \bigcap_{i \in I_0} \tau(q_i) \subseteq \tau(q'_j). \tag{+}$$

This is proved by constructing f as above, but with

$$f(\tau(p'_j)) = \tau(p'_j) \cap \bigcap_{i \in I_j} \tau(q_i),$$

which defines an element of $\mathsf{CK}_{\mathsf{Id}}(X)$. From j satisfying $(+)$, one obtains an instance of (CI) that proves Γ. As a consequence, cut is admissible in $\mathbf{GR_{CK}}$ and $\mathbf{GR_{CKId}}$.

5 Cut Elimination, Syntactically

In the previous section, we have seen that one-step cut-free completeness is a sufficient criterion to ensure that an ensuing sequent calculus enjoys cut-free completeness, and we have deduced admissibility of contraction on the way. We now complement these results and give a purely syntactic criterion for admissibility of both cut and contraction. As we will see, conditions imposed on the set of modal rules under scrutiny will be equivalent to one-step cut-free completeness.

We start with admissibility of contraction, which is – unlike weakening and inversion – not automatic, and only holds if the underlying rule set satisfies an additional property. Recall that $\mathbf{GC}_0$ consists of all propositional sequent rules and the cut rule, but restricted to purely propositional formulas.

Definition 5.1. A set $\mathbf{R}$ of one-step rules *absorbs contraction* if, for every rule $\Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}$ and every renaming $\sigma : V \to V$ there exists a rule $\Delta_1 \dots \Delta_m/\Delta_0 \in \mathbf{R}$ and a renaming $\rho : V \to V$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$ and

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid 1 \leq i \leq n\} \vdash \Delta_j\rho$$

for all $j = 1, \dots, m$.

In other words, the result of identifying two or more literals in the conclusion of a rule r can always be simulated using a (generally different) rule s such that all premises of s are propositionally (i.e. with the help of cut) derivable from the premises of r.

The definition of absorption of cut is modelled on the same idea: an application of cut to the conclusions of two one-step rules r_1, r_2 can be replaced by a different one-step rule r_0 such that all the premises of r_0 are propositional consequences (can be derived with the help of cut) from the premises of r_1, r_2.

Definition 5.2. A set $\mathbf{R}$ of one-step rules *absorbs cut*, if for all $\Gamma_1 \dots \Gamma_n/\Gamma_0$ and all $\Delta_1 \dots \Delta_m/\Delta_0 \in \mathbf{R}$ and all renamings $\sigma, \rho : V \to V$ such that $\Gamma_0\sigma = \Gamma, A$ and $\Delta_0\tau = \Delta, \neg A$ there exists a rule $\Sigma_1 \dots \Sigma_l/\Sigma_0$ and a substitution $\kappa : V \to V$ such that $\mathrm{supp}(\Sigma_0\rho) \subseteq \Gamma, \Delta$ and

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid 1 \leq i \leq n\} + \{\Delta_i\rho \mid 1 \leq i \leq m\} \vdash \Sigma_j\kappa$$

for all $j = 1, \dots, l$.

Taken together, absorption of cut and contraction already imply the admissibility of cut and contraction in the associated sequent calculus. We note that both properties are local in the sense that they can be checked by considering just the set of modal (one-step) rules without considering cuts that arise through propositional rules or between modal and propositional rules. In particular, there is no need for a fully fledged cut elimination proof, and cuts between conclusions of modal rules and propositional rules are automatically admissible. We first establish this fact for derivability in $\mathbf{GR}_n$.

Proposition 5.3. *If* $\mathbf{R}$ *absorbs cut and contraction, then*

- $\mathbf{GR}_n \vdash \Gamma, A$ *whenever* $\mathbf{GR}_n \vdash \Gamma, A, A$
- $\mathbf{GR}_n \vdash \Gamma$ *whenever* $\mathbf{GRC}_n \vdash \Gamma$

for all $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$.

Proof. We proceed by induction on n, where there is nothing to show for $n = 0$. For $n > 0$, we note that, as a consequence of Lemma 3.6, $\mathbf{GR}_n \vdash \Gamma$ iff $\mathbf{G}_n + H \vdash \Gamma$ where

$$H = \{\Gamma_0\Sigma, \Delta \mid \Gamma_1 \dots \Gamma_k/\Gamma_0 \in \mathbf{R}, \sigma : v \to \mathcal{F}_{n-1}(\lambda), \forall_{1 \le i \le k}(\mathbf{GR}_{n-1} \vdash \Gamma_i\sigma)\}$$

for all $\Gamma \in \mathcal{S}(\mathcal{F}_n(\Lambda))$. We deal with contraction first. So suppose that $\mathbf{GR} \vdash \Gamma$, or equivalently, $\mathbf{G}_n + H \vdash \Gamma$. We show that $\mathbf{GR}_n \vdash \mathrm{supp}(\Gamma)$ by induction on the $\mathbf{G}_n$-proof of Γ from the additional assumptions in H. In case $\Gamma = \Gamma_0\sigma, \Delta \in H$ for $\Gamma_1 \dots \Gamma_k/\Gamma_0 \in \mathbf{R}$ and $\sigma : V \to \mathcal{F}_{n-1}(\Lambda)$ we use absorption of contraction to find a rule $\Delta_1, \dots, \Delta_l/\Delta_0$ and a substitution $\rho : V \to \mathcal{F}_{n-1}(\Lambda)$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$ and $\mathbf{GC}_{n-1} + \{\Gamma_i\sigma \mid i \le i \le k\} \vdash \Delta_j\rho$ for all $j = 1, \dots, l$ by the Substitution Lemma 3.9. As $\mathbf{GR}_{n-1} \vdash \Gamma_i\sigma$, we have that $\mathbf{GRC}_{n-1} \vdash \Gamma_i\sigma$ for all $i = 1, \dots, k$. Therefore $\mathbf{GRC}_{n-1} \vdash \Delta_j\rho$ whence, by outer induction hypothesis, $\mathbf{GR}_{n-1} \vdash \Delta_j\rho$ for all $j = 1, \dots, n$. Applying the rule $\Delta_1\rho \dots \Delta_l\rho/\Delta_0\rho$ therefore gives $\mathbf{GR}_n \vdash \Delta_0\rho$ by Lemma 3.6 and the fact $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$ gives $\mathbf{GR}_n \vdash \mathrm{supp}(\Gamma_0\sigma, \Delta)$ by the Weakening Lemma 3.7.

The remaining cases, where $\Gamma_0\sigma$ has been proved using rules of $\mathbf{G}_n$ are readily established inductively.

We turn to admissibility of cut, where it suffices to show that $\mathbf{GR}_n \vdash \Gamma, \Delta$ whenever $\mathbf{GR}_n \vdash \Gamma, A$ and $\mathbf{GR}_n \vdash \Delta, \neg A$. If this is the case, we find that $\mathbf{G}_n + H \vdash \Gamma, A$ and $\mathbf{G}_n + H \vdash \Delta, \neg A$ with H as above. We show that $\mathbf{G}_n + H \vdash \Gamma, \Delta$ using the classical double induction method, with outer induction on the size of the cut formula A and inner induction on the sum of the size of the proof trees of $\mathbf{G}_n + H \vdash \Gamma, A$ and $\mathbf{G}_n + H \vdash \Delta, A$. We distinguish three different types of cut: (a) cuts between elements of H, (b) cuts between elements of H and conclusions of $\mathbf{G}_n$-rules and (c) cuts between conclusions of $\mathbf{G}_n$-rules. As regards (a), we have that $\Gamma, A = \Gamma_0\sigma, \Gamma'$ and $\Delta, \neg A = \Delta_0\rho, \Delta'$ for two substitutions $\sigma, \rho : V \to \mathcal{F}_{n-1}(\Lambda)$ and two rules $\Gamma_1 \dots \Gamma_k/\Gamma_0$ and $\Delta_1 \dots \Delta_l/\Delta_0 \in \mathbf{R}$. In case $A \in \Gamma'$ or $\neg A \in \Delta'$ there is nothing to show, so suppose that $A \in \Gamma_0\sigma$ and $\neg A \in \Delta_0\rho$. As $\mathbf{R}$ absorbs cut, may use the Substitution Lemma 3.9 to find a rule $\Sigma_1 \dots \Sigma_m/\Sigma_0$ and a substitution $\kappa : V \to \mathcal{F}_{n-1}(\Lambda)$ such that $\mathrm{supp}(\Sigma_0\kappa) \subseteq \Gamma_0\sigma, \Delta_0\tau \setminus \{\neg A, A\} \subseteq \Gamma, \Delta$ and,

$$\mathbf{GC}_{n-1} + \{\Gamma_i\sigma \mid i = 1, \dots, k\} + \{\Delta_i\rho \mid \rho = 1, \dots, l\} \vdash \Sigma_j\kappa$$

for all $j = 1, \dots, m$. As all assumptions are $\mathbf{GR}_{n-1}$-derivable and cut is admissible in $\mathbf{GR}_{n-1}$, we have that $\mathbf{GR}_{n-1} \vdash \Sigma_j\kappa$ for all $j = 1, \dots, m$ and as contraction is admissible in $\mathbf{GR}_n$, we finally obtain $\mathbf{GR}_n \vdash \mathrm{supp}(\Sigma_0\kappa_0) \subseteq \Gamma, \Delta$ and $\mathbf{GR} \vdash \Gamma, \Delta$ follows from the relativised Weakening Lemma 3.7.

We now look at cuts of type (b), that is, cuts between propositional rules and additional assumptions in H. So suppose that $\Gamma, A = \Gamma_0\sigma, \Gamma' \in H$ and $\Delta, \neg A$ has been derived using a propositional rule. If $A \in \Gamma'$ there is nothing to show, so suppose that $A \in \Gamma_0\sigma$, i.e. $\Gamma_0\sigma = A, \Gamma''$. We only deal with the case that Δ, A has been derived using $(\wedge)$; all other cases are analogous and even simpler. As $A \in \Gamma_0\sigma$ we know that A cannot be a conjunction, so that $\mathbf{GR}_n \vdash \Sigma, C, A$ and $\mathbf{GR}_n \vdash \Sigma, D, A$ with shorter proofs, and $\Delta, A = \Sigma, C \wedge D, A$. As both Σ, C, A and Σ, D, A have been derived using shorter proofs, the inner induction hypothesis gives $\mathbf{GR}_n \vdash \Sigma, C, \Gamma''$ and $\mathbf{GR}_n \vdash \Sigma, D, \Gamma''$ and an application of $(\wedge)$ yields $\mathbf{GR} \vdash \Sigma, C \wedge D, \Gamma'' = \Gamma, \Delta$.

The elimination of cuts between conclusions of propositional rules is standard, and follows from the $\mathbf{GR}_n$-admissibility of contraction (that we have already established) and the inversion lemma 3.8. $\square$

The following theorem, which readily follows from Proposition 5.3 and Proposition 3.10, therefore provides a purely syntactic counterpart of Theorem 4.9.

Theorem 5.4. *If* $\mathbf{R}$ *absorbs cut and contraction, then all instances of the cut and contraction rules*

$$\frac{\Gamma, A \qquad \Delta, \neg A}{\Gamma, \Delta} \qquad \frac{\Gamma, A, A}{\Gamma, A}$$

where $\Gamma, \Delta \in \mathcal{S}$ *and* $A \in \mathcal{F}(\Lambda)$, *are admissible in* $\mathbf{GR}$.

Our last main result in this section is that both properties are actually equivalent in the presence of one-step completeness, which we split into two separate lemmas.

Proposition 5.5. *Let* $\mathbf{R}$ *be one-step complete. Then* $\mathbf{R}$ *is one-step cut-free complete if* $\mathbf{R}$ *absorbs cut and contraction.*

Proof. Consider the set

$$\Psi = \{\Gamma_0\sigma, \Delta \mid \Delta \in \mathcal{S}(\Lambda(V)), \frac{\Gamma_1 \ldots \Gamma_n}{\Gamma_0} \in \mathbf{R}, \sigma : V \to \mathsf{Prop}(V), X, \tau \models \Gamma_i\sigma\}$$

where we require $X, \tau \models \Gamma_i\sigma$ to hold for all $i = 1, \ldots, n$. Clearly Ψ is closed under weakening, i.e. $\Gamma \in \Psi$ implies that $\Gamma, \Delta \in \Psi$ for $\Delta \in \mathcal{S}(\Lambda(V))$.

We now establish that Ψ is closed under contraction, i.e. $\Gamma \in \Psi$ implies that $\mathrm{supp}(\Gamma) \in \Psi$. If $\Gamma \in \Psi$, we can find a rule $\Gamma_1 \ldots \Gamma_n/\Gamma_0 \in \mathbf{R}$, a substitution $\sigma : V \to \mathsf{Prop}(V)$ such that $X, \tau \models \Gamma_i\sigma$ for all $i = 1, \ldots, n$ and

$\Gamma = \Gamma_0\sigma, \Gamma'$ for some $\Gamma' \in \mathcal{S}(\Lambda(V))$. It suffices to show that $\mathrm{supp}(\Gamma_0\sigma) \in \Psi$ as Ψ is closed under weakening. By choosing prositional variables $p_\sigma(p) \in V$ for all $p \in V$ such that $p_A \neq p_B$ for $A \neq B$ and considering the renaming $\sigma_0(p) = p_{\sigma(p)}$ and the substutition σ_1 such that $\sigma_1(p_A) = A$, the fact that **R** absorbs contraction gives us a rule $\Delta_1 \dots \Delta_m/\Delta_0 \in \mathbf{R}$ and a substitution $\rho : V \to \mathsf{Prop}(V)$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$ and, for all $i = 1, \dots, m$ we have that

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid i = 1, \dots, n\} \vdash \Delta_j\rho$$

for all $j = 1, \dots, m$. As a consequence, $X, \tau \models \Delta_i\rho$ for all $i = 1, \dots, m$ whence $\Delta_0\rho \in \Psi$. This establishes that Ψ is closed under contraction, since $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$.

We now claim that Ψ is closed under cut, that is if Γ, A and $\Delta, \neg A \in \Psi$ then $\Gamma, \Delta \in \Psi$. So suppose that Γ, A and $\Delta, \neg A \in \Psi$. By definition, we have two rules $\Gamma_1 \dots, \Gamma_n/\Gamma_0$ and $\Delta_1 \dots \Delta_m/\Delta_0 \in \mathbf{R}$ and two substitutions $\sigma, \rho : V \to \mathsf{Prop}(V)$ such that $X, \tau \models \Gamma_i\sigma$ and $X, \tau \models \Delta_j\tau$ for all $i = 1, \dots n$ and all $j = 1, \dots, m$. Moreover,

$$\Gamma, A = \Gamma_0\sigma, \Gamma' \text{ and } \Delta, \neg A = \Delta_0\tau, \Delta'$$

for some $\Gamma', \Delta' \in \mathcal{S}(\Lambda(V))$. In case $A \in \Gamma'$ or $A \in \Delta'$ there is nothing to show. So suppose that $A \in \Gamma_0\sigma$ and $\neg A \in \Delta_0\tau$. Hence $\Gamma_0\sigma = A, \Gamma''$ and $\Delta_0\rho = \neg A, \Delta''$ for $\Gamma'', \Delta'' \in \mathcal{S}(\Lambda(V))$. As Ψ is closed under weakening, it suffices to show that $\Gamma'', \Delta'' \in \Psi$.

By choosing propositional variables $p_{\sigma(p)}$ and $p_{\rho(p)}$ as before, the fact that **R** absorbs cut provides us with a rule $\Sigma_1, \dots, \Sigma_k/\Sigma_0$ and a substitution $\kappa : V \to \mathsf{Prop}(V)$ such that $\mathrm{supp}(\Sigma_0\kappa) \subseteq \Gamma'', \Delta''$ and

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid i = 1, \dots, n\} + \{\Delta_i\rho \mid i = 1, \dots, m\} \vdash \Sigma_j\kappa$$

for all $j = 1, \dots, l$. By soundness of $\mathbf{GC}_0$, we have $X, \tau \models \Sigma_i\kappa$ for all $i = 1, \dots, n$ whence $\Sigma_0\kappa \in \Psi$. Since Ψ is closed under contraction, we have that $\mathrm{supp}(\Sigma_0\kappa) \in \Psi$, and in summary

$$\mathrm{supp}(\Sigma_0\kappa) \subseteq \Gamma'', \Delta''$$

so that $\Gamma'', \Delta'' \in \Psi$ as claimed.

Finally, we establish that **R** is one-step cut-free complete. So let $\Gamma \in \mathcal{S}(\Lambda(V))$ and let $\tau : V \to \mathcal{P}(X)$ such that $TX, \tau \models \Gamma$. We need to show that there exist $\Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}$ and a renaming $\sigma : V \to V$ such that $\Gamma_0\sigma \subseteq \Gamma$ and $X, \tau \models \Gamma_i, \sigma, i = 1, \dots, n$. As **R** is one-step complete, $\mathbf{GC}_1 + \Psi \vdash \Gamma$. As a consequence of Lemma 5.6 below, we have $\Gamma \in \Psi$ which establishes that **R** is one-step cut-free complete. $\square$

To complete the proof of Proposition 5.5 we need to supply the following lemma.

Lemma 5.6. *Let $\Psi \subseteq \mathcal{S}(\Lambda(V))$ be closed under cut, contraction, weakening, and inversion. Then $\mathbf{GC}_1 + \Psi \vdash \Gamma$ iff $\mathbf{G}_1 + \Psi \vdash \Gamma$. In particular, if $\Gamma \in \mathcal{S}(\mathcal{F}(\Lambda))$ we have $\mathbf{GC}_1 + \Psi \vdash \Gamma$ iff $\Gamma \in \Psi$.*

Proof. This is a standard cut-elimination proof for $\mathbf{G}$ where the fact that Ψ is closed under cut, contraction, weakening, and inversion allows propagating instances of the respective rules to the leaves; see [22, Section 4.4] for details. □

The converse of Proposition 5.5 requires more semantic considerations.

Proposition 5.7. *Let $\mathbf{R}$ be one-step sound and one-step cut-free complete. Then $\mathbf{R}$ absorbs cut and contraction.*

Proof. We first establish that $\mathbf{R}$ absorbs contraction. So suppose $\Gamma_1 \dots \Gamma_n/\Gamma_0 \in \mathbf{R}$ and $\sigma : V \to V$ is a renaming. We have to show that there exists a rule $\Delta_1 \dots \Delta_m/\Delta_0$ and a renaming $\rho : V \to V$ such that $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$ and

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid i = 1, \dots, n\} \vdash \Delta_j\rho$$

for all $j = 1, \dots, m$. Consider $X_0 = \mathcal{P}(V)$ and let $\tau_0 : V \to \mathcal{P}(X_0)$ be the canonical valuation $\tau_0(p) = \{A \in X_0 \mid p \in A\}$. If $X = \bigcap_{i=1,\dots,n} [\![\Gamma_i]\!]_{X_0}^{\tau_0}$ and $\tau(p) = \tau_0(p) \cap X$, inverse image along the inclusion $i : X \to X_0$ is a boolean algebra morphism that satisfies $i^{-1} \circ \tau_0 = \tau$ whence $X, \tau \models \Gamma_i\sigma$ for $i = 1, \dots, n$ and, by one-step soundness, $TX, \tau \models \Gamma_0\sigma$, and, a fortiori, $TX, \tau \models \mathrm{supp}(\Gamma_0\sigma)$. Since $\mathbf{R}$ is one-step cut-free complete, we can find a rule $\Delta_1 \dots \Delta_m/\Delta_0$ and a renaming $\rho : V \to V$ such that $X, \tau \models \Delta_i\rho$ for $i = 1, \dots, m$ and $\Delta_0\rho \subseteq \mathrm{supp}(\Gamma_0\sigma)$. As $\Delta_j\rho$ is a semantic consequence of $\{\Gamma_i\sigma \mid i = 1, \dots, n\}$ for all $j = 1, \dots, m$ by construction, we have that

$$\mathbf{GC}_0 + \{\Gamma_i\sigma \mid i = 1, \dots, n\} \vdash \Delta_j\tau$$

for all $j = 1, \dots, m$, or, in other words, $\mathbf{R}$ absorbs contraction.

We use a very similar argument to show that $\mathbf{R}$ absorbs cut. If $\Gamma_1 \dots \Gamma_n/\Gamma_0$ and $\Delta_1 \dots \Delta_m/\Delta_0 \in \mathbf{R}$ and $\sigma, \rho : V \to V$ are renamings with $\Gamma_0\sigma = \Gamma, A$ and $\Delta_0\rho = \Delta, \neg A$, we pick X_0 and τ_0 as above and let $X = \bigcap_{i=1,\dots,n} [\![\Gamma_i]\!]_{X_0}^{\tau_0} \cap \bigcap_{j=1,\dots,m} [\![\Delta_j\rho]\!]_{X_0}^{\tau_0}$ and fix $\tau : V \to \mathcal{P}(X)$, given by $\tau(p) = \tau_0(p) \cap X$. Using one-step soundness, we note that $TX, \tau \models \Gamma, \Delta$, and – using the same argument as above – by one-step cut-free completeness we find a rule $\Sigma_1 \dots \Sigma_l / \Sigma_0$ and a renaming $\kappa : V \to V$ such that $X, \tau \models \Sigma_i \kappa$ for $i = 1, \dots, l$ and $\Sigma_0 \kappa \subseteq \Gamma, \Delta$. Since $\Sigma_j \kappa$ is a semantic consequence of the $\Gamma_i \tau$ and the $\Delta_i \rho$, we have that

$$\mathbf{GC}_0 + \{\Gamma_i \sigma \mid i = 1, \dots, n\} + \{\Delta_i \rho \mid i = 1, \dots, m\} \vdash \Sigma_j \kappa$$

for all $j = 1, \dots, l$ which shows that $\mathbf{R}$ absorbs cut. □

We conclude the section with a short methodological digression on the construction of cut-free complete rule sets.

Remark 5.8. We have seen in Theorems 4.9 and 5.4 that cut-free completeness, or equivalently the absoprtion of cut and contraction, give rise to a cut-free sequent system for a large range of coalgebraic logics. The syntactic approach to cut elimination provides us with a methodology to construct cut-free rule sets. To turn a one-step complete system of rules into a one-step cut-free complete system, we add instances of cut and contraction to the rule set in question until both cut and contraction are absorbed. It is evident that this preserves one-step soundness.

6 Applications

This section presents, from a syntactic viewpoint, some applications of cut-free completeness of $\mathbf{GR}$ for a one-step cut-free complete set $\mathbf{R}$ of one-step rules. The first application, the subformula property, is immediate:

Theorem 6.1. *Let $\mathbf{R}$ be a set of one-step rules. Then $\mathbf{GR}$ has the subformula property, i.e. every deduction $\mathbf{GR} \vdash \Gamma$ only mentions subformulas, or negations thereof, of formulas occurring in Γ.*

Proof. By induction on the derivation of $\mathbf{GR} \vdash \Gamma$, where both the case of propositional connectives and the application of an instance of a one-step rule are immediate by the rule format. □

As a consequence, we obtain alternative proofs of two results of [20] regarding conservativity and complexity of coalgebraic logics.

Corollary 6.2 (Conservativity)**.** *Let $\Lambda_0 \subseteq \Lambda$ be a sub-similarity type, and let $\mathbf{R}$ be one-step sound and one-step cut-free complete for a Λ-structure T. If $\mathbf{R}_0$ consists of those $(\Gamma_1, \dots, \Gamma_n, \Gamma_0) \in \mathbf{R}$ for which $\Gamma_0 \in \mathcal{S}(\Lambda_0(V))$ then $\mathbf{GR}_0$ is complete for T.*

Proof. Let Γ be a valid sequent over $\mathcal{F}(\Lambda_0)$. Then $\mathbf{GR} \vdash \Gamma$. By the subformula property, all rules used in this derivation belong to $\mathbf{R}_0$. $\square$

As the design of the system **GR** is such that the logical complexity of the formula strictly decreases when passing from conclusion to premise, these systems can be used to establish both decidability and complexity of the satisfiability problem. Simply put, proof search in **GR** terminates if for every sequent Γ there are only finitely many substitution instances of rule conclusions equal to Γ with properly different premises. Polynomial bounds on the size of such rules imply decidability in polynomial space using depth-first search. This allows us to re-prove the main theorem of [20] (to which we refer for the definition of *PSPACE*-tractable) in the setting of sequent calculi:

Theorem 6.3. *Let* $\mathbf{R}$ *be one-step sound and one-step cut-free complete. If moreover* $\mathbf{R}$ *is PSPACE-tractable, then the satisfiability problem for* $\mathcal{F}(\Lambda)$ *w.r.t.* $\mathsf{Mod}(T)$ *is decidable in polynomial space.*

Proof. As $\mathbf{R}$ is *PSPACE*-tractable, there are only finitely many (rule, substitution)-pairs of polynomial size that allow deriving any given sequent, and these pairs can be represented in polynomial space. Moreover, the depth of the search tree is linear in the size of the input formula, as every backwards rule application removes either a propositional connective or a layer of modal operators. $\square$

Cut-free proof calculi also provide all the necessary scaffolding to prove Craig interpolation by induction on cut-free proofs. To aid the formulation of the interpolation property, we write $\mathsf{FV}(A)$ for the set of propositional variables occurring in $A \in \mathcal{F}(\Lambda)$, and extend this to sequents by $\mathsf{FV}(\Gamma) = \bigcup\{\mathsf{FV}(A) \mid A \in \Gamma\}$. Interpolation then takes the following form:

Definition 6.4. $\mathcal{F}(\Lambda)$ has the *Craig Interpolation Property* (CIP) with respect to $\mathsf{Mod}(T)$ if whenever $\mathsf{Mod}(T) \models A \to B$ for $A, B \in \mathcal{F}(\Lambda)$, then there exists an *interpolant* $F \in \mathcal{F}(\Lambda)$ such that $\mathsf{Mod}(T) \models A \to F$, $\mathsf{Mod}(T) \models F \to B$ and $\mathsf{FV}(F) \subseteq \mathsf{FV}(A) \cap \mathsf{FV}(B)$.

Syntactic proofs of the CIP proceed by induction on cut-free proofs. The following definition introduces the necessary terminology.

Definition 6.5. A *split sequent* is a pair (Γ_0, Γ_1) of sequents, written $\Gamma_0 \mid \Gamma_1$. We say that $\Gamma_0 \mid \Gamma_1$ is a *splitting* of Γ if $\Gamma = \Gamma_0, \Gamma_1$. A formula F is an *interpolant* of a split sequent $\Gamma_0 \mid \Gamma_1$ if $\mathsf{FV}(F) \subseteq \mathsf{FV}(\Gamma_0) \cap \mathsf{FV}(\Gamma_1)$, $\mathbf{GR} \vdash$

Γ_0, F, and $\mathbf{GR} \vdash \neg F, \Gamma_1$. We say that a sequent Γ *admits interpolation* if every splitting of Γ has an interpolant. The system $\mathbf{GR}$ has the *Craig interpolation property* (CIP) if every derivable sequent admits interpolation.

The idea of the syntactic proof of Craig interpolation [22, Chapter 4], in contrast to the semantic proofs via amalgamation (see [11] for the case of normal modal logics and [8] for monotone modal logic) is to construct interpolants inductively – clearly this fails in the presence of the cut-rule. Completeness gives the link between both the syntactic and the semantic versions of the CIP.

Proposition 6.6. *Let $\mathbf{R}$ be one-step sound and one-step cut-free complete w.r.t the Λ-structure T. Then $\mathbf{GR}$ has the CIP iff $\mathcal{F}(\Lambda)$ has the CIP with respect to $\mathsf{Mod}(T)$.*

Proof. Straightforward using soundness and cut-free completeness (Section 4). □

Inductive proofs of the CIP for $\mathbf{GR}$ are often straightforward. Below, we show that the systems used in our running examples, coalition logic and conditional logic have the CIP. For coalition logic, this is not a new result [7] but our proof is shorter due to the smaller number of modal proof rules. For the conditional logics CK and $CK + ID$ the CIP is – to the best of our knowledge – a new result which was explicitly left as future work in [13], where a substantially different proof calculus is used.

The proof of the CIP in both examples benefits from the following notions.

Definition 6.7. A sequent rule $\Gamma_1 \ldots \Gamma_n / \Gamma_0$ *supports interpolation* if Γ_0 admits interpolation provided all of $\Gamma_1, \ldots, \Gamma_n$ admit interpolation. A set $\mathbf{S}$ of sequent rules *supports interpolation* if all rules in $\mathbf{S}$ support interpolation.

As it is well known (and shown e.g. in [22]) that all (instances of) rules of $\mathbf{G}$ support interpolation, the following is evident.

Lemma 6.8. *If $\mathcal{S}(\mathbf{R})$ supports interpolation, then $\mathbf{GR}$ has the CIP.*

Moreover, we may restrict ourselves to rule instances without context formulas:

Lemma 6.9. *The set $\mathcal{S}(\mathbf{R})$ supports interpolation iff for every rule $\Gamma_1 \ldots \Gamma_n / \Gamma_0$ in $\mathbf{R}$ and every substitution $\sigma : V \to \mathcal{F}(\Lambda)$, the sequent rule $\Gamma_1\sigma \ldots \Gamma_n\sigma / \Gamma_0\sigma$ supports interpolation.*

Proof. Let $\Gamma_1 \dots \Gamma_n/\Gamma_0$ be a one-step rule in $\mathbf{R}$, let $\sigma : V \to \mathcal{F}(\Lambda)$ be a substitution, and let Δ be a sequent. Moreover, let $\Gamma_i\sigma$ admit interpolation for all $i = 1, \dots, n$; we have to show that the arising rule conclusion $\Gamma_0\sigma, \Delta$ admits interpolation. Every splitting of $\Gamma_0\sigma, \Delta$ is of the form $\Gamma_0^0\sigma, \Delta_0 \mid \Gamma_0^1\sigma, \Delta_1$, where $\Gamma_0^0\sigma \mid \Gamma_0^1\sigma$ is a splitting of $\Gamma_0\sigma$ and $\Delta_0 \mid \Delta_1$ is a splitting of Δ. By assumption, $\Gamma_0\sigma$ admits interpolation, so that there exists an interpolant F for the splitting $\Gamma_0^0\sigma \mid \Gamma_0^1\sigma$. By admissibility of weakening, F is also an interpolant for the given splitting of $\Gamma_0\sigma, \Delta$. □

We turn to our running examples:

Theorem 6.10. *Coalition logic, i.e. the system* **GC**, *has the CIP.*

Proof. By the above lemmas, we only have to check that the given one-step rules support interpolation.

Rule (A). If $S = \neg[\mathbf{C}_0]\mathbf{A}_0 \mid \neg[\mathbf{C}_1]\mathbf{A}_1$ is a splitting of the (substituted) rule conclusion (recall the notation of Example 2.1) and F is an interpolant of $\neg\mathbf{A}_0 \mid \neg\mathbf{A}_1$, then $G = [\cup\mathbf{C}_0]F$ is an interpolant of S: From $\neg F, \neg\mathbf{A}_1$, we deduce $\neg G, \neg[\mathbf{C}_1]\mathbf{A}_1$ by rule (A), and from $\neg\mathbf{A}_0, F$, we deduce $\neg[\mathbf{C}_0]\mathbf{A}_0, G$ by rule (B).

Rule (B). There are two cases to distinguish, depending on which part of the splitting the literal $[D]B$ belongs to. First consider splittings of the rule conclusion of the form

$$S = \neg[\mathbf{C}_0]\mathbf{A}_0, [D]B, [\mathbf{N}]\mathbf{B}_0 \mid \neg[\mathbf{C}_1]\mathbf{A}_1, \mid [\mathbf{N}]\mathbf{B}_1.$$

If F is an interpolant of $\neg\mathbf{A}_0, B, \mathbf{B}_0 \mid \neg\mathbf{A}_1, \mathbf{B}_1$, then $\neg[\cup\mathbf{C}_1]\neg F$ is an interpolant of S.

Now consider a splitting of the rule conclusion of the form

$$S = \neg[\mathbf{C}_0]\mathbf{A}_0, [\mathbf{N}]\mathbf{B}_0 \mid \neg[\mathbf{C}_1]\mathbf{A}_1, [D]B, [\mathbf{N}]\mathbf{B}_1.$$

In this case, if F is an interpolant of $\neg\mathbf{A}_0, \mathbf{B}_0 \mid \neg\mathbf{A}_1, B, \mathbf{B}_1$, then $[\cup\mathbf{C}_0]F$ is an interpolant of S. □

By a similar argument we establish the CIP for the conditional logics CK and $CK + ID$.

Theorem 6.11. *The conditional logics CK and $CK + ID$ have the CIP.*

Proof. First consider **GCK**; we have to show that rule (C) supports interpolation. First consider splittings of the rule conclusion of the form $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0), A \Rightarrow B \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1)$. If F is an interpolant of

$\neg\mathbf{B}_0, B \mid \neg\mathbf{B}_1$, then $\neg(A \Rightarrow \neg F)$ is an interpolant of S. Now consider splittings of the form $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0) \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1), A \Rightarrow B$. If F interpolates $\neg\mathbf{B}_0 \mid \neg\mathbf{B}_1, B$ then $A \Rightarrow F$ interpolates S.

We now consider interpolation for **GCKId**, which follows the same pattern. To show that the rule (CI) supports interpolation, first consider a splitting of the conclusion of (CI) of the form $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0), A \Rightarrow B \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1)$. If F is an interpolant of $\neg A_0, \neg\mathbf{B}_0, B \mid \neg\mathbf{B}_1$, then $\neg(A \Rightarrow \neg F)$ is an interpolant of S. Similarly, if $S = \neg(\mathbf{A}_0 \Rightarrow \mathbf{B}_0) \mid \neg(\mathbf{A}_1 \Rightarrow \mathbf{B}_1), A \Rightarrow B$ and F interpolates $\neg\mathbf{B}_0 \mid \neg\mathbf{B}_1, B, \neg A$ then $A \Rightarrow F$ interpolates S. $\square$

7 Conclusions

We have argued that strict one-step completeness of a system of one-step rules automatically results in a sequent system that is cut free and complete. Cut free sequent systems are the key to a number of typical applications, including in particular proofs of the Craig interpolation property (CIP) which plays an important role in the modularisation of proofs. We have thus established the CIP for our two running examples; here, the CIP for the conditional logics CK and $CK + ID$ is apparently a new result. It remains an open problem to find a quickly verifiable general criterion for a set of rules, or, semantically, a coalgebraic modal logic, to have the CIP. It is worthwhile to point out that for coalition logic, the inductive step in the proof of the CIP is not entirely straightforward as the newly constructed interpolant uses a modality that does not necessarily appear in the rule at hand. We phrase this problem explicitly as

Open Problem 7.1. *Find easily verifiable and general semantic or syntactic criteria for a coalgebraic modal logic to have the CIP.*

Our second observation pertains to our proof of cut-free completeness, which is heavily based on semantic notions. While we strongly believe that this theorem could also have been obtained purely syntactically, i.e. by comparison of different proof systems, we are as of yet unsure whether these methods extend beyond rank 1. In particular, can cut always be absorbed into the modal proof rules? We formulate this as

Open Problem 7.2. *To what extent can resolution closure be used to absorb the cut rule into a system of modal proof rules?*

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