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CUT-FREE SINGLE-SUCCEDENT SYSTEMS REVISITED

Abstract

It is shown that LJ with a generalized Dummett rule, LJ with a generalized Peirce rule and LJ with a specialized Peirce rule have the cut-elimination property. It is also shown that the third system has a weak subformula property and Craig's interpolation property. The systems presented are versions of single-succedent sequent systems for classical logic. The cut-elimination results for the single-succedent systems can also be extended to modal logics.

1. Introduction

It is known that some sequent calculi for classical logic are obtained from LJ (a sequent calculus for intuitionistic logic) by adding the following inference rules:

$$\frac{\alpha{\rightarrow}\beta, \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha}\ (\text{Peirce}), \qquad \frac{\neg\alpha, \Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha}\ (\text{r-Peirce}), \qquad \frac{\neg\alpha, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow \alpha}\ (\text{Raa}),$$

$$\frac{\neg\alpha, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{Gem}), \qquad \frac{\neg p, \Gamma \Rightarrow \gamma \quad p, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{Gem-at})$$

where p is an atomic formula.

The rule (Peirce) was introduced and studied by Curry [2], Felscher [3], Gordeev [4] and Africk [1]. It was shown that the cut-elimination theorem holds for LJ + (Peirce). The subformula property for a version of LJ + (Peirce) without the falsity constant $\bot$ was shown by Gordeev, i.e. it was shown that β in (Peirce) can be restricted to being a subformula of some formulas in (Γ, α). The rule (r-Peirce) was introduced and studied by Curry [2], Gordeev [4] and Africk [1]. It was shown that the cut-elimination theorem holds for LJ + (r-Peirce). Thus, LJ + (r-Peirce) has a weak subformula property allowing negation formulas. A very simple proof of the cut-elimination theorem for LJ + (Peirce) and LJ + (r-Peirce) was given by Africk.

The rule (Raa) was studied by Negri and von Plato in [5]. As mentioned in [5], it is difficult to give a direct proof of the cut-elimination theorem for some systems with (Raa). It was shown in [5] that the structural rules (including the cut rule) are admissible in a sequent calculus G3ip (for intuitionistic logic) with the rule of the form:

$$\frac{\neg p, \Gamma \Rightarrow \bot}{\Gamma \Rightarrow p} \text{ (Raa-at).}$$

where p is an atomic formula. [1] It was also shown in [5] that G3ip + (Raa-at) is not a system of classical logic, but a system of an intermediate logic called *stable logic*.

The rules (Gem) and (Gem-at) were introduced by von Plato. It was shown in [5] that the cut rule and (Gem) are admissible in some versions of cut-free LJ with (Gem-at). By using this result, it was proved that a weak subformula property allowing atomic formulas and negations of atomic formulas holds for such systems.

In this paper, we consider three alternative versions of the rules discussed above. The first and second ones are generalized versions (g-Dmt) and (g-Peirce) of the presented rules, called here a *generalized Dummett rule* and a *generalized Peirce rule*, respectively. The third one is a specialized version (s-Peirce) of (r-Peirce) and (Raa), called here a *specialized Peirce rule*. The rule (s-Peirce) was studied by Gordeev [4] based on a different cut-free formulation of LJ with a specialized negation-cut rule.

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[1] In [5], various sequent calculi were introduced and studied for propositional intuitionistic logic. For example, G3ip and G3ipm are a structural-rule-free single-succedent system, in which weakening and contraction are bulit into the logical rules, and a cut-free multiple-succedent system, respectively.

2. Single-succedent systems

Prior to the precise discussion, the usual propositional language with the constants is introduced below. [2] *Formulas* are constructed from propositional variables, $\top$ (truth constant), $\bot$ (falsity constant), $\to$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction) and $\neg$ (negation). Lower case Greek letters $\alpha, \beta, ...$ are used to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ are used to represent finite (possibly empty) sequences of formulas. An *intuitionistic sequent* is an expression of the form $\Gamma \Rightarrow \gamma$ where γ is a formula or an empty sequence. A *classical sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. The sequent calculi LJ and LK are assumed in the following discussion. The precise definitions of LJ and LK are in Appendix.

DEFINITION 1. (LJ_{gd}, LJ_{gp}, LJ_{sp}) LJ_{gd}, LJ_{gp} and LJ_{sp} are obtained from LJ by adding the inference rule respectively of the form:

$$\frac{\alpha{\to}\beta, \Gamma \Rightarrow \gamma \quad \delta{\to}\alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{g-Dmt}) \qquad \frac{\alpha{\to}\beta, \Gamma \Rightarrow \gamma \quad \alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{g-Peirce})$$

$$\frac{\neg\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \alpha}\ (\text{s-Peirce})$$

where γ is a formula or an empty sequence.

It is remarked that β in (g-Dmt) and (g-Peirce) can be restricted to $\bot$, and δ in (g-Dmt) can be restricted to $\top$, i.e. the provability is not changed by these restrictions, and the cut-elimination theorem holds for the systems with such restricted rules.

The rule (g-Dmt) is also a generalized version of the rule of the form:

$$\frac{\alpha{\to}\beta, \Gamma \Rightarrow \gamma \quad \beta{\to}\alpha, \Gamma \Rightarrow \gamma}{\Gamma \Rightarrow \gamma}\ (\text{Dmt}),$$

which corresponds to the law of linearity: $(\alpha{\to}\beta) \vee (\beta{\to}\alpha)$ characterizing the intermediate logic called *Dummett's LC*. The rule (Dmt) and a multiple-succedent sequent calculus G3ipm with (Dmt) for Dummett's logic were discussed in [5].

[2] The constants $\bot$ and $\top$ are used to simplify some lemmas and theorems.

The rule (s-Peirce) of Gordeev is not very different from (Raa) of Negri and von Plato, but (s-Peirce) has a little advantage, i.e. it can derive a weak subformula property without $\bot$.

In the following, it is shown that LJ_{gd}, LJ_{gp} and LJ_{sp} are single-succedent sequent calculi for classical logic.

Proposition 2. *The rules* (g-Dmt)*,* (g-Peirce) *and* (s-Peirce) *are derivable in* LK*.*

Proof.

$$\dfrac{\begin{array}{c}\vdots\\ \Rightarrow (\alpha\to\beta)\vee(\delta\to\alpha)\end{array} \quad \dfrac{\alpha\to\beta,\Gamma\Rightarrow\gamma \quad \delta\to\alpha,\Gamma\Rightarrow\gamma}{(\alpha\to\beta)\vee(\delta\to\alpha),\Gamma\Rightarrow\gamma}}{\Gamma\Rightarrow\gamma}$$

where $\Rightarrow (\alpha\to\beta)\vee(\delta\to\alpha)$ is provable in LK by using the right-contraction rule.

$$\begin{array}{c}\dfrac{\dfrac{\dfrac{\alpha,\Gamma\Rightarrow\gamma}{\alpha,\Gamma\Rightarrow\gamma,\beta}}{\Gamma\Rightarrow\gamma,\alpha\to\beta}\quad \alpha\to\beta,\Gamma\Rightarrow\gamma}{\Gamma,\Gamma\Rightarrow\gamma,\gamma}\\ \vdots\\ \Gamma\Rightarrow\gamma\end{array}\qquad \dfrac{\dfrac{\alpha\Rightarrow\alpha}{\Rightarrow\alpha,\neg\alpha}\quad \neg\alpha,\Gamma\Rightarrow}{\Gamma\Rightarrow\alpha}\ .$$

Q.E.D.

Theorem 3. (Equivalence to Classical Logic) LJ_{gd}*,* LJ_{gp} *and* LJ_{sp} *are sequent calculi for classical logic.*

Proof. By Proposition 2, it is obvious that LJ_{gd}, LJ_{gp} and LJ_{sp} are subsystems of LK. Thus, it is enough to show that $\Rightarrow ((\alpha\to\beta)\to\alpha)\to\alpha$ (Peirce's law) or $\Rightarrow \alpha\vee\neg\alpha$ (the law of excluded middle) is provable in the systems. The following are examples of such proofs in LJ_{gd} and LJ_{sp}, respectively.

$$\dfrac{\dfrac{\dfrac{\alpha{\to}\beta \Rightarrow \alpha{\to}\beta \quad \alpha \Rightarrow \alpha}{(\alpha{\to}\beta){\to}\alpha, \alpha{\to}\beta \Rightarrow \alpha}}{\alpha{\to}\beta, (\alpha{\to}\beta){\to}\alpha \Rightarrow \alpha} \qquad \dfrac{\Rightarrow \top \quad \dfrac{\alpha \Rightarrow \alpha}{\alpha, (\alpha{\to}\beta){\to}\alpha \Rightarrow \alpha}}{\top{\to}\alpha, (\alpha{\to}\beta){\to}\alpha \Rightarrow \alpha}}{\dfrac{(\alpha{\to}\beta){\to}\alpha \Rightarrow \alpha}{\Rightarrow ((\alpha{\to}\beta){\to}\alpha){\to}\alpha}} \;\text{(g-Dmt)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\alpha \Rightarrow \neg\alpha}{\neg\alpha \Rightarrow \alpha \vee \neg\alpha}}{\neg(\alpha \vee \neg\alpha), \neg\alpha \Rightarrow}}{\neg\alpha, \neg(\alpha \vee \neg\alpha) \Rightarrow}}{\neg(\alpha \vee \neg\alpha) \Rightarrow \alpha} \;\text{(s-Peirce)}}{\neg(\alpha \vee \neg\alpha) \Rightarrow \alpha \vee \neg\alpha}}{\neg(\alpha \vee \neg\alpha), \neg(\alpha \vee \neg\alpha) \Rightarrow}}{\dfrac{\neg(\alpha \vee \neg\alpha) \Rightarrow}{\Rightarrow \alpha \vee \neg\alpha} \;\text{(s-Peirce)}.}$$

Q.E.D.

3. Cut-elimination

In the following, all the cut-elimination proofs use Africk's method introduced in [1]. This method may also be applied to some systems with (Raa). First, in order to prove the cut-elimination theorems for LJ_{gd} and LJ_{gp}, we show the following lemma, which is a slight modification of the lemma introduced by Africk. An expression $\Delta{\to}\bot$ means the sequence $\langle \delta{\to}\bot \mid \delta \in \Delta \rangle$ if Δ is non-empty, and means the empty sequence if Δ is empty. An expression $\neg\Delta$ means the sequence $\langle \neg\delta \mid \delta \in \Delta \rangle$.

Lemma 4. (Key Lemma for LJ_{gd} and LJ_{gp}) *Let L be LJ_{gd} or LJ_{gp}. If a sequent $\Gamma \Rightarrow \Delta$ is provable in cut-free* LK, *then the sequent $\Delta{\to}\bot, \Gamma \Rightarrow$ is provable in cut-free L.*

Proof. By induction on the proof P of $\Gamma \Rightarrow \Delta$ in cut-free LK. We only show the case of LJ_{gd}, since the case of LJ_{gp} can be shown similarly. We distinguish the cases according to the last inference of P. We only show the following case. The other cases can be shown similarly or straightforwardly.

The last inference rule of P is of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}.$$

By the hypothesis of induction, the sequent $\Delta\to\bot, \alpha\to\bot, \Gamma \Rightarrow$ is provable in cut-free LJ_{gd}. Then, we obtain the required fact:

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}\Delta\to\bot, \alpha\to\bot, \Gamma \Rightarrow \\ \vdots \\ \alpha\to\bot, \Delta\to\bot, \Gamma \Rightarrow \alpha\end{array} \quad \dfrac{\Rightarrow \top \quad \begin{array}{c}\alpha \Rightarrow \alpha \\ \vdots \\ \alpha, \Delta\to\bot, \Gamma \Rightarrow \alpha\end{array}}{\top\to\alpha, \Delta\to\bot, \Gamma \Rightarrow \alpha}}{\Delta\to\bot, \Gamma \Rightarrow \alpha}\ (\text{g-Dmt})}{\Delta\to\bot, \Gamma \Rightarrow \alpha \vee \beta} \quad \bot \Rightarrow}{\begin{array}{c}(\alpha \vee \beta)\to\bot, \Delta\to\bot, \Gamma \Rightarrow \\ \vdots \\ \Delta\to\bot, (\alpha \vee \beta)\to\bot, \Gamma \Rightarrow\end{array}}
$$

Q.E.D.

THEOREM 5. (Cut-Elimination for LJ_{gd} and LJ_{gp}) *Let L be LJ_{gd} or LJ_{gp}. The rule*

$$
\dfrac{\Gamma \Rightarrow \alpha \quad \alpha, \Delta \Rightarrow \gamma}{\Gamma, \Delta \Rightarrow \gamma}\ (\text{cut})
$$

is admissible in cut-free L.

PROOF. We only show the case of LJ_{gd}, since the case of LJ_{gp} can be shown similarly. Suppose that a sequent $\Gamma \Rightarrow \gamma$ is provable in LJ_{gd}. By Proposition 2, it is provable in LK. It is also provable in cut-free LK by the well-known cut-elimination theorem for LK. By Lemma 4, the sequent $\gamma\to\bot, \Gamma \Rightarrow$ is provable in cut-free LJ_{gd}. Therefore $\Gamma \Rightarrow \gamma$ is provable in cut-free LJ_{gd}:

$$
\dfrac{\dfrac{\gamma\to\bot, \Gamma \Rightarrow}{\gamma\to\bot, \Gamma \Rightarrow \gamma} \quad \dfrac{\Rightarrow \top \quad \begin{array}{c}\gamma \Rightarrow \gamma \\ \vdots \\ \gamma, \Gamma \Rightarrow \gamma\end{array}}{\top\to\gamma, \Gamma \Rightarrow \gamma}}{\Gamma \Rightarrow \gamma}\ (\text{g-Dmt}).
$$

Q.E.D.

LEMMA 6. (Key Lemma for LJ_{sp}) *If a sequent* $\Gamma \Rightarrow \Delta$ *is provable in cut-free* LK*, then the sequent* $\neg\Delta, \Gamma \Rightarrow$ *is provable in cut-free* LJ_{sp}*.*

PROOF. By induction on the proof P of $\Gamma \Rightarrow \Delta$ in cut-free LK. We distinguish the cases according to the last inference of P. We only show the following case. The other cases can be proved similarly or straightforwardly.

The last inference rule of P is of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\ .$$

By the hypothesis of induction, $\neg\Delta, \neg\beta, \alpha, \Gamma \Rightarrow$ is provable in cut-free LJ_{sp}. Then, we obtain the required fact:

$$\begin{array}{c}
\neg\Delta, \neg\beta, \alpha, \Gamma \Rightarrow \\
\vdots \\
\dfrac{\neg\beta, \neg\Delta, \alpha, \Gamma \Rightarrow}{\neg\Delta, \alpha, \Gamma \Rightarrow \beta}\ \text{(s-Peirce)} \\
\vdots \\
\dfrac{\dfrac{\alpha, \neg\Delta, \Gamma \Rightarrow \beta}{\neg\Delta, \Gamma \Rightarrow \alpha \to \beta}}{\neg(\alpha \to \beta), \neg\Delta, \Gamma \Rightarrow} \\
\vdots \\
\neg\Delta, \neg(\alpha \to \beta), \Gamma \Rightarrow
\end{array}$$

Q.E.D.

Using Lemma 6, we obtain the following theorem.

THEOREM 7. (Cut-Elimination for LJ_{sp}) *The rule* (cut) *is admissible in cut-free* LJ_{sp}*.*

COROLLARY 8. (Weak Subformula Property for LJ_{sp}) *If a sequent S is provable in cut-free* LJ_{sp}*, then there is a proof P of S in cut-free* LJ_{sp} *such that all formulas that appear in P are subformulas or negated-subformulas of some formula in S.*

4. Interpolation

We will prove Craig's interpolation theorem for LJ_{sp} by using Maehara's method based on the cut-elimination theorem. [3] In the following, $V(\alpha)$ denotes the set of all propositional variables in a formula α. When Γ is a sequence of formulas $\alpha_1, ..., \alpha_m$, we define $V(\Gamma) = V(\alpha_1) \cup \cdots \cup V(\alpha_m)$. For any given finite sequence Γ of formulas, we call a pair $\langle \Gamma_1; \Gamma_2 \rangle$ of (possibly empty) sequences Γ_1 and Γ_2 of formulas, a *partition* of Γ, if the multiset union of Γ_1 and Γ_2 is equal to Γ when regarding Γ, Γ_1 and Γ_2 as multisets of formulas.

LEMMA 9. *Let Δ be a formula or an empty sequence. Suppose that a sequent $\Gamma \Rightarrow \Delta$ is provable in LJ_{sp} and that $\langle \Gamma_1; \Gamma_2 \rangle$ is any partition of Γ. Then, there is a formula γ such that both $\Gamma_1 \Rightarrow \gamma$ and $\gamma, \Gamma_2 \Rightarrow \Delta$ are provable in LJ_{sp}, and moreover that $V(\gamma) \subseteq V(\Gamma_1) \cap V(\Gamma_2, \Delta)$.*

PROOF. By Theorem 7, we can take a cut-free proof P of $\Gamma \Rightarrow \Delta$. We prove this lemma by induction on P. We distinguish the cases according to the last inference of P. Since the cases that the last rule of P is an LJ-rule are the same as that for LJ, [4] it is sufficient to prove the following case.

The last inference rule of P is of the form:

$$\frac{\neg\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \text{ (s-Peirce).}$$

Let $\langle \Gamma_1; \Gamma_2 \rangle$ be any partition of Γ. We take a partition $\langle \Gamma_1; \neg\alpha, \Gamma_2 \rangle$ of $\neg\alpha, \Gamma$. Then, by induction hypothesis, we have a formula γ such that (1) both $\Gamma_1 \Rightarrow \gamma$ and $\gamma, \neg\alpha, \Gamma_2 \Rightarrow$ are provable, and (2) $V(\gamma) \subseteq V(\Gamma_1) \cap V(\neg\alpha, \Gamma_2)$. Since we obtain the provable sequent $\gamma, \Gamma_2 \Rightarrow \alpha$ by (1) and (s-Peirce), we obtain one of the required conditions: (1′) both $\Gamma_1 \Rightarrow \gamma$ and $\gamma, \Gamma_2 \Rightarrow \alpha$ are provable. Since $V(\neg\alpha, \Gamma_2) = V(\Gamma_2, \alpha)$, we also obtain the other condition: (2′) $V(\gamma) \subseteq V(\Gamma_1) \cap V(\Gamma_2, \alpha)$. Q.E.D.

In Lemma 9, taking $\alpha \Rightarrow \beta$ for $\Gamma \Rightarrow \Delta$ and taking the partition $\langle \alpha; \emptyset \rangle$ of α, we can obtain the following theorem.

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[3] Craig's interpolation theorem is really for a logic rather than a sequent calculus, and the proof using the sequent calculus is considered to be not so simple. It is thus only appealing that this theorem is an application of the cut-elimination theorem for LJ_{sp}.

[4] For a proof, see, e.g. [6].

THEOREM 10.(Craig's Interpolation Theorem for LJ_{sp}) *If a sequent $\alpha \Rightarrow \beta$ is provable in LJ_{sp}, then there is a formula γ such that both $\alpha \Rightarrow \gamma$ and $\gamma \Rightarrow \beta$ are provable in LJ_{sp}, and moreover that $V(\gamma) \subseteq V(\alpha) \cap V(\beta)$.*

5. Final remarks

First, we remark that the cut-elimination and interpolation results presented can be extended to the first-order predicate versions in a natural way. We also remark that the results which are related to (g-Peirce) and (s-Peirce) can be adapted for the system G3ip in [5]. [5]

Second, we remark that Glivenko's theorem with respect to LJ_{sp} and LJ, i.e. "a sequent $\Rightarrow \alpha$ is provable in LJ_{sp} iff $\Rightarrow \neg\neg\alpha$ is provable in LJ", can easily be proved by showing the following statement which is similar to Lemma 6: If a sequent $\Gamma \Rightarrow \Delta$ is provable in LJ_{sp}, then $\neg\Delta, \Gamma \Rightarrow$ is provable in LJ.

Finally, we mention about some modal extensions of our frameworks. In the following, we assume the language with the modal operator $\Box$. An expression $\Box\Gamma$ means the sequence $\langle \Box\gamma \mid \gamma \in \Gamma \rangle$. We can extend the frameworks based on LJ_{gd}, LJ_{gp} and LJ_{sp} to some standared modal logics such as K, KT and S4. As an example, we only address the case for S4 based on LJ_{sp} as follows. A single-succedent sequent calculus JS4 for S4 is obtained from LJ_{sp} by adding the inference rules of the form:

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\Box\alpha, \Gamma \Rightarrow \gamma} \qquad \frac{\Box\Gamma \Rightarrow \alpha}{\Box\Gamma \Rightarrow \Box\alpha}$$

where γ is a formula or an empty sequence. Then, it can be shown that the cut-elimination theorem and Craig's interpolation theorem (as formulated in Theorem 10) hold for JS4.

6. Appendix

DEFINITION 11. (LK) The initial sequents of LK are of the form:

$$\alpha \Rightarrow \alpha \qquad \bot \Rightarrow \qquad \Rightarrow \top.$$

[5] For the case of (g-Dmt), the constant $\top$ must be added to the framework of G3ip.

The cut rule of LK is of the form:

$$\frac{\Delta \Rightarrow \Pi, \alpha \quad \alpha, \Sigma \Rightarrow \Gamma}{\Delta, \Sigma \Rightarrow \Pi, \Gamma}.$$

The inference rules of LK are of the form:

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha} \quad \frac{\alpha, \alpha, \Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \alpha}{\Gamma \Rightarrow \Delta, \alpha}$$

$$\frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \Sigma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \Sigma} \quad \frac{\Gamma \Rightarrow \Sigma, \alpha, \beta, \Delta}{\Gamma \Rightarrow \Sigma, \beta, \alpha, \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi} \quad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\neg\alpha, \Gamma \Rightarrow \Delta} \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\alpha}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \quad \frac{\beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}.$$

Definition 12. (LJ) Let γ be a formula or an empty sequence.
The initial sequents of LJ are the same as that of LK.
The cut rule of LJ is of the form:

$$\frac{\Delta \Rightarrow \alpha \quad \alpha, \Sigma \Rightarrow \gamma}{\Delta, \Sigma \Rightarrow \gamma}.$$

The inference rules of LJ are of the form:

$$\frac{\Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \quad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \quad \frac{\alpha, \alpha, \Gamma \Rightarrow \gamma}{\alpha, \Gamma \Rightarrow \gamma} \quad \frac{\Gamma, \alpha, \beta, \Delta \Rightarrow \gamma}{\Gamma, \beta, \alpha, \Delta \Rightarrow \gamma}$$

$$\frac{\Gamma \Rightarrow \alpha \quad \beta, \Sigma \Rightarrow \gamma}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \gamma} \quad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta} \quad \frac{\neg\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \alpha} \quad \frac{\alpha, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg\alpha}$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \quad \frac{\beta, \Gamma \Rightarrow \gamma}{\alpha \wedge \beta, \Gamma \Rightarrow \gamma} \quad \frac{\Gamma \Rightarrow \alpha \quad \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \wedge \beta}$$

$$\frac{\alpha, \Gamma \Rightarrow \gamma \quad \beta, \Gamma \Rightarrow \gamma}{\alpha \vee \beta, \Gamma \Rightarrow \gamma} \quad \frac{\Gamma \Rightarrow \alpha}{\Gamma \Rightarrow \alpha \vee \beta} \quad \frac{\Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \vee \beta}.$$

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