CUT-OFF FREQUENCY IN RADIALLY INHOMOGENEOUS SINGLE-MODE FIBRE

Indexing term: Optical fibres

The limit for single-mode operation in a graded-index fibre has been obtained by calculating the normalised cut-off frequency of the TE_{01} mode. The effect of diffusion at the core-cladding boundary has been estimated.

Introduction: Single-mode fibres offer the potentiality of very large bandwidths, particularly when operated at wavelengths of about $1{\cdot}2\,\mu m$, where the loss[1] and dispersion[2] are low. Microbending, and thus cabling problems, can be reduced considerably by appropriate choice of core diameter, although the problem of jointing remains. Nevertheless, single-mode fibres merit careful consideration as potential long-distance transmission lines. A major factor is the limit for single-mode operation, which is determined by the onset of the 2nd-order TE_{01} mode. In a perfect step-index fibre, this occurs at a normalised cut-off frequency $V_c = 2{\cdot}4$, although the apparent value[3] may be appreciably higher. However, during the fabrication process, whether by the concentric crucible or the homogeneous vapour-deposition techniques, interdiffusion of the components can occur at the core–cladding boundary, causing a significant rounding of the stepped profile, particularly with soft glasses.[4] It is necessary, therefore, to consider the effect of such refractive-index grading on V_c. We therefore calculate, by a series method, the cut-off frequency V_c for the TE_{01} mode in a radially inhomogeneous fibre for a wide range of profiles, and assess the effect of diffusion at the core–cladding boundary.

Theory: Consider a fibre with a relative-permittivity distribution

$$\varepsilon(R) = \begin{cases} \varepsilon_1(1-2\Delta R^\alpha) = n_1^2(1-2\Delta R^\alpha) & 0 \leqslant R \leqslant 1 \\ \varepsilon_1(1-2\Delta) = n_2^2 & R \geqslant 1 \end{cases} \quad (1)$$

where R = radial distance r/core radius a,

$$\Delta = (n_1^2 - n_2^2)/2n_1^2$$

and $1 \leqslant \alpha \leqslant \infty$. The extremes $\alpha = 1, \infty$ denote triangular and stepped profiles, respectively.

For the TE_{01} mode we seek solutions in cylindrical (r, θ, z) co-ordinates for the electric- and magnetic-field components of the type

$$\left.\begin{aligned} E_\theta &= -je_\theta(R) \\ H_r &= h_r(R) \\ H_z &= h_z(R) \end{aligned}\right\} \exp(-j\beta z) \quad (2)$$

Substitution in Maxwell's equations gives

$$\left.\begin{aligned} e_\theta(R) &= \frac{\text{In core}}{\text{see eqn. 4}} && \frac{\text{In cladding}}{AK_1(WR)} \\ h_r(R) &= \frac{j\beta}{\omega\mu} e_\theta R && \frac{j\beta}{\omega\mu} AK_1(WR) \\ h_z(R) &= \frac{1}{\omega\mu a}\frac{1}{R}\frac{d}{dR}[Re_\theta(R)] && -\frac{AW}{\omega\mu a} K_0(WR) \end{aligned}\right\} \quad (3)$$

while, for $R \leqslant 1$, we obtain for $e_\theta(R)$ the equation

$$\frac{d^2 e_\theta}{dR^2} + \frac{1}{R}\frac{de_\theta}{dR} + (U^2 - V^2 R^\alpha - R^{-2})e_\theta = 0 \quad (4)$$

where $K_n(W)$ = modified Hankel function and

$$V^2 = U^2 + W^2 = \omega^2 \mu\varepsilon_1 a^2 2\Delta \quad (5)$$

Applying eqn. 3, the condition of continuity of e_θ and h_z at the core–cladding boundary gives

$$\frac{d}{dR}[Re_\theta(R)] = -\frac{WK_0(W)}{K_1(W)} e_\theta \quad \text{at} \quad R = 1 \quad (6)$$

The cut-off condition is obtained by setting $W = 0$ into eqns. 3, 4 and 6. Thus

$$\frac{d^2 e_\theta}{dR^2} + \frac{1}{R}\frac{de_\theta}{dR} + [V_c^2(1-R^\alpha) - R^{-2}]e_\theta = 0 \quad (7)$$

and at $R = 1$

$$\frac{de_\theta}{dR} + e_\theta = 0 \quad (8)$$

Eqn. 7 may be solved by V_c by a series solution in which the coefficients a_n are selected such that eqn. 9 is a solution of eqn. 7. Thus

$$e_\theta(R) = \sum_{n=0}^{\infty} a_n R^{n+1} \qquad a \neq 0 \quad (9)$$

Note that the convergence is greatly increased, and the calculation correspondingly simplified, by taking the exponent of R as $n+1$ rather than n.

Substituting eqn. 9 into eqn. 7 and equating the coefficients of terms in each power of R produces a recurrence relation for the a_n:

$$\begin{aligned} &2 \times 4a_2 + V_c^2 a_0 = 0 \\ &4 \times 6a_4 + V_c^2 a_2 = 0 \\ &\cdots\cdots \\ &(\alpha+1)(\alpha+3)a_{\alpha+1} + V_c^2 a_{\alpha-1} = 0 \\ &(\alpha+2)(\alpha+4)a_{\alpha+2} + V_c^2(a_\alpha - a_0) = 0 \\ &(\alpha+3)(\alpha+5)a_{\alpha+3} + V_c^2(a_{\alpha+1} - a_1) = 0 \\ &\cdots\cdots \end{aligned} \quad (10)$$

where $a_1 = 0$. From the boundary condition (eqn. 8) the condition for cut-off is obtained, namely

$$\sum_{n=0}^{\infty} (n+2)a_n = 0 \quad (11)$$

Finally, substitution of eqn. 10 into eqn. 11 gives an algebraic equation with real coefficients for V_c, which may be solved by the Newton–Raphson method.

Results: The calculated cut-off frequency V_c is shown in Fig. 1 for the whole range of α-profiles, in a single-mode fibre,

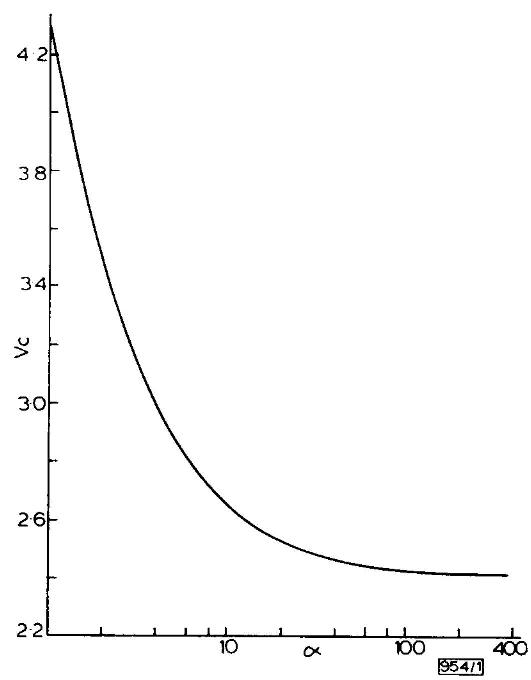

Fig. 1 *Variation with profile parameter α of cut-off frequency V_c in single-mode graded-index fibre*

from a step-index to a triangular-index profile. As expected, for a step-index fibre ($\alpha = \infty$) $V_c = 2{\cdot}405$ and rises progressively with smaller α to $4{\cdot}381$ at $\alpha = 1$.

Comparison with other results is possible for the particular case of $\alpha = 2$, for which Fig. 1 gives $V_c = 3{\cdot}518$. Thus, for the same parabolic-index profile, Dil and Blok,[5] who also use a series method of solution (although a different form of series), obtain $V_c = 3{\cdot}530$ for the HE_{21} mode. Since the latter

59

has a slightly higher cut-off frequency than that of the TE_{01} mode, the agreement is satisfactory. Other results are 3·401 for the variational[6] method and 3·642 for a more approximate perturbation[7] method.

Fig. 1 gives V_c for any value of α, and it can be used to estimate the effect of diffusion at the core–cladding interface by plotting the refractive-index profiles for various α from eqn. 1. Thus the equivalent of diffusion from the cladding by up to 10% of the core radius has practically no effect on V_c. Similarly, a fall in refractive index over 25% of the core radius causes V_c to change by only 5%. Strictly, the corresponding small increase in the refractive index in the cladding should also be taken into account, but the effect will be small compared with that in the core.

It appears that the variational method should be used with care. For example, when used to deduce the E_y component of the HE_{11} mode, it gives

$$E_y = A_1 \sum_{k=1}^{\infty} \frac{1}{(U^2 - j_k)^2} \frac{J_0(j_k R)}{J_0(j_k)} \quad . \quad . \quad . \quad . \quad . \quad (12)$$

where

J_n = Bessel function of order n

$j_k = (k-1)$th root of $J_1 = 0$

$A_{1,2}$ = constants

U = normalised propagation constant, as in eqn. 5.

However, it may be shown via the Dini expansion (see Appendix) that eqn. 12 reduces to $E_y = A_2 J_0(UR)$, whereas, in a graded-index, as distinct from a step-index, fibre a Bessel function is not an accurate approximation to the field distribution. In particular, the field distribution far from cut-off in a parabolic-index fibre is more correctly described by a Laguerre polynomial.

Further, calculations of spot size and its variation with V_c by using the variational method differ from those obtained elsewhere.[8] It is also worth noting that the series method, in the form of eqn. 9, reduces the computing time by a factor of about 25.

Acknowledgments: Grateful acknowledgment is made to the Pirelli General Cable Company for the award of research fellowships, and to the UK Science Research Council for supporting the work.

Appendix: The Bessel function $J_m(UR)$ can be shown from the Dini expansion to be given by

$$J_m(UR) = 2\sum_{k=1}^{\infty} \frac{j_k^2 J_m(j_k R)[U J_m(j_k) J_{m+1}(U) - j_k J_{m+1}(j_k) J_m(U)]}{(U^2 - j_k^2)\left[j_k^2 \frac{d}{dj_k}\{J_m(j_k)\}\right]^2 + (j_k^2 - m^2)[J_m(j_k)]^2} \quad . \quad . \quad . \quad (13)$$

where j_1, j_2 etc. are the positive zeros of the equation

$$z\frac{d}{dz}\{J_m(z)\} + hJ_m(z) \qquad (h = \text{constant}) \quad . \quad . \quad . \quad (14)$$

For $h = 0$ and $m = 0$, the j_k become the positive zeros of $J_1(z)$, so that eqn. 13 may be rewritten as

$$J_0(UR) = 2UJ_1(U)\sum_{k=1}^{\infty} \frac{J_0(j_k R)}{(U^2 - j_k^2)J_0(j_k)} \quad . \quad . \quad (15)$$

W. A. GAMBLING
D. N. PAYNE
H. MATSUMURA

9th February 1977

Department of Electronics
University of Southampton
Southampton SO9 5NH, England

References

1 HORIGUCHI, M., and OSANAI, H.: 'Spectral losses of low-OH-content optical fibres', *Electron. Lett.*, 1976, **12**, pp. 310–312
2 PAYNE, D. N., and GAMBLING, W. A.: 'Zero material dispersion in optical fibres', *ibid.*, 1975, **11**, pp. 176–178
3 MATSUMURA, H. *et al.*: (to be published)
4 KRUMPHOLZ, O.: 'Mode propagation in fibres: Discrepancies between theory and experiment' *in* 'Trunk telecommunications by guided waves'. IEE Conf. Publ. 71, 1970, pp. 56–61
5 DIL, J. G., and BLOK, H.: 'Propagation of electromagnetic surface waves in a radially inhomogeneous optical waveguide', *Opto-Electron.*, 1973, **5**, pp. 415–428
6 OKAMOTO, K., and OKOSHI, T.: 'Analysis of wave propagation in optical fibres having core with α-power refractive index distribution and uniform cladding', *IEEE Trans.*, 1976, **MTT-2**, pp. 416–421
7 SOMEDA, C. G., and ZOBOLI, M.: 'Cut-off wavelengths of guided modes in optical fibres with square-law profile', *Electron. Lett.* 1975, **11**, pp. 602–603
8 PETERMANN, K.: Private communication