# Cut Systems with Relational Morphisms for Semiring-Valued Fuzzy Structures

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**Abstract:** Many of the new $MV$-valued fuzzy structures, including intuitionistic, neutrosophic, or fuzzy soft sets, can be transformed into so-called almost $MV$-valued fuzzy sets, or, equivalently, fuzzy sets with values in dual pair of semirings (in symbols, $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets). This transformation allows any construction of almost $MV$-valued fuzzy sets to be retransformed into an analogous construction for these new fuzzy structures. In that way, approximation theories for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, rough $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets theories, or $F$-transform theories for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets have already been created and then retransformed for these new fuzzy structures. In this paper, we continue this trend and define, on the one hand, the theory of extensional $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets defined on sets with fuzzy similarity relations with values in dual pair of semirings and power sets functors related to this theory and, at the same time, the theory of cuts with relational morphisms of these structures. Illustratively, the reverse transformations of some of these concepts into new fuzzy structures are presented.

**Keywords:** almost $MV$-valued fuzzy sets; dual pair of semirings; cut systems; extensional fuzzy sets; power set functors

**MSC:** 03E72

## 1. Introduction

In the current development of fuzzy sets theory, and especially in the connection with the development of applicability in soft fields, there is a significant tendency to use completely new fuzzy structures and their mutual combinations in addition to the classic $L$-valued fuzzy sets. Let us mention, for example, intuitionistic fuzzy sets [1], fuzzy soft sets [2,3], or neutrosophic fuzzy sets [4] and their mutual combinations, such as intuitionistic fuzzy soft sets [5], etc.

Given that the individual fuzzy structures created in this way are defined relatively independently, the basic operations with these structures are also defined independently, with only a partial relationship to the operations of fuzzy structures of another type. Moreover, it very often happens that different variants of these operations are defined for one type of new fuzzy structure, which complicates the creation of one consistent and generally accepted theory of a new fuzzy structure. For example, fuzzy soft sets were introduced in [2] and modified in many other papers. The consequence of this is that there are a number of different definitions of basic operations with fuzzy soft sets, and some of these definitions do not correspond to the standards for these operations. For example, some of the operations introduced in that way do not meet DeMorgan's laws. The consequence of this state, among other things, is that each of these new fuzzy structures creates its own independent theory, including the necessary theoretical results and their proofs.

Therefore, it is understandable that efforts are being made to unify at least part of these new fuzzy structures and to create a theory that would allow us to work with these fuzzy structures in a manner analogous to classic $L$-fuzzy sets. One of the possible approaches to

this unification consists of the transformation of some of these $L$-valued fuzzy structures into a new structure called $AMV$-valued fuzzy sets or, equivalently, $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, where $(\mathcal{R}, \mathcal{R}^*)$ is a dual pair of semirings (see, e.g., [6]). Although most of the new $L$-valued fuzzy structures in the set $X$ do not represent mappings $X \to L$, after this transformation, individual fuzzy structures represent mappings $X \to R$, where only the dual pair of semirings $(\mathcal{R}, \mathcal{R}^*)$ with the underlying set $R$ changes depending on the type of fuzzy structure. With the help of this transformation, not only can their basic operations and terms be consistently defined for these new fuzzy structures, but the reverse transformation to the original fuzzy structures can be applied to the entire theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. This makes it possible to define $(\mathcal{R}, \mathcal{R}^*)$-sets theory analogies for new fuzzy structures without the need for special definitions and new proofs.

Since the reverse transformation of the results related to the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets to the results related to the new fuzzy structure is relatively simple, it is advisable to develop the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets as much as possible so that it is possible to subsequently transform these results on analogical theory in new fuzzy structures. In previous papers, we dealt with, e.g., definitions and properties of the theory of approximations of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, as well as definitions and properties of rough $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets [7] or $F$-transform theory for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets [6]. All these notions can be relatively simply transformed (without any additional proofs) into analogical notions with analogical properties into new fuzzy structures that can be transformed into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets; moreover, due to the existence of two monads, the results are mostly defined in two adjoint variants.

In this paper, we want to continue the process of defining concepts and theories for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, well known in the standard theories of classic $L$-valued fuzzy set theory. Naturally, the question may arise as to why the methods of classic $L$-fuzzy sets cannot be mechanically applied to define the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. The reason is in the formal tools that are used to work with $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. These tools are based on the use of two isomorphic monads in the category **Set**, which are created from a dual pair of semirings $(\mathcal{R}, \mathcal{R}^*)$. The result of this approach is, among other things, that most of the constructions defined in this way are much clearer and simpler than if we tried to directly apply the classical tools of $L$-valued fuzzy theory sets to $(\mathcal{R}, \mathcal{R}^*)$-sets.

In this paper, we focus on two concepts from the theory of classic $L$-fuzzy sets that are often used and the possibilities of their conversion into the concepts of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets.

The first concept concerns the so-called $L$-sets, whose origin is in the Wyler category **Set**$(L)$ [8] and was further developed in detail in the publications of Hohle [9] and other authors. This category represents a generalization of the classical category **Set** of sets, with the difference that the more general $L$-valued similarity relation is used instead of the classical identity. Among recent applications of $L$-sets theory is, for example, the theory of multilevel fuzzy sets, introduced by Šostak [10,11].

In the paper, for each dual pair of semirings $(\mathcal{R}, \mathcal{R}^*)$, we introduce two terms of similarity $R$-relations and two related terms of $\mathcal{R}$-sets and $\mathcal{R}^*$-sets, as a pair $(X, Q)$, where $X$ is a set and $Q$ is an $\mathcal{R}$- or $\mathcal{R}^*$-similarity relation. For these objects, we define two isomorphic categories **Set**$(\mathcal{R})$ and **Set**$(\mathcal{R}^*)$, whose morphisms are again $(\mathcal{R}, \mathcal{R}^*)$-relations, defined by the two monads already mentioned. Thus, these categories represent generalizations of the category **Set**$(L)$. We introduce the concept of fuzzy sets defined over objects of these categories, which are, in fact, analogies of extensional fuzzy sets, and we show the construction of power sets of these extensional fuzzy sets. All these constructions can be applied to new fuzzy structures by using the transformation of new $L$-fuzzy structures into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets.

Another area of the theory of $L$-fuzzy sets, whose analogy we want to define in the paper on the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, concerns cut systems. It is well known that any $L$-fuzzy set $s : X \to L$ can be equivalently defined as the cut system $(C_\alpha)_{\alpha \in L}$, satisfying some natural axioms. Between the cut systems in $X$ and the fuzzy sets in $X$, there are some interesting relationships, and from some point of view, an investigation of the $L$-valued fuzzy sets can be substituted by the investigation of cut systems. Cut systems

play a significant role in fuzzy topology [11], fuzzy algebra [12], fuzzy measure, fuzzy analysis [13] or [14], rule-based systems [15], and many other areas. For more information on the relationships between fuzzy sets and cut systems, see [16] or [17]. The cut systems are the natural bridge between the fuzzy sets and the classical sets.

In this paper, we define the theory of $(\mathcal{R}, \mathcal{R}^*)$-cuts represented by two categories **Cut**$(\mathcal{R})$ and **Cut**$(\mathcal{R}^*)$, where the morphisms are again $(\mathcal{R}, \mathcal{R}^*)$-relations defined using the mentioned monads. As the main result, we show that these categories are isomorphic to the categories **Set**$(\mathcal{R})$ and **Set**$(\mathcal{R}^*)$, respectively. Subsequently, this makes it possible to define the theory of cut systems in a universal and consistent way for any new fuzzy system that can be transformed into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets.

## 2. Preliminary Notions

The basic value structures that we use in the paper are dual pairs of semirings as the equivalent form of *AMV*-algebras. All these notions have been introduced in a recent paper [7], and it is, therefore, appropriate to repeat the basic definitions and properties of these new structures. In this section, we present the basic definitions and properties of these structures.

Recall that a complete monoid is a monoid $(R, +, 0)$ of type $(2, 0)$ such that for an arbitrary set $\{r_i : i \in I\} \subseteq R$ there exists a sum $\sum_{i \in I} r_i$ such that for an arbitrary partition $\{J_k : k \in K\}$ of $I$, the equality $\sum_{k \in K}(\sum_{i \in J_k} r_i) = \sum_{i \in I} r_i$ holds.

We recall the definition of an idempotent semiring.

**Definition 1** ([18])**.** *A complete commutative idempotent semiring (or a semiring, shortly) $\mathcal{R} = (R, +, \times, 0, 1)$ is an algebraic structure with the following properties:*

- *$(R, +, 0)$ is a complete idempotent commutative monoid;*

- *$(R, \times, 1)$ is a commutative monoid;*

- *$x \times \sum_{i \in I} y_i = \sum_{i \in I}(x \times y_i)$ holds for all $x, y_i \in R$;*

- *$0 \times x = 0$ holds for all $x \in R$.*

The definition of dual pairs of semirings was introduced in [7], which is presented below.

**Definition 2.** *Let $\mathcal{R} = (R, +, \times, 0, 1)$ and $\mathcal{R}^* = (R, +^*, \times^*, 0^*, 1^*)$ be complete idempotent commutative semirings with the same underlying set $R$. The pair $(\mathcal{R}, \mathcal{R}^*)$ is called the dual pair of semirings if there exists a semiring isomorphism $\neg : \mathcal{R} \to \mathcal{R}^*$ and the following axioms hold:*

1. *$\neg : \mathcal{R} \to \mathcal{R}^*$ is the involutive isomorphism;*

2. *$\forall a \in R, S \subseteq R \quad a \times^* (\sum_{b \in S} b) = \sum_{b \in S}(a \times^* b)$;*

3. *$\forall a \in R, S \subseteq R \quad a + (\sum^*_{b \in S} b) = \sum^*_{b \in S}(a + b)$, where $\sum^*$ is the complete operation $+^*$ in $\mathcal{R}^*$;*

4. *$\forall a, b \in R, \quad a + b = a \Leftrightarrow a +^* b = b$.*

Some properties of dual pairs of semirings are described in the following lemma:

**Lemma 1** ([7])**.** *Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings defined in Definition 3.*

1. *Let the relations $\leq$ and $\leq^*$ be defined by*

$$x, y \in R, \quad x \leq y \Leftrightarrow x + y = y, \quad x \leq^* y \Leftrightarrow x +^* y = y.$$

*The following statements hold:*

 *(a)* *$\leq$ and $\leq^*$ are the order relations on R;*

*(b)*  $x \leq^* y \Leftrightarrow x \geq y;$

*(c)*  $x \geq y \Leftrightarrow \neg x \leq \neg y;$

*(d)*  $x \leq y \Rightarrow x + z \leq y + z, \quad x +^* z \leq y +^* z;$

*(e)*  $x \leq y \Rightarrow x \times z \leq y \times z, \quad x \times^* z \leq y \times^* z.$

2.  $(R, +, \times, \leq)$ *and* $(R, +^*, \times^*, \leq^*)$ *are lattice-ordered semirings, where, for arbitrary* $S \subseteq R$,

$$\sup S = \sum_{x \in S} x, \quad \inf S = \sum_{x \in S}^* x, \text{ in } (R, \leq),$$

$$\sup S = \sum_{x \in S}^* x, \quad \inf S = \sum_{x \in S}, \text{ in } (R, \leq^*).$$

*where* $\sum_{x \in S}^* x$ *is the sum of elements with respect to* $+^*$.

Using dual pairs of semirings $(\mathcal{R}, \mathcal{R}^*)$, we can also introduce $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets and operations with these fuzzy sets.

**Definition 3** ([7])**.** *Let* $(\mathcal{R}, \mathcal{R}^*)$ *be a dual pair of semirings.*

1.  *A mapping* $s : X \to R$ *is called a* $(\mathcal{R}, \mathcal{R}^*)$-*fuzzy set in a set* $X$.

2.  *Operations with* $(\mathcal{R}, \mathcal{R}^*)$-*fuzzy sets are defined by*

    *(a)*  *The intersection* $s \sqcap t$ *is defined by* $(s \sqcap t)(x) = s(x) +^* t(x)$, $x \in X$;

    *(b)*  *The union* $s \sqcup t$ *is defined by* $(s \sqcup t)(x) = s(x) + t(x)$, $x \in X$;

    *(c)*  *Complement* $\neg s$ *is defined by* $(\neg s)(x) = \neg(s(x))$;

    *(d)*  *The external multiplication* $\star$ *by elements of R is defined by* $(a \star s)(x) = a \times s(x)$;

    *(e)*  *The order relation* $\leq$ *between* $s, t$ *is defined by* $s \leq t \Leftrightarrow (\forall x \in X)s(x) \leq t(x)$ *where* $\leq$ *is the order relation defined in Lemma 1.*

Another tool we use is the elementary theory of monads in categories, as introduced in [19]. This theory allows us to use a monad to define the concept of a monadic relation, which can be used to construct the general theory of upper and lower approximations and many other constructions. We use this theory in the context of dual pairs of semirings.

**Definition 4** ([19])**.** *The structure* $\mathbf{T} = (T, \Diamond, \eta)$ *is a monad in the category* **Set***, where*

1.  $T : \mathbf{Set} \to \mathbf{Set}$ *is the mapping of objects;*

2.  $\eta$ *is a system of mappings* $\{\eta_X : X \to T(X) | X \in \mathbf{Set}\}$;

3.  *For each pair of mappings* $f : X \to T(Y)$, $g : Y \to T(Z)$, *there exists a composition (called a Kleisli composition)* $g \Diamond f : X \to T(Z)$, *which is associative;*

4.  *For every mapping* $f : X \to T(Y)$, $\eta_Y \Diamond f = f$ *and* $f \Diamond \eta_X = f$ *hold;*

5.  $\Diamond$ *is compatible with the composition of mappings, i.e., for mappings* $f : X \to Y$, $g : Y \to T(Z)$, *we have* $g \Diamond (\eta_Y . f) = g . f$.

In the following proposition, it is shown that any dual pair of semirings defines two monads in the category **Set**:

**Proposition 1** ([7])**.** *Let* $\mathcal{R} = (R, +, \times, 0_R, 1_R)$ *be a complete commutative idempotent semiring and let the structure* $\mathbf{T}_{\mathcal{R}} = (T, \Diamond, \eta)$ *be defined by*

1. *The mapping $T : \mathbf{Set} \to \mathbf{Set}$ of objects is defined by $T(X) = R^X$,*

2. *For the mappings $f : X \to T(Y)$ and $g : Y \to T(Z)$ their composition $g \lozenge f : X \to T(Z)$ is defined by*

$$x \in X, z \in Z, \quad (g \lozenge f)(x)(z) = \sum_{y \in Y} f(x)(y) \times g(y)(z).$$

3. *$\eta_X$ is the mapping $X \to T(X)$ defined by*

$$\eta_X(x)(y) = \begin{cases} 1_R, & x = y, \\ 0_R, & x \neq y, \end{cases}.$$

*Then, $\mathbf{T}_{\mathcal{R}}$ is a monad in the category $\mathbf{Set}$.*

**Remark 1.** *If $(\mathcal{R}, \mathcal{R}^*)$ is a dual pair of semirings, according to Proposition 1, there exists another monad $\mathbf{T}_{\mathcal{R}^*} = (T, \lozenge^*, \eta^*)$, where for $f : X \to T(Y)$ and $g : Y \to T(Z)$,*

$$g \lozenge^* f(x)(z) = \sum_{y \in Y}^{*} f(x)(y) \times^* g(y)(z),$$

$$\eta_X^*(x)(y) = \begin{cases} 1_R^*, & x = y, \\ 0_R^*, & x \neq y. \end{cases}.$$

The last definition we need is the notion of a monadic relation. This notion was introduced by Manes [19]. For simplicity, we define this notion for monads $\mathbf{T}_{\mathcal{R}}$ and $\mathbf{T}_{\mathcal{R}^*}$ only.

**Definition 5** ([19]). *Let $\mathbf{T}_{\mathcal{R}} = (T, \lozenge, \eta)$ be the monad in category $\mathbf{Set}$ from Proposition 1 and let $X, Y$ be sets. A $\mathbf{T}_{\mathcal{R}}$-relation $Q$ from $X$ to $Y$ (denoted $Q : X \rightsquigarrow Y$) is a mapping $Q : X \to R^Y$. If $Q : X \rightsquigarrow Y$ and $S : Y \rightsquigarrow Z$ are $\mathbf{T}_{\mathcal{R}}$-relations, their composition is the $\mathbf{T}_{\mathcal{R}}$-relation $S \lozenge Q : X \rightsquigarrow Z$. A $\mathbf{T}_{\mathcal{R}}$-relation $Q : X \rightsquigarrow X$ is called a $\mathcal{R}$-similarity relation if*

1. *It is reflexive, that is, $\eta_X \preceq Q$;*

2. *it is transitive, that is, $Q \lozenge Q \preceq Q$;*

3. *it is symmetric, that is, $Q(x)(y) = Q(y)(x)$, for arbitrary $x, y \in X$.*

**Remark 2.** *1. From Observation 1 it follows that for a dual pair of semirings $(\mathcal{R}, \mathcal{R}^*)$ we can introduce two types of monadic relations, namely $\mathbf{T}_{\mathcal{R}}$-relation and $\mathbf{T}_{\mathcal{R}^*}$-relation. From Definition 5, it follows that these monadic relations are identical objects. For this reason, we sometimes include these two objects under the common name $(\mathcal{R}, \mathcal{R}^*)$-relations.*

2. *If the composition of $(\mathcal{R}, \mathcal{R}^*)$-relations is considered, we need to distinguish between $\mathcal{R}$-relations and $\mathcal{R}^*$-relations, depending on the compositions $\lozenge$ and $\lozenge^*$, respectively.*

3. *Since the notion of the $\mathcal{R}$-similarity relation is defined using terms from the monad $\mathbf{T}_{\mathcal{R}}$, it is necessary to distinguish between the notions of the $\mathcal{R}$-similarity relation and the $\mathcal{R}^*$-similarity relation.*

## 3. $(\mathcal{R}, \mathcal{R}^*)$-Sets

In the classical theory of $L$-valued fuzzy sets, where $L$ is a complete, distributive lattice with possible other operations, there are two basic categories of objects for which $L$-valued fuzzy sets are defined. The first of these categories is the classic category $\mathbf{Set}$ with sets as objects and mappings as morphisms. The second type of basic category is category $\mathbf{Set}(L)$, which represents a generalization of category $\mathbf{Set}$. This category has its origin in Wyles' category $\mathbf{Set}(L)$ [8], developed in the work of Höhle [9] and other authors. Objects of the

category $\mathbf{Set}(L)$ are the so-called $L$-sets, that is, pairs $(X, \delta)$, where $X$ is a set, and delta is the equality valued $L$ on $X$, that is, $\delta : X \times X \to L$ with natural axioms. Morphisms are structures that preserve mapping.

As we already mentioned in the introduction, many of the new fuzzy structures, traditionally called $L$-fuzzy structures, are, in fact, $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets for suitably dual pairs of semirings $(\mathcal{R}, \mathcal{R}^*)$. Just as $L$-sets are a generalization of classic sets, we can introduce the general concept of $(\mathcal{R}, \mathcal{R}^*)$-sets, which will be a generalization of $L$-sets and it will be possible to apply them to all new fuzzy structures that can be transformed into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets.

Based on the current trend in fuzzy set theory categories, we introduce new categories $\mathbf{Rel}(\mathcal{R})$ and $\mathbf{Set}(\mathcal{R})$, respectively, which generalize the category of sets $\mathbf{Set}$ and the category of $L$-sets $\mathbf{Set}(L)$, respectively, with suitable types of relations such as morphisms instead of mappings. According to Remark 2, with any dual pair of semirings $(\mathcal{R}, \mathcal{R}^*)$ two types of relations are used, namely, $\mathbf{T}_{\mathcal{R}}$ and $\mathbf{T}_{\mathcal{R}^*}$-relations and it follows that we obtain two types of these categories, namely $\mathbf{Rel}(\mathcal{R})$ and $\mathbf{Rel}(\mathcal{R}^*)$ and $\mathbf{Set}(\mathcal{R})$ and $\mathbf{Set}(\mathcal{R}^*)$, respectively.

The analogies of the category $\mathbf{Set}$ but with $(\mathcal{R}, \mathcal{R}^*)$-relations as morphisms are the following categories $\mathbf{Rel}(\mathcal{R})$ and $\mathbf{Rel}(\mathcal{R}^*)$.

**Definition 6.** *Let* $(\mathcal{R}, \mathcal{R}^*)$ *be a dual pair of semirings.*

1. *The category* $\mathbf{Rel}(\mathcal{R})$ *of sets with* $\mathbf{T}_{\mathcal{R}}$*-relations as morphisms is defined by*

    (a) *Objects are sets;*

    (b) *Morphisms from $X$ to $Y$ are $(\mathcal{R}, \mathcal{R}^*)$-relations $f : X \rightsquigarrow Y$;*

    (c) *The composition of morphisms is defined by $\Diamond$;*

    (d) *For arbitrary object $X$, $1_X = \eta_X$.*

2. *The category* $\mathbf{Rel}(\mathcal{R}^*)$ *of sets with* $\mathbf{T}_{\mathcal{R}^*}$*-relations as morphisms is defined by*

    (a) *Objects are sets;*

    (b) *Morphisms from $X$ to $Y$ are $(\mathcal{R}, \mathcal{R}^*)$-relations $f : X \rightsquigarrow Y$;*

    (c) *The composition of morphisms is defined by $\Diamond^*$;*

    (d) *For arbitrary object $X$, $1_X^* = \eta_X^*$.*

The analogies of the category $\mathbf{Set}(L)$ are the categories $\mathbf{Set}(\mathcal{R})$ and $\mathbf{Set}(\mathcal{R}^*)$ with sets as objects and with $\mathcal{R}$ or $\mathcal{R}^*$-similarity relations, respectively.

**Definition 7.** *Let* $(\mathcal{R}, \mathcal{R}^*)$ *be a dual pair of semirings.*

1. *The category* $\mathbf{Set}(\mathcal{R})$ *of sets with $\mathcal{R}$-similarity relations is defined by*

    (a) *The objects are pairs $(X, Q)$, where $X$ is a set and $Q : X \rightsquigarrow X$ is a $\mathcal{R}$-similarity relation;*

    (b) *Morphisms $f : (X, Q) \rightsquigarrow (Y, S)$ are $(\mathcal{R}, \mathcal{R}^*)$-relations $f : X \rightsquigarrow Y$ such that $f \Diamond Q \leq f$, $S \Diamond f \leq f$, where the order relation $\leq$ is defined point-wise;*

    (c) *The composition of the morphisms $f : (X, Q) \rightsquigarrow (Y, S)$ and $g : (Y, S) \rightsquigarrow (Z, V)$ is defined by $g \Diamond f$;*

    (d) *For arbitrary object $(X, Q)$, $1_{(X,Q)} = Q : (X, Q) \rightsquigarrow (X, Q)$.*

2. *The category* $\mathbf{Set}(\mathcal{R}^*)$ *of sets with $\mathcal{R}^*$-similarity relations is defined by*

    (a) *The objects are pairs $(X, Q)$, where $X$ is a set and $Q : X \rightsquigarrow X$ is a $\mathcal{R}^*$-similarity relation;*

(b) *Morphisms* $f : (X, Q) \rightsquigarrow (Y, S)$ *are* $(\mathcal{R}, \mathcal{R}^*)$*-relations* $f : X \rightsquigarrow Y$ *such that* $f \Diamond^* Q \leq^* f$, $S \Diamond^* f \leq^* f$, *where the order relation* $\leq^*$ *is defined point-wise;*

(c) *The composition of the morphisms* $f : (X, Q) \rightsquigarrow (Y, S)$ *and* $g : (Y, S) \rightsquigarrow (Z, V)$ *is defined by* $g \Diamond^* f$;

(d) *For the arbitrary object* $(X, Q)$, $1^*_{(X,Q)} = Q$.

**Remark 3.** *1.　It should be noted that objects of categories* **Set**$(\mathcal{R})$ *and* **Set**$(\mathcal{R}^*)$ *are different. In fact, objects of* **Set**$(\mathcal{R})$ *are pairs* $(X, Q)$, *such that* $Q : X \rightsquigarrow X$ *are* $(\mathcal{R}, \mathcal{R}^*)$*-relations such that* $Q \Diamond Q \leq Q$ *and* $Q \geq \eta_X$, *but objects of* **Set**$(\mathcal{R}^*)$ *are pairs* $(X, Q)$, *such that* $Q$ *are* $(\mathcal{R}, \mathcal{R}^*)$*-relations such that* $Q \Diamond^* Q \leq^* Q$ *and* $Q \geq^* \eta_X^*$.

*2.　According to the traditional designation of L-sets, the objects of the categories* **Set**$(\mathcal{R})$ *and* **Set**$(\mathcal{R}^*)$, *respectively, will be called* $\mathcal{R}$*-sets and* $\mathcal{R}^*$*-sets, respectively.*

*3.　It should be observed that conditions* $f \Diamond Q \leq f$, $S \Diamond f \leq f$ *and* $f \Diamond^* Q \leq^* f$, $S \Diamond^* \leq^* f$, *respectively, are equivalent to conditions* $f \Diamond Q = f = S \Diamond f$ *and* $f \Diamond^* Q = f = S \Diamond^* f$, *respectively.*

**Proposition 2.** *All categories* **Rel**$(\mathcal{R})$, **Rel**$(\mathcal{R}^*)$, **Set**$(\mathcal{R})$, *and* **Set**$(\mathcal{R}^*)$ *are correctly defined.*
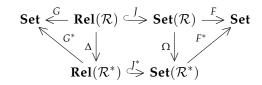
　　The **proof** follows directly from the definitions of operations in $(\mathcal{R}, \mathcal{R}^*)$ and the corresponding monads, and will be omitted.

　　As we have already stated, both types of newly defined pairs of categories can be considered certain generalizations of the classical category **Set**. In the classical *L*-fuzzy set theory, the power set structure $L^X$ is the principal structure, which is the basis for many theoretical results. In the following part, we will, therefore, show how analogies of this power set structure can also be defined for new analogies of the category **Set**, mentioned in the previous definitions.

　　In what follows, we introduce new types of power set structures based on generalized categories of sets, which are defined as functors from these categories of sets to the classical category of sets. However, as mentioned above, we have two types of generalization of the category **Set**, ie, category **Rel**$(\mathcal{R})$ and category **Set**$(\mathcal{R})$. For this reason, for each set $X \in$ **Set**, we obtain two types generalizing the classical power set objects $L^X$, namely, objects $R^X$ and objects $R^{(X,Q)}$. These two power set structures are represented by two pairs of functors

$$\mathbf{Rel}(\mathcal{R}) \to \mathbf{Set}, \quad \mathbf{Rel}(\mathcal{R}^*) \to \mathbf{Set},$$
$$\mathbf{Set}(\mathcal{R}) \to \mathbf{Set}, \quad \mathbf{Set}(\mathcal{R}^*) \to \mathbf{Set}.$$

**Theorem 1.** *Let* $(\mathcal{R}, \mathcal{R}^*)$ *be a dual pair of semirings. There exist power set functors* $F, F^*, G, G^*$ *such that the following diagram commutes, where* $\Delta$ *and* $\Omega$ *are isomorphic functors.*

$$\mathbf{Set} \xleftarrow{G} \mathbf{Rel}(\mathcal{R}) \xhookrightarrow{J} \mathbf{Set}(\mathcal{R}) \xrightarrow{F} \mathbf{Set}$$

$$G^* \diagdown \quad \Delta \downarrow \qquad \quad \Omega \downarrow \quad \diagup F^*$$

$$\mathbf{Rel}(\mathcal{R}^*) \xhookrightarrow{J^*} \mathbf{Set}(\mathcal{R}^*)$$

**Proof.** (1) Let $f : X \rightsquigarrow Y$ be a morphism in the category **Rel**$(\mathcal{R})$. The functor $G$ is defined by

$$G(X) = R^X, \quad G(f) := f \Diamond 1_{R^X} : R^X \to R^Y.$$

From Definition 6, it follows that $G(1_X) = G(\eta_X) = \eta_X \Diamond 1_{R^X} = 1_{R^X} = 1_{G(X)}$ and for another morphism $g : Y \rightsquigarrow Z$ we obtain $G(g \Diamond f) = g \Diamond f \Diamond 1_{R^X} = (g \Diamond 1_{R^Y}).(f \Diamond 1_{R^X}) = G(g).G(f)$, where "." is the composition of mappings. Therefore, $G$ is the functor.

(2) Let $f : X \rightsquigarrow Y$ be a morphism in the category $\mathbf{Rel}(\mathcal{R}^*)$. The functor $G^*$ is defined by

$$G^*(X) = G(X), \quad G^*(f) := \neg(f \Diamond^* \neg 1_{R^X}) : R^X \to R^Y.$$

$G^*$ is the functor. In fact, we have $G^*(1_X^*) = G^*(\eta_X^*) = \neg(\eta_X^* \Diamond^* \neg 1_{R^X}) = \neg(\neg 1_{R^X}) = 1_{R^X} = 1_{G^*(X)}$. For arbitrary morphisms $f : X \rightsquigarrow Y$ and $g : Y \rightsquigarrow Z$, we have

$$G^*(g \Diamond^* f) = \neg(g \Diamond^* f \Diamond^* \neg 1_{R^X}) = \neg((g \Diamond^* \neg 1_{R^Y}).(f \Diamond^* \neg 1_{R^X})) =$$
$$\neg(g \Diamond^* \neg 1_{R^Y}).\neg(f \Diamond^* \neg 1_{R^X}) = G^*(g).G^*(f),$$

(3) Let $f : X \rightsquigarrow Y$ be a morphism in the category $\mathbf{Rel}(\mathcal{R})$. The functor $\Delta$ is defined by

$$\Delta(X) = X, \quad \Delta(f) = \neg f.$$

It is straightforward to show that $\Delta$ is the functor. We show that $G^*.\Delta = G$. In fact, $G^*.\Delta(X) = G^*(X) = G(X)$. In addition, we have

$$G^*.\Delta(f) = G^*(\neg f) = \neg(\neg f \Diamond^* \neg 1_{R^X}) = f \Diamond 1_{R^X} = G(f)$$

and the diagram commutes.

(4) Let $f : (X, Q) \rightsquigarrow (Y, S)$ be a morphism in $\mathbf{Set}(\mathcal{R})$. The functor $F$ is defined by

$$F(X, Q) = \{s \in R^X : Q^\uparrow(s) := (Q \Diamond 1_{R^X})(s) \le s\},$$
$$F(f) : F(X, Q) \to F(Y, S), \quad F(f) = S \Diamond f \Diamond 1_{R^X} = f \Diamond 1_{R^X}.$$

We show that this definition is correct. We have $F(f)(s) \in F(Y, S)$. In fact, for arbitrary $s \in F(X, Q)$,

$$(S \Diamond 1_{R^Y})(F(f)(s)) = (S \Diamond 1_{R^Y})(S \Diamond f \Diamond 1_{R^X})(s) = S \Diamond 1_{R^Y}.(S \Diamond f \Diamond 1_{R^X})(s) =$$
$$(S \Diamond S \Diamond f \Diamond 1_{R^X})(s) \le (S \Diamond f \Diamond 1_{R^X})(s) = F(f)(s),$$

and $F(f)(s) \in F(Y, S)$. Furthermore, for $1_{(X,Q)} = Q$ and $s \in F(X, Q)$, we have

$$F(Q) = Q \Diamond Q \Diamond 1_{R^X} = Q \Diamond 1_{R^X},$$
$$s \le (Q \Diamond 1_{R^X})(s) \le s,$$

and it follows that $(Q \Diamond 1_{R^X})(s) = s$. Therefore, $F(1_{(X,Q)}) = (Q \Diamond 1_{R^X}) = 1_{F(X,Q)}$. Finally, let $f : (X, Q) \rightsquigarrow (Y, S)$ and $g : (Y, S) \rightsquigarrow (Z, W)$ be morphisms in $\mathbf{Set}(\mathcal{R})$. Therefore, $f \Diamond Q = f = S \Diamond f$ and $g \Diamond S = g = W \Diamond g$ and we obtain the following equation:

$$F(g).F(f) = (W \Diamond g \Diamond 1_{R^Y}).(S \Diamond f \Diamond 1_{R^X}) = W \Diamond g \Diamond S \Diamond f \Diamond 1_{R^X} =$$
$$W \Diamond g \Diamond f \Diamond 1_{R^X} = F(g \Diamond f),$$

as follows from identities $W \Diamond W = W$ and composition rules for $\Diamond$. Therefore, $F$ is the functor.

(5) Let $f : (X, Q) \rightsquigarrow (Y, S)$ be a morphism in $\mathbf{Set}(\mathcal{R}^*)$. The functor $F^*$ is defined by

$$F^*(X, Q) = \{s \in R^X : Q^\downarrow(s) = (Q \Diamond^* 1_{R^X})(\neg s) \le^* \neg s\},$$
$$F^*(f) = \neg(S \Diamond^* f \Diamond^* \neg 1_{R^X}) = \neg(f \Diamond^* \neg 1_{R^X}) : F^*(X, Q) \to F^*(Y, S).$$

We show that this definition is correct. First, the following equalities hold for an arbitrary object $(X, Q) \in \mathbf{Set}(\mathcal{R}^*)$:

$$F^*(X, Q) = \{s \in R^X : (Q \Diamond^* 1_{R^X})(\neg s) = \neg s\} = \{s \in R^X : \neg(Q \Diamond^* \neg 1_{R^X})(s) = s\}. \quad (1)$$

In fact, for $s \in F^*(X, Q)$ we have $(Q \Diamond^* 1_{RX})(\neg s) \leq^* \neg s$. On the other hand, $(Q \Diamond^* 1_{RX})(\neg s) \geq^* (\eta^*_X \Diamond^* 1_{RX})(\neg s) = \neg s$ and the first equality is proved. If $(Q \Diamond^* 1_{RX})(\neg s) = \neg s$, we obtain $(Q \Diamond^* \neg 1_{RX})(s) = (Q \Diamond^* 1_{RX})(\neg s) = \neg s$ and it follows that $\neg (Q \Diamond^* \neg 1_{RX})(s) = s$.

We prove that $F^*$ preserves unity morphisms. According to (1), for arbitrary $s \in F^*(X, Q)$ we have

$$F^*(1^*_{(X,Q)})(s) = F^*(Q)(s) = \neg(Q \Diamond^* Q \Diamond^* \neg 1_{RX})(s) = \neg(Q \Diamond^* \neg 1_{RX})(s) = s,$$

and it follows that $F^*(1^*_{(X,Q)}) = 1_{F^*(X,Q)}$.

Let $f : (X, Q) \rightsquigarrow (Y, S)$ and $g : (Y, S) \rightsquigarrow (Z, W)$ be morphisms of **Set**$^*(\mathcal{R}, \mathcal{R}^*)$. Using the identities $f \Diamond^* Q = f = S \Diamond^* f$ and $g \Diamond^* S = g = W \Diamond^* g$, we obtain the following:

$$F^*(g).F^*(f) = \neg(W \Diamond^* g \Diamond^* \neg 1_{RY}).\neg(S \Diamond^* f \Diamond^* \neg 1_{RX}) =$$
$$(\neg W \Diamond \neg g \Diamond 1_{RX}).(\neg S \Diamond \neg f \Diamond 1_{RX}) =$$
$$\neg W \Diamond \neg g \Diamond \neg S \Diamond \neg f \Diamond 1_{RX} =$$
$$\neg(W \Diamond^* g \Diamond^* S \Diamond^* f \Diamond^* \neg 1_{RX}) =$$
$$\neg(W \Diamond^* g \Diamond^* f \Diamond^* \neg 1_{RX}) = F^*(g \Diamond^* f).$$

Finally, we show that for arbitrary $s \in F^*(X, Q)$ and arbitrary morphism $f : (X, Q) \rightsquigarrow (Y, S)$, $F^*(f)(s) \in F^*(Y, S)$ holds. In fact, we have

$$(S \Diamond^* 1_{RY})(\neg F^*(f)(s)) = (S \Diamond^* 1_{RY}).(S \Diamond^* f \Diamond^* \neg 1_{RX})(s) =$$
$$(S \Diamond^* S \Diamond^* f \Diamond^* \neg 1_{RX})(s) = (S \Diamond^* f \Diamond^* \neg 1_{RX})(s) = \neg F^*(f)(s),$$

and, according to (1), we have $F^*(f)(s) \in F^*(Y, S)$.

(6) Let $f : (X, Q) \rightsquigarrow (Y, S)$ be a morphism in **Set**$(\mathcal{R})$. The functor $\Omega : \textbf{Set}(\mathcal{R}) \rightarrow \textbf{Set}(\mathcal{R}^*)$ is defined by

$$\Omega(X, Q) = (X, \neg Q), \quad \Omega(f) = \neg f$$

Since $\neg Q$ is the $\mathcal{R}^*$-similarity relation, the definition is correct. We prove that the diagram commutes, that is, $F^*.\Omega = F$. In fact, according to (1), we have

$$F^*.\Omega(X, Q) = F^*(X, \neg Q) = \{s \in R^X : (\neg Q \Diamond^* 1_{RX})(\neg s) = \neg s\} =$$
$$\{s \in R^X : \neg(Q \Diamond \neg 1_{RX})(\neg s) = \neg s\} = \{s \in R^X : (Q \Diamond \neg 1_{RX})(\neg s) = s\}$$
$$= \{s \in R^X : (Q \Diamond 1_{RX})(s) = s\} = F(X, Q).$$

Further, we have $\neg f : (X, \neg Q) \rightsquigarrow (Y, \neg S)$ and obtain

$$F^*.\Omega(f) = F^*(\neg f) = \neg(\neg S \Diamond^* \neg f \Diamond^* \neg 1_{RX}) = S \Diamond f \Diamond 1_{RX} = F(f).$$

(7) Let $f : X \rightsquigarrow Y$ be a morphism in the category **Rel**$(\mathcal{R})$ (or **Rel**$(\mathcal{R}^*)$, equivalently). The embedding functors $J$ and $J^*$ are defined by

$$J(X) = (X, \eta_X), \quad J^*(X) = (X, \eta^*_X),$$
$$J(f) = f = J^*(f).$$

It is easy to see that $J, J^*$ are functors and that $\Omega.J = J^*.\Delta$. $\square$

**Remark 4.** *As in the case of classic power set structures $L^X$, the objects of $F(X, Q)$ and $F^*(X, S)$, respectively, will be called $(X, Q)$-extensional and $(X, S)$-extensional $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, respectively, where $(X, Q) \in \textbf{Set}(\mathcal{R})$ and $(X, S) \in \textbf{Set}(\mathcal{R}^*)$.*

## 4. $(\mathcal{R}, \mathcal{R}^*)$-Cut Systems

It is well known that the theory of classical *L*-valued fuzzy sets can be equivalently replaced by the theory of *α*-cuts. In many theories, and especially in applications, this substitution is a more acceptable solution because it allows one to explicitly interpret the approximate solutions of a number of problems. A typical example can be problems related to fuzzy rough set theory, the goal of which is to approximate indeterminate (i.e., fuzzy) phenomena *s* with the help of the so-called upper and lower approximations $Q^\uparrow(s)$ and $Q^\downarrow(s)$ based on a suitable similarity *Q*. However, this approximation again represents *L*-valued fuzzy sets, and therefore, its visualization is somewhat problematic. Therefore, a suitable solution is to replace these approximations with their *α*-cuts, which allows us to explicitly define sets of elements $Q^\downarrow(s)_\alpha \subseteq s_\alpha \subseteq Q^\uparrow(s)_\alpha$, representing the lower and upper estimate of the set of objects satisfying the given concept *s* with the degree at least *α*.

Since in addition to classic *L*-fuzzy sets, other types of fuzzy structures are currently used in applications, it is appropriate to extend this more illustrative form of fuzzy sets and its theory to these new fuzzy structures as well. For this reason, in this section, we will focus on introducing the theory of cut systems for $(\mathcal{R}, \mathcal{R}^*)$-sets and extensional $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. For individual types of fuzzy structures transformable to $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, their cut system theories are only special examples of the cut system theory for general $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. In this way, the theory of *α*-cuts can be obtained, for example, for neutrosophic, intuitionistic, or *L*-fuzzy soft sets.

As we mentioned in the previous section, the basic categories for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets we use are the categories $\mathbf{Set}(\mathcal{R})$ and $\mathbf{Set}(\mathcal{R}^*)$. Our goal in this section is to show that these categories can be equivalently expressed by cut systems, which are introduced in the following definitions.

**Definition 8.** *Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings and let $\mathbf{C} = (C_r)_{r\in R}$, where for arbitrary $r \in R$, $C_r : X \rightsquigarrow X$ is a $(\mathcal{R}, \mathcal{R}^*)$-relation.*

1. $\mathbf{C}$ *is called an $\mathcal{R}$-cut system in a set X, if for arbitrary $r, s \in R, x, x' \in X$,*

   (a) *$C_r$ is a crisp $\mathcal{R}$-relation, that is, $C_r(x)(x') \in \{1_R, 0_R\}$;*

   (b) *$C_r \lozenge C_s \leq C_{r\times s}$;*

   (c) *$C_r^{-1} = C_r$;*

   (d) *$C_r \geq \eta_X$;*

   (e) *$r \leq s \Rightarrow C_s \leq C_r$;*

   (f) *For arbitrary $x, y \in X$, the set $\{r \in R : C_r(x)(y) = 1_R\}$ has the greatest element in $\mathcal{R}$.*

2. $\mathbf{C}$ *is called am $\mathcal{R}^*$-cut system in a set X, if for arbitrary $r, s \in R, x, x' \in X$,*

   (a) *$C_r$ is a crisp $\mathcal{R}^*$-relation, that is, $C_r(x)(x') \in \{1_R^*, 0_R^*\}$;*

   (b) *$C_r \lozenge^* C_s \leq^* C_{r\times^* s}$;*

   (c) *$C_r^{-1} = C_r$;*

   (d) *$C_r \geq^* \eta_X^*$;*

   (e) *$r \leq^* s \Rightarrow C_s \leq^* C_r$;*

   (f) *For arbitrary $x, y \in X$, the set $\{r \in R : C_r(x)(y) = 1_R^*\}$ has the greatest element in $\mathcal{R}^*$.*

**Remark 5.** *It should be noted that conditions (e) and (f) are equivalent to the following implications:*

$$\sum_{\{r\in R:C_r(x)(x')=1_R\}} r \geq s \Rightarrow C_s(x)(x') = 1_R,$$

$$\sum_{\{r\in R:C_r(x)(x')=1_R^*\}}^{*} r \geq^* s \Rightarrow C_s(x)(x') = 1_R^*,$$

*In fact, if $q \in R$ is the greatest element of $A = \{r : C_r(x)(x') = 1_R\}$, for arbitrary $r \in A$ we have $r \leq q$ and it follows that $s \leq \sum_{r\in A} r \leq q$ and we obtain $C_q \leq C_s$. Therefore, $C_s(x)(x') = 1_R$.*

In the following definition, we introduce two categories of cut systems:

**Definition 9.** *Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings.*

1. *The category $\mathbf{Cut}(\mathcal{R})$ of $\mathcal{R}$-cut systems is defined by*

   (a) *Objects are pairs $(X, \mathbf{C})$, where $\mathbf{C}$ is an $\mathcal{R}$-cut system in a set $X$;*

   (b) *Morphisms $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ are $(\mathcal{R}, \mathcal{R}^*)$-relations $X \rightsquigarrow Y$ such that for all $r \in R$, $f \Diamond (r \star C_r) \leq f$, $(r \star D_r) \Diamond f \leq f$ holds;*

   (c) *The composition of the morphisms $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ and $g : (Y, \mathbf{D}) \rightsquigarrow (Z, \mathbf{G})$ is defined by $g \Diamond f$;*

   (d) *For the arbitrary object $(X, \mathbf{C})$, the unit morphism $1_{(X,\mathbf{C})} : (X, \mathbf{C}) \rightsquigarrow (X, \mathbf{C})$ is defined by*

   $$1_{(X,\mathbf{C})} = \bigsqcup_{r\in R} r \star C_r.$$

2. *The category $\mathbf{Cut}(\mathcal{R}^*)$ of $\mathcal{R}^*$-cut systems is defined by*

   (a) *Objects are pairs $(X, \mathbf{C})$ representing $\mathcal{R}^*$-cut systems in a set $X$;*

   (b) *Morphisms $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ are $(\mathcal{R}, \mathcal{R}^*)$-relations $X \rightsquigarrow Y$ such that for all $r \in R$, $f \Diamond^* (r \star^* C_r) \leq^* f$, $(r \star^* D_r) \Diamond^* f \leq^* f$ holds;*

   (c) *The composition of the morphisms $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ and $g : (Y, \mathbf{D}) \rightsquigarrow (Z, \mathbf{G})$ is defined by $g \Diamond^* f$;*

   (d) *For arbitrary object $(X, \mathbf{C})$, the unit morphism $1_{(X,\mathbf{C})}^* : (X, \mathbf{C}) \rightsquigarrow (X, \mathbf{C})$ is defined by*

   $$1_{(X,\mathbf{C})}^* = \bigsqcup_{r\in R}^{*} r \star^* C_r.$$

The main reason for introducing cut systems categories is to take advantage of the approximate representation of $\mathcal{R}$- and $\mathcal{R}^*$-sets, including methods for working with this approximate representation. Generally speaking, rather than working with the $\mathcal{R}$-set $(X, Q)$ that is not very intuitive, we can use its approximation $(X, C_r)$, where $r \in R$ is a level of approximation, and $C_r$ is a classical crisp relation in a set $X$. The basis for defining this relation is the following theorem, which proves the existence of isomorphisms between the categories of $\mathcal{R}$-sets and $\mathcal{R}$-cut systems.

**Theorem 2.** *Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings. There exist isomorphic functors $H, H^*$ and $\Phi$ such that the following diagram commutes, where $\Omega$ is from Theorem 1:*

$$
\begin{array}{ccc}
\mathbf{Set}(\mathcal{R}) & \xrightarrow{H} & \mathbf{Cut}(\mathcal{R}) \\
\Omega \downarrow & & \downarrow \Phi \\
\mathbf{Set}(\mathcal{R}^*) & \xrightarrow{H^*} & \mathbf{Cut}(\mathcal{R}^*).
\end{array}
$$

**Proof.** For arbitrary $r \in R$ we define the mappings $\Delta_r, \Delta_r^* : R^X \to R^X$ by

$$
s \in R^X, x \in X, \quad \Delta_r(s)(x) = \begin{cases} 1_R, & s(x) \geq r, \\ 0_R, & s(x) \not\geq r. \end{cases} \quad \Delta_r^*(s) = \neg \Delta_{\neg r}(s).
$$

(1) First, we show that the definition of the category $\mathbf{Cut}(\mathcal{R})$ is correct. We show that $1_{(X,\mathbf{C})}$ is a morphism. For $x, x' \in X$, and $r \in R$, we have

$$
1_{(X,\mathbf{C})} \Diamond r \star C_r(x)(x') = \sum_t r \star C_r(x)(t) \times \left( \sum_{s \in R} s \star C_s \right)(t)(x') =
$$

$$
\sum_{t \in X, C_r(x)(t) = 1_R} r \times \sum_{s \in R, C_s(t)(x') = 1_R} s = \sum_{\{t \in X, s \in R : C_r(x)(t) = C_s(t)(x') = 1_R\}} r \times s \leq
$$

$$
\sum_{\{s \in R : C_s \Diamond C_r(x)(x') = 1_R\}} r \times s \leq \sum_{\{s \in R : C_{s \times r}(x)(x') = 1_R\}} r \times s \leq
$$

$$
\sum_{\{p \in R : C_r(x)(x') = 1_R\}} p = 1_{(X,\mathbf{C})},
$$

as follows from the inequality $C_{s \times r} \geq C_s \Diamond C_r$. Therefore, $1_{(X,\mathbf{C})}$ is the morphism in $\mathbf{Cut}(\mathcal{R})$. For an arbitrary morphism $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ we have $f \Diamond 1_{(X,\mathbf{C})} = f$ and $1_{(Y,\mathbf{D})} \Diamond f = f$. In fact, for $x \in X, y \in Y$ we have

$$
f \Diamond 1_{(X,\mathbf{C})}(x)(y) = \sum_{t \in X} 1_{(X,\mathbf{C})}(x)(t) \times f(t)(y) = \sum_{t \in X} \sum_{\{s \in R, C_s(x)(t) = 1_R\}} s \times f(t)(y) =
$$

$$
\sum_{s \in R} \sum_{\{t \in X : C_s(x)(t) = 1_R\}} s \times f(t)(y) = \sum_{s \in R} (f \Diamond s \star C_s)(x)(y) = f.
$$

The other identity can be proven analogously and $\mathbf{Cut}(\mathcal{R})$ is defined correctly.

Let $(X, Q) \rightsquigarrow (Y, S)$ be a morphism in $\mathbf{Set}(\mathcal{R})$. The functor $H$ is defined by

$$
H(X, Q) = (X, \mathbf{C}), \quad \mathbf{C} = (C_r)_r, \quad C_r = \Delta_r.Q : X \rightsquigarrow X,
$$
$$
H(f) : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D}), \quad H(f) = f.
$$

We prove that this definition is correct. It is clear that $C_r$ is a crisp $\mathcal{R}$-relation. Furthermore, using the equality $Q \Diamond Q = Q$, by a simple calculation we obtain

$$
r, s \in R, \quad C_{r \times s} = \Delta_{s \times r}.Q \geq (\Delta_s.Q) \Diamond (\Delta_r.Q) = C_r \Diamond C_s.
$$

Analogously, we have $C_r^{-1} = (\Delta_r.Q)^{-1} = \Delta_r.Q^{-1} = \Delta_r.Q = C_r$. For $r \leq s$, we have $C_s = \Delta_s.Q \leq \Delta_r.Q = C_r$. For $x \in X$, $C_r(x)(x) = \Delta_r(Q(x))(x) = 1_R$ and it follows that $C_r \geq \eta_X$. Finally, the greatest element of the set $\{r \in R : C_r(x)(y) = 1_R\} = \{r \in R : \Delta_r.Q(x)(y) \geq r\}$ equals $Q(x)(y)$. Hence, $(X, \mathbf{C})$ is the object in the category $\mathbf{Cut}(\mathcal{R})$.

We prove that $H(f) = f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ is a morphism in the category $\mathbf{Cut}(\mathcal{R})$. In fact, we have

$$
f \Diamond (r \star C_r) = f \Diamond (r \star (\Delta_r.Q)) \leq f \Diamond Q \leq f.
$$

Therefore, $H$ is defined correctly. To prove that $H$ is a functor, we only need to show that $H(1_{(X,Q)}) = 1_{H(X,Q)} = 1_{(X,\mathbf{C})}$. We have $H(1_{(X,Q)}) = H(Q) = Q$ and for $x, x' \in X$ we have

$$1_{(X,\mathbf{C})}(x)(x') = \sum_{s \in R, C_s(x)(x')=1_R} s = \sum_{s \in R, Q(x)(x') \geq s} s = Q(x)(x') = 1_{H(X,Q)}(x)(x')$$

(2) We define the inverse functor $H^{-1} : \mathbf{Cut}(\mathcal{R}) \to \mathbf{Set}(\mathcal{R})$. Let $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ be a morphism in $\mathbf{Cut}(\mathcal{R})$. The functor $H^{-1}$ is defined by

$$H^{-1}(X, \mathbf{C}) = (X, Q), \quad Q = \bigsqcup_{r \in R} r \star C_r,$$

$$H^{-1}(f) : (X, Q) = H^{-1}(X, \mathbf{C}) \rightsquigarrow H^{-1}(Y, \mathbf{D}) = (Y, S), \quad H^{-1}(f) = f.$$

We show that this definition is correct. We have

$$Q = \bigsqcup_{r \in R} r \star C_r \geq \bigsqcup_{r \in R} r \star \eta_X \geq 1_R \star \eta_X = \eta_X.$$

Further, we have

$$Q \Diamond Q = (\bigsqcup_{r \in R} r \star C_r) \Diamond (\bigsqcup_{s \in R} s \star C_s) = \bigsqcup_{r,s \in R} (r \star C_r) \Diamond (s \star C_s) =$$

$$\bigsqcup_{r,s \in R} (r \times s) \star (C_r \Diamond C_s) \leq \bigsqcup_{r,s \in R} (r \times s) \star C_{r \times s} \leq$$

$$\bigsqcup_{q \in R} q \star C_q = Q.$$

Since $C_r^{-1} = C_r$, $Q$ is the symmetric $(\mathcal{R}, \mathcal{R}^*)$-relation and it follows that $Q$ is the $\mathcal{R}$-similarity relation in $X$.

We show that $f : (X, Q) \rightsquigarrow (Y, S)$ is the morphism in $\mathbf{Set}(\mathcal{R})$. We have

$$Q \Diamond f = (\bigsqcup_{r \in R} r \star C_r) \Diamond f = \bigsqcup_{r \in R} (r \star C_r \Diamond f) \leq \bigsqcup_{r \in R} f = f.$$

Finally, we have

$$H^{-1}(1_{(X,\mathbf{C})}) = \bigsqcup_{r \in R} r \star C_r = Q = 1_{H^{-1}(X,\mathbf{C})},$$

and $H^{-1}$ is the functor. We show that $H, H^{-1}$ are inverse functors. In fact, for $(X, \mathbf{C}) \in \mathbf{Cut}(\mathcal{R})$ and $(X, Q) \in \mathbf{Set}(\mathcal{R})$, we obtain the following.

$$H.H^{-1}(X, \mathbf{C}) = H(H^{-1}(X, \mathbf{C})) = H(X, \bigsqcup_{r \in R} r \star C_r) = (X, (\Delta_s(\bigsqcup_{r \in R} r \star C_r))_s) =$$

$$(X, (C_s)_s) = (X, \mathbf{C}),$$

$$H^{-1}.H(X, Q) = H^{-1}(X, (\Delta_r.Q)_r) = (X, \bigsqcup_{s \in R} s \star (\Delta_s.Q)) = (X, Q),$$

as follows from a simple calculation and using Remark 5.

(3) We show that the definition of the category $\mathbf{Cut}(\mathcal{R}^*)$ is correct. Most of the proof can be carried out similarly to the previous case. For illustration, we prove only the following identity:

We show that for an arbitrary object $(X, \mathbf{C}) \in \mathbf{Cut}(\mathcal{R}^*)$, $1^*_{(X,\mathbf{C})}$ is a morphism. For $x, x' \in X$ and $r \in R$ we have

$$1^*_{(X,\mathbf{C})} \lozenge^* r \times^* C_r(x)(x') = \sum_{t \in X}^* r \times^* C_r(x)(t) \times^* \sum_{s \in R, C_s(t)(x)=1^*_R}^* s =$$

$$\sum_{t \in X, C_r(x)(t)=1^*_R}^* r \times^* \sum_{s \in R, C_s(t)(x')=1^*_R}^* s = \sum_{\{t \in X, s \in R: C_r(x)(t)=C_s(t)(x')=1^*_R\}}^* r \times^* s \leq^*$$

$$\sum_{\{s \in R: C_s \lozenge^* C_r(x)(x')=1^*_R\}}^* r \times^* s \leq^* \sum_{\{s \in R: C^*_{s \times^* r}(x)(x')=1^*_R\}}^* r \times^* s \leq^*$$

$$\sum_{p \in R: C_r(x)(x')=1^*_R}^* p = 1^*_{(X,\mathbf{C})'}$$

as follows from the inequality $C_{s \times^* r} \geq^* C_s \lozenge^* C_r$. Therefore, $1^*_{(X,\mathbf{C})}$ is the morphism in $\mathbf{Cut}(\mathcal{R}^*)$. The proof that $1^*_{(X,\mathbf{C})}$ is the unit morphism in $\mathbf{Cut}(\mathcal{R}^*)$ is analogous to the proof that $1_{(X,\mathbf{C})}$ is the unit morphism in $\mathbf{Set}(\mathcal{R})$ and will be omitted.

Let $(X, Q) \rightsquigarrow (Y, S)$ be a morphism in $\mathbf{Set}(\mathcal{R}^*)$. The functor $H^*$ is defined by

$$H^*(X, Q) = (X, \mathbf{C}), \quad \mathbf{C} = (C_r)_r, \quad C_r = \Delta^*_r . Q : X \rightsquigarrow X,$$
$$H^*(f) : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D}), \quad H^*(f) = f.$$

We show that $H^*$ is the functor. We have $H^*(X, Q) \in \mathbf{Cut}(\mathcal{R}^*)$. In fact,

$$C_{r \times^* s} = \Delta^*_{r \times^* s} . Q = \neg \Delta_{\neg(r \times^* s)} . Q = \neg \Delta_{\neg r \times \neg s} . Q \geq^* \neg (\Delta_{\neg r} . Q \lozenge \Delta_{\neg s} . Q) =$$
$$\Delta^*_r . Q \lozenge^* \Delta^*_s . Q = C_r \lozenge^* C_s,$$
$$\Delta^*_r . Q = \neg \Delta_{\neg r} . Q \geq^* \neg \eta_X = \eta^*_X,$$
$$r \leq^* s \Rightarrow \neg r \leq \neg s \Rightarrow \Delta_{\neg r} . Q \geq \Delta_{\neg s} . Q \Rightarrow C_r \geq^* C_s.$$

Therefore, $(X, \mathbf{C})$ is the $\mathcal{R}^*$-cut system. Furthermore, $f : H^*(X, Q) \rightsquigarrow H^*(Y, S)$ is a morphism in $\mathbf{Cut}(\mathcal{R}^*)$, as can be verified by a simple calculation. Finally, we have $1^*_{H^*(X,Q)} = H^*(1^*_{(X,Q)})$, In fact, we have

$$1^*_{H^*(X,Q)} = 1^*_{(X,(\Delta^*_r . Q)_r)} = \bigsqcup_r^* r \star^* (\Delta^*_r . Q) = Q = H^*(1^*_{(X,Q)}).$$

(4) We define the inverse functor $H^{*-1} : \mathbf{Cut}(\mathcal{R}^*) \to \mathbf{Set}(\mathcal{R}^*)$. For a morphism $(X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ we set

$$H^{*-1}(X, \mathbf{C}) = (X, Q), \quad Q = \bigsqcup_{r \in R}^* r \star^* C_r,$$
$$H^{*-1}(f) : (X, Q) = H^{*-1}(X, \mathbf{C}) \rightsquigarrow H^{*-1}(Y, \mathbf{D}) = (Y, S), \quad H^{*-1}(f) = f.$$

The proofs that $H^{*-1}$ is the functor and $H^*, H^{*-1}$ are inverse functors are similar to these proofs for the functors $H^{-1}$ and $H$ of (2) and will be omitted.

(5) We define the isomorphic functor $\Phi$. Let $f : (X, \mathbf{C}) \rightsquigarrow (Y, \mathbf{D})$ be a morphism in the category $\mathbf{Set}(\mathcal{R})$ and $g : (X', \mathbf{C}') \rightsquigarrow (Y', \mathbf{D}')$ in the category $\mathbf{Cut}(\mathcal{R}^*)$. We set

$$\Phi(X, \mathbf{C}) = (X, (\neg C_{\neg r})_r), \quad \Phi^{-1}(X', \mathbf{C}') = (X', (\neg C'_{\neg r})_r)$$

$$\Phi(f) = \begin{cases} \bigsqcup_{r \in R}^* r \star^* \neg C_{\neg r}, & f = 1_{(X,\mathbf{C})}, \\ \neg f, & f \neq 1_{(X,\mathbf{C})}, \end{cases}$$

$$\Phi^{-1}(g) = \begin{cases} \bigsqcup_{r \in R} r \star \neg C'_{\neg r}, & g = 1^*_{(X,\mathbf{C}')}, \\ \neg g, & g \neq 1^*_{(X,\mathbf{C}')}, \end{cases}$$

We prove that $\Phi$ and $\Phi^{-1}$ are defined correctly, that is, we show that $\Phi(X, \mathbf{C}) \in \mathbf{Cut}(\mathcal{R}^*)$ and $\Phi^{-1}(X, \mathbf{C}') \in \mathbf{Cut}(\mathcal{R})$. We set $D_r := \neg C_{\neg r}$ and we have

$$r \leq^* s \Rightarrow \neg r \leq \neg s \Rightarrow C_{\neg r} \geq C_{\neg s} \Rightarrow D_r \geq^* D_s,$$
$$D_r \lozenge^* D_s = (\neg C_{\neg r}) \lozenge^* (\neg C_{\neg s}) = \neg(C_{\neg r} \lozenge C_{\neg s}) \leq^* \neg C_{\neg r \times \neg s} = \neg C_{\neg(r \times^* s)} = D_{r \times^* s},$$
$$C_{\neg r} \geq \eta_X \Rightarrow D_r \geq^* \neg \eta_X = \eta_X^*,$$

and, finally, if $\sum_{r \in R : D_r(x)(x') = 1_R^*}^* r \geq^* s$, we have

$$\neg \left( \sum_{\{r \in R : \neg C_{\neg r}(x)(x') = 1_R^*\}}^* r \right) \geq \neg s \Rightarrow$$
$$\sum_{\{r \in R : \neg C_{\neg r}(x)(x') = 1_R^*\}} r \geq \neg s \Rightarrow$$
$$\sum_{\{r \in R : C_{\neg r}(x)(x') = 1_R\}} r \geq \neg s \Rightarrow$$
$$C_{\neg s}(x)(x') = 1_R \Rightarrow D_r(x)(x') = 1_R^*.$$

Therefore, from Remark 5 it follows that $\Phi(X, \mathbf{C}) \in \mathbf{Cut}(\mathcal{R}^*)$. The proof for $\Phi^{-1}$ is similar and will be omitted.

We have $\Phi(1_{(X,\mathbf{C})}) = 1^*_{\Phi(X,\mathbf{C})}$ and $\Phi^{-1}(1^*_{(X',\mathbf{C}')}) = 1_{\Phi^{-1}(X',\mathbf{C}')}$. The functors $\Phi$ and $\Phi^{-1}$ are mutually inverse. For $(X, \mathbf{C}) \in \mathbf{Cut}(\mathcal{R}^*)$ we have

$$\Phi^{-1}.\Phi(X, \mathbf{C}) = \Phi^{-1}(X, (\neg C_{\neg r})_r) = \Phi^{-1}(X, (D_r)_r) = (X, (\neg D_{\neg r})_r) =$$
$$(X, (\neg(\neg C_r))_r) = (X, (C_r)_r) = (X, \mathbf{C}),$$
$$\Phi^{-1}.\Phi(f) = f, \quad \Phi.\Phi^{-1}(g) = g.$$

(6) We prove that the diagram commutes, that is, $\Phi.H = H^*.\Omega$. In fact, let $(X, Q)$ be an object of $\mathbf{Set}(\mathcal{R})$. We have

$$H^*.\Omega(X, Q) = H^*(X, \neg Q) = (X, (\Delta_r^*.(\neg Q))_r) = (X, (\neg \Delta_{\neg r}.(\neg Q))_r) =$$
$$\Phi.H(X, Q),$$
$$H^*.\Omega(f) = H^*(\neg f) = \neg f = \Phi(f) = \Phi.H(f),$$

and the diagram commutes.  $\square$

Just as we define the power set structure $F(X, Q)$ of all $(X, Q)$-extensional fuzzy sets, we can define the power set structure $E(X, \mathbf{D})$ of $(X, \mathbf{D})$-extensional cuts. To define the functor $E : \mathbf{Cut}(\mathcal{R}) \to \mathbf{Set}$, we start with the definition of the object function of $E$.

Recall that for a $(\mathcal{R}, \mathcal{R}^*)$-relation $T : X \rightsquigarrow X$, the upper approximation mapping $T^\uparrow : R^X \to R^X$ is defined by $T^\uparrow = T \lozenge 1_{R^X}$. For an arbitrary subset $A \subseteq X$, we define the characteristic $(\mathcal{R}, \mathcal{R}^*)$-fuzzy set $\Gamma(A) \in R^X$ of $A$ by $\Gamma(A) = \bigsqcup_{a \in A} \eta_X(a)$.

**Definition 10.** *Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings.*

1. *Let $(X, \mathbf{D} = (D_r)_r) \in \mathbf{Cut}(\mathcal{R})$. A system $\mathbf{C} = (C_r)_{r \in R}$ is called an extensional $(X, \mathbf{D})$-cut, if*

   *(a)* $C_r \subseteq X$;

   *(b)* $(\forall x \in X) \sum_{r \in R, x \in C_r} r \geq q \Rightarrow x \in C_q$;

   *(c)* $(\forall r, q \in R)(q \star D_q)^{\uparrow}(r \star \Gamma(C_r)) \leq (r \times q) \star \Gamma(C_{r \times q})$.

2. *The object function $E : \mathbf{Cut}(\mathcal{R}) \to \mathbf{Set}$ is defined by*

$$E(X, \mathbf{D}) = \{\mathbf{C} : \mathbf{C} \text{ is the extensional } (X, \mathbf{D}) - cut\}.$$

We first show that there is a strong relationship between $(X, \mathbf{D})$-extensional cuts and $(X, Q)$-extensional fuzzy sets.

**Proposition 3.** *Let $(X, Q) \in \mathbf{Set}(\mathcal{R})$ be an $\mathcal{R}$-set. There exist mutually inverse bijections*

$$E.H(X, Q) \xrightarrow{\sigma_{(X,Q)}} F(X, Q) \xrightarrow{\tau_{(X,Q)}} E.H(X, Q).$$

**Proof.** According to the proof of Theorem 2, we have the following: $H(X, Q) = (X, \mathbf{D})$, where $\mathbf{D} = (D_q)_q = (\Delta_q.Q)_q$. Let $\mathbf{C} = (C_r)_r \in E.H(X, Q) = E(X, \mathbf{D})$. The mapping $\sigma_{(X,Q)}$ is defined by

$$\sigma_{(X,Q)}(\mathbf{C}) = \bigsqcup_{r \in R} r \star \Gamma(C_r).$$

We show that this definition is correct, that is, $\sigma_{(X,Q)}(\mathbf{C})$ is an $(X, Q)$-extensional fuzzy set. Using the identity $(X, Q) = H^{-1}.H(X, Q) = (X, \bigsqcup_q q \star D_q)$ from Theorem 2, we have

$$Q^{\uparrow}(\sigma_{(X,Q)}(\mathbf{C})) = Q^{\uparrow}(\bigsqcup_r r \star \Gamma(C_r)) = \bigsqcup_r Q^{\uparrow}(r \star \Gamma(C_r)) =$$

$$\bigsqcup_r (\bigsqcup_q q \star D_q)^{\uparrow}(r \star \Gamma(C_r)) \leq \bigsqcup_{r,q} (r \times q) \star \Gamma(C_{r \times q}) \leq$$

$$\bigsqcup_s s \star \Gamma(c_s) = \sigma_{(X,Q)}(\mathbf{C}).$$

According to the definition of functor $F$, we have $\sigma_{(X,Q)}(\mathbf{C}) \in F(X, Q)$, and the definition of $\sigma$ is correct. In contrast, let $s \in F(X, Q)$. The mapping $\tau_{(X,Q)}$ is defined by

$$\tau_{(X,Q)}(s) = (s_r)_{r \in R}, \quad s_r = \{x \in X : s(x) \geq r\}.$$

We show that $\tau_{(X,Q)}(s)$ is an extensional $(X, \mathcal{D})$-cut. In fact, for arbitrary $x \in X$, we have

$$(q \star D_q)^{\uparrow}(r \star \Gamma(s_r))(x) = (q \star \Delta_q.Q)^{\uparrow}(r \star \Gamma(s_r))(x) =$$

$$q \times \sum_{t \in X} r \times \Gamma(s_r)(t) \times \Delta_q.Q(t)(x) = q \times \sum_{t \in X, s(t) \geq t, Q(t)(x) \geq q} r =$$

$$\begin{cases} q \times r = (r \times q) \star \Gamma(s_{r \times q})(x), & \exists t \in X, s(t) \geq r, Q(t)(x) \geq q, \\ 0, & \text{otherwise} \end{cases} \leq$$

$$(r \times q) \star \Gamma(s_{r \times q}),$$

as follows from the inequality $s(x) \geq s(t) \times Q(t)(x) \geq r \times q$ for arbitrary $t \in X, s(t) \geq r, Q(t)(x) \geq q$. Therefore, $\tau_{(X,Q)}(s) \in E.H(X, Q)$.

We show that $\sigma$ and $\tau$ are mutually inverse mappings. For $s \in F(X, Q)$ and $\mathbf{C} \in E.H(X, Q)$, we have

$$\sigma_{(X,Q)}.\tau_{(X,Q)}(s) = \sigma_{(X,Q)}((s_r)_r) = \bigsqcup_r r \times \Gamma(s_r) = s,$$

$$\tau_{(X,Q)}.\sigma_{(X,Q)}(\mathbf{C}) = \tau_{(X,Q)}\left(\bigsqcup_r r \star \Gamma(C_r)\right) = (v_r)_r,$$

$$v_r = \{x \in X : \bigsqcup_p p \star \Gamma(C_p) = \sum_{p,x \in C_p \geq r} p \geq r\} \subseteq C_r,$$

$$x \in C_p \Rightarrow r \in \{p : x \in C_p\} \Rightarrow x \in v_r,$$

as follows from the axioms of extensional $(X, \mathbf{D})$-cut.  □

The object function $E$ can be extended to the powerset functor

$$E : \mathbf{Cut}(\mathcal{R}) \to \mathbf{Set}.$$

Let $f : (X, \mathbf{D}) \rightsquigarrow (Y, \mathbf{G})$ be a morphism in $\mathbf{Cut}(\mathcal{R})$, and let us consider the following diagram:

$$
\begin{array}{ccc}
E(X, \mathbf{D}) & \xdashrightarrow{\quad E(f) \quad} & E(Y, \mathbf{G}) \\
{\scriptstyle =}\Big\downarrow & & \Big\uparrow{\scriptstyle =} \\
E.H(H^{-1}(X, \mathbf{D})) & & E.H(H^{-1}(Y, \mathbf{G})) \\
{\scriptstyle \sigma_{H^{-1}(X,\mathbf{D})}}\Big\downarrow & & \Big\uparrow{\scriptstyle \tau_{H^{-1}(Y,\mathbf{G})}} \\
F(H^{-1}(X, \mathbf{D})) & \xrightarrow{\ F.H^{-1}(f)\ } & F(H^{-1}(Y, \mathbf{G}))
\end{array}
$$

To make this diagram commutative, we can set

$$E(f) = \tau_{H^{-1}(Y,\mathbf{G})}.F.H^{-1}(f).\sigma_{H^{-1}(X,\mathbf{D})}.$$

From the construction of $E(f)$, it follows that $E$ respects the composition of morphisms and unit morphisms. Therefore, we obtain the theorem below, which expresses the equivalence between $\mathcal{R}$-extensional fuzzy sets and extensional cuts.

**Theorem 3.** *Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings. The functors $E.H$ and $F$ are naturally isomorphic, that is, $\sigma = \{\sigma_{(X,Q)} : (X, Q) \in \mathbf{Set}(\mathcal{R})\}$ and $\tau = \{\tau_{(X,Q):(X,Q \in \mathbf{Set}(\mathcal{R}))}\}$ are natural inverse isomorphisms between these funtors,*

$$E.H \underset{\tau}{\overset{\sigma}{\rightleftarrows}} F.$$

**Proof.** Let $f : (X, Q) \rightsquigarrow (Y, S)$ be a morphism in $\mathbf{Set}(\mathcal{R})$. We show that $\sigma$ is the natural transformation, that is, the following diagram commutes:

$$
\begin{array}{ccc}
E.H(X, Q) & \xrightarrow{\ \sigma_{(X,Q)}\ } & F(X, Q) \\
{\scriptstyle E(f)=E.H(f)}\Big\downarrow & & \Big\downarrow{\scriptstyle F(f)} \\
E.H(Y, S) & \xrightarrow{\ \sigma_{(Y,S)}\ } & F(Y, S).
\end{array}
$$

If we set $H(X, Q) = (X, \mathbf{D}), H(Y, S) = (Y, \mathbf{G})$, we obtain the following.

$$\sigma_{(Y,S)}.E(f) = \sigma_{H^{-1}(Y,\mathbf{G})}.\tau_{H^{-1}(Y,\mathbf{G})}.F.H^{-1}(f).\sigma_{H^{-1}(X,\mathbf{D})} =$$

$$F(f).\sigma_{H^{-1}(X,\mathbf{D})} = F(f).\sigma_{(X,Q)},$$

and the diagram commutes. Therefore, $\sigma$ is the natural transformation and also the natural isomorphism. By analogy, it can be proved that $\tau$ is the natural isomorphism. $\square$

## 5. Examples

In this section, we show illustrative examples of how $\mathcal{R}$-cut systems can be used to approximate a standard construction used in the theory of fuzzy sets. As we have already stated in the previous sections, the advantage of the theory of $\mathcal{R}$-cut systems is the possibility to use this theory for arbitrary fuzzy systems that can be transformed into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. Therefore, as an illustration of this procedure and its application in various fuzzy systems, in the following examples, we focus on classical $MV$-valued fuzzy sets, $MV$-valued intuitionistic fuzzy sets, and $MV$-valued fuzzy soft sets.

As an example of the method on which we will illustrate the possibility of using $\mathcal{R}$-cut systems, we chose the upper approximation defined by the $\mathcal{R}$-similarity relation. Recall that if $Q : X \rightsquigarrow X$ is an $\mathcal{R}$-similarity relation on the set $X$, the upper approximation defined by $Q$ is the mapping $Q^{\uparrow} : R^X \to R^X$ which is defined by the formula

$$Q^{\uparrow} = Q \Diamond 1_{R^X} : R^X \to R^X.$$

It is obvious that if we fully know the $\mathcal{R}$-similarity relation $Q$, there is no need to perform any approximation of the output $Q^{\uparrow}(s)$ of the fuzzy structure $s$. However, a problem can arise if we only know approximately the $\mathcal{R}$-similarity relation. For example, instead of knowing the complete $\mathcal{R}$-similarity relation $Q : X \to R^X$, we sometimes only know its approximations represented by several classical equivalence relations $E_r \subseteq X \times X$ for some values $r_1, \ldots, r_n \in R$. This classical equivalence relation $E_r$ then represents those pairs of points $(x, x') \in X \times X$, for which we "estimate" $Q(x)(x') \geq r$. Therefore, it is natural to ask whether from these various local equivalence relations $E_{r_1}, \ldots, E_{r_n}$ at least some approximation of the global upper approximation $Q^{\uparrow}(s)$ can be obtained for any $s \in R^X$. In this section, we will first show how this global approximation can be obtained for the case of general $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, and then we will illustrate the concrete procedure for the case of three fuzzy structures that can be transformed into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, i.e., classical fuzzy sets, intuitionistic fuzzy sets, and fuzzy soft sets.

**Example 1.** *General approach: Approximation of $Q^{\uparrow}(s)$ for $(\mathcal{R}, \mathcal{R}^*)$-sets.*

Let $(\mathcal{R}, \mathcal{R}^*)$ be a dual pair of semirings and let $X$ be a set. Suppose that instead of an $\mathcal{R}$-similarity relation $Q : X \rightsquigarrow X$ we have at our disposal only a finite number of classical equivalence relations $E_{r_i} \subseteq X \times X$, where $r_i \in R$, $i = 1, \ldots, N$, where $(x, x') \in E_{r_i}$ represents our local opinions that the points $x$ and $x'$ are at least in a degree $r_i \in R$ similar to $Q$ (which we *do not know* as a global $\mathcal{R}$-similarity relation $X \rightsquigarrow X$).

Instead of the equivalence relations $E_{r_i}$ we can consider the $\mathcal{R}$-similarity relations $C_{r_i} : X \rightsquigarrow X$, defined by $C_{r_i}(x)(x') = 1_R$ iff $(x, x') \in E_{r_i}$. Let us further assume that our local equivalence relations $E_{r_i}$ are consistent, i.e., that it holds.

$$r_i \leq r_j \Rightarrow E_{r_j} \subseteq E_{r_i}.$$

In that case, we obtain an approximation $\mathbf{C}' = (C_{r_i})_i$ of an $\mathcal{R}$-cut in $X$, and using the functor $H^{-1}$ of Theorem 2, we obtain the "approximation" $(X, Q') = H^{-1}(\mathbf{C}')$ of the unknown $\mathcal{R}$-set $(X, Q)$, where $H(X, Q) = (X, \mathbf{C}) = (X, (C_r)_r)$. Hence,

$$Q = \bigsqcup_{r \in R} r \star C_r \geq \bigsqcup_{i=1}^{N} r_i \star C_{r_i} = Q'.$$

In general, $Q'$ does not need to be an $\mathcal{R}$-similarity relation, although $Q' \geq \eta_X$ and $Q'$ are symmetric. To obtain an "almost" $\mathcal{R}$-similarity relation, we can calculate $Q' \Diamond \ldots \Diamond Q'$ for several copies of $Q'$ and this $\mathcal{R}$-relation can be considered a reasonable approximation of

$Q$. Using this approximation $Q'$, for arbitrary $s \in R^X$, we can calculate at least the lower estimate of the approximation $Q^\uparrow(s)$, that is,

$$Q'^\uparrow(s) = (\bigsqcup_{i=1}^{n} r_i \star C_{r_i})^\uparrow(s) \leq Q^\uparrow(s).$$

Analogously, as the $\mathcal{R}$-similarity relation $Q$, sometimes we are not able to fully describe $(\mathcal{R}, \mathcal{R}^*)$-fuzzy set $s \in R^X$ as the mapping $s : X \to R$. Instead of that, we know (or estimate) only a few cuts $s_{q_k} = \{x \in X : s(x) \geq q_k\} \subseteq X, k = 1, \dots, K$. Using the inequality

$$s = \bigsqcup_{q \in R} q \star \Gamma(s_q) \geq \bigsqcup_{k=1}^{K} q_k \star \Gamma(s_{q_k}),$$

for $x \in X$, the lower estimate of $Q^\uparrow(s)$ can be calculated by

$$Q^\uparrow(s)(x) \geq Q^\uparrow(\bigsqcup_{k=1}^{K} q_k \star \Gamma(s_{q_k}))(x) = \bigsqcup_{k=1}^{K} q_k \star Q^\uparrow(\Gamma(s_{q_k}))(x) \geq$$

$$\bigsqcup_{k=1}^{K} q_k \star Q'^\uparrow(\Gamma(s_{q_k}))(x) = \bigsqcup_{k=1}^{K} q_k \star \bigsqcup_{i=1}^{N} (r_i \star C_{r_i})^\uparrow(\Gamma(s_{q_k}))(x) =$$

$$\sum_{\{(k,i): \exists t \in s_{q_k}, (x,t) \in E_{r_i}\}} q_k \times r_i. \tag{2}$$

In the following examples, we show how the calculation of the lower estimate of $Q^\uparrow(s)$ looks for concrete examples of $\mathcal{L}$-fuzzy sets, $\mathcal{L}$-intuitionistic fuzzy sets, and $L$-fuzzy soft sets. In these examples, we assume that $\mathcal{L}$ is the complete Łukasiewicz algebra $\mathcal{L}_L = ([0,1], \vee, \wedge, \oplus, \otimes, \neg, 0)$, where

$$x \otimes y = 0 \vee (x + y - 1), \quad \neg x = 1 - x, \quad x \oplus y = 1 \wedge (x + y).$$

We use the notation of Example 1.

**Example 2.** *Lower approximation of $Q^\uparrow(s)$ for $\mathcal{L}$-fuzzy sets.*

It is easy to see that $\mathcal{L}$-fuzzy sets can be transformed to $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, where $(\mathcal{R}, \mathcal{R}^*)$ is a dual pair of semirings, where

$$\mathcal{R} = (L, \vee, \otimes, 0 = 0_L, 1 = 1_L), \quad \mathcal{R}^* = (L, \wedge, \oplus, 0^* = 1_L, 1^* = 0_L),$$

and $\neg : [0,1] \to [0,1]$ is the involutive negation $\mathcal{R} \to \mathcal{R}^*$. In that case, the algebraic structure of $\mathcal{L}$-fuzzy sets is isomorphic to the algebraic structure of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. Using the notation of Example 1 and the operations of $(\mathcal{R}, \mathcal{R}^*)$, formula (2) is expressed by

$$Q^\uparrow(s)(x) \geq \bigvee_{\{(k,i): \exists t \in s_{q_k}, (t,x) \in E_{r_i}\}} q_k \otimes r_i.$$

**Example 3.** *Lower approximation of $Q^\uparrow(s)$ for $\mathcal{L}$-intuitionistic fuzzy sets.*

In [6], we showed that $\mathcal{L}$-intuitionistic fuzzy sets can be transformed into $(\mathcal{R}, \mathcal{R}^*)$ fuzzy sets, where $R = \{(\alpha, \beta) \in [0,1]^2 : \neg \alpha \geq \beta\} \subseteq [0,1]^2$, and

1.  $(\alpha, \beta) + (\alpha_1, \beta_1) := (\alpha \vee \alpha_1, \beta \wedge \beta_1)$;

2.  $(\alpha, \beta) \times (\alpha_1, \beta_1) := (\alpha \otimes \alpha_1, \beta \oplus \beta_1)$;

3.  $0_R = (0, 1), 1_R = (1, 0)$;

4. $(\alpha, \beta) +^* (\alpha_1, \beta_1) := (\alpha \wedge \alpha_1, \beta \vee \beta_1)$;

5. $(\alpha, \beta) \times^* (\alpha_1, \beta_1) := (\alpha \oplus \alpha_1, \beta \otimes \beta_1)$;

6. $0_R^* = (1, 0)$, $1_R^* = (0, 1)$;

7. $\neg R \to R$ is defined by $\neg(\alpha, \beta) = (\beta, \alpha)$.

Then, $(\mathcal{R}, \mathcal{R}^*)$ is the dual pair of semirings and the algebraic structure of $\mathcal{L}$-intuitionistic fuzzy sets is isomorphic to the algebraic structure of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. Using the notation of (2), for elements $r_i, q_i \in R$ we can set

$$r_i = (a_i, b_i), \quad q_k = (c_k, d_k) \in R.$$

In that case, formula (2) can be transformed into the following formula for elements of $R$:

$$Q^{\uparrow}(s)(x) \geq \sum_{\{(k,i): \exists t \in s_{(c_k,d_k)}, (x,t) \in E_{(a_i,b_i)}\}} q_k \times r_i =$$

$$\sum_{\{(k,i): \exists t \in s_{(c_k,d_k)}, (x,t) \in E_{(a_i,b_i)}\}} (c_k, d_k) \times (a_i, b_i) =$$

$$\left( \bigvee_{\{(k,i): \exists t \in s_{(c_k,d_k)}, (x,t) \in E_{(a_i,b_i)}\}} (c_k \otimes a_i), \bigwedge_{\{(k,i): \exists t \in s_{(c_k,d_k)}, (x,t) \in E_{(a_i,b_i)}\}} (d_k \oplus b_i) \right) \in R.$$

**Example 4.** *Lower approximation of $Q^{\uparrow}(s)$ for $\mathcal{L}$-fuzzy soft sets.*

Recall [2] that an $\mathcal{L}$-fuzzy soft set in a set $X$ is a pair $(E, s)$, where $E \subseteq K$ and $s : K \to [0,1]^X$, such that for arbitrary $k \in K \setminus E$, $s(k)(x) = 0$ for all $x \in X$. Basic operations with $L$-fuzzy soft sets are defined by

$$(E, s) = (F, t) \Leftrightarrow \forall k \in K, x \in X, s(k)(x) = t(k)(x),$$
$$(E, s) \cup (F, t) = (E \cup F, s \vee t), \quad (E, s) \cap (F, t) = (E \cap F, s \wedge t),$$
$$\neg(E, s) = (K, \neg s), \quad (\neg s)(k)(x) = \neg(s(k)(x)).$$

The dual pair of semirings $(\mathcal{R}, \mathcal{R}^*)$ that transforms $\mathcal{L}$ fuzzy soft sets into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets is defined by

$$R = \{[E, s] | E \subseteq K, s \in [0,1]^K, s(k) = 0, k \in K \setminus E\} \subseteq [0,1]^K;$$

where $[E, s] \in L^K$ is defined by $[E, s](k) = \begin{cases} s(k), & k \in E, \\ 0_L, & k \notin E \end{cases}$ and $[E, s] = [F, t] \Leftrightarrow \forall k \in K, [E, s](k) = [F, t](k)$.

1. The semiring $\mathcal{R} = (R, +, \times, 0, 1)$ is defined by

   (a) $[E, t] + [F, s] := [E \cup F, t \vee s] \in R$, where $t \vee s$ is the supremum in $[0,1]^K$;

   (b) $[E, t] \times [F, s] = [E \cap F, t \otimes s] \in R$, where $t \otimes s \in [0,1]^K$ is defined by $t \otimes s(k) = t(k) \otimes s(k)$;

   (c) $0_R = [K, \underline{0}]$, $1_R = [K, \underline{1}]$, where $\underline{\alpha}(k) = \alpha$ for arbitrarily $k \in K$, $\alpha \in [0,1]$.

2. The semiring $\mathcal{R}^* = (R, +^*, \times^*, 0^*, 1^*)$ is defined by

   (a) $[E, t] +^* [F, s] := [E \cap F, t \wedge s]$, where $t \wedge s$ is the infimum in $[0,1]^K$;

   (b) $[E, t] \times^* [F, s] = [E \cup F, s \oplus t]$, where $\oplus$ in $[0,1]^K$ is defined component-wise;

   (c) $0_R^* = [K, \underline{1}]$, $1_R^* = [K, \underline{0}]$, where $\underline{\alpha}(k) = \alpha$ for arbitrarily $k \in K$, $\alpha \in [0,1]$.

3.     $\neg : R \to R$ is defined by $\neg[E, s] = [K, \neg s]$ where $\neg s$ is defined point-wise in $L^K$.

Then, $(\mathcal{R}, \mathcal{R}^*)$ is the dual pair of semirings, and the algebraic structure of $\mathcal{L}$-fuzzy soft sets is isomorphic to the algebraic structure of the $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. Using the notation of (2), for elements $r_i, q_i \in R$ we can set

$$r_i = [A_i, p_i], \quad q_k = [B_k, w_k] \in R,$$

where $A_i, B_k \subseteq K$ and $p_i, q_k \in [0, 1]^K$, such that $p_i(x) = 0, q_k(z) = 0$ for $x \in K \setminus A_i, z \in K \setminus B_k$. In that case, formula (2) can be transformed into the following formula for elements of $R$:

$$Q^{\uparrow}(s)(x) \geq \sum_{\{(k,i) : \exists t \in s_{q_k}, (x,t) \in E_{r_i}\}} q_k \times r_i =$$

$$\sum_{\{(k,i) : \exists t \in s_{[B_k, w_k]}, (x,t) \in E_{[A_i, p_i]}\}} [B_k, w_k] \times [A_i, p_i] =$$

$$\sum_{\{(k,i) : \exists t \in s_{[B_k, w_k]}, (x,t) \in E_{[A_i, p_i]}\}} [B_k \cap A_i, w_k \otimes p_i] =$$

$$\left[ \bigcup_{\{(k,i) : \exists t \in s_{[B_k, w_k]}, (x,t) \in E_{[A_i, p_i]}\}} B_k \cap A_i, \quad \bigvee_{\{(k,i) : \exists t \in s_{[B_k, w_k]}, (x,t) \in E_{[A_i, p_i]}\}} w_k \otimes p_i \right] \in R$$

## 6. Discussion

A known disadvantage of the method that uses $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets to unify new fuzzy structures is these new fuzzy structures need to be *MV*-valued. It is, therefore, appropriate to try to modify the concept of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy set in such a way as to enable the transformation of fuzzy structures based on complete residual lattices, for example. The second disadvantage is the fact that some of the new *MV*-valued fuzzy structures cannot be transformed into $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets because the algebraic systems of the fuzzy sets of both structures are not isomorphic. An example can be hesitant fuzzy sets, where certain variants of the definitions of basic operations with hesitant fuzzy sets lead to the non-distributive nature of these operations, which is not possible for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. The question, therefore, arises whether it would be appropriate to modify either some of these operations related to new fuzzy structures or, on the contrary, to define the operations with $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets in a different way.

## 7. Conclusions

With the gradual development of fuzzy set applications, many new fuzzy structures were also developed, based on the theory of fuzzy sets, whose primary goal was to be a suitable tool, especially for certain types of applications. Over time, however, own theories began to be built even for these new fuzzy structures, which created a number of parallel theories, only loosely connected by some common methods from the general theory of fuzzy sets. Therefore, parallel to this development of theoretical tools for new fuzzy structures, there is naturally also an effort to unify these new fuzzy structures and their theories.

One of these unifications is the theory of fuzzy sets defined by a dual pair of semirings, briefly $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. For this reason, it was appropriate to create new theoretical foundations of the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, and, for example, the theory of approximations for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, the theory of rough $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, or the *F*-transform theory for $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets has already been created.

In this paper, we continued this development of the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets and, as an analogy of the so-called *L*-sets, we defined the category $\mathcal{R}$- and $\mathcal{R}^*$-sets and a power set functor for these categories, formed by the so-called extensional $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. Due to the monadic structure used for the theory of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets, two

isomorphic variants of this concept could be automatically created, i.e., upper and lower extensional $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets. In parallel with these concepts, we defined cut systems of these structures, which can be used, among other things, as certain approximations of these structures. At the end of the paper, we presented several simple examples of how these notions of $(\mathcal{R}, \mathcal{R}^*)$-fuzzy sets can be transformed into new original fuzzy structures.

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