Progress of Theoretical Physics, Vol. 87, No. 1, January 1992

Cutoff Dependence of Self-Consistent Solutions in Unquenched QED_3

Kei-ichi KONDO and Hajime NAKATANI*

Department of Physics, Chiba University, Chiba 260
**Ishikawa Polytechnic College, Anamizu, Fugeshi-gun 927*

(Received August 19, 1991)

In the leading order of the $1/N$ expansion for the vacuum polarization, we obtain the chiral-symmetry-breaking solutions of the Schwinger-Dyson equation in three-dimensional QED with N four-component Dirac fermions. Special attention is paid on the critical number of the fermion flavor N_c and the scaling law for the dynamical fermion mass. In this framework we introduce the infrared cutoff as well as the ultraviolet one. We show the relation between the vertex ansatz and the scaling law which is consistent with our numerical results that the scaling law is affected by the infrared cutoff. Especially in the case of the Pennington and Webb's vertex ansatz, the mean-field type scaling law and cutoff-dependent finite N_c are obtained in the large infrared cutoff, while the exponential type scaling and infinite N_c are given in the sufficiently small infrared cutoff. The connection with the Monte Carlo simulation is discussed.

§ 1. Introduction

Quantum electrodynamics in three dimensions, QED_3 is an interesting model in the gauge field theory, which is plausible to be related to the recent study for the high T_c superconductor, the quark confinement and so on. From the viewpoint of the strong coupling gauge theory, it exhibits a non-trivial critical behavior, even though it is superrenormalizable. Furthermore the three-dimensional theory is easy to analyze the critical behavior beyond the ladder approximation in the Schwinger-Dyson(SD) equation.

So far, many people have discussed this model in the strong coupling gauge theories.[1)~4)] There are two claims about the question, whether or not there exists the finite critical point of the fermion number N_c in QED_3. Appelquist, Nash and Wijewardhana[2)] (ANW) pointed out that there exists the finite critical point, $N_c=32/\pi^2$, using Appelquist et al.'s assumption[1)] that the wave-function renormalization would be negligible in the large N limit. Matsuki, Miao and Visbwanathan[5)] also obtained the same result from the viewpoint of the effective potential.*) The existence of the critical point is also supported by the Monte Carlo(MC) calculations by Dagotto, Kogut and Kocić[10)] and its physical meaning is explained from the viewpoint of the similarity with QED_4. On the other hand, Pennington and Webb[3)] (PW) and Atkinson, Johnson and Pennington[4)] (AJP) claimed that if one takes into account the one loop correction to the wave-fuction renormalization and the corresponding vertex modifications, the critical point N_c goes away to infinity in the infinite ultraviolet(UV) cutoff limit. This means that only the symmetry-breaking-phase survives in QED_3 against ANW's result.

*) Recent study for the effective potential by Matsuki shows that the effective potential in the direction of fermion wave-function renormalization is always unstable for any flavor number N, which reconfirms that chiral symmetry is broken for any N.[6)]

In the approximation of the bare vertex and no fermion loop in the photon propagator, that is called the quenched planar approximation, it is proved that there is no wave-function renormalization in the Landau gauge.[9] In this case, we have already succeeded in reproducing the MC result[10] of quenched QED_3 based on the analysis of the SD equation.[8] Generally, however, in QED the wave-function renormalization is unavoidable if the vacuum polarization is included. In fact, the one loop correction to the photon propagator leads to the non-trivial wave-function renormalization even in the Landau gauge.

In contrast with the perturbative expansion in the coupling constant up to some finite orders, the SD equation can give non-perturbative results,[7] since it is a self-consistent equation. In the framework of the SD equation defined in this paper, the parameter $1/N$ plays the role of the coupling constant. Hence the SD equation for unquenched QED_3 can extract the *large* $1/N$ *effect* without restricting to the small $1/N$ region. Therefore we incorporate the $1/N$ correction to the wave-function renormalization self-consistently through the coupled SD equation.

In this paper, we study the SD equation with the one loop correction in the photon propagator and the wave-function renormalization for a fermion. Due to the Ward-Takahashi identity, we also consider the vertex correction. However, in the present stage, we do not know how the vertex correction should be. So here, we simply take the ansatz and see how its choice effects the critical behavior of the theory. As a result, we can point out that the vertex ansatz is closely related to the critical behavior of the theory.

In the next section, we give the SD equation with the one loop correction in the photon propagator in the arbitrary covariant gauge. However, the SD equation is the integral equation which is not easy to solve. Then we transform it to the differential equations with the boundary conditions under some approximations.

The main result is contained in §§ 3 and 4, where the critical behavior of QED_3 is studied in the Landau gauge. First we give the analytic proof that, in the case of the bare or PW's vertex, there is no finite critical point. Next we give some explanation for the relation between the form of the vertex and the critical point. Then we confirm these analytic results by numerical calculations. In them, we pay special attention to the effect of the cutoff on the critical value of the fermion flavor and the scaling law. Actually we obtained the numerical results that the scaling law becomes the different type depending on the value of the cutoff. For example, using PW's vertex ansatz, we have the scaling law of the exponential type for sufficiently small infrared(IR) cutoff, while we have the mean-field type as the IR cutoff becomes larger. These results are very interesting because the former type is the one which was obtained by PW and AJP and the latter seems to agree with the one obtained by the MC calculations.*) It may give the possibility that the relation of the conflicting results in QED_3 is explained by the analysis of the SD equation.

The purpose of this paper is to summarize the present achievement of relevant investigations and clarify the root of the confused situation, and then to state the

*) According to the MC results, the mean-field type scaling was obtained in the region $N>N_c$, with the dimensionless parameter $\beta=1/e^2a$ where a is a lattice spacing. We also obtained the mean-field type scaling by the calculations of the SD equation with the corresponding parameter $\beta=\Lambda/e^2$.

points which should be taken into account in the future investigation. In the final section, we summarize the discussion at the present and give some comments for further study.

§ 2. SD equation

Here we give the SD equation for the massless fermion self-energy with the one loop vacuum polarization. In order to discuss the chiral symmetry breaking in QED_3, we use the formalism of the four-component Dirac fermion.[1] In this section, we consider the SD equation in the covariant gauge.

The fermion propagator is written in the form of

$$S(p)=[\not{p}A(p)+B(p)]^{-1}. \tag{1}$$

The SD equation for the fermion propagator $S(p)$ is

$$S(p)^{-1}=S_0(p)^{-1}+e^2\int\frac{d^3k}{(2\pi)^3}\gamma_\mu S(k)\Gamma_\nu(p,k)D^{\mu\nu}(p-k), \tag{2}$$

where $S_0(p)^{-1}=\not{p}$ is the bare fermion propagator, $D^{\mu\nu}(p-k)$ the photon propagator, $\Gamma_\nu(p,k)$ the vertex function and e the bare coupling constant.

We use the ordinary covariant gauge-fixing term,

$$L_{\text{gauge}}=\frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{3}$$

Then the photon propagator is written as

$$D_{\mu\nu}(p)=\frac{1}{p^2}\frac{1}{1+\Pi(p)}\left(g_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)+\xi\frac{p_\mu p_\nu}{p^4}, \tag{4}$$

where the vacuum polarization $\Pi(p)$ for photon is defined as

$$\Pi(p)=-\frac{e^2N}{2p^2}\int\frac{d^3k}{(2\pi)^3}\mathrm{Tr}[\gamma_\mu S_0(k)\gamma^\mu S_0(k-p)]. \tag{5}$$

In this paper we consider the leading correction in the $1/N$ expansion, i.e., the one loop correction, in the photon propagator for massless fermion,[11]

$$\Pi(p)=\frac{\tilde{\alpha}}{p}, \tag{6}$$

where

$$\tilde{\alpha}:=\frac{e^2N}{8}. \tag{7}$$

Supposing the ansatz that the vertex $\Gamma_\mu(p,k)$ is a function of p^2 and k^2, i.e.,

$$\Gamma_\mu(p,k)=\gamma_\mu G(p^2,k^2), \tag{8}$$

we can perform the angular integration. After the angular integration, the SD equation (2) is decomposed into two equations for $A(p)$ and $B(p)$,

$$A(p)=1-\frac{\tilde{\alpha}}{\pi^2 N}\frac{1}{p^3}\int_0^\infty dk\frac{kA(k)G(p^2,k^2)}{k^2A(k)^2+B(k)^2}\Big[\tilde{\alpha}^2\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}$$

$$-\tilde{\alpha}(p+k-|p-k|)+2pk-\frac{1}{\tilde{\alpha}}|p^2-k^2|(p+k-|p-k|)$$

$$-\frac{1}{\tilde{\alpha}^2}(p^2-k^2)^2\Big\{\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}-\ln\frac{p+k}{|p-k|}\Big\}-\xi\Big\{(p^2+k^2)\ln\frac{p+k}{|p-k|}-2pk\Big\}\Big],\tag{9a}$$

$$B(p)=\frac{\tilde{\alpha}}{\pi^2 N}\frac{1}{p}\int_0^\infty dk\frac{kB(k)G(p^2,k^2)}{k^2A(k)^2+B(k)^2}\Big\{4\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}+2\xi\ln\frac{p+k}{|p-k|}\Big\}.\tag{9b}$$

The integral equation can be transformed into the equivalent differential equation, if the kernel of the integral equation has the *degenerate* form (i.e., separated in p and k). This procedure greatly simplifies the following numerical analysis. In order to obtain the degenerate form, we take two approximations, the expansion of the logarithmic function and the vertex ansatz, $G(p^2,k^2)=G(k^2)$.

Note the following expansion,

$$\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}=2\Big[\frac{k}{p+\tilde{\alpha}}+\frac{1}{3}\Big(\frac{k}{p+\tilde{\alpha}}\Big)^3+\cdots\Big]\theta(p-k)$$

$$+2\Big[\frac{p}{k+\tilde{\alpha}}+\frac{1}{3}\Big(\frac{p}{k+\tilde{\alpha}}\Big)^3+\cdots\Big]\theta(k-p).\tag{10a}$$

This expansion is also used for the term, $\ln(p+k/|p-k|)$ by putting $\tilde{\alpha}=0$. In the 2+1 dimensional space, $\tilde{\alpha}$ can be considered as the UV cutoff, since the coupling constant $\tilde{\alpha}$ is dimensionful and in the region $p>\tilde{\alpha}$ the integration is damped quickly.[1] Therefore, the logarithmic term is replaced by the following approximate form,

$$\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}\simeq 2\Big[\frac{k}{\tilde{\alpha}}-\frac{kp}{\tilde{\alpha}^2}+\frac{3kp^2+k^3}{3\tilde{\alpha}^3}\Big]\theta(p-k)$$

$$+2\Big[\frac{p}{\tilde{\alpha}}-\frac{pk}{\tilde{\alpha}^2}+\frac{3pk^2+p^3}{3\tilde{\alpha}^3}\Big]\theta(k-p).\tag{10b}$$

Then we obtain the following equations,

$$A(p)=1+K\int_\epsilon^{\tilde{\alpha}} dk\frac{kA(k)G(k^2)}{k^2A(k)^2+B(k)^2}$$

$$\times\Big[\frac{k^3}{p^3}\Big\{-\frac{1}{3}+\frac{\xi}{4}\frac{\tilde{\alpha}}{p}\Big\}\theta(p-k)+\Big\{-\frac{1}{3}+\frac{\xi}{4}\frac{\tilde{\alpha}}{k}\Big\}\theta(k-p)\Big],\tag{11a}$$

$$B(p)=K\int_\epsilon^{\tilde{\alpha}} dk\frac{kB(k)G(k^2)}{k^2A(k)^2+B(k)^2}$$

$$\times\Big[\frac{k}{p}\Big\{1+\frac{\xi}{2}\frac{\tilde{\alpha}}{p}\Big\}\theta(p-k)+\Big\{1+\frac{\xi}{2}\frac{\tilde{\alpha}}{k}\Big\}\theta(k-p)\Big],\tag{11b}$$

where

$$K:=\frac{8}{\pi^2 N}.$$

Here we have introduced the IR cutoff ϵ for numerical calculations. Note that these equations show that the kernels of the integrations in Eqs. (9a, b) are singular, but the results integrated out are not.

Introducing the variable,

$$t=\ln(p/\tilde{\alpha}), \tag{12}$$

we can rewrite the integral equations (11a, b) with the equivalent differential equations,

$$\frac{d^2A}{dt^2}+\frac{3-4\xi\frac{\tilde{\alpha}}{p}}{1-\xi\frac{\tilde{\alpha}}{p}}\frac{dA}{dt}-K\left(1-\xi\frac{\tilde{\alpha}}{p}\right)\frac{G(p^2)A}{A^2+B^2/p^2}=0, \tag{13a}$$

$$\frac{d^2B}{dt^2}+\frac{1+2\xi\frac{\tilde{\alpha}}{p}}{1+\xi\frac{\tilde{\alpha}}{p}}\frac{dB}{dt}+K\left(1+\xi\frac{\tilde{\alpha}}{p}\right)\frac{G(p^2)B}{A^2+B^2/p^2}=0 \tag{13b}$$

with the following boundary conditions,

$$\frac{p^3}{1-\xi\frac{\tilde{\alpha}}{p}}\frac{dA}{dt}\bigg|_{t=t_0}=0, \qquad \frac{p}{1+\xi\frac{\tilde{\alpha}}{p}}\frac{dB}{dt}\bigg|_{t=t_0}=0, \tag{14}$$

$$\frac{\frac{1}{3}-\frac{1}{4}\xi\frac{\tilde{\alpha}}{p}}{1-\xi\frac{\tilde{\alpha}}{p}}\frac{dA}{dt}+A\bigg|_{t=t_{\tilde{\alpha}}}=1, \qquad \frac{1+\frac{1}{2}\xi\frac{\tilde{\alpha}}{p}}{1+\xi\frac{\tilde{\alpha}}{p}}\frac{dB}{dt}+B\bigg|_{t=t_{\tilde{\alpha}}}=0, \tag{15}$$

where $t_{\tilde{\alpha}}=0$ and $t_0=\ln(\epsilon/\tilde{\alpha})$.

Almost all the results of numerical calculations reported in the following section are done for the differential equations (13a, b). As a matter of fact, we have checked that the results are qualitatively unchanged by solving numerically the original integral equations (9a, b), at least for not so large cutoff.*)

§ 3. Analytical study

In this section we consider the case of the Landau gauge.**) Then the SD equations for $A(p)$ and $B(p)$ are written by[3]

$$A(p)=1-\frac{\tilde{\alpha}}{\pi^2 N}\frac{1}{p^3}\int_0^\infty dk\frac{kA(k)G(p^2,k^2)}{k^2A(k)^2+B(k)^2}\left[\tilde{\alpha}^2\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}\right.$$

*) For a large cutoff, the integral equations are harder to be calculated numerically than the differential equations.

**) We give a comment for gauge dependence in the final section.

$$-\tilde{\alpha}(p+k-|p-k|)+2pk-\frac{1}{\tilde{\alpha}}|p^2-k^2|(p+k-|p-k|)$$

$$-\frac{1}{\tilde{\alpha}^2}(p^2-k^2)^2\left\{\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}-\ln\frac{p+k}{|p-k|}\right\}\Bigg], \tag{16a}$$

$$B(p)=\frac{4\tilde{\alpha}}{\pi^2 N}\frac{1}{p}\int_0^\infty dk\frac{kB(k)G(p^2,k^2)}{k^2A(k)^2+B(k)^2}\ln\frac{p+k+\tilde{\alpha}}{|p-k|+\tilde{\alpha}}. \tag{16b}$$

Note that, even in the Landau gauge, $A(p)$ is no longer equal to one due to the vacuum polarization in the photon propagator. In the quenched planar QED where $\Pi(p)\equiv 0$ and $\Gamma_\mu(p,k)\equiv\gamma_\mu$, it is proved[9] that $A(p)\equiv 1$ in any space-time dimensions. In the presence of the vacuum polarization, this is not the case.

Putting $\xi=0$ in Eqs. (11a, b), we obtain the following equations,[4]

$$A(p)=1-\frac{K}{3}\int_\epsilon^{\tilde{\alpha}} dk\frac{kA(k)G(k^2)}{k^2A(k)^2+B(k)^2}\left[\frac{k^3}{p^3}\theta(p-k)+\theta(k-p)\right], \tag{17a}$$

$$B(p)=K\int_\epsilon^{\tilde{\alpha}} dk\frac{kB(k)G(k^2)}{k^2A(k)^2+B(k)^2}\left[\frac{k}{p}\theta(p-k)+\theta(k-p)\right]. \tag{17b}$$

These are the basic equations in the following discussion.

3.1. *Inequality*

The ansatz for the vertex taken by PW is

$$\Gamma_\mu(p,k)=\gamma_\mu A(k), \tag{18}$$

which is supposed to be the consequence of the Ward-Takahashi identity. They claimed, mainly based on numerical calculations, that the chiral symmetry breaking occurs for any number of fermion species, i.e., $N_c=\infty$. Here we derive an inequality which gives the analytical proof for the result that in the limit $\tilde{\alpha}\to\infty$, the critical point N_c goes to infinity.

In the physical situation, the functions $A(p)$ and $B(p)$ are restricted to the region,

$$A(p)>0 \quad \text{and} \quad B(p)>0, \tag{19}$$

since they are related to the wave-function renormalization factor,

$$Z(p):=\frac{1}{A(p)}, \tag{20}$$

and the fermion mass function,

$$M(p):=\frac{B(p)}{A(p)}. \tag{21}$$

Note that the second term of the right-hand side in Eq. (17a) is negative, and then $1>A(p)$.

Differentiating Eqs. (17a, b), we obtain

$$\frac{dA(p)}{dp}=K\frac{1}{p^4}\int_\epsilon^p dk\frac{k^4A(k)^2}{k^2A(k)^2+B(k)^2}>0, \tag{22a}$$

$$\frac{dB(p)}{dp}=-K\frac{1}{p^2}\int_\epsilon^p dk\frac{k^2B(k)A(k)}{k^2A(k)^2+B(k)^2}<0\,, \tag{22b}$$

which mean that $A(p)$ is a monotonically increasing function and $B(p)$ a monotonically decreasing function. Therefore $M(p)$ is monotonically decreasing in p. Now substituting $p=\epsilon$ in Eq. (17a), we have a series of inequalities,

$$\frac{3}{K}(1-A(\epsilon))=\int_\epsilon^{\tilde{\alpha}} dk\frac{k}{k^2+M(k)^2}>\int_\epsilon^{\tilde{\alpha}} dk\frac{k}{k^2+M(\epsilon)^2}$$
$$=\frac{1}{2}\ln\left(\frac{\tilde{\alpha}^2+M(\epsilon)^2}{\epsilon^2+M(\epsilon)^2}\right)>\frac{1}{2}\ln\left(\frac{\tilde{\alpha}^2}{\epsilon^2+M(\epsilon)^2}\right). \tag{23}$$

Then finally we obtain the inequality, using $A(\epsilon)>0$,

$$\ln\frac{M(\epsilon)}{\tilde{\alpha}}>\frac{1}{2}\ln\left[\exp\left(-\frac{3\pi^2}{4}N\right)-\frac{\epsilon^2}{\tilde{\alpha}^2}\right]. \tag{24a}$$

In the limit $\epsilon\to 0$,

$$\frac{3\pi^2}{8}N>\ln\left(\frac{\tilde{\alpha}}{M(0)}\right). \tag{24b}$$

This inequality tells us that in the limit of infinite $\tilde{\alpha}$, the critical number of flavors must go to infinity, $N_c\to\infty$. Numerical results are presented in the next section.

For the simplest choice, i.e., the bare vertex

$$\Gamma_\mu(p,k)=\gamma_\mu$$

or the *symmetric* vertex

$$\Gamma_\mu(p,k)=\gamma_\mu[A(p)\theta(p-k)+A(k)\theta(k-p)]\,, \tag{25}$$

we have the same inequality as the above.

If $M(p)$ is almost constant, the same argument leads to the following scaling law *in the absence of the IR cutoff*,

$$\frac{M(0)}{\tilde{\alpha}}\sim\exp\left(\frac{-3\pi^2}{8}N\right), \tag{26}$$

where $3\pi^2/8=3.701\cdots$. This is the scaling relation which PW and AJP have obtained by numerical calculations. The above proof gives us a kind of the no-go theorem for having the finite critical point, in the absence of the IR cutoff.*)

3.2. *Vertex ansatz and effective coupling*

As we mentioned earlier, we do not know how the vertex ansatz should be taken in actual investigation. So in this subsection, in order to see the relation between the vertex ansatz and the critical behavior, we consider the more general ansatz**)

---

*) This type of argument was done for QED$_4$ with the negative gauge parameter by Nonoyama and Tanabashi.[15]

**) Although we should consider the symmetric ansatz,[22] $\Gamma_\mu(p,k)=\gamma_\mu[A(p)^n\theta(p-k)+A(k)^n\theta(k-p)]$, this is not so easy to handle with analytically. Our numerical solutions show that the qualitative behavior of the effective coupling is unchanged for the ansatz (27) and this symmetric one.

$$\Gamma_\mu(p,k)=\gamma_\mu A(k)^n\,. \tag{27}$$

The cases $n=0$ and 1 have been already considered in the previous subsection. Especially the case $n=1$ is likely to be a physical one due to the Ward-Takahashi identity.*) Not only the $n=1$ or $n=2$ case, but also these generalized cases are very interesting since it is shown that, depending on n, they correspond to the asymptotically free or asymptotically non-free theory, in the sense explained below.[17]

Now we give the analysis on the SD equation (17a, b) by the bifurcation method, that $B(k)^2$ in the denominator is neglected.[16] We obtain a solution for $A(p)$, by the approximation that $(k^3/p^3)\theta(p-k)$, the higher order term in k/p, is negligible in Eq. (17a).

$$A(t)=1-\frac{K}{3}\int_t^0 ds A(s)^{n-1}\,. \tag{28}$$

The solution of this equation is easily obtained as follows: For $n\neq 2$,

$$A(t)=\left(1+\frac{2-n}{3}Kt\right)^{1/(2-n)}. \tag{29}$$

Note that for $n<2$, this solution should be taken in the region $A(t)>0$, i.e., $t>t_\infty := -3/(2-n)K$. In the IR region $t_0<t<t_c$ where t_c is slightly larger than t_∞, $A(t)$ keeps the small value, since $A(t)$ is positive and monotonically increasing. This behavior of $A(t)$ is confirmed by direct numerical calculations by solving the full SD equations (13a, b).

Substituting this solution into the equation for $B(p)$, we obtain

$$B(p)=K\int_\epsilon^{\tilde{a}} dk \frac{1}{\left(1+\frac{2-n}{3}Kt\right)}\frac{B(k)}{k}\left[\frac{k}{p}\theta(p-k)+\theta(k-p)\right]. \tag{30}$$

This equation is valid for any n, including $n=2$.**) Then we can read Eq. (30) as the "improved" SD equation.[19] In other words, we can see that the correction to $A(p)$ by the vacuum polarization is equivalent to replacing the fixed coupling by the "running coupling" constant,

$$K(t)=\frac{K}{\left(1+\frac{2-n}{3}Kt\right)}. \tag{31}$$

The case $n<2$, in which the effective coupling $K(t)$ diverges in the IR region, is what we called "asymptotically free".[17),19] In this case, $K(t)$ becomes strong enough to cause the chiral symmetry breaking and make the bound state in sufficiently small region for t. However, note that the asymptotic freedom defined here is valid only

*) Recent study for the anomalous dimension by Atkinson, Johnson and Maris[12] proposes the $n=2$ vertex as a physical one, while new calculations under the Ball-Chiu ansatz[13] by Pennington and Walsh[14] show that the scaling law is qualitatively similar to the $n=1$ case.

**) For $n=2$, the solution (29) does not give a correct solution of Eq. (28) for A. However, in this case, Eqs. (17a) and (17b) are decoupled with each other in the bifurcation method, and the function A disappears in (17b). As a result, Eq. (30) for $n=2$, which is just the linearized form of Eq. (17b) with $A=1$, is correct.

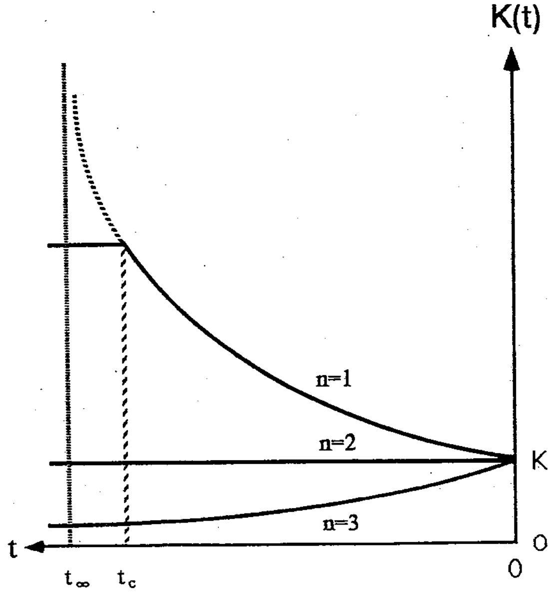

Fig. 1. The effective coupling $K(t)$, for $n>2$, $n=2$, $n<2$.

if we take the cutoff t_0 such that $t_0=\ln(\epsilon/\tilde{\alpha})<t_\infty$. Therefore this asymptotic freedom would disappear for sufficiently large IR cutoff ϵ or small UV cutoff $\tilde{\alpha}$.[19] This gives a possible explanation for the numerical results given below. The case $n>2$ is what we call "asymptotically non-free" case,[17] in which $K(t)$ remains finite and tends to zero in the IR region. The case $n=2$ is a special one in which the coupling constant does not run and the situation is just like ANW's case.

§ 4. Numerical calculations

We solve the integral equation (17a, b) to obtain the critical point for N, and calculate the dynamical mass, which is obtained as the solution of the following equation,

$$m=M(m).$$

Since the mass function $M(p)$ is almost constant in the IR region (see the boundary condition (14)), we approximate $m=M(m)\sim M(\epsilon)$ for simplicity, which does not alter essential feature of the scaling behavior. From the dimensional analysis, the dynamical mass m can be written as

$$m=\tilde{\alpha}f(N)\,. \tag{32}$$

We call the function $f(N)$ *the scaling function.*

The critical coupling is plotted on the $(\ln(m/\tilde{\alpha}), N)$ plane in Figs. 2 and 3 for $\tilde{\alpha}=10^4$, 10^{11}. All the analytical results obtained from the inequality and the approximate solution of Eq. (29) are confirmed by numerical calculations.

The numerical results are summarized as follows:

4.1. *The case of large $\tilde{\alpha}/\epsilon$*

For sufficiently large UV cutoff $\tilde{\alpha}$ (or small IR cutoff ϵ), we have three types of the scaling law in respect of the vertex ansatz (27) $\Gamma_\mu(p, k)=\gamma_\mu A(k)^n$ as follows:

i) $n<2$: The choice $n=1$ is taken by PW and the case $n=0$ is nothing but the bare vertex case. The scaling law for these cases is given by the exponential type,

$$f(N)\propto\exp(-CN) \tag{33}$$

with a constant value C. Actually $C=3.70(\sim 3\pi^2/8)$ for $n=1$, which confirms PW and AJP's results, and $C=1.84(\sim 3\pi^2/16)$ for $n=0$.

ii) $n=2$: This is the case in which two equations are approximately decoupled with each other. We have the solution for $B(p)$ with the criticality at $N_c=3.1$ $(\sim 32/\pi^2)$ which is the critical point obtained by ANW. In this case the scaling law is given by the essential-singularity type,

$$f(N)\propto\exp\left(\frac{-2\pi}{\sqrt{N_c/N-1}}\right). \tag{34}$$

iii) $n>2$: In this case, there is also the finite critical point but the scaling law does not fit the essential-singularity type. Rather it seems to be of the power-law type,

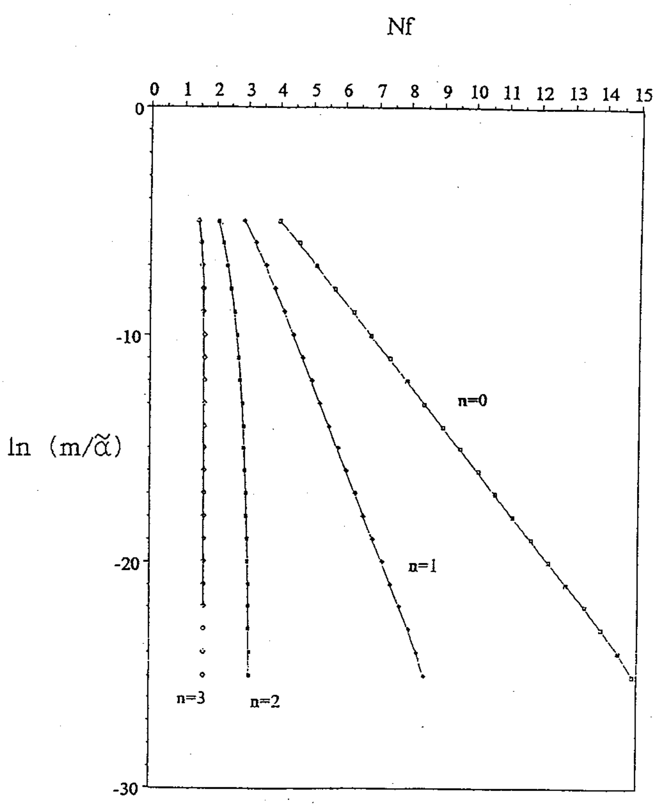

Fig. 2. The scaling behavior in the Landau gauge with $\tilde{\alpha}=10^{11}$ and $\epsilon=1$; $\ln(m/\tilde{\alpha})$ vs N.

$$f(N) \propto (N_c - N)^{\lambda}, \tag{35}$$

where λ is an exponent, whose value is, for example, $\lambda \sim .7$ for $n=3$. (Fig. 2)

4.2. *The case of small* $\tilde{\alpha}/\epsilon$

For relatively small UV cutoff $\tilde{\alpha}$ (or large IR cutoff ϵ), the scaling law is remarkably different from the above three types. For any value of n, it becomes of the mean-field type (Fig. 3(b)),

$$f(N) \propto (N_c - N)^{1/2}, \tag{36}$$

where N_c is cutoff-dependent.

In $n=1$ case, we can expect this mean-field type scaling from the discussion of the previous section. From Eq. (24a), the scaling function could be approximately written by

$$\frac{M(\epsilon)}{\tilde{\alpha}} = f(N) \sim \left[\exp\left(-\frac{3\pi^2}{4}N\right) - \frac{\epsilon^2}{\tilde{\alpha}^2}\right]^{1/2}. \tag{37}$$

The critical point N_c is defined by the zero of the scaling function. We can expand the function in the square-root of Eq. (37) around N_c, since the function is regular at the point $N=N_c$. Then in the neighborhood of N_c, where the higher order terms of $N_c - N$ is negligible, the scaling function $f(N)$ approximately behaves like the function, $(N_c - N)^{1/2}$, which means the mean-field type scaling.[18)]

(a)

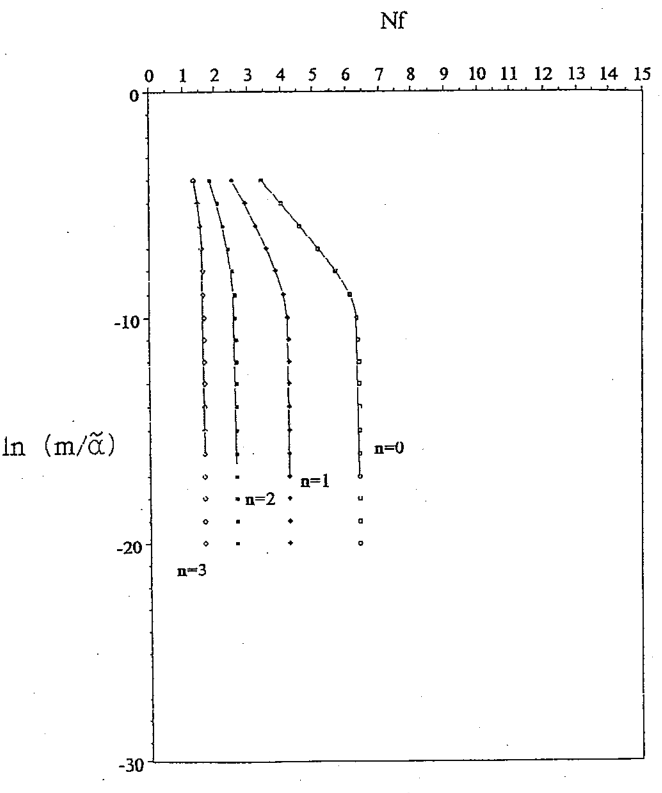

Fig. 3. (continued)

(b)

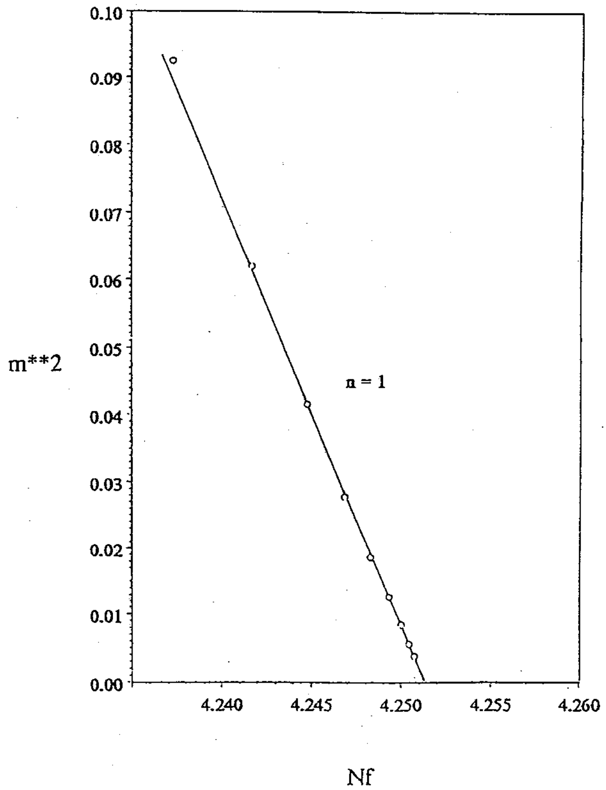

Fig. 3. The scaling behavior in the Landau gauge with $\tilde{\alpha}=10^4$ and $\epsilon=1$;
(a) ln $(m/\tilde{\alpha})$ vs N,
(b) m^2 vs N in the neighborhood of the critical point for the $n=1$ case.

§ 5. Conclusion and discussion

In this paper we have solved the SD gap equation in QED$_3$ combined with the $1/N$ expansion for the vacuum polarization without using the assumption of no wave-function renormalization.[1),2)] We have paid special attention to the critical value N_c of the fermion flavor and the scaling law in the neighborhood of N_c. We showed that the scaling behavior of the dynamical mass is restricted by the inequality and discussed the relation between the "generalized vertex ansatz" and the scaling law.

Our numerical results show that the scaling law depends on the IR cutoff (or the ratio, $\epsilon/\tilde{\alpha}$). Actually in the limit of vanishing IR cutoff $\epsilon\to 0$, we have three types of the scaling law depending on the vertex ansatz, i.e., the exponential type (33), the essential-singularity type (34) and the power-law type (35). We have given an explanation on this difference based on the concept of the effective coupling. The result of the exponential type would be the physical one and agrees with the previous result obtained by PW and AJP.

On the other hand, in the presence of the finite IR cutoff, the scaling obeys the mean-field type (36) independent of the vertex ansatz, and N_c has a finite value which depends on the value of the IR cutoff. According to MC results, there exists a finite critical value for fermion flavor. It is, however, still an open question what the scaling type really is in QED$_3$. It should be remarked that IR cutoff introduced in our framework may correspond to the lattice size in MC simulation, while the UV cutoff corresponds to the lattice spacing. It appears that our framework is able to provide a possible explanation to apparently conflicting results previously obtained, based on the SD equation[3),4)] and the MC simulation.[10)]

Finally we would like to point out some remarks for the further investigation. The first important point is gauge invariance. Of course, MC result is gauge-independent. In our opinion the correction in the $1/N$ expansion should be incorporated so that the physical results become gauge-independent. Therefore we expect that the correct truncation-procedure of the SD equation in the $1/N$ expansion may lead to gauge-independent results. However all the above investigations are limited to the case of the Landau gauge. In this context, we will have to mention the recent work by Nash.[20)] Actually, he has shown, for the non-local gauge-fixing which was used by one of the authors (H.N.),[21)] that the critical number of fermion flavors becomes gauge-independent, if one takes into account all the diagrams of the $1/N^2$ order in the function B of the SD equation, while the function A includes only the correction from the $1/N$ order diagram. Then he reproduced surprisingly the ANW's result, i.e., $N_c=32/\pi^2<\infty$ and the scaling law of the essential singularity type in "Feynman gauge". However, to accept this result literally we must further clarify the nature of the non-local gauge-fixing taken there and the consistency for taking the different order in the $1/N$ expansion for the functions A and B.

Thus we can check whether the truncation of the $1/N$ expansion series is consistent, using the criterion of gauge independence for the self-consistent solutions. From this viewpoint, the numerical calculations of the SD equation for the arbitrary gauge (9a, b) are in progress.[23)]

Another point is the criticality for the fermion number in QED$_4$. We note that, in the framework of the SD equation nobody has succeeded in showing the existence of the finite critical number of flavors in QED$_4$. In QED$_4$, under a similar ansatz for the vertex, the gauge independence of the critical coupling e_c and the scaling law for the dynamical mass near e_c are proved by the numerical calculations.[22)] However the effect of the vacuum polarization is included up to one loop level there. We are performing the numerical calculations of the SD equation beyond one loop correction to the vacuum polarization of the photon propagator, including the improvement of the vertex.[24)]

Acknowledgements

Numerical calculations were done on FACOM M-780 and VP-200 at Nagoya University Computation Center.

References

1) T. Appelquist, M. Bowick, D. Karabali and L. C. R. Wijewardhana, Phys. Rev. **D33** (1986), 3704.
2) T. Appelquist, D. Nash and L. C. R. Wijewardhana, Phys. Rev. Lett. **60** (1988), 2575.
3) M. R. Pennington and S. P. Webb, BNL Preprint-40886 (January, 1988).

4) D. Atkinson, P. W. Johnson and M. R. Pennington, BNL Preprint-41615 (August, 1988).
5) T. Matsuki, L. Miao and K. S. Visbwanathan, Simon Fraser Univ. Preprint, June 1987 (revised version: May 1988).
6) T. Matsuki, *Proceeding of 1989 Workshop on Dynamical Symmetry Breaking, Nagoya, 21-23 December 1989,* ed. T. Muta and K. Yamawaki (Nagoya University, 1990), p. 198.
7) T. Maskawa and H. Nakajima, Prog. Theor. Phys. **52** (1974), 1326; **54** (1975), 860.
R. Fukuda and T. Kugo, Nucl. Phys. **B117** (1976), 250.
V. A. Miransky, Nuovo Cim. **90A** (1985), 149.
8) K. -I. Kondo and H. Nakatani, Mod. Phys. Lett. **A5** (1990), 407.
9) K. -I. Kondo and H. Nakatani, Mod. Phys. Lett. **A4** (1989), 2155.
10) E. Dagotto, J. B. Kogut and A. Kocić, Phys. Rev. Lett. **62** (1989), 1083; ILL-(TH)-89-#40, 1989.
11) R. Pisarski, Phys. Rev. **D29** (1984), 2423.
12) D. Atkinson, P. W. Johnson and P. Maris, Phys. Rev. **D42** (1990), 602.
13) J. S. Ball and T. W. Chiu, Phys. Rev. **D22** (1980), 2542.
14) M. R. Pennington and D. Walsh, Phys. Lett. **B253** (1991), 246.
15) T. Nonoyama and M. Tanabashi, Prog. Theor. Phys. **81** (1989), 209.
16) D. Atkinson, J. Math. Phys. **28** (1987), 2494.
17) K-I. Aoki, M. Bando, T. Kugo, K. Hasebe and H. Nakatani, Prog. Theor. Phys. **81** (1989), 866.
K-I. Aoki, M. Bando, K. Hasebe, T. Kugo and H. Nakatani, Prog. Theor. Phys. **82** (1989), 1151.
18) K. -I. Kondo, Phys. Lett. **B226** (1989), 329.
19) K. Higashijima, Phys. Rev. **D29** (1984), 1228.
20) D. Nash, Phys. Rev. Lett. **62** (1989), 3024.
21) H. Nakatani, Comment given at the 1988 International Workshop on New Trend in Strong Coupling Gauge Theories, Nagoya, Aug. 24-27.
22) K. -I. Kondo, Y. Kikukawa and H. Mino, Phys. Lett. **B220** (1989), 270.
23) K. -I. Kondo and H. Nakatani, in preparation.
24) K. -I. Kondo, H. Mino and H. Nakatani, Chiba University Preprint, CHIBA-EP-51.