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CUTS IN CYCLICALLY ORDERED SETS

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1. PRELIMINARY REMARKS

An *ordered set* is a pair $(G, <)$ where G is a set and $<$ is an order on G, i.e. an irreflexive and transitive binary relation on G. We write briefly G instead of $(G, <)$ if the order $<$ is given. If $<$ is an order on G, then the dual relation $<^* = >$ is an order on G. An element $x \in G$ is called the *least element* of $(G, <)$ iff $x < y$ for any $y \in G - \{x\}$; the *greatest element* is defined dually. If $(G, <)$ is an ordered set and $H \subseteq G$, then $< \cap H^2$ is an order on H; this order is denoted by $<|_H$ or, briefly, also $<$, and the subset $H = (H, <)$ is called an *ordered subset* of the ordered set $G = (G, <)$. An order $<$ on a set G is *linear* iff $x < y$ or $y < x$ for any $x, y \in G$, $x \neq y$; in this case $(G, <)$ is called a *linearly ordered set.*

1.1. Definition. Let $(G, <_G), (H, <_H)$ be ordered sets with $G \cap H = \emptyset$. An *ordinal sum* $G \oplus H$ of ordered sets G, H is the set $G \cup H$ with the binary relation $<$ defined by $x < y$ iff either $x, y \in G$, $x <_G y$ or $x, y \in H$, $x <_H y$ or $x \in G$, $y \in H$.

It is known ([1]; but it is trivial to prove it) that $<$ is an order on $G \cup H$ so that $G \oplus H$ is an ordered set. Further, the operation $\oplus$ is associative so that the symbol $G_1 \oplus G_2 \oplus \ldots \oplus G_n$ is defined, whenever $G_1, \ldots, G_n$ are pairwise disjoint ordered sets.

1.2. Definition. Let $(G, <)$ be a linearly ordered set. A subset $I \subseteq G$ is called an *interval* in G iff there exist subsets A, B of G with $G = A \oplus I \oplus B$. A subset $A \subseteq G$ is called an *initial interval* in G iff there exists a subset B of G with $G = A \oplus B$. A *final interval* is defined dually.

The following assertion is known; however, it is not difficult to prove it directly:

1.3. Theorem. *Let $(G, <)$ be a linearly ordered set. A subset $I \subseteq G$ is an interval in G iff it has the following property*: $x, y \in I$, $z \in G$, $x < z < y \Rightarrow z \in I$. *A subset $A \subseteq G$ is an initial interval in G iff it has the following property*: $x \in A$, $y \in G$, $y < x \Rightarrow y \in A$. *A subset $B \subseteq G$ is a final interval in G iff it has the following property*: $x \in B$, $y \in G$, $x < y \Rightarrow y \in B$.

1.4. Definition. Let G be a set, T a ternary relation on G. This relation is called:

asymmetric, iff $(x, y, z) \in T \Rightarrow (z, y, x) \notin T$,
cyclic, iff $(x, y, z) \in T \Rightarrow (y, z, x) \in T$,
transitive, iff $(x, y, z) \in T$, $(x, z, u) \in T \Rightarrow (x, y, u) \in T$,
linear, iff $x, y, z \in G$, $x \neq y \neq z \neq x \Rightarrow (x, y, z) \in T$ or $(z, y, x) \in T$.

1.5. Definition. Let G be a set, C a ternary relation on G which is asymmetric, cyclic and transitive. Then C is called a *cyclic order* on G and the pair (G, C) is called a *cyclically ordered set*. If, moreover, card $G \geqq 3$ and C is linear, it is called a *linear cyclic order* on G and (G, C) is called a *linearly cyclically ordered set* or a *cycle*.

If we define a dual relation T^* to a ternary relation T by $(x, y, z) \in T^* \Leftrightarrow (z, y, x) \in T$, then the following remark obviously holds:

1.6. Remark. If C is a cyclic order on a set G, then C^* is a cyclic order on G.

1.7. Theorem. *Let (G, C) be a cyclically ordered set, let $x \in G$. For any $y, z \in G$ put $y <_{C,x} z$ iff either $(x, y, z) \in C$ or $x = y \neq z$. Then $<_{C,x}$ is an order on G with the least element x.*

Proof. [4], 3.1.

1.8. Remark. Analogously we can define, for a cyclically ordered set (G, C) and $x \in G : y <^{C,x} z \Leftrightarrow$ either $(y, z, x) \in C$ or $y \neq z = x$. Then $<^{C,x}$ is an order on G with the greatest element x.

1.9. Lemma. *If C is a linear cyclic order on a set G, then $<_{C,x}$ is a linear order on G.*

Proof. Trivial; see also [4], 3.4.

1.10. Theorem. *Let $(G, <)$ be an ordered set. Define a ternary relation $C_<$ on G by $(x, y, z) \in C_<$ iff either $x < y < z$ or $y < z < x$ or $z < x < y$. Then $C_<$ is a cyclic order on G.*

Proof. [4], 3.5.

1.11. Lemma. *Let $(G, <)$ be a linearly ordered set with* card $G \geqq 3$. *Then $C_<$ is a linear cyclic order on G.*

Proof. Trivial; see also [4], 3.7.

1.12. Lemma. *Let $<$ be an order on a set G. Then $C_{<^*} = C_<^*$.*

Proof. Trivial.

2. DEFINITION OF A CUT

From now on, we shall deal only with linearly cyclically ordered sets. For the sake of brevity, we shall omit the adjective "linear"; thus, "cyclically ordered set" means always "linearly cyclically ordered set".

A cut in a linearly ordered set is defined as a couple of its subsets. An analogue in a cyclically ordered set is impossible. Intuitively, a "section" of an oriented circle determines a linear ordering of points of that circle. This is a motivation for the following

2.1. Definition. Let (G, C) be a cyclically ordered set. A *cut* on this set is a linear order $<$ on G with the property $x < y < z \Rightarrow (x, y, z) \in C$.

In 2.5 we shall see that cuts exist. Now we derive some simple properties of theirs.

2.2. Lemma. *Let (G, C) be a cyclically ordered set, let $<$ be a cut on (G, C), let $x, y, z \in G$, $(x, y, z) \in C$. Then either $x < y < z$ or $y < z < x$ or $z < x < y$.*

Proof. Any of the remaining possibilities $z < y < x$, $y < x < z$, $x < z < y$ implies $(z, y, x) \in C$ by definition of a cut, which contradicts $(x, y, z) \in C$.

2.3. Theorem. *Let (G, C) be a cyclically ordered set, let $<$ be a linear order on G. The order $<$ is a cut on (G, C) if and only if $C_< = C$.*

Proof. 1. Let $<$ be a cut on (G, C) and let $(x, y, z) \in C_<$. Then either $x < y < z$ or $y < z < x$ or $z < x < y$, which implies (by the definition of a cut) $(x, y, z) \in C$. Thus $C_< \subseteq C$. As $C_<$ is a linear cyclic order by 1.11 and C is linear, we have $C_< = C$.

2. Let $C_< = C$. If $x, y, z \in G$, $x < y < z$, then $(x, y, z) \in C_< = C$. Thus $<$ is a cut on (G, C).

2.4. Theorem. *Let $(G, <)$ be a linearly ordered set with* card $G \geqq 3$. *Then there exists just one cyclic order C on G such that $<$ is a cut on (G, C).*

Proof. Existence: Put $C = C_<$. By 1.11, C is a cyclic order on G and by 2.3, $<$ is a cut on (G, C).

Unicity: Let C_1, C_2 be cyclic orders on G for which $<$ is a cut. Let $(x, y, z) \in C_1$. By 2.2 we have either $x < y < z$ or $y < z < x$ or $z < x < y$, which implies $(x, y, z) \in C_2$ by 2.1. Thus $C_1 \subseteq C_2$ and as the both relations C_1, C_2 are linear, we obtain $C_1 = C_2$.

2.5. Theorem. *Let (G, C) be a cyclically ordered set, let $x \in G$. Then $<_{C,x}$ is a cut on (G, C).*

Proof. By 1.9, $<_{C,x}$ is a linear order on G. Let $u, v, w \in G$, $u <_{C,x} v <_{C,x} w$. First assume $x \notin \{u, v, w\}$. Then $(x, u, v) \in C$, $(x, v, w) \in C$, thus $(v, w, x) \in C$, $(v, x, u) \in C$ and by transitivity of C, $(v, w, u) \in C$ and hence $(u, v, w) \in C$. If $x \in \{u, v, w\}$, then $x = u$ and as $v <_{C,x} w$, we have $(x, v, w) \in C$, i.e. $(u, v, w) \in C$.

Thus we always have $u <_{C,x} v <_{C,x} w \Rightarrow (u, v, w) \in C$ and $<_{C,x}$ is a cut on (G, C).
Dually, we can prove:

2.6. Remark. Let (G, C) be a cyclically ordered set, let $x \in G$. Then $<^{C,x}$ is a cut on (G, C).

The both orders $<_{C,x}$, $<^{C,x}$ are thus cuts on (G, C) and by their definitions, $<_{C,x}$ has the least element, $<^{C,x}$ the greatest element. Other cuts with this property do not exist, for:

2.7. Theorem. *Let (G, C) be a cyclically ordered set, let $<$ be a cut on (G, C) with the least element x. Then $< \, = \, <_{C,x}$.*

Proof. Let $y, z \in G - \{x\}$, $y < z$. Then $x < y < z$ and, by definition of a cut, $(x, y, z) \in C$. Hence $y <_{C,x} z$. Further, x is the least element in both $(G, <)$ and $(G, <_{C,x})$. We have shown that $< \, \subseteq \, <_{C,x}$ and as the both orders are linear, we have $< \, = \, <_{C,x}$.

Of course, dually we have:

2.8. Remark. Let (G, C) be a cyclically ordered set, let $<$ be a cut on (G, C) with the greatest element x. Then $< \, = \, <^{C,x}$.

3. PROPERTIES OF CUTS

3.1. Definition. Let (G, C) be a cyclically ordered set, let $<$ be a cut on (G, C). This cut is called:
a *jump*, iff $(G, <)$ has both the least and the greatest element,
a *gap*, iff $(G, <)$ has neither the least nor the greatest element,
Dedekind, iff $(G, <)$ has just one of the boundary elements.

3.2. Definition. A cyclically ordered set (G, C) is called *dense* iff there exists no jump on (G, C).

As one can expect, it holds:

3.3. Theorem. *A cyclically ordered set (G, C) is dense iff it has the following property: $x, y \in G$, $x \neq y \Rightarrow$ there exists $z \in G$ with $(x, z, y) \in C$.*

Proof. 1. Assume that for any $x, y \in G$, $x \neq y$ there exists $z \in G$ with $(x, z, y) \in C$ and let $<$ be a jump on (G, C) with the least element y and the greatest element x. By 2.7. we obtain $< \, = \, <_{C,y}$ and by the assumption an element $z \in G$ exists with $(x, z, y) \in C$. Then $(y, x, z) \in C$ which implies $x <_{C,y} z$, i.e. $x < z$ and this is a contradiction, for x is the greatest element in $(G, <)$. Thus, (G, C) contains no jumps and it is dense.

2. Let elements $x, y \in G$, $x \neq y$ exist so that $(x, z, y) \in C$ holds for no $z \in G$. Then $<_{C,y}$ is a cut on (G, C) with the least element y; we show that x is its greatest element. When an element $z \in G$ exists with $x <_{C,y} z$, then $(y, x, z) \in C$ and also

$(x, z, y) \in C$ which contradicts our assumption. Thus $<_{C,y}$ is a jump on (G, C) and (G, C) is not dense.

3.4. Definition. Let (G, C) be a cyclically ordered set, let $x, y \in G$, $x \neq y$. The ordered pair (x, y) is called a *pair of consecutive elements* in (G, C) iff there exists no $z \in G$ with $(x, z, y) \in C$.

Note that by 3.3, (G, C) is dense iff it contains no pair of consecutive elements.

3.5. Lemma. *Let (G, C) be a cyclically ordered set, let (x, y) be a pair of consecutive elements in (G, C) nad let $<$ be any cut on (G, C). Then just one of the following possibilities occurs:*
(1) y is the least and x is the greatest element in $(G, <)$;
(2) y covers x in $(G, <)$.

Proof. If $< \; = \; <_{C,y}$ or $< \; = \; <^{C,x}$, then by the same argument as in the proof of 3.3 we find that (1) holds. In all the other cases x is not the greatest element in $(G, <)$. Suppose $y < x$; then there exists $z \in G$ with $y < x < z$, which implies $(y, x, z) \in C$ and $(x, z, y) \in C$, a contradiction. Hence $x < y$ and there exists no $z \in G$ with $x < z < y$, for otherwise $(x, z, y) \in C$. This means that y covers x in $(G, <)$.

3.6. Theorem. *Let (G, C) be a cyclically ordered set and let $<_1$, $<_2$ be two distinct cuts on (G, C). Then there exist nonempty disjoint subsets A, B of G such that $A \cup B = G$, $<_1|_A = <_2|_A$, $<_1|_B = <_2|_B$ and $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$.*

Proof. First observe that $<_2 \; = \; <_1^*$ is impossible for in that case $C_{<_2} = C^*_{<_1}$ by 1.12, while necessarily $C_{<_1} = C = C_{<_2}$ by 2.3. Thus there exist elements $x, y \in G$ such that $x <_1 y$, $x <_2 y$ so that there exist nonempty subsets $H \subseteq G$ with $<_1|_H = <_2|_H$. Denote this property of subsets of G by (P). If $\mathscr{S}$ is a chain (with respect to set inclusion) of (P)-subsets of G, then the set-theoretic union $\cup\mathscr{S}$ is a (P)-subset; so, by Zorn's lemma, there exists a maximal (P)-subset $A \subseteq G$. We show that A is an interval in $(G, <_1)$. Let $x, y \in A$, $z \in G$, $x <_1 z <_1 y$. Then $(x, z, y) \in C$ so that either $x <_2 z <_2 y$ or $z <_2 y <_2 x$ or $y <_2 x <_2 z$. The second and the third cases are impossible, since $x <_2 y$. Thus $x <_2 z <_2 y$. Let $u \in A$ be any element with $u <_1 z$. If $u = x$, then $u <_2 z$. If $u <_1 x$, then $u <_1 x <_1 z$, thus $(u, x, z) \in C$, which implies either $u <_2 x <_2 z$ or $x <_2 z <_2 u$ or $z <_2 u <_2 x$. The second case is impossible, since $u <_2 x$ ($u, x \in A$ and $<_1|_A = <_2|_A$), the third one is also impossible, since $x <_2 z$. If $x <_1 u$, then $u <_1 z <_1 y$, thus $(u, z, y) \in C$ and hence either $u <_2 z <_2 y$ or $z <_2 y <_2 u$ or $y <_2 u <_2 z$. The second and the third cases are impossible, since $u <_2 y$ and $z <_2 y$. We have shown $u <_1 z \Rightarrow u <_2 z$. By a similar argument we find $u \in A$, $z <_1 u \Rightarrow z <_2 u$. It follows that $A \cup \{z\}$ is a (P)-subset and the maximality of A implies $z \in A$. Note that for the same reason A is an interval also in $(G, <_2)$.

As A is an interval in $(G, <_1)$, we have $x \in G - A$, $x <_1 y$ for some $y \in A \Rightarrow x <_1 <_1 z$ for each $z \in A$; the same holds for $<_2$. This yields:

$x \in G - A$, $x <_1 y$ for some $y \in A \Rightarrow y <_2 x$. (*)

Otherwise there would exist $x \in G - A$, $y \in A$ with $x <_1 y$, $x <_2 y$ and then $x <_1 z$, $x <_2 z$ for each $z \in A$, thus $A \cup \{x\}$ is a (P)-subset, which contradicts the maximality of A.

Suppose now that A is neither an initial nor a final interval in $(G, <_1)$. Then $(G, <_1) = (H, <_1) \oplus (A, <_1) \oplus (K, <_1)$ with $H \neq \emptyset$, $K \neq \emptyset$. Choose $x \in H$, $y \in A$, $z \in K$. Then $x <_1 y <_1 z$ and (*) implies $z <_2 y <_2 x$. This is a contradiction, for $x <_1 y <_1 z$ implies $(x, y, z) \in C$ and $z <_2 y <_2 x$ implies $(z, y, x) \in C$. Thus A is an initial or a final interval in $(G, <_1)$ and for the same reason it is an initial or a final interval also in $(G, <_2)$.

Put $B = G - A$; B is a final or an initial interval both in $(G, <_1)$ and in $(G, <_2)$, and we show that $<_1|_B = <|_B$. Assume the existence of elements $x, y \in B$ with $x <_1 y$, $y <_2 x$. Choose any $z \in A$; if A is an initial interval in $(G, <_1)$, then $z <_1 <_1 x <_1 y$ and from (*) we have $y <_2 x <_2 z$. This is a contradiction, for $z <_1 x <_1 <_1 y$ implies $(z, x, y) \in C$ and $y <_2 x <_2 z$ implies $(y, x, z) \in C$. If A is a final interval in $(G, <_1)$, then $x <_1 y <_1 z$ and $z <_2 y <_2 x$, which leads to a contradiction as well.

Assume that A is an initial interval both in $(G, <_1)$ and in $(G, <_2)$. Then $(G, <_1) = (A, <_1) \oplus (B, <_1)$, $(G, <_2) = (A, <_2) \oplus (B, <_2)$ and as $(A, <_1) = (A, <_2)$, $(B, <_1) = (B, <_2)$, we have $<_1 = <_2$, which is a contradiction. Thus, if A is an initial interval in $(G, <_1)$, it is a final interval in $(G, <_2)$ and $(G, <_1) = (A, <_1) \oplus \oplus (B, <_1)$, $(G, <_2) = (B, <_2) \oplus (A, <_2)$. If A is a final interval in $(G, <_1)$, it is an initial interval in $(G, <_2)$ and the given equality holds after interchanging the sets A, B.

3.7. Remark. The sets A, B from 3.6 are unique.

Proof. Assume $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$ and, at the same time, $(G, <_1) = A_1 \oplus B_1$, $(G, <_2) = B_1 \oplus A_1$. As A, A_1 are initial intervals of the linearly ordered set $(G, <_1)$, either $A \subseteq A_1$ or $A_1 \subseteq A$ holds; let the first possibility occur. Suppose $A \neq A_1$; if we choose arbitrary elements $x \in A_1 - A$ and $y \in B_1$, then $x <_2 y$ in $(B, <_2) \oplus (A, <_2)$ and $y <_2 x$ in $(B_1, <_2) \oplus (A_1, <_2)$. This is a contradiction and hence $A = A_1$.

3.8. Lemma. *Let G be a set with* card $G \geqq 3$. *Let* $<_1, <_2$ *be linear orders on G such that there exist disjoint subsets A, B of G with* $A \cup B = G$, $<_1|_A = <_2|_A$, $<_1|_B = <_2|_B$ *and* $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$. *Then there exists just one cyclic order C on G such that* $<_1, <_2$ *are cuts on* (G, C).

Proof. The uniqueness follows from 2.4. For the existence it suffices to prove $C_{<_1} = C_{<_2}$. Let $(x, y, z) \in C_{<_1}$. Then either $x <_1 y <_1 z$ or $y <_1 z <_1 x$ or $z <_1 <_1 x <_1 y$. We investigate only the first case; the second and the third one are similar. We have the following possibilities:

$$x, y, z \in A \Rightarrow x <_2 y <_2 z \Rightarrow (x, y, z) \in C_{<_2};$$

$$x, y \in A, \quad z \in B \Rightarrow z <_2 x <_2 y \Rightarrow (x, y, z) \in C_{<_2};$$

$$x \in A, \quad y, z \in B \Rightarrow y <_2 z <_2 x \Rightarrow (x, y, z) \in C_{<_2};$$

$$x, y, z \in B \Rightarrow x <_2 y <_2 z \Rightarrow (x, y, z) \in C_{<_2}.$$

Thus we have shown $C_{<_1} \subseteq C_{<_2}$ and as both cyclic orders $C_{<_1}, C_{<_2}$ are linear, we conclude $C_{<_1} = C_{<_2}$.

3.9. Corollary. *Let G be a set with* card $G \geqq 3$, *let $<_1, <_2$ be distinct linear orders on G. Then $C_{<_1} = C_{<_2}$ holds if and only if there exist nonempty disjoint subsets A, B of G with $A \cup B = G$, $<_1|_A = <_2|_A$, $<_1|_B = <_2|_B$ and $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$.*

Proof. If $C_{<_1} = C_{<_2}$, then, by 2.3, $<_1, <_2$ are two distinct cuts on a cyclically ordered set (G, C) where $C = C_{<_1} = C_{<_2}$. By 3.6, the orders $<_1, <_2$ have the desired properties. Conversely, if the condition of Corollary is satisfied, then, by 3.8 and 2.3, $C_{<_1} = C_{<_2}$ holds.

If $(G, <)$ is a linearly ordered set and $x \in G$, then we denote by $(G, <)_x$ or, briefly, G_x, the open initial interval in $(G, <)$ determined by the element x, i.e. $G_x = \{y \in G; y < x\}$.

3.10. Lemma. *Let (G, C) be a cyclically ordered set, let $<$ be a cut on (G, C) and let $x \in G$. Then $(G, <_{C,x}) = (G - (G, <)_x, <) \oplus (G, <)_x$.*

Proof. If x is the least element in $(G, <)$, then $< \; = \; <_{C,x}$ and the formula holds, since $(G, <)_x = \emptyset$. Otherwise $<, <_{C,x}$ are distinct cuts on (G, C) and by 3.6 there exist nonempty disjoint subsets A, B of G with $A \cup B = G$, $<|_A = <_{C,x}|_A$, $<|_B = <_{C,x}|_B$ and $(G, <) = A \oplus B$, $(G, <_{C,x}) = B \oplus A$. Then A is an initial interval in $(G, <)$ and $(G, <_{C,x}) = B \oplus A$ implies that B has the least lement x. Thus $A = (G, <)_x$ and $B = G - (G, <)_x$.

3.11. Theorem. *Let (G, C) be a cyclically ordered set and let $<_1, <_2, <_3$ be three pairwise distinct cuts on (G, C). Then there exist three nonempty pairwise disjoint subses A, B, D of G such that $A \cup B \cup D = G$, $<_1|_A = <_2|_A = <_3|_A$, $<_1|_B = <_2|_B = <_3|_B$, $<_1|_D = <_2|_D = <_3|_D$, and either $(G, <_1) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_3) = D \oplus A \oplus B$ or $(G, <_3) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_1) = D \oplus A \oplus B$ holds.*

Proof. By 3.6 there exist nonempty disjoint subsets A_1, B_1 of G with $A_1 \cup B_1 = G$, $<_1|_{A_1} = <_2|_{A_1}$, $<_1|_{B_1} = <_2|_{B_1}$, $(G, <_1) = A_1 \oplus B_1$, $(G, <_2) = B_1 \oplus A_1$, and there exist nonempty disjoint subsets A_2, B_2 of G with $A_2 \cup B_2 = G$, $<_1|_{A_2} = <_3|_{A_2}$, $<_1|_{B_2} = <_3|_{B_2}$, $(G, <_1) = A_2 \oplus B_2$, $(G, <_3) = B_2 \oplus A_2$. As A_1, A_2 are initial intervals of the linearly ordered set $(G, <_1)$, we have either $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. The inclusion here is proper, for if $A_1 = A_2$, then $B_1 = B_2$ so that $<_2 = <_3$, which contradicts our assumption.

1. Let $A_1 \subset A_2$. Consider the sets $A_1, A_2 - A_1, B_2$. As $<_1|_{A_1} = <_2|_{A_1}$, $<_1|_{A_2} = <_3|_{A_2}$ and $A_1 \subset A_2$, we have $<_1|_{A_1} = <_2|_{A_1} = <_3|_{A_1}$. Further, $<_1|_{A_2 - A_1} = <_3|_{A_2 - A_1}$ and as $A_2 - A_1 \subseteq B_1$, we have $<_1|_{A_2 - A_1} = <_2|_{A_2 - A_1}$. Thus $<_1|_{A_2 - A_1} = <_2|_{A_2 - A_1} = <_3|_{A_2 - A_1}$. Finally, we have $B_2 \subseteq B_1$ and hence $<_1|_{B_2} = <_2|_{B_2}$, $<_1|_{B_2} = <_3|_{B_2}$. Consequently, $<_1|_{B_2} = <_2|_{B_2} = <_3|_{B_2}$. Now, we have

$$(G, <_1) = A_1 \oplus (A_2 - A_1) \oplus B_2 ,$$

$$(G, <_2) = (A_2 - A_1) \oplus B_2 \oplus A_1 ,$$

$$(G, <_3) = B_2 \oplus A_1 \oplus (A_2 - A_1) .$$

2. Let $A_2 \subset A_1$. By an analogous reasoning we find

$$<_1|_{A_2} = <_2|_{A_2} = <_3|_{A_2} , \quad <_1|_{A_1 - A_2} , \; <_2|_{A_1 - A_2} = <_3|_{A_1 - A_2} ,$$

$$<_1|_{B_1} = <_2|_{B_1} = <_3|_{B_1}$$

and

$$(G, <_3) = (A_1 - A_2) \oplus B_1 \oplus A_2 ,$$

$$(G, <_2) = B_1 \oplus A_2 \oplus (A_1 - A_2) ,$$

$$(G, <_1) = A_2 \oplus (A_1 - A_2) \oplus B_1 .$$

4. CYCLIC ORDERING OF CUTS

4.1. Definition. Let (G, C) be a cyclically ordered set, let $<_1, <_2, <_3$ be three pairwise distinct cuts on (G, C). Put $(<_1, <_2, <_3) \in \mathscr{C}$ iff there exist three nonempty pairwise disjoint subsets A, B, D of G such that $A \cup B \cup D = G$, $<_1|_A = <_2|_A = <_3|_A$, $<_1|_B = <_2|_B = <_3|_B$, $<_1|_D = <_2|_D = <_3|_D$, and $(G, <_1) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_3) = D \oplus A \oplus B$.

4.2. Theorem. *Let (G, C) be a cyclically ordered set and let $\mathscr{G}$ be the set of all cuts on (G, C). Then $\mathscr{C}$ is a cyclic order on the set $\mathscr{G}$.*

Proof. Suppose that there exist pairwise distinct cuts $<_1, <_2, <_3$ on (G, C) with $(<_1, <_2, <_3) \in \mathscr{C}$, $(<_3, <_2, <_1) \in \mathscr{C}$. Then there exist nonempty pairwise disjoint subsets A, B, D of G with $A \cup B \cup D = G$, $<_1|_A = <_2|_A = <_3|_A$, $<_1|_B = <_2|_B = <_3|_B$, $<_1|_D = <_2|_D = <_3|_D$, $(G, <_1) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_3) = D \oplus A \oplus B$, and nonempty pairwise disjoint subsets A_1, B_1, D_1 of G with $A_1 \cup B_1 \cup D_1 = G$, $<_1|_{A_1} = <_2|_{A_1} = <_3|_{A_1}$, $<_1|_{B_1} = <_2|_{B_1} = <_3|_{B_1}$, $<_1|_{D_1} = <_2|_{D_1} = <_3|_{D_1}$, $(G, <_3) = A_1 \oplus B_1 \oplus D_1$, $(G, <_2) = B_1 \oplus D_1 \oplus A_1$, $(G, <_1) = D_1 \oplus A_1 \oplus B_1$. Then $B \oplus D \oplus A = B_1 \oplus D_1 \oplus A_1 = (G, <_2)$, and hence either $B \subseteq B_1$ or $B_1 \subseteq B$. Let $B \subseteq B_1$; if $B \subset B_1$, choose $x \in B$, $y \in (B_1 - B) \cap D$. Then $(G, <_3) = A_1 \oplus B_1 \oplus D_1$ implies $x <_3 y$ and $(G, <_3) = D \oplus A \oplus B$ implies $y <_3 x$. This is a contradiction. Analogously $B_1 \subset B$ is impossible and thus $B = B_1$. Now we have $(G, <_1) = A \oplus B \oplus D$, $(G, <_1) = D_1 \oplus A \oplus B$, which implies $D = \emptyset$, $D_1 = \emptyset$ and this is a contradiction. The relation $\mathscr{C}$ is thus asymmetric.

Assume $<_1, <_2, <_3, <_4 \in \mathscr{G}$, $(<_1, <_2, <_3) \in \mathscr{C}$, $(<_1, <_3, <_4) \in \mathscr{C}$. Then there exist nonempty disjoint subsets A, B, D of G with $A \cup B \cup D = G$, $<_1|_A = <_2|_A = <_3|_A$, $<_1|_B = <_2|_B = <_3|_B$, $<_1|_D = <_2|_D = <_3|_D$, $(G, <_1) = A \oplus B \oplus D$, $(G, <_2) = B \oplus D \oplus A$, $(G, <_3) = D \oplus A \oplus B$, and nonempty disjoint subsets A_1, B_1, D_1 of G with $A_1 \cup B_1 \cup D_1 = G$, $<_1|_{A_1} = <_3|_{A_1} = <_4|_{A_1}$, $<_1|_{B_1} = <_3|_{B_1} = <_4|_{B_1}$, $<_1|_{D_1} = <_3|_{D_1} = <_4|_{D_1}$, $(G, <_1) = A_1 \oplus B_1 \oplus D_1$, $(G, <_3) = B_1 \oplus D_1 \oplus A_1$, $(G, <_4) = D_1 \oplus A_1 \oplus B_1$. As $A \oplus B \oplus D = A_1 \oplus B_1 \oplus D_1 = (G, <_1)$, we have either $A \subseteq A_1$ or $A_1 \subseteq A$. The equality $A = A_1$ is impossible, for in that case $D \oplus A \oplus B = B_1 \oplus D_1 \oplus A = (G, <_3)$, which implies $B = \emptyset$, a contradiction. Suppose $A_1 \subset A$; if we choose $x \in A_1$, $y \in A - A_1$, then $(G, <_3) = D \oplus A \oplus B$ implies $x <_3 y$ and $(G, <_3) = B_1 \oplus D_1 \oplus A_1$ implies $y <_3 x$. This is a contradiction and thus $A \subset A_1$. Further, we have either $A_1 \subseteq A \oplus B$ or $A \oplus B \subseteq A_1$. If $A_1 \subset A \oplus B$, choose $x \in A$, $y \in B - A_1$. Then $(G, <_3) = D \oplus A \oplus B$ implies $x <_3 y$ and $(G, <_3) = B_1 \oplus D_1 \oplus A_1$ implies $y <_3 x$, which is impossible. If $A \oplus B \subset A_1$, choose $x \in A \oplus B$, $y \in A_1 - (A \oplus B)$. Then $(G, <_3) = B_1 \oplus D_1 \oplus A_1$ implies $x <_3 y$ and $(G, <_3) = D \oplus A \oplus B$ implies $y <_3 x$, which is a contradiction. Thus $A_1 = A \oplus B$ and from $A \oplus B \oplus D = A_1 \oplus D = A_1 \oplus B_1 \oplus D_1 = (G, <_1)$ we have $D = B_1 \oplus D_1$. Now, we have $(G, <_1) = A \oplus (B \oplus B_1) \oplus D_1$, $(G, <_2) = (B \oplus B_1) \oplus D_1 \oplus A$, $(G, <_4) = D_1 \oplus A \oplus (B \oplus B_1)$. This implies $(<_1, <_2, <_4) \in \mathscr{C}$ and the relation $\mathscr{C}$ is transitive. It follows directly from the definition that $\mathscr{C}$ is cyclic. Finally, if $<_1, <_2, <_3 \in \mathscr{G}$ are pairwise distinct, then 3.11 implies either $(<_1, <_2, <_3) \in \mathscr{C}$ or $(<_3, <_2, <_1) \in \mathscr{C}$. Thus $\mathscr{C}$ is linear and it is a cyclic order on $\mathscr{G}$.

4.3. Lemma. *Let (G, C) be a cyclically ordered set and let $<_1, <_2, <_3 \in \mathscr{G}$. Then $(<_1, <_2, <_3) \in C$ holds if and only if there exist elements $x, y, z \in G$ with $x <_1 y <_1 z$, $y <_2 z <_2 x$, $z <_3 x <_3 y$.*

Proof. Let $(<_1, <_2, <_3) \in \mathscr{C}$. If A, B, D are subsets of G with the properties from 4.1, choose $x \in A$, $y \in B$, $z \in C$. Then $x <_1 y <_1 z$, $y <_2 z <_2 x$, $z <_3 x <_3 y$. Conversely, let there exist elements $x, y, z \in G$ with $x <_1 y <_1 z$, $y <_2 z <_2 x$, $z <_3 x <_3 y$. Then the cuts $<_1, <_2, <_3$ are pairwise distinct and thus there exist subsets A, B, D of G with the properties from 3.11. Elements x, y, z must lie in the distinct sets A, B, D, since the orders $<_1, <_2, <_3$ coincide on these sets. If the second case from 3.11 occurred, we should obtain in all possible situations always a contradiction. Thus the first case of 3.11 occurs and $(<_1, <_2, <_3) \in \mathscr{C}$.

4.4. Theorem. *Let (G, C) be a cyclically ordered set and let $x, y, z \in G$, $x \neq y \neq z \neq x$. Then $(x, y, z) \in C$ holds if and only if $(<_{C,x}, <_{C,y}, <_{C,z}) \in \mathscr{C}$.*

Proof. $(x, y, z) \in C$ implies $x <_{C,x} y <_{C,x} z$, $y <_{C,y} z <_{C,y} x$, $z <_{C,z} x <_{C,z} y$ and from 4.3 we have $(<_{C,x}, <_{C,y}, <_{C,z}) \in \mathscr{C}$. Conversely, let $(<_{C,x}, <_{C,y}, <_{C,z}) \in \mathscr{C}$ and assume $(x, y, z) \bar{\in} C$. Then $(z, y, x) \in C$ and from the first step of the proof we have $(<_{C,z}, <_{C,y}, <_{C,x}) \in \mathscr{C}$, which is a contradiction. Thus $(x, y, z) \in C$.

4.5. Corollary. *Let (G, C) be a cyclically ordered set. Then $(\{<_{C,x};\ x \in G\}, \mathscr{C})$ is a cyclically ordered set isomorphic with (G, C).*

Proof. $(\{<_{C,x};\ x \in G\}, \mathscr{C})$ is – as a subset of $(\mathscr{G}, \mathscr{C})$ – cyclically ordered. The mapping $G \to \{<_{C,x};\ x \in G\}$ assigning to any $x \in G$ the cut $<_{C,x}$ is evidently a bijection; by 4.4 it is an isomorphism.

5. COMPLETION BY CUTS

5.1. Definition. A cyclically ordered set is called *complete*, iff it contains no gaps.

Note that "complete" has another meaning here than in [4].

5.2. Theorem. *Let (G, C) be a cyclically ordered set. Then the cyclically ordered set $(\mathscr{G}, \mathscr{C})$ is complete.*

Proof. Let $\prec$ be a cut on $(\mathscr{G}, \mathscr{C})$. Define a linear order $<$ on G by $x < y \Leftrightarrow <_{C,x} \prec <_{C,y}$. The relation $<$ is indeed a linear order on G, for $\prec$ is a linear order on $\mathscr{G}$, thus also on $\{<_{C,x};\ x \in G\}$ and as a consequence of the bijection $x \to <_{C,x}$, $<$ is a linear order. We show that $<$ is a cut on (G, C). Let $x, y, z \in G$, $x < y < z$. Then $<_{C,x} \prec <_{C,y} \prec <_{C,z}$, thus $(<_{C,x}, <_{C,y}, <_{C,z}) \in \mathscr{C}$ and by 4.4, $(x, y, z) \in C$. Thus $< \in \mathscr{G}$.

Suppose that $<$ is neither the least nor the greatest element in $(\mathscr{G}, \prec)$. Then there exist $<_1, <_2 \in \mathscr{G}$ such that $<_1 \prec < \prec <_2$. This implies $(<_1, <, <_2) \in \mathscr{C}$ and by 4.1 there exist nonempty disjoint subsets A, B, D of G such that $A \cup B \cup D = G$, $<_1|_A = <|_A = <_2|_A$, $<_1|_B = <|_B = <_2|_B$, $<_1|_D = <|_D = <_2|_D$ and $(G, <_1) = A \oplus B \oplus D$, $(G, <) = B \oplus D \oplus A$, $(G, <_2) = D \oplus A \oplus B$. Choose elements $x \in A$, $y \in B$. We show that $<_{C,x} \prec <_{C,y}$. If $<_{C,x} = <_1$ and $<_{C,y} = <_2$, then the desired relation holds. Let $<_{C,x} \neq <_1$. Then $A = A_x \oplus (A - A_x)$ and as $(G, <_1)_x = A_x$, 3.10 implies $(G, <_1) = A_x \oplus (A - A_x) \oplus (B \oplus D)$, $(G, <_{C,x}) = (A - A_x) \oplus (B \oplus D) \oplus A_x$, $(G, <) = (B \oplus D) \oplus A_x \oplus (A - A_x)$. Consequently, $(<_1, <_{C,x}, <) \in \mathscr{C}$ and hence either $<_1 \prec <_{C,x} \prec <$ or $<_{C,x} \prec < \prec <_1$ or $< \prec <_1 \prec <_{C,x}$. The second and the third case are impossible for $<_1 \prec <$. Thus $<_1 \prec <_{C,x} \prec <$. Analogously, if $<_{C,y} \neq <_2$, then we find $< \prec <_{C,y} \prec <_2$. Thus in all cases we have $<_1 \precsim <_{C,x} \prec < \prec <_{C,y} \precsim <_2$ and hence $<_{C,x} \prec <_{C,y}$. But $(G, <) = B \oplus D \oplus A$ and $y \in B$, $x \in A$, thus $y < x$. This contradicts the definition of the order $<$.

5.3. Corollary. *Let (G, C) be a cyclically ordered set. Then there exists a complete cyclically ordered set (H, D) containing an isomorphic subset with (G, C).*

Proof follows from 5.2 and 4.5.

5.4. Lemma. *Let (G, C) be a cyclically ordered set, let $x \in G$. Then $(<_{C,x}, <^{C,x})$ is a pair of consecutive elements in $(\mathscr{G}, \mathscr{C})$.*

Proof. Let $<$ be any cut on (G, C) distinct from both $<_{C,x}$ and $<^{C,x}$. By 3.6 there exist nonempty disjoint subsets A, B of G with $A \cup B = G$, $<_{C,x}|_A = <|_A$, $<_{C,x}|_B = <|_B$ and $(G, <_{C,x}) = A \oplus B$, $(G, <) = B \oplus A$. As $< \neq <^{C,x}$, we have $A \neq \{x\}$. Now we have $(G, <_{C,x}) = \{x\} \oplus (A - \{x\}) \oplus B$, $(G, <^{C,x}) = (A - \{x\}) \oplus \oplus B \oplus \{x\}$, $(G, <) = B \oplus \{x\} \oplus (A - \{x\})$ so that $(<_{C,x}, <^{C,x}, <) \in \mathscr{C}$. Thus $(<_{C,x}, <, <^{C,x}) \in \mathscr{C}$ holds for no cut $< \in \mathscr{G}$ and, therefore, $(<_{C,x}, <^{C,x})$ is a pair of consecutive elements in $(\mathscr{G}, \mathscr{C})$.

Note that 5.4 implies that $(\mathscr{G}, \mathscr{C})$ is never dense.

5.5. Notation. Let (G, C) be a cyclically ordered set. Denote $\mathscr{G}_r = \{< \in \mathscr{G};\ < \text{ is a gap}\} \cup \{<_{C,x};\ x \in G\}$; the elements of $\mathscr{G}_r$ will be called *regular cuts.*

$\mathscr{G}_r$ thus contains all jumps and all gaps in (G, C) and from Dedekind cuts it contains only those which have the least element. As a subset of $\mathscr{G}$, $(\mathscr{G}_r, \mathscr{C})$ is a cyclically ordered set and by 4.5, $x \to <_{C,x}$ is an isomorphic embedding of (G, C) into $(\mathscr{G}_r, \mathscr{C})$.

5.6. Theorem. *Let (G, C) be a cyclically ordered set. Then the cyclically ordered set $(\mathscr{G}_r, \mathscr{C})$ is complete.*

Proof. Let $\prec$ be a cut on $(\mathscr{G}_r, \mathscr{C})$. This cut in a natural way determines a cut on $(\mathscr{G}, \mathscr{C})$, which we denote by the same symbol $\prec$: any cut from $\mathscr{G} - \mathscr{G}_r$ is of the form $<^{C,x}$; for such a cut we put $<_{C,x} \prec <^{C,x}$; if $y \in G$, $y \neq x$, then $<_{C,y} \prec <^{C,x} \Leftrightarrow \Leftrightarrow <_{C,y} \prec <_{C,x}$, $<^{C,x} \prec <_{C,y} \Leftrightarrow <_{C,x} \prec <_{C,y}$, $<^{C,x} \prec <^{C,y} \Leftrightarrow <_{C,x} \prec <_{C,y}$ and for $< \in \mathscr{G}$, which is a gap, we put $< \prec <^{C,x} \Leftrightarrow < \prec <_{C,x}$, $<^{C,x} \prec < \Leftrightarrow <_{C,x} \prec <$. It is not difficult to show that $\prec$ is indeed a cut on $(\mathscr{G}, \mathscr{C})$. By 5.2 there exists a cut $< \in \mathscr{G}$ which is either the least or the greatest element in $(\mathscr{G}, \prec)$. If $< \in \mathscr{G}_r$, then $\mathscr{G}_r$ has either the least or the greatest element. If $< \bar{\in} \mathscr{G}_r$, then $< = <^{C,x}$ for some $x \in G$. In this case, by 5.4, $(<_{C,x}, <^{C,x})$ is a pair of consecutive elements in $(\mathscr{G}, \mathscr{C})$ and by 3.5 we have: (1) either $<^{C,x}$ is the least and $<_{C,x}$ the greatest element in $(\mathscr{G}, \prec)$, (2) or $<^{C,x}$ covers $<_{C,x}$ in $(\mathscr{G}, \prec)$. If (1) holds, then $<_{C,x}$ is the greatest element in $(\mathscr{G}_r, \prec)$. If (2) holds, then $<^{C,x}$ cannot be the least element in $(\mathscr{G}, \prec)$, for it covers $<_{C,x}$. Therefore $<^{C,x}$ is the greatest element in $(\mathscr{G}, \prec)$ and then $<_{C,x}$ is the greatest element in $(\mathscr{G}_r, \prec)$. Thus no cut on $(\mathscr{G}_r, \mathscr{C})$ is a gap and $(\mathscr{G}_r, \mathscr{C})$ is complete.

If (G, C) is a cyclically ordered set, then $(\mathscr{G}_r, \mathscr{C})$ will be called its *completion by cuts.*

5.7. Theorem. *Let (G, C) be a cyclically ordered set. If (G, C) is dense, then $(\mathscr{G}_r, \mathscr{C})$ is dense.*

Proof. Let $<_1, <_2 \in \mathscr{G}_r$, $<_1 \neq <_2$. If $<_1 = <_{C,x}$, $<_2 = <_{C,y}$ for some $x, y \in G$, then $x \neq y$ and by the assumption, there exists $z \in G$ such that $(x, z, y) \in C$. Then 4.4 yields $(<_1, <_{C,z}, <_2) \in \mathscr{C}$. Assume now that at least one of the cuts $<_1$, $<_2$ is a gap. By 3.6 there exist nonempty disjoint subsets A, B of G with $A \cup B = G$, $<_1|_A = <_2|_A$, $<_1|_B = <_2|_B$ and $(G, <_1) = A \oplus B$, $(G, <_2) = B \oplus A$. The subset A is necessarily infinite: otherwise A would have both the least and the greatest element and then $(G, <_1)$ would have the least, $(G, <_2)$ the greatest element. Choose any element $x \in A$ which is neither its least nor its greatest element and put $< = <_{C,x}$. Then $A = A_x \oplus (A - A_x)$ and by 3.10 we have $(G, <_1) = A_x \oplus (A - A_x) \oplus B$, $(G, <) = (A - A_x) \oplus B \oplus A_x$, $(G, <_2) = B \oplus A_x \oplus (A - A_x)$. This implies $(<_1, <, <_2) \in \mathscr{C}$. Thus $(\mathscr{G}_r, \mathscr{C})$ is dense.

5.8. Definition. A cyclically ordered set (G, C) is called *continuous* iff any cut on (G, C) is Dedekind.

In other words, (G, C) is continuous iff it is dense and complete. From 5.6 and 5.7 we directly obtain

5.9. Theorem. *Let (G, C) be a dense cyclically ordered set. Then its completion by cuts $(\mathscr{G}_r, \mathscr{C})$ is continuous.*

5.10. Corollary. *For any dense cyclically ordered set (G, C) there exists a continuous cyclically ordered set (H, D) and an isomorphic embedding of (G, C) into (H, D).*

5.11. Definition. Let (G, C) be a cyclically ordered set, let $H \subseteq G$. H is called *dense in* (G, C) iff for any elements $x, y \in G$, $x \neq y$ there exists $z \in H$ with $(x, z, y) \in C$.

Note that if (G, C) contains a dense subset, then (G, C) itself is dense.

Let (G, C) be a cyclically ordered set, let $(\mathscr{G}_r, \mathscr{C})$ be its completion by cuts. Let us identify the set G with its image by the canonical isomorphism given in 4.5, i.e. let us identify the element $x \in G$ with the element $<_{C,x} \in \mathscr{G}_r$. Thus any cyclically ordered set is a subset of a complete cyclically ordered set.

5.12. Theorem. *Let (G, C) be a dense cyclically ordered set. Then G is dense in $(\mathscr{G}_r, \mathscr{C})$.*

Proof. In the proof of 5.7 we have shown that for any distinct elements $<_1, <_2 \in \mathscr{G}_r$ there exists $x \in G$ such that $(<_1, <_{C,x}, <_2) \in \mathscr{C}$, i.e., after identifying the elements $y \in G$ with the cuts $<_{C,y}$, $(<_1, x, <_2) \in \mathscr{C}$. Thus G is dense in $(\mathscr{G}_r, \mathscr{C})$.

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