Cycles Through Particular Subgraphs of Claw-Free Graphs

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ABSTRACT

Let G be a 2-connected claw-free graph on n vertices, and let H be a subgraph of G. We prove that G has a cycle containing all vertices of H whenever $\alpha_3(H) \leq \kappa(G)$, where $\alpha_3(H)$ denotes the maximum number of vertices of H that are pairwise at distance at least three in G, and $\kappa(G)$ denotes the connectivity of G. This result is an analog of a result from the thesis of Fournier, and generalizes the result of Zhang that G is hamiltonian if the degree sum of any $\kappa(G) + 1$ pairwise nonadjacent vertices is at least $n - \kappa(G)$.

1. TERMINOLOGY

We use Bondy and Murty [3] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph of order n. The connectivity of G is denoted by $\kappa(G)$, the number of vertices in a maximum independent set of G by $\alpha(G)$, the set of vertices adjacent to a vertex $v \in V(G)$ by $N(v)$, and the degree of v by $d(v) = |N(v)|$. If H is a subgraph of G, we denote by $\alpha_3(H)$ the maximum number of vertices of H that are pairwise at distance at least three in G. Let $S \subseteq V(G)$. We denote by $\sigma_k(S)$ the minimum value of the degree sum (in G) of any k pairwise nonadjacent vertices of S if $k \leq \alpha(G[S])$, where $G[S]$ is the subgraph of G induced by S; if $k > \alpha(G[S])$, we set

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$\sigma_k(S) = k(n-1)$. Instead of $\sigma_k(V(G))$ we use $\sigma_k(G)$, and instead of $\sigma_1(S)$ we use the more common notation $\delta(S)$. If H is a subgraph of G or a subset of $V(G)$, an *H-cycle* of G is a cycle containing all vertices of H; a G-cycle is a *Hamilton cycle,* and a graph G containing a G-cycle is called *hamiltonian.* A graph G is *claw-free* if G has no induced subgraph isomorphic to $K_{1,3}$.

2. RESULTS

There are many results showing a graph G is hamiltonian if G satisfies a particular degree condition. The disadvantage of such results is that there is no result left even when only a few vertices have "small" degree, while recent results show that in these cases the graphs under consideration can still have "long" cycles containing particular sets of vertices with "large" degree. We give an example.

Theorem 1 (Dirac [5]). let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{1}{2}n$, then G is hamiltonian.

Recently, Theorem 1 was generalized independently by Bollobás and Brightwell, and by Shi.

Theorem 2 ([2,8]). Let G be a 2-connected graph of order n and let $S \subseteq V(G)$. If $\delta(S) \geq \frac{1}{2}n$, then G has an S-cycle.

In a similar fashion, a generalization of the following well-known result of Chvátal and Erdös appeared in the thesis of Fournier.

Theorem 3 [4]. Let G be a graph of order $n \geq 3$. If $\alpha(G) \leq \kappa(G)$, then G is hamiltonian.

Theorem 4 [6]. Let G be a 2-connected graph and let H be a subgraph of G. If $\alpha(H) \leq \kappa(G)$, then G has an H-cycle.

Extremal graphs for Theorems 3 and 4 can be found, e.g., within the class of complete bipartite graphs with unequal cardinalities of the two color classes. Most of these graphs contain many induced subgraphs isomorphic to $K_{1,3}$ and have $\alpha_3 = 1$, while the connectivity can be arbitrarily large. The situation is quite different if we consider claw-free graphs. In Section 3 we give a proof of the following analog of Theorem 4 for claw-free graphs.

Theorem 5. Let G be a 2-connected claw-free graph and let H be a subgraph of G. If $\alpha_3(H) \leq \kappa(G)$, then G has an H-cycle.

Corollary 6. Let G be a k-connected claw-free graph ($k \geq 2$) on n vertices and let $S \subseteq V(G)$. If $\sigma_{k+1}(S) \geq n - k$, then G has an S-cycle.

Corollary 6 is an immediate consequence of Theorem 5 and the following lemma.

Lemma 7. Let G be a 2-connected graph on n vertices and let $S \subseteq V(G)$. If $\sigma_{k+1}(S) \geq n - k$, then $\alpha_3(G[S]) \leq k$.

Proof. Suppose $\alpha_3(G[S]) \geq k + 1$. Consider a subset $T \subseteq S$ with $|T| = k + 1$ vertices that are pairwise at distance at least three in G. Then

$$\sum_{v \in T} d(v) \leq n - (k + 1) < n - k .$$

Therefore, $\sigma_{k+1}(S) \geq n - k$ implies $\alpha_3(G[S]) \leq k$. ∎

Corollary 6 generalizes the following result of Zhang.

Theorem 8 [9]. Let G be a k-connected claw-free graph ($k \geq 2$) on n vertices. If $\sigma_{k+1}(G) \geq n - k$, then G is hamiltonian.

By the way, as observed in [1], implicit in the proof of Theorem 8 in [9] is a proof of the following stronger result, which is a special case of Theorem 5 and an analog of Theorem 3.

Theorem 9. Let G be a 2-connected claw-free graph. If $\alpha_3(G) \leq \kappa(G)$, then G is hamiltonian.

The proof of Theorem 5 given in Section 3 is based on proof ideas used in [1] and [9]. The main idea to use so-called vertex-insertion was originally introduced in [9], and later presented more clearly in [1]. This idea is used throughout the proof of Theorem 5 in a similar way, restricted to vertices of H.

It is clear that Theorem 5 is more general than Theorem 4 within the class of claw-free graphs.

Theorem 5 is best possible in the following sense. Let G_1 be a graph obtained from three disjoint complete graphs on at least three vertices by joining two disjoint triples of vertices by a triangle, where each triple contains one vertex of each complete graph. Then G_1 has no hamilton cycle, is claw-free, while $\alpha_3(G_1) = \kappa(G_1) + 1 = 3$. We do not know whether Theorem 5 is best possible for graphs G with $\kappa(G) \geq 3$. Matthews and Sumner [7] conjectured that any 4-connected claw-free graph is hamiltonian.

3. PROOF OF THEOREM 5

Let G be a 2-connected claw-free graph, let H be a subgraph of G such that $\alpha_3(H) \leq \kappa(G)$, and let C be a cycle of G containing as many vertices of H as possible, and longest subject to this condition. Fix an orientation on C. If $u, v \in V(C)$, then $u\vec{C}v$ denotes the consecutive vertices on C from u to v in the direction specified by the orientation of C. The same vertices, in reverse order, are given by $v\overleftarrow{C}u$. We will consider $u\vec{C}v$ and $v\overleftarrow{C}u$ both as paths and as vertex sets. We use u^+ to denote the successor of u on C and u^- to denote its predecessor.

Assuming C is no H-cycle, let $a_0 \in V(H) - V(C)$. By a variation on Menger's Theorem, there are at least $\kappa(G)$ paths connecting a_0 to vertices of C having only the vertex a_0 in common and having no internal vertices on C. Choose a maximum number of paths with this property, and denote by $X = \{x_i \mid i = 1, 2, \ldots, t\}$ the end vertices of the paths on C, occurring on C in the order of their indices. Obviously, $|X| = t \geq \kappa(G)$. First note that any pair of vertices $x_i, x_j \in X$ is joined by a path containing a_0 with all internal vertices in $G - V(C)$. We denote such a path by $x_i \pi x_j$. Let S_i be the set of vertices of $x_i^+\vec{C}x_{i+1}^-$, and choose $u_i \in S_i$ such that $u_i x_i \in E(G)$ and $|x_i\vec{C}u_i^-|$ is as large as possible ($i = 1, 2, \ldots, t$; indices mod t). A vertex $u \in S_i \cap V(H)$ is *insertible* if there exist vertices $v, v^+ \in V(C) - S_i$ such that $uv, uv^+ \in E(G)$; then $vv^+ \in E(C)$ is an *insertion edge* of u. We denote the set of insertion edges of u by $I(u)$, and we denote the first noninsertible vertex of $V(H)$ occurring on $u_i^+\vec{C}x_{i+1}^-$ by a_i. Using standard techniques in hamiltonian graph theory, it is straightforward to show

$$x_i^- x_i^+ \in E(G), \tag{1}$$

$$V(H) \cap u_i^+\vec{C}x_{i+1}^- \neq \emptyset, \tag{2}$$

and

$$a_i \text{ exists.} \tag{3}$$

In the rest of the proof we show that a_i and a_j are at distance at least three in G ($i, j \in \{1, 2, \ldots, t\}; i \neq j$), implying $\alpha_3(H) \geq t + 1 \geq \kappa(G) + 1$, and proving Theorem 5.

Suppose there exists a vertex $w \in u_i^+\vec{C}x_{i+1}^- \cap V(H) \cap N(x_j)$ and assume $a_i \notin u_i^+\vec{C}w^-$. Then every vertex of $u_i^+\vec{C}w^- \cap V(H)$ is insertible. Let w_1 be the first vertex of $u_i^+\vec{C}w \cap V(H) \cap N(x_j)$ with the property that $x_j^- x_j$ and $x_j x_j^+$ are not insertion edges of the vertices in $u_i^+\vec{C}w^- \cap V(H)$

(or $w_1 = u_i^+$). Using the cycle $x_j w_1 \vec{C} x_j^- x_j^+ \vec{C} x_i^- x_i^+ \vec{C} u_i x_i \pi x_j$ and inserting the vertices of $u_i^+ \vec{C} w_1^- \cap V(H)$ in a straightforward way, we derive a contradiction. Hence

$$\text{If } w \in u_i^+ \vec{C} x_{i+1}^- \cap V(H) \cap N(x_j), \text{ then } a_i \in u_i^+ \vec{C} w^-. \tag{4}$$

By (4) and the definition of u_i, we have

$$N(a_i) \cap X = \varnothing. \tag{5}$$

From the definition of X and the fact that $a_i \notin X$, we obtain

$$a_0 \notin N(a_i). \tag{6}$$

Next we show

$$\text{There is no edge joining a vertex of } u_i^+ \vec{C} a_i \text{ and a vertex of } u_j^+ \vec{C} a_j, \text{ and } I(x) \cap I(y) = \varnothing \text{ for all } x \in u_i^+ \vec{C} a_i^- \cap V(H) \text{ and } y \in u_j^+ \vec{C} a_j^- \cap V(H). \tag{7}$$

To prove (7), suppose to the contrary there are vertices $x \in u_i^+ \vec{C} a_i$ and $y \in u_j^+ \vec{C} a_j$ such that $xy \in E(G)$ or x and y have a common insertion edge $vv^+ \in E(C)$ (and $x, y \in V(H)$). Choose x as close to u_i along $u_i^+ \vec{C} a_i$ as possible subject to the above conditions. Define

$$E_i = \begin{cases} \bigcup_{z \in u_i^+ \vec{C} x^- \cap V(H)} I(z) & \text{if } x \neq u_i^+ \\ \varnothing & \text{if } x = u_i^+ \end{cases}$$

Analogously define E_j. By the choice of x, we have

$$\text{There is no edge joining a vertex of } u_i^+ \vec{C} x^- \text{ and a vertex of } u_j^+ \vec{C} y^-, \tag{8}$$

and

$$E_i \cap E_j = \varnothing. \tag{9}$$

Consider two cases.

Case 1. $xy \in E(G)$.

By (4), there is no edge joining a vertex of $u_i^+ \vec{C} x^- \cap V(H)$ to x_j, since $a_i \notin u_i^+ \vec{C} x^-$. Thus $E_i \subseteq E(C - x_j)$. Similarly, $E_j \subseteq E(C - x_i)$. By (8), $E_i \subseteq E(C - u_j \vec{C} y)$ and $E_j \subseteq E(C - u_i \vec{C} x)$. The vertices of $V(C) - (u_i^+ \vec{C} x^- \cup u_j^+ \vec{C} y^-)$ are contained in a cycle $D =$

$x_i u_i \overleftarrow{C} x_i^+ x_i^- \overleftarrow{C} y x \overrightarrow{C} x_j^- x_j^+ \overrightarrow{C} u_j x_j \pi x_i$. Clearly, $E_i \cup E_j \subseteq E(D)$. Using (9) and similar arguments as before, we can insert the vertices of $(u_i^+ \overrightarrow{C} x^- \cup u_j^+ \overrightarrow{C} y^-) \cap V(H)$ in D to obtain a cycle containing more vertices of H than C, a contradiction.

Case 2. x and y have a common insertion edge $vv^+ \in E(C)$.

Assuming Case 1 does not apply, choose y as close to u_j along $u_j^+ \overrightarrow{C} a_j$ as possible subject to the other conditions. By the definition of insertion edges, $vv^+ \notin S_i \cup S_j$. Furthermore, the choice of x and y implies

$$vv^+ \notin E_i \cup E_j . \tag{10}$$

Combining (4), (8), (9), and (10), we get $E_i \cup E_j \subseteq E(C - (\{x_i, x_j\} \cup u_i \overrightarrow{C} x \cup u_j \overrightarrow{C} y)) - \{vv^+\}$ and $E_i \cap E_j = \varnothing$. If $v \in x_i^+ \overrightarrow{C} x_j^-$, define $D = x_i u_i \overleftarrow{C} x_i^+ x_i^- \overleftarrow{C} y v \overleftarrow{C} x v^+ \overrightarrow{C} x_j^- x_j^+ \overrightarrow{C} u_j x_j \pi x_i$; if $v \in x_j^+ \overrightarrow{C} x_i^-$, define $D = x_i u_i \overleftarrow{C} x_i^+ x_i^- \overleftarrow{C} v^+ y \overrightarrow{C} v x \overrightarrow{C} x_j^- x_j^+ \overrightarrow{C} u_j x_j \pi x_i$. Clearly, in both cases $E_i \cup E_j \subseteq E(D)$, and we can derive a contradiction with the choice of C by inserting the vertices of $(u_i^+ \overrightarrow{C} x^- \cup u_j^+ \overrightarrow{C} y^-) \cap V(H)$ in D as before. This completes the proof of (7).

By (5) and (6), $a_0 a_i \notin E(G)$ and $N(a_0) \cap N(a_i) = \varnothing$, and, by (7), $a_i a_j \notin E(G)$. It suffices to show $N(a_i) \cap N(a_j) = \varnothing$. Suppose $v \in N(a_i) \cap N(a_j)$. Then similar arguments as in the proof of Case 1 of (7) show $v \in V(C)$. By (5) and (7), $v \notin u_i^+ \overrightarrow{C} a_i^- \cup u_j^+ \overrightarrow{C} a_j^- \cup \{x_i, x_j\}$. If $v \in x_{i+1}^+ \overrightarrow{C} x_j^- \cup x_{j+1}^+ \overrightarrow{C} x_i^-$, then the noninsertibility of a_i and a_j, and (7) imply that $G[\{v, v^+, a_i, a_j\}] \cong K_{1,3}$, a contradiction. If $v \in a_i^+ \overrightarrow{C} x_{i+1} \cup a_j^+ \overrightarrow{C} x_{j+1}$, assume without loss of generality, $v \in a_i^+ \overrightarrow{C} x_{i+1}$. Then, considering $G[\{v, v^+, a_i, a_j\}]$ and using (7) and the noninsertibility of a_j, we obtain $a_i v^+ \in E(G)$. Let w be the first vertex of $u_j^+ \overrightarrow{C} a_j$ adjacent to v. As before we can insert the vertices of $(u_i^+ \overrightarrow{C} a_i^- \cup u_j^+ \overrightarrow{C} w^-) \cap V(H)$ in the cycle $x_i u_i \overleftarrow{C} x_i^+ x_i^- \overleftarrow{C} w v \overleftarrow{C} a_i v^+ \overrightarrow{C} x_j^- x_j^+ \overrightarrow{C} u_j x_j \pi x_i$ to derive a contradiction with the choice of C. (Note that $v \neq x_j^-, x_j$ by (5).) Hence $v \in x_i^+ \overrightarrow{C} u_i \cup x_j^+ \overrightarrow{C} u_j$. Assume without loss of generality $v \in x_i^+ \overrightarrow{C} u_i$. Considering $G[\{v^-, v, a_i, a_j\}]$ and using (7) and the noninsertibility of a_j, we obtain $a_i v^- \in E(G)$ and $v^- \neq x_i$ (by (5)). Choose $b_j \in u_j^+ \overrightarrow{C} a_j$ as close to u_j along $u_j^+ \overrightarrow{C} a_j$ as possible such that $vb_j \in E(G)$. As before we can insert the vertices of $(u_i^+ \overrightarrow{C} a_i^- \cup u_j^+ \overrightarrow{C} b_j^-) \cap V(H)$ in the cycle $x_i u_i \overleftarrow{C} v b_j \overrightarrow{C} x_i^- x_i^+ \overrightarrow{C} v^- a_i \overrightarrow{C} x_j^- x_j^+ u_j \overleftarrow{C} x_j \pi x_i$ to derive a contradiction with the choice of C. Hence the set $\{a_0, a_1, \ldots, a_t\}$ consists of at least $\kappa(G) + 1$ vertices of H which are pairwise at distance at least three in G, showing that $\alpha_3(H) > \kappa(G)$, our final contradiction. ∎

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