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CYCLIC HOMOLOGY AND EQUIVARIANT THEORIES

by Jean-Luc BRYLINSKI (*)

Introduction.

We discuss several equivariant theories for the action of a compact Lie group G on a C^∞ manifold X. Two such theories are well-known : equivariant cohomology and equivariant K-theory ; the relation between them is given by a theorem of Atiyah and Segal [1]. For the case G abelian, P. Baum, R. MacPherson and I have introduced in [2] a new equivariant cohomology theory, indexed by integers modulo 2. This theory is based on a mixture of de Rham theory and representation rings. Its main advantage is that this theory is isomorphic, via a Chern character, to equivariant K-theory tensored with **C**.

In this article, we discuss this theory for $G = S^1$, where one may present the somewhat obtruse constructions of [2] in a much simpler and more geometric form.

Another sort of equivalent theory is obtained by considering the periodic cyclic homology of the *smooth crossed-product algebra* $\mathcal{A} = C_c^\infty(G \times X)$, endowed with a convolution product (see § 2). We have proven [6] that this cyclic homology is the same as a « scalar extension » of equivariant K-theory (the precise result is stated in § 3). A (future) advantage in knowing such a result is to use the rich and powerful formalism of cyclic homology for the study of compact group actions.

Since the crossed-product algebra makes sense even if G is non-compact, its cyclic homology is then a candidate for an equivariant

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theory (equivariant K-theory does not extend well to non-compact groups). The results already obtained in [6], on the Hochschild and cyclic homology of that algebra, seem quite promising.

1. Delocalized equivariant cohomology.

Let X be a locally compact space, countable at infinity, and let G be a compact group which acts on it. There are two well-known theories in this context.

(*a*) The equivariant cohomology groups $H^k_G(X)$ are defined by Borel as $H^k_G(X) = H^k(EG \overset{G}{\times} X)$, where EG is a contractible space on which G acts freely, and $EG \overset{G}{\times} X$ is the quotient of $EG \times X$ by the diagonal action of G. The cohomology ring $H^*_G(X)$ is an algebra over the cohomology ring $H^*(BG)$ of $BG = (EG)\backslash G$, the classifying space for G [5]. We will actually use the notation $H^*_G(X)$ for the corresponding theory, *with compact supports.*

(*b*) The equivariant K-theory group $K^i_G(X)$. One definition of $K^0_G(X)$ is as the Grothendieck group for pairs $E_0 \xrightarrow{h} E_1$, where E_0, E_1 are G-equivariant vector bundles, h is a G-equivariant morphism of vector bundles, which is an isomorphism outside some *compact subset of* X. To be precise, the class of $E_0 \xrightarrow{\text{id}} E_0$ is decreed to be 0, and homotopically equivalent pairs are identified. For $i \geqslant 0$, one puts: $K^i_G(X) = K^0_G(X \times \mathbf{R}^i)$. As shown by Segal [13], Bott periodicity holds for equivariant K-theory, so we have a theory indexed by integers mod 2. Each group $K^i_G(X)$ is a module over the ring R(G) of (virtual) representations of G. Indeed, for each (finite-dimensional) representation of G in a vector space V, one has a corresponding G-equivariant vector bundle $\tilde{V}$ on X. The class of V acts on the class $E_0 \xrightarrow{h} E_1$ in $K^0_G(X)$ by transforming it to $E_0 \otimes \tilde{V} \xrightarrow{h \otimes \text{id}} E_1 \otimes \tilde{V}$.

There is a Chern character $K^0_G(X) \xrightarrow{\text{ch}} H^{\text{even}}_G(X)$ and $K^1_G(X) \xrightarrow{\text{ch}} H^{\text{odd}}_G(X)$. For X a point, this Chern character is an algebra morphism $R(G) \xrightarrow{\text{ch}} H^*(BG, \mathbf{Q})$; this identifies

$$H^{**}(BG, \mathbf{Q}) = H^{\text{even}}(BG, \mathbf{Q})$$

with the completion of $R(G) \otimes \mathbf{Q}$ with respect to the maximal ideal I consisting of representations of virtual dimension 0. For instance, $R(S^1)$ is the Laurent polynomial ring $Z[t, t^{-1}]$, $H^{**}(BS^1, \mathbf{Q})$ is the formal power series ring $\mathbf{Q}[[X]]$, and $\mathrm{ch}(t) = e^X$. Now we may recall the following :

THEOREM (Atiyah-Segal [1]). — *As an* $R(G)$*-module,* $H_G^{\mathrm{even}}(X, \mathbf{Q})$ *is isomorphic, via* ch, *to the completion of the* $R(G)$*-module* $K_G^0(X) \otimes \mathbf{Q}$ *(resp.* $H_G^{\mathrm{odd}}(X)$ *to the completion of* $K_G^1(X) \otimes \mathbf{Q}$*) with respect to the ideal* I.

Now one must beware that $\mathrm{ch}: K_G^0(X) \to H_G^{\mathrm{even}}(X)$ need not be injective; indeed, for G a finite group, X a point, it will not be injective. The reason is that, according to the above theorem, $H_G^*(X)$ is a localization (and completion) at the element 1 of G, of the R(G)-module $K_G^0(X)$. If we think of R(G) as a ring of functions on G, the ideal I corresponds to the origin of the group.

In this sense, equivariant cohomology is *localized* at the origin of G. In [2], Baum, MacPherson and I introduced a theory $H^*(G, X)$, indexed by integers mod 2, for the following situation : X is a finite-dimensional manifold, G an *abelian* compact Lie group acting smoothly on X. Since [2] is hard to fathom, I will unravel the construction for $G = S^1$. First, we have the general notion of a *G-basic differential form* ω on X ; ω is basic if and only if it is G-invariant and satisfies $i(\xi)\omega = 0$, for any $\xi \in \mathrm{Lie}(G)$, which gives rise to a vector field on X, denoted by the same letter.

The following basic theorem is due to J.-L. Koszul.

THEOREM (Koszul [11]). — *The cohomology of the complex* $\Omega^*_{\mathrm{bas},c}(X)$ *of compactly-supported, basic differential forms on* X, *is isomorphic to the compactly-supported cohomology of the quotient space* X/G.

Here a confession is in order : we stated this in [2] as if it was a new result. The fact of the matter is we proved this 30 years after Koszul ; we were very happy with it, and the specialists we talked to did not know of this article of Koszul. But he himself did remember it, and pointed it out to me during my lecture in Grenoble. I now realize that I am even more dependent on Koszul's work than I thought.

Now $H^*(S^1, X)$ will be the cohomology of a 2-term complex $A^0 \underset{d}{\overset{d}{\rightleftarrows}} A^1$, which is constructed from an S^1-invariant fixed riemannian metric on X. I need some notation: let $Y = X^{S^1}$ be the submanifold of fixed points, and for each $\zeta \in S^1$ of finite order, let X^ζ be the fixed point manifold of ζ. Using the metric, for T a small tubular neighborhood of Y in X, there is a projection $p: T - Y \to Y$; let $p_\zeta : T^\zeta - Y \to Y$ be the restriction of p to $T^\zeta - Y = (T - Y) \cap X^\zeta$. Let ξ be the vector field on X deduced from the action of S^1. On $X - Y$, ξ does not vanish, and we may construct a 1-form β such that $i(\xi)\beta = 1$ and $i(\eta)\beta = 0$ if the vector field η is orthogonal to ξ in the riemannian metric. It is easy to check that the 2-form $\alpha = d\beta$ is S^1-basic on $X - Y$. Now an element of the complex $A^\bullet$ is a family of basic differential forms ω_ζ on $X^\zeta - Y$, for $\zeta \in S^1$ of finite order, together with a finite sum $\sum_{n \in \mathbf{Z}} \psi_n . t^n$ (where t is a formal parameter, and each ψ_n is a differential form on Y), with the following properties:

(i) only finitely many ω_ζ are $\neq 0$; the support of each ω_ζ has compact closure in X^ζ;

(ii) each ψ_n.is compactly-supported on Y;

(iii) for each ζ, there exists a neighborhood U of Y in X^ζ, contained in T^ζ, such that ω_ζ is equal to $\sum_n \zeta^n . p_\zeta^*(\psi_n) \wedge e^{n.\alpha}$ on $X - Y$.

Now A^0 (resp. A^1) consists of families with all ω_ζ and ψ_n of even (resp. odd) degree. The differential of the complex acts by exterior differentiation on the ω_ζ and on the ψ_n.

Notice that the ω_ζ have singularities along Y, because the 2-form α is singular along Y. We next define an action of $R(S^1) = \mathbf{Z}[t, t^{-1}]$ on the complex $A^\bullet$. The generator t acts by $t.(\omega_\zeta, \Sigma\psi_n . t^n) = (\zeta . \omega_\zeta \wedge e^\alpha, \Sigma\psi_n . t^{n+1})$. So the cohomology groups $H^i(G,X)$ of A^* are $R(S^1)$-modules.

Although this might look like an esoteric construction, it affords the construction of a Chern Character ch : $K^0_{S^1}(X) \to H^0(S^1, X)$. So consider $E_0 \xrightarrow{h} E_1$, where h is an isomorphism outside of a compact subset K of X. First, the restriction of each $E_i (i=0,1)$ to Y splits as a direct sum $E_i = \bigoplus_{n \in \mathbf{Z}} (E_{i,n})t^n$, where $E_{i,n}$ is simply a vector bundle on Y. Now $h_{|Y}$ maps $E_{0,n}$ to $E_{1,n}$ and is an isomorphism outside $K \cap Y$. We choose connections on $E_{0,n}$ and $E_{1,n}$, which correspond, outside of some compact subset of Y, via $h_{|Y}$. Now we have the Chern characters ch $(E_{i,n})$ computed from these connections; hence $\psi_n = \mathrm{ch}(E_{0,n}) - \mathrm{ch}(E_{1,n})$ has compact support on Y.

There are only finitely many $\zeta \in S^1$, of finite order, for which $K \cap X^\zeta$ is strictly bigger than $K \cap Y$. For each such ζ, pick a neighborhood U_ζ of Y in X^ζ such that $(E_i)_{|U_\zeta}$ is isomorphic to $\oplus\, p_\zeta^*(E_{i,n}) \otimes t^n$ (since t^n is a 1-dimensional character of S^1, it gives a rank 1-vector bundle, S^1-equivariant, on X). Choose these isomorphisms compatibly with h, outside of some compact subset of U_ζ. Endow $p_\zeta^*(E_{i,n})$ with the pull-back connection, and the line bundle t with the connection given by the 1-form β (since t is trivial as a vector bundle); then $p_\zeta^*(E_{i,n}) \otimes t^n$ is given the tensor product connection, and we have a connection on the direct sum $\bigoplus_n p_\zeta^*(E_{i,n}) \otimes t^n$, hence on $(E_i)_{|U^\zeta - Y}$. Now say that $\zeta \in S^1$ is of order exactly m. Since ζ acts trivially on X^ζ, $(E_i)_{|X_\zeta}$ decomposes as a direct sum $\bigoplus_{\lambda^m=1} (E_i)_\lambda$, where ζ acts on $(E_i)_\lambda$ as λ. Our connections are compatible with these decompositions. For each λ, we put a connection $\nabla_{i,\lambda}$ on $(E_i)_\lambda$, so that:

(i) $\nabla_{i,\lambda}$ coincides near Y with the previous (pull-back) connection;

(ii) $\nabla_{i,\lambda}$ is basic, i.e. $\nabla_{i,\lambda}(\xi)$ acts in the same way as ξ viewed as an element of Lie (S^1);

(iii) $\nabla_{0,\lambda}$ and $\nabla_{1,\lambda}$ correspond, via h, outside of some set, which is relatively compact in X^ζ.

Then we put $\omega_\zeta = \sum_{\lambda^m=1} \lambda.(\mathrm{ch}\,(\nabla_{0,\lambda}) - \mathrm{ch}\,(\nabla_{1,\lambda}))$.

From (i), we get $\omega_\zeta = \sum_n \xi^n . p_\zeta^*(\psi_n) \wedge e^{n.\alpha}$ near Y.

From (ii), ω_ζ is a basic differential form on $X^\zeta - Y$.

From (iii), its support has compact closure in X^ζ.

This gives the Chern character $K^0_{S^1}(X) \xrightarrow{\mathrm{ch}} H^0(S^1, X)$. If we endow **R** with the standard metric, and $X \times \mathbf{R}$ with the product metric, integration along the fibers gives a map of complexes

$$\begin{array}{ccc} A^0_{X\times\mathbf{R}} & \overset{d}{\underset{d}{\rightleftarrows}} & A^1_{X\times\mathbf{R}} \\ \downarrow & & \downarrow \\ A^1_X & \overset{d}{\underset{d}{\rightleftarrows}} & A^0_X . \end{array}$$

This is a quasi-isomorphism. Therefore, the « odd » Chern character may be defined by

$$K^1_{S^1}(X) = K^1_{S^1}(X, \mathbf{R}) \to H^1(S^1, X) = H^0(S^1, X \times \mathbf{R}).$$

THEOREM [2]. — ch *induces an isomorphism of* $K^i_{S^1}(X) \underset{\mathbf{Z}}{\otimes} \mathbf{C}$ *with* $H^i(S^1, X)$, *for* $i = 0, 1$.

Let us give an idea of the proof. We use the 6-term exact sequences for $Y \hookrightarrow X \hookleftarrow X - Y$, both for equivariant K-theory and the delocalized theory. Using the lemma, we are basically reduced to 2 cases :

(*a*) $Y = \varphi$, i.e. $A^\bullet$ is the direct sum, indexed by $\zeta \in S^1$ of finite order, of the complex $\Phi^*_{c,\mathrm{bas}}(X^\zeta)$. Therefore we have (using Koszul's theorem)

$$H^0(S^1, X) = \bigotimes_{\substack{\zeta \in S^1 \\ \text{of finite order}}} H^{\mathrm{even}}_c(X^\zeta, \mathbf{C}).$$

It is also well-known (since all stabilizer subgroups are finite) that $K^0_{S^1}(X) = \bigoplus_\zeta K^0(X^\zeta)$, so the Chern character is given by $\zeta.\mathrm{ch}$ on $K^0(X^\zeta)$, where $\mathrm{ch} : K^0(X^\zeta) \to H^{\mathrm{even}}_c(X^\zeta, \mathbf{C})$ is the ordinary Chern character. So we reduce to ordinary K-theory.

(*b*) $X = Y$, i.e. S^1 acts trivially on X.

Then $H^0(S^1, X) = H^{\mathrm{even}}(X) \otimes R(S^1)$, $K^0_{S^1}(X) = K^0(X) \underset{\mathbf{Z}}{\otimes} R(S^1)$. The Chern character is $\mathrm{ch} \otimes \mathrm{id}$, where again $\mathrm{ch} : K^0(X) \to H^{\mathrm{even}}(X, \mathbf{C})$ is the ordinary Chern character.

Because of this theorem, we may think of the delocalized theory $H^i(S^1, X)$ as a concrete way to compute $K^i_{S^1}(X) \otimes \mathbf{C}$. The construction of the complex $A^\bullet$ gives in fact a complex of fine sheaves on the quotient space $X \backslash S^1$. It is of course a very good thing, to have a complex of sheaves computing a given cohomology theory. Many properties of equivariant K-theory, like Bott periodicity, functoriality for maps $X \xrightarrow{p} Y$ such that $TX \oplus p^{-1}TY$ has an S^1-invariant Spin^c-structure, may be derived on the sheaf-theoretic level. Also, I have established a form of Poincaré duality for the delocalized theory.

Let us prove, in this context, the *concentration theorem* of G. Segal. Let $\theta \in S^1$ be an element of infinite order.

PROPOSITION. – *The restriction map* $H^*(S^1, X) \to H^*(S^1, Y)$ *induces an isomorphism, after localization at* $\theta \in S^1$.

Proof. – It is enough to show that the $R(S^1)$-module $H^*(S^1, X-Y)$ becomes 0 after localization at θ. This is the cohomology of the

direct sum
$$\underset{\substack{\zeta \in S^1 \\ \text{of finite order}}}{\zeta} \quad \Omega^*_{c,\,\mathrm{bas}}(X^\zeta - Y).$$

For each ζ, the character t of S^1 acts on $\Omega^*_{c,\,\mathrm{bas}}(X^\zeta - Y)$ by multiplication by $\zeta . e^\alpha$. Let $N = \left[\dfrac{\dim(X)-1}{2}\right]$. Then $(t-\zeta)^N$ acts trivially on the complex $\Omega^*_{c,\,\mathrm{bas}}(X^\zeta - Y)$. Since $\theta \neq \zeta$, this $R(S^1)$-module has 0 localization at θ. So the same holds for the cohomology of this complex.

This delocalized theory should have applications to the index theorem for transversally elliptic operators. For this purpose, it will be necessary to extend it to singular spaces. Since equivariant-cohomology for the action of S^1 on loop spaces was shown by Atiyah, Bismut, Witten and others to be closely related to the Atiyah-Singer index theorem, it may be worthwhile to mention that the delocalized theory makes sense for an S^1-action on a hilbertian manifold (or, rather, a non-compactly supported version of the theory, based on a projective limit of complexes of sheaves).

2. Cyclic homology of crossed product algebras.

To a compact Lie group G acting smoothly on a C^∞-manifold X, one associates the *smooth crossed-product algebra* $\mathcal{A} = C_c^\infty(X \times G)$, with the convolution product

$$f_* h(x, \gamma) = \int_\Phi (f(x,g) . h(g.x, \gamma . g^{-1})\, dg$$

where dg is a fixed left Haar measure on G.

Even though $\mathcal{A}$ is an algebra without unit (in general), it is a complete locally convex topological algebra, therefore one may consider its Hochschild homology groups $HH_*(\mathcal{A})$ and cyclic homology group $HC_*(\mathcal{A})$, using completed tensor products as Connes does [7, II, § 5], and the Connes exact sequence, relating the two theories, still holds.

We recall, for the reader's convenience, that $HH_*(\mathcal{A})$ is defined as the homology of the standard complex $(\hat{\mathcal{A}}^{\otimes(n+1)}, b)$, where b is the standard Hochschild boundary. Here $\hat{\mathcal{A}}^{\otimes(n+1)}$ denotes the completed projective tensor product of $\mathcal{A}$ with itself, $n+1$ times. Since $\mathcal{A}$ is without unit, $HC_*(\mathcal{A})$ is the total homology of a double complex, which is described in [8], in [9] and in [12]. Cyclic homology comes with a periodicity operator $S : HC_l(\mathcal{A}) \to HC_{l-2}(\mathcal{A})$, a natural map $I : HH_l(\mathcal{A}) \to HC_l(\mathcal{A})$ and a map $B : HC_{l-1}(\mathcal{A}) \to HH_l(\mathcal{A})$. These fit into the exact sequence of Connes

$$\ldots \to HC_{l-1}(\mathcal{A}) \xrightarrow{B} HH_l(\mathcal{A}) \xrightarrow{I} HC_l(\mathcal{A}) \xrightarrow{S} HC_{l-2}(\mathcal{A}) \to \ldots$$

There is also a spectral sequence of Connes, from Hochschild to cyclic homology, which we will describe later. We use the notation $HC_i^{per}(\mathcal{A})$ for the periodic cyclic homology, indexed by integers i modulo 2, which is defined as the inverse limit of $HC_{i+2n}(\mathcal{A})$, under the operator S. We refer to [7], [8], [9] and [12] for details and further information.

If our group G is trivial, $\mathcal{A}$ is just $C_c^\infty(X)$; in that case, $HH_i(C_c^\infty(X)) = \Omega_c^i(X)$ and the periodic cyclic homology is the compactly-supported de Rham cohomology; this is due to Connes, who proves it in [7] for the case X compact.

There are several *a priori* indications that the periodic cyclic homology of $\mathcal{A}$ is related to the equivariant K-theory $K_G^*(X)$. First, in case $G = \{1\}$, Connes showed {7, II, § 5} that $HC_*^{per}(\mathcal{A}) = H_c^*(X, \mathbf{C})$ and we have the usual isomorphism between $H_c^*(X, \mathbf{C})$ and $K^*(X) \otimes \mathbf{C}$. For G finite, A. Wasserman (unpublished) proved that $HC_*^{per}(\mathcal{A})$ is isomorphic to $K_G^*(X) \otimes \mathbf{C}$. If the action of G on X is free, the quotient space X/G is a manifold, the algebra $\mathcal{A}$ is Morita-equivalent to $C_c^\infty(X/G)$, so it has the same cyclic homology; also $K_G^*(X) = K^*(X/G)$, so again the equality $HC_*^{per}(\mathcal{A}) = K_G^*(X) \otimes \mathbf{C}$ holds.

Another strong hint is the theorem of P. Julg [10] which gives an isomorphism $K_i(\mathcal{A}) \cong K_G^i(X)$ $(i=0,1)$. Let us just explain here that the Chern character of Connes is a map $K_i(\mathcal{A}) \to HC_*^{per}(\mathcal{A})$ [7], and that in our case, using Julg's theorem, it may be viewed as a map from $K_G^i(X)$ to $HC_i(\mathcal{A})$.

The following theorem was obtained by us in August 1985 (see [6] for a proof), and later re-derived by J. Block, by a somewhat different method [4].

THEOREM. — *Recall* G *is a compact Lie group acting smoothly on the smooth manifold* X, *and* $\mathcal{A} = C_c^\infty(X \times G)$ *is the convolution algebra.*

(i) $HC_c^{per}(\mathcal{A})$ *has a natural structure of* $R_\infty(G)$*-module, where* $R_\infty(G)$ *is the algebra, under pointwise multiplication, of central* C^∞*-functions on* G.

(ii) $ch : K_G^i(X) \to HC_i^{per}(\mathcal{A})$ *is* R(G)*-linear and induces an isomorphism :* $K_G^i(X) \underset{R(G)}{\otimes} R_\infty(G) \xrightarrow{\sim} HC_i^{per}(\mathcal{A})$.

This theorem is not too hard to prove, once Mayer-Vietoris has been established for the cyclic homology of the crossed-product algebra. It is not clear *a priori* whether the theorem sheds any new light on equivariant K-theory. After all, cyclic homology is not such an easily computable gadget. What happens here is that the Hochschild homology groups have a very nice description, in terms of differential forms on an auxiliary space $\hat{X}$ I will now describe. $\hat{X}$ is a closed subspace of $X \times G$; it consists of all pairs (x, g) such $g.x = x$. In other words, we have a projection map $\hat{X} \to G$, with fibre over g equal to the fixed point at X^g. G acts on $\hat{X}$ by $\gamma.(x,g) = (\gamma.x, \gamma.g.\gamma^{-1})$.

We will need the notion of relative differential form, relative to the projection $\hat{X} \to G$. Intuitively, this is a C^∞-family ω_g of differential forms on the manifold X^g, parametrized by $g \in G$. More precisely, let $\Omega^i_{X \times G \to G}$ be the space of differential forms on $X \times G$, relative to the projection $X \times G \to G$; an element of $\Omega^i_{X \times G \to G}$ is a smooth family ω_g of differential forms on X, parametrized by $g \in G$. Let I be the ideal of $C^\infty(X \times G)$ consisting C^∞-functions which vanish on $\hat{X}$. Then define the graded differential algebra $\Omega^*_{\hat{X} \to G}$ as the quotient of $\Omega^*_{X \times G \to G}$ by the ideal generated by I and by dI (i.e., by the differential ideal generated by I). An element ω of $\Omega^*_{\hat{X} \to G}$ gives rise, for each $g \in G$, to a differential form ω_g on X^g.

DEFINITION. — *A relative differential form* $\omega = (\omega_g)_{g \in G}$ in $\Omega^*_{\hat{X} \to G}$ *is called* G*-basic if*

(*a*) ω *is* G*-invariant,*

(*b*) *for* $\forall \gamma \in G$, $\forall \xi \in \mathrm{Lie}(G_\gamma)$, $i(\xi)(\omega_\gamma) = 0$.

Remark. — G_γ is the centralizer of γ in G; any ξ in Lie (G_γ) induces a vector field on X^γ. Condition (*b*) could be replaced by (*b*′) for $\gamma \in G$, ω_γ is a G_γ-basic differential form on X^γ. One might wish to impose an apparently stronger condition than (*b*), replacing G_γ by the normalizer of the closed subgroup T of G generated by γ. However, as is well-known, the quotient group, isomorphic to N(T)/Z(T), is finite, so again one would get an equivalent notion of basic differential form.

The Hochschild homology of the algebra $\mathcal{A}$ is computed as follows.

PROPOSITION [6, Proposition 3.4]. — *For each* $p \geqslant 0$, $HH_p(\mathcal{A})$ *is isomorphic to the space of compactly-supported,* G*-basic elements of* $\Omega^p_{\hat{X}\to G}$.

Examples.

(1) If the action of G on X is free, then $\hat{X} = X$. The proposition says that $HH_p(\mathcal{A})$ is isomorphic to the space of compactly supported basic differential forms on X, i.e. to $\Omega_c^p(X/G)$. So $HH_p(\mathcal{A})$ is isomorphic to $HH_p(C_c^\infty(X/G))$. This isomorphism follows from the fact that $\mathcal{A}$ is Morita-equivalent to $C_c^\infty(X/G))$, and from the computation of $HH_p(X_c^\infty(Y))$, for Y a manifold, mentioned earlier.

(2) If G acts trivially on X, then $\hat{X} = X \times G$, and Proposition gives an isomorphism between $HH_p(\mathcal{A})$ and $\Omega_c^p(X) \hat{\otimes} C^\infty(G)^G$. Now we have $HH_p(C_c^\infty(X)) = \Omega_c^p(X)$, and for the convolution algebra $C^\infty(G)$, we have :

$$HH_0(C^\infty(G)) = C^\infty(G)^G \quad \text{and} \quad HH_i(C^\infty(G)) = 0 \quad \text{for} \quad i > 0$$

(note that $C^\infty(G)$ is more or less a direct sum of matrix algebras). This isomorphism is a special case of the Künneth theorem for Hochschild homology.

See [6] for the example of S^1 acting on S^2 by rotations. We have a complex of basic (relative) differential forms. We need to compute the cohomology of this complex. The result imitates Koszul's theorem in [11] (recalled in § 1). To state it, we need to introduce some weird-looking sheaf of algebras $\mathcal{B}$ on the quotient space $\hat{X}/G$.

DEFINITION. — $\mathcal{B}$ *is the subsheaf of* C^∞*-functions on* $\hat{X}/G$ *consisting on functions which are locally constant in the* X*-direction.*

So a section of $\mathcal{B}$ is a G-invariant function $F(x,g)$ of $(x,g) \in \hat{X}$ such that for any $g \in G$, $F(x,g)$ is constant on each connected component of X^g. If the action of G is free, so that $\hat{X} = X$, then $\mathcal{B}$ is just the constant sheaf C on X/G. If G acts trivially on X, so that $\hat{X} = X \times G$, $\mathscr{B}$ is the sheaf of G-invariant functions on X × G, which (locally) depend on G alone.

PROPOSITION. – *The cohomology of the complex of compactly-supported G-basic relative differential forms on* $\hat{X}$ *is equal to the cohomology* $H^*_c(\hat{X}/G, \mathscr{B})$ *of the sheaf* $\mathscr{B}$, *with compact supports.*

The relevance of this is that it computes the E^2-term of the Connes spectral sequence from Hochschild homology to cyclic homology. The E^1-term of the spectral sequence is $E^1_{p,q} = HH_{q-p}(\mathscr{A})$; the d^1-differential is induced by exterior differentiation of basic differential forms. Therefore we get :

COROLLARY. – *The* E^2*-term of the Connes spectral sequence is given by* $E^2_{p,q} = H^{q-p}_c(\hat{X}/G, \mathscr{B})$.

At this point, one might observe a close analogy of this E^2-term with the E^2-term of a spectral sequence of Segal, converging to $K^*_G(X)$. The E^2-term is $H^*_c(X/G, R(H))$, where $R(H)$ is a « constructible » sheaf on X/G, with stalk at the image of $x \in X$, equal to the representation ring $R(G_x)$. Let $\hat{X}/G \xrightarrow{\pi} X/G$ be the projection. Then $R(H)$ may be viewed as a subsheaf of $\pi_*(\mathscr{B})$, consisting of functions which are « algebraic » in the G-direction (and, of course, still locally constant in the X-direction). Notice that π is a proper map and the restriction of $\mathscr{B}$ to each fibre is a fine sheaf, hence $R^i\pi_!(\mathscr{B}) = 0$ for $i > 0$, and

$$H^k_c(\hat{X}/G, \mathscr{B}) = H^k_c(X/G, \pi_*\mathscr{B})$$

is isomorphic to $H^k_c(X/G, R(H)) \otimes_{R(G)} R_\infty(G)$. What happens here (although I did not write down a complete proof) is that the Connes spectral sequence is obtained, from the Segal spectral sequence, by scalar extension from $R(G)$ to $R_\infty(G)$.

In one spectral sequence, G is treated as a C^∞-manifold, in the other one, it is endowed with the Zariski topology (for which a basis of open sets is described by $\{g \in G; F(g) \neq 0\}$, where F is a matrix coefficient of a finite-dimensional representation of G).

Observation. – Endow X with its manifold topology, G with the Zariski topology. Then $\hat{X}$ is a closed subset of X × G.

Indeed, the question is local near each $x \in X$; then the appropriate piece of $\hat{X}$ is contained in $X \times G_x$. But the action of G_x may be linearized on a neighborhood of x. Then $\hat{X}$ is defined by the equation $g.x - x = 0$, which is algebraic as a function of g.

From the computation of $HH_p(\mathscr{A})$, one gets $HH_p(\mathscr{A}) = 0$ for $p > \dim(X/G)$. It follows from Connes spectral sequence that the periodicity operator $S : HC_p(\mathscr{A}) \to HC_{p-2}(\mathscr{A})$ is an isomorphism for $p > \dim(X/G) + 1$.

To explain why $HC_*(\mathscr{A})$ has a $R_\infty(G)$-module structure, one may point out, as above, that the sheaf $\pi_*(\mathscr{B})$ on X/G is a sheaf of $R_\infty(G)$-algebras. One may also, more directly, show that $HC_*(\mathscr{B})$ is the homology of a cyclic object, in the sense of Connes [8], which is $p \mapsto C_c^\infty(G \times X^{p+1})^G$, where for instance the cyclic permutation operator t is given by:

$$(t.F)(x_0, \ldots, x_p; g) = (-1)^p F(g.x_p, x_0, x_1, \ldots, x_{p-1}, g)$$

$R_\infty(G)$ acts on $C_c^\infty(X^{p+1} \times G)^G$ simply by point-wise multiplication. This action is compatible with t, and with the simplicial face maps. This cyclic object is somewhat simpler than the standard one (since it involves only one copy of G).

Since cyclic homology is a very concrete object (a cycle is a function of several variables, satisfying some functional equations), the cyclic homology of the crossed-product algebra $\mathscr{A}$ may be viewed as a very concrete de Rham sort of equivariant theory. Even in the case of a free G-action, it is fascinating to investigate geometric objects, like the curvature of a principal G-bundle, in terms of cyclic homology. For the case $G = S^1$, one may map directly the double complex of Connes, for the algebra $\mathscr{B}$, to a complex homotopically equivalent to the complex $A^\bullet$ of § 1, which defines delocalized equivariant cohomology. This will be discussed elsewhere. In the case where G is a non-compact Lie group, the crossed-product algebra $\mathscr{A}$ still makes sense. It seems reasonable to ask what its periodic cyclic homology is, and whether it is a reasonable equivariant theory for smooth actions of G on manifolds.

We obtained a number of results in [6]. We will explain what we know in the case G is discrete. As in the case of compact groups, one obtains a very good description of Hochschild homology. First one has a decomposition of $HH_p(\mathscr{A})$ into direct summands $HH_p(\mathscr{A})_{\mathcal{O}}$, indexed by conjugacy classes $\mathcal{O}$ in G. For the case X is a point, this is due to Burghelea [14]. If $\mathcal{O}$ is the class of γ, $HH_p(\mathcal{A})_{\mathcal{O}}$ is the abutment of a spectral sequence

$$E^2_{p,q} = H_q(G_\gamma, H_p(C^\infty_c(X), (C^\infty_c(X)(\gamma))$$

where $H_p(C^\infty_c(X), (C^\infty_c(X)(\gamma))$ is the Hochschild homology of the algebra $C^\infty_c(X)$, with values in the bimodule $C^\infty_c(X)(\gamma)$; as a vector space, $C^\infty_c(X)(\gamma)$ is isomorphic to $C^\infty_c(X)$, with left $C^\infty_c(X)$-module structure twisted by γ.

This is basically an unraveling of a spectral sequence due to Feigin and Tsygan [9]. Geometrically, something very interesting happens when γ has the property that the diagonal of X and the graph of the diffeomorphism γ intersect cleanly inside $X \times X$ (i.e., at every point of their intersection, the intersection of the tangent spaces is the tangent space to the intersection X^γ). Then $H_p(C^\infty_c(X), (C^\infty_c(X)(\gamma))$ is isomorphic to $\Omega^p_c(X^\gamma)$, and the above spectral sequence degenerates.

The cyclic homology itself decomposes into groups $HC_p(\mathcal{A})_{\mathcal{O}}$, one for each conjugacy class $\mathcal{O}$ in G. The groups $HC_*(\mathcal{A})$, are computable from Van Est complexes, which are double complexes of inhomogeneous chains of the group G with values in $\Omega^q_c(X)$. In general precise results are obtained in loc.-cit. if $\mathcal{O}$ is the conjugacy class of a nice γ in the sense discussed above. The case where γ is not so nice is much more difficult. However, complete results may be obtained when $X = S^2$, and G is any discrete subgroup of $SL(2, \mathbf{C})$.

It is clear that the results fit very well with conjectures of Baum and Connes [3], and indeed cyclic homology might be a good approach to these. To conclude, let us reformulate the computation of the cyclic homology of $\mathcal{A}$ (assuming all $\gamma \in G$ give nice diffeomorphisms) in terms of the subspace $\hat{X} \subset X \times G$ which was so important for compact group actions. Using $\hat{X} = \coprod_{\gamma \in G} X^\gamma \times \{\gamma\}$ and the Shapiro lemma, we obtain a spectral sequence $E^2_{p,q} = H_q(G, \Omega^p_c(\hat{X}))$, which converges to $HH_*(\mathcal{A})$ and degenerates at E^2.

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