Cyclic matching sequencibility of graphs

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Abstract

We define the cyclic matching sequencibility of a graph to be the largest
integer d such that there exists a cyclic ordering of its edges so that every
d consecutive edges in the cyclic ordering form a matching. We show that
the cyclic matching sequencibility of K_{2m} and K_{2m+1} equals $m - 1$.

1   Introduction

Let $G \subseteq K_n$ be a graph of order n (without loops or multiple edges) with p edges.
Alspach [1] defined the following notions. The *matching number* of a linear ordering
$[1], [2], \ldots, [p]$ of the edges of G (we use $[i]$ to denote the ith edge in an ordering of
the edges of G) is the largest number d such that every d consecutive edges in the
ordering form a matching of G. The *matching sequencibility* of G, denoted $\mathrm{ms}(G)$, is
the maximum matching number of a linear ordering of the edges of G. Clearly, the
matching sequencibility of G is bounded above by the largest number of edges in a
matching of G and this in turn is bounded above by $\lfloor \frac{n-1}{2} \rfloor$ if n is odd. If n is even and
G itself is a perfect matching, then $\mathrm{ms}(G) = \frac{n}{2}$. If n is even and G is not a matching,
then $\mathrm{ms}(G)$ cannot equal $\frac{n}{2}$ because the edge $[1]$ and the edge $[\frac{n}{2} + 1]$ would have to
be identical. Thus, if n is even, provided G is not a matching, $\mathrm{ms}(G) \leq \lfloor \frac{n-1}{2} \rfloor$.

The matching sequencibility of K_2 is clearly 1. Using the Walecki decomposition of the complete graph K_n with odd $n \geq 3$ into Hamilton cycles and the decomposition of K_n with even $n \geq 4$ into Hamilton paths, Alspach [1] showed how to order the edges of K_n $(n \geq 3)$ to get a matching number equal to $\lfloor \frac{n-1}{2} \rfloor$. Thus

$$\mathrm{ms}(K_n) = \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{1}$$

Let $G \subseteq K_{m,n}$ be a bipartite graph with p edges, and let A be an $m \times n$ biadjacency matrix of G. Then an ordering of the p edges of G corresponds to a bijective replacement of the p 1s of A with the integers $1, 2, \ldots, p$ resulting in a matrix $\widehat{A}$. The matching sequencibility of this ordering is the largest integer k such that every set of k consecutive integers lie in different rows and columns.

For a complete bipartite graph $K_{m,n}$ with $m \leq n$, it is not difficult to show that

$$\mathrm{ms}(K_{m,n}) = \begin{cases} n-1 & \text{if } m = n \\ m & \text{if } m < n. \end{cases}$$

The matching sequencibility is certainly bounded above by these numbers. Using the biadjacency matrix A, it is straightforward to order the edges (the 1s of A) to show that these upper bounds can be attained. In this representation, a matching of size k corresponds to a set of k 1s of A no two from the same row and column. We illustrate this for $K_{4,4}$ and $K_{4,6}$ whose biadjacency matrices are the 4×4 and 4×6 matrices $J_{4,4}$ and $J_{4,6}$ of all 1s, respectively.

$$(K_{4,4}) : \widehat{A} = \begin{bmatrix} 1 & 5 & 9 & 13 \\ 14 & 2 & 6 & 10 \\ 11 & 15 & 3 & 7 \\ 8 & 12 & 16 & 4 \end{bmatrix} \text{ and } (K_{4,6}) : \widehat{A} = \begin{bmatrix} 1 & 13 & 9 & 21 & 5 & 17 \\ 18 & 2 & 14 & 10 & 22 & 6 \\ 7 & 19 & 3 & 15 & 11 & 23 \\ 24 & 8 & 20 & 4 & 16 & 12 \end{bmatrix}.$$

Henceforth, to avoid trivialities, we assume that $n \geq 3$. We define a variation of the matching sequencibility of G by considering cyclic orderings of the edges of G. The *cyclic matching number* of a cyclic ordering $[1], [2], \ldots, [p], [1]$ of the edges of G is the largest number d such that every d consecutive edges in the cyclic ordering form a matching of G. The *cyclic matching sequencibility* of G, denoted $\mathrm{cms}(G)$, is the maximum cyclic matching number of a cyclic ordering of the edges of G. We have

$$\mathrm{cms}(G) \leq \mathrm{ms}(G) \leq \left\lfloor \frac{n-1}{2} \right\rfloor. \tag{2}$$

Clearly, $\mathrm{cms}(K_3) = 1$. The main result of this note is the following theorem.

Theorem 1.1 *For $n \geq 4$,*

$$\mathrm{cms}(K_n) = \begin{cases} \lfloor \frac{n-1}{2} \rfloor & \text{if } n \text{ is even} \\ \lfloor \frac{n-3}{2} \rfloor & \text{if } n \text{ is odd.} \end{cases}$$

Combining Theorem 1.1 and equation (1), we have:

Corollary 1.2 *For $n \geq 4$,*

$$\text{cms}(K_n) = \left\{ \begin{array}{ll} \text{ms}(K_n) & \text{if } n \text{ is even} \\ \text{ms}(K_n) - 1 & \text{if } n \text{ is odd.} \end{array} \right.$$

2   Proof of the Main Result

The proof is in three parts: constructions for the even case (I), constructions for the odd case (II), and a nonexistence argument for the odd case (III).
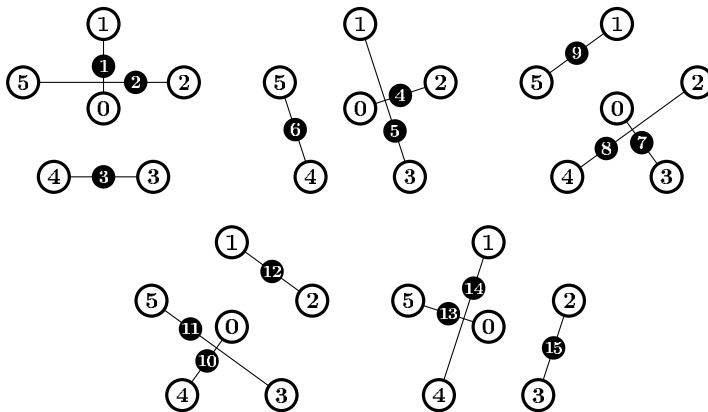

Figure 1

(I) We first show that if $m \geq 2$, then there is a cyclic labeling of the edges of K_{2m} whose cyclic matching number is $m - 1$. In view of (2), this will establish the first equality in Theorem 1.1.

Let the vertices of K_{2m} be labeled $0, 1, 2, \ldots, 2m - 1$, and consider the vertices $1, 2, \ldots, 2m-1$ as equally spaced points on the unit circle centered at a point labeled 0. Consider the perfect matching

$$M = \{\{0, 1\}, \{2, 2m - 1\}, \{3, 2m - 2\}, \ldots, \{m, m + 1\}\},$$

of K_{2m}. We label the edges of M as listed in the order $[1], [2], \ldots, [m]$. Let ϕ be the rotation about the origin by $360/(2m - 1)$ degrees. Then

$$M, \phi(M), \phi^2(M), \ldots, \phi^{2m-2}(M)$$

is a 1-factorization of K_{2m}, that is, $(2m - 1)$ perfect matchings of K_{2m} partitioning the edges of K_{2m}. We note that $\phi^{2m-1}(M) = M$. For $k = 1, 2, \ldots, 2m - 2$, the edges

$$\phi^k(\{0, 1\}), \phi^k(\{2, 2m - 1\}), \phi^k(\{3, 2m - 2\}), \ldots, \phi^k(\{m, m + 1\})$$

receive labels

$$[1 + mk], [2 + mk], [2 + mk] \ldots, [m + mk],$$

respectively. In this way we obtain an ordering $[1], [2], \ldots, [2m^2 - m]$ of the edges of K_{2m} which we regard as a cyclic ordering with $[1]$ following $[2m^2 - m]$. To verify that this cyclic labeling has cyclic matching number $m - 1$, it suffices to show that each set of $m - 1$ consecutive edges of $\phi^k(M), \phi^{k+1}(M)$ is a matching for each $k = 1, 2, \ldots, 2m - 2$. But this is straightforward to check using the definitions of M and ϕ. We illustrate this construction for $m = 3$, that is, K_6, in Figure 1.

(II) We next show that if $m \geq 2$, then there is a cyclic labeling of the edges of K_{2m+1} whose cyclic matching number is $m - 1$.

A *near-perfect matching* of K_{2m+1} is a matching of m edges. Such a matching meets all but one vertex of K_{2m+1}, and we call the missing vertex the *isolated vertex of the matching*. Let the vertices of K_{2m+1} be labeled $0, 1, 2, \ldots, 2m$, and consider the vertices $0, 1, 2, \ldots, 2m$ as equally spaced points on the unit circle centered at the origin. The near-perfect matching

$$M = \{\{1, 2m\}, \{2, 2m - 1\}, \ldots, \{m, m + 1\}\}$$

has vertex 0 as its isolated vertex. We label the edges of M as listed in the order $[1], [2], \ldots, [m]$ (according to the smallest vertex the edge contains). Let ϕ be the rotation about the origin by $360/(2m + 1)$ degrees. Then

$$M, \phi(M), \phi^2(M), \ldots, \phi^{2m}(M)$$

are $(2m+1)$ near-perfect matchings of K_{2m+1} and they partition the edges of K_{2m+1}. Each vertex of K_{2m+1} is an isolated vertex of one of these near-perfect matchings. For $k = 1, 2, \ldots, 2m$, the edges

$$\phi^k(\{1, 2m\}), \phi^k(\{2, 2m - 1\}), \ldots, \phi^k(\{m, m + 1\})$$

receive labels

$$[1 + mk], [2 + mk], \ldots, [m + mk],$$

respectively. In this way we obtain an ordering $[1], [2], \ldots, [2m^2 + m]$ of the edges of K_{2m+1} which we regard as a cyclic ordering with $[1]$ following $[2m^2 + m]$. (We remark that the Walecki decomposition of K_{2m+1} into Hamilton cycles proceeds with $2m$ vertices equally spaced on the unit circle and one at the origin and with one Hamilton cycle C. Then with θ equal to the rotation about the origin through $360/(2m)$ degrees $C, \theta(C), \ldots, \theta^{m-1}(C)$ is a Hamilton decomposition of K_{2m+1}.)

To verify that this cyclic labeling has cyclic matching number (at least) $m-1$, we first note that each $\phi^k(M)$ is a near-perfect matching and thus each of the two sets of $m-1$ consecutive edges of $\phi^k(M)$ (that is, its first $m-1$ edges and its last $m-1$ edges) is a matching. Also since $\phi(\phi^k(M)) = \phi^{k+1}(M)$ including $\phi(\phi^{2m}(M)) = \phi^{2m+1}(M) = M$, it is enough to show that any $m-1$ consecutive edges from $M \cup \phi(M)$ form a matching. But again this is straightforward to check using the definitions of M and ϕ. We illustrate this construction for $m = 3$, that is, K_7, in Figure 2.
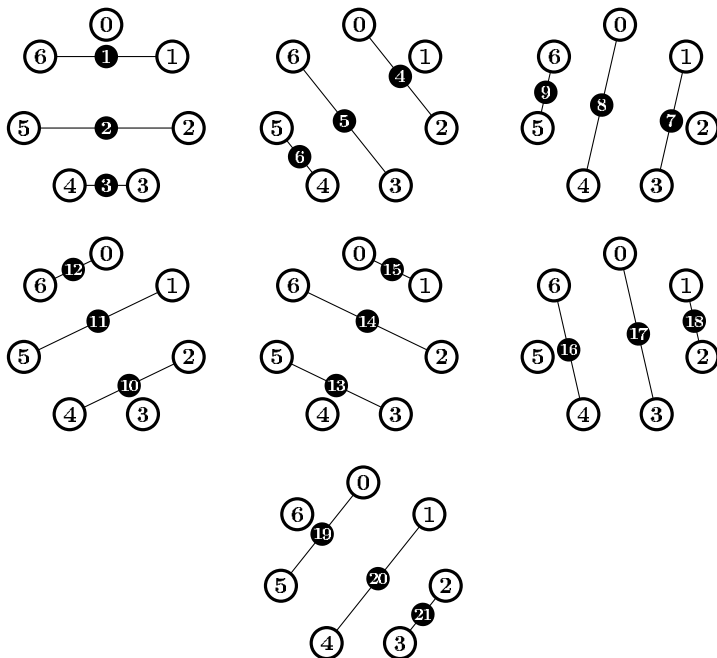

Figure 2

(III) Finally we show that if $m \geq 2$, then there does not exist a cyclic ordering of the edges of K_{2m+1} with cyclic matching number m. In view of (II) we conclude that $\mathrm{cms}(K_{2m+1}) = m - 1$.

Assume to the contrary that there is a cyclic ordering $[1], [2], \dots, [2m^2 + m], [1]$ of the edges of K_{2m+1} with cyclic matching number equal to m. Each set of m consecutive edges in the cyclic ordering is a near-perfect matching of K_{2m+1} with a unique isolated vertex. Let

$$M_{[i]} = \{[i], [i+1], [i+2], \dots, [i+m-1]\} \quad (i = 1, 2, \dots, 2m^2 + m)$$

where the edges in the matching $M_{[i]}$ are taken in the cyclic ordering. Since

$$M_{[i+1]} = \{[i+1], [i+2], \ldots, [i+m-1], [i+m]\},$$

the edge $[i+m]$ joins the isolated vertex of $M_{[i]}$ to one of the vertices of edge $[i]$. For each $i = 1, 2, \ldots, m$, we get a partition

$$\mathcal{F}_{[i]} = \{M_{[i]}, M_{[i+m]}, M_{[i+2m]}, \ldots, M_{[i+(2m)m]}\}$$

of the edges of K_{2m+1} into $2m+1$ near-perfect matchings; here the subscripts are taken modulo $2m^2 + m$. Since the degree of each vertex of K_{2m+1} equals $2m$, each vertex of K_{2m+1} is the isolated vertex of exactly one near-perfect matching of $\mathcal{F}_{[i]}$.

For each $i = 1, 2, \ldots, m$, let $G_{[i]}$ be the spanning subgraph of K_{2m+1} consisting of the $2m+1$ edges

$$\{[i], [i+m], [i+2m], \ldots, [i+2m^2]\}. \tag{3}$$

Note that $[i+2m^2+m] = [i]$ in our cyclic ordering, since K_{2m+1} has exactly $2m^2+m$ edges. Each graph $G_{[i]}$ has exactly $2m+1$ vertices and exactly $2m+1$ edges.

Claim 1: Each $G_{[i]}$ is a connected unicyclic graph consisting of a cycle with pendent edges (possibly none) at its vertices.

In fact, since the number of edges of $G_{[i]}$ equals the number of its vertices, if $G_{[i]}$ is connected and the edges span all its vertices, $G_{[i]}$ must be unicyclic. In constructing $G_{[i]}$ in the order of the edges listed, each edge of $G_{[i]}$ has a vertex in common with the edge of $G_{[i]}$ preceding it in the order (interpreted cyclically) given in (3); thus, starting with one of the vertices of $G_{[i]}$, we include one new vertex at a time, giving an ordering of the vertices for each $G_{[i]}$, and attach an edge from the new vertex to an old vertex. It follows that $G_{[i]}$ is connected, and since each vertex of K_{2m+1} is an isolated vertex of one of the near-perfect matchings of $\mathcal{F}_i$, the edges of $G_{[i]}$ span its vertices. Thus $G_{[i]}$ is a connected unicyclic graph, and so $G_{[i]}$ consists of a cycle with a tree (possibly an empty tree) rooted at each of its vertices. Moreover, since each edge of $G_{[i]}$ has a vertex in common with the previous edge, these trees cannot have paths of length greater than 1. This establishes Claim 1.

Consider the connected unicyclic graph $G_{[1]}$. Let k be the length of its cycle γ where we label the vertices so that $\gamma = (1, 2, \ldots, k, 1)$. We can assume that $\{1, 2\}$ is edge $[1]$ in the cyclic ordering of the edges of K_{2m+1} and that edge $[1+m]$ is adjacent to vertex 2 (rather than vertex 1). Let there be $p_i \geq 0$ pendent edges of $G_{[1]}$ at vertex i. Since each edge of $G_{[1]}$ has a vertex in common with the previous edge, it follows that the cyclic ordering of the edges of $G_{[1]}$ is of the form

$$\{1, 2\}, \{2, v_{2,1}\}, \ldots, \{2, v_{2,p_2}\}, \{2, 3\}, \{3, v_{3,1}\}, \ldots, \{3, v_{3,p_3}\}, \ldots$$

$$\ldots, \{k, 1\}, \{1, v_{1,1}\}, \ldots, \{1, v_{1,p_1}\}, \{1, 2\},$$

where $v_{i,1}, \ldots, v_{i,p_i}$ are the pendent vertices of $G_{[i]}$ joined to vertex i ($i = 1, 2, \ldots, k$).

Assuming, without loss of generality, that vertex 1 (as opposed to vertex 2) is the isolated vertex of the near-perfect matching $M_{[2]}$, then it follows that the isolated vertices of the near-perfect matchings in the partition $\mathcal{F}_{[2]}$ are in the order

$$1, v_{2,1}, \ldots, v_{2,p_2}, 2, v_{3,1}, \ldots, v_{3,p_3} \ldots, k, v_{1,1}, \ldots, v_{1,p_1}, 1. \tag{4}$$

Thus, in constructing $G_{[2]}$ in the manner described above, the vertices are added in this order with an edge from a vertex to some preceding vertex.

Claim 2: The edges $\{v_{t,i}, v_{t,i+1}\}$ $(1 \leq i < p_t)$ and the edges $\{t-1, v_{t,1}\}$ $(1 \leq t \leq k)$ are not edges of $G_{[2]}$.

Suppose $\{v_{t,i}, v_{t,i+1}\}$ is an edge of $G_{[2]}$. Then $v_{t,i+2}$ must be joined to either $v_{t,i}$ or $v_{t,i+1}$ in $G_{[2]}$, and it follows that v_{t,p_t} is joined by an edge to one of $v_{t,i}, \ldots, v_{t,p_t-1}$. Now vertex t is joined by an edge in $G_{[2]}$ to one of $v_{t,i}, \ldots, v_{t,p_t-1}, v_{t,p_t}$, a contradiction since all such edges belong to $G_{[1]}$. Similar reasoning shows that $\{t-1, v_{t,1}\}$ cannot be an edge of $G_{[2]}$. This establishes Claim 2.

The vertices of $G_{[2]}$ are added in the order given in (4) with its edge $[2+m]$ equal to $\{1, w\}$ for some vertex w. The edge $[2+2m]$ is the edge $\{v_{2,1}, w\}$, since by Claim 2, it cannot be the edge $\{1, v_{2,1}\}$. Similarly, by Claim 2, the next edge is $\{v_{2,2}, w\}$ since it cannot be the edge $\{v_{2,1}, v_{2,2}\}$. Continued application of Claim 2 shows that all subsequent vertices must be joined by an edge to w in $G_{[2]}$. This implies that $G_{[2]}$ is a graph $K_{1,2m}$ and thus is not unicyclic, a contradiction.

This completes the proof of Theorem 1.1.

3   Remarks and Open Questions

Let $m \geq 2$. By Theorem 1.1, $\text{cms}(K_{2m+1}) = m - 1$. As already remarked, in [1], the Walecki decomposition of K_{2m+1} into Hamilton cycles is used to show that $\text{ms}(K_{2m+1}) = m$. If one uses that construction twice, one obtains that

$$\text{cms}(2K_{2m+1}) = m \quad (m \geq 2),$$

where $2K_{2m+1}$ is the multigraph obtained from K_{2m+1} by doubling each edge. This is illustrated in Figure 3 for $m = 3$. In this figure, the ordering of the edges of K_7 given by the integers $1, 2, \ldots, 21$ is the ordering from the Walecki decomposition used by Alspach [1] to show that the matching sequencibility of K_7 equals 3; note that this ordering does not give that the cyclic matching sequencibility of K_7 equals 3.

If C_n is a cycle of n vertices with n odd, then it is easy to show that $\text{cms}(C_n) = \text{ms}(C_n) = \frac{n-1}{2}$, the maximum possible. This follows by assigning the integers $1, 2, \ldots, n$ to the edges, starting with any edge and going around the cycle twice taking alternate edges. This is illustrated in Figure 4 for $n = 7$.
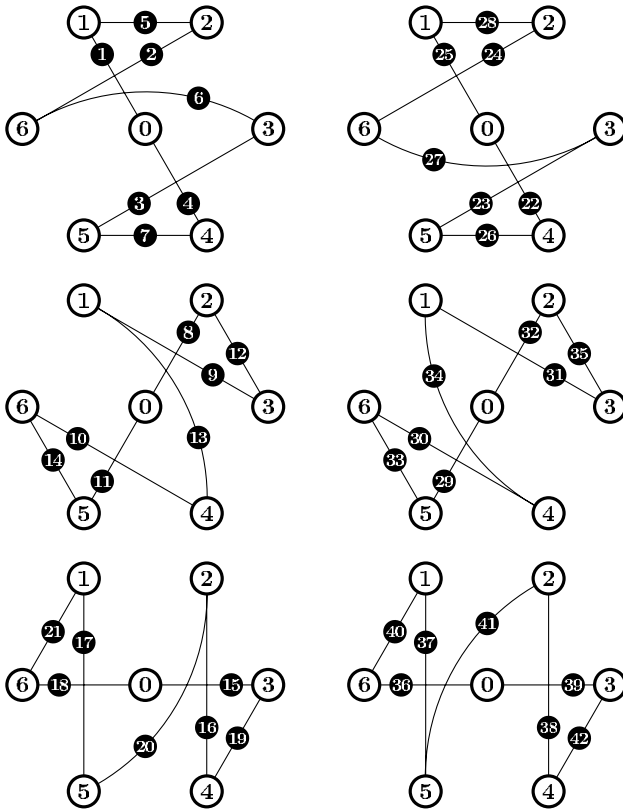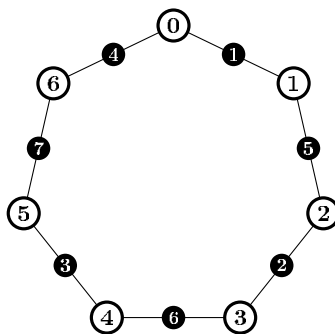
Figure 3

Figure 4

Now consider an even length cycle C_{2q}. Then $C_{2q} \subseteq K_{q,q}$ and C_{2q} has as a biadjacency matrix $A = I + P_q$ where P_q is the permutation matrix with 1s in positions $(1,2), (2,3), \ldots, (q-1,q), (q,1)$.

First suppose that q is even, and let a cyclic ordering of the edges of C_{2q} be given by the matrix $\widehat{A}$ in Figure 5.

1	q											
	$2q-1$	$q-2$										
		$2q-3$	$q-4$									
			$2q-5$	$q-6$								
				$\ddots$	$\ddots$							
					$q+3$	2						
						$q+1$	$2q$					
							$q-1$	$2q-2$				
								$\ddots$	$\ddots$			
									9	$q+8$		
										7	$q+6$	
											5	$q+4$
$q+2$												3

Figure 5

Now the difference of two positive integers in the same row or column of $\widehat{A}$ is $\pm(q-1)$ modulo $2q$. This implies that every (cyclical) set of $q-1$ consecutive positive integers lie in different rows and columns. Since $\mathrm{cms}(C_{2q})$ and $\mathrm{ms}(C_{2q})$ are at most $q-1$, it follows that

$$\mathrm{cms}(C_{2q}) = \mathrm{ms}(C_{2q}) = q-1 \quad (q \text{ equal } 0 \text{ modulo } 4).$$

The matrix $\widehat{A}$ when $q = 4$ is

1	8						
	15	6					
		13	4				
			11	2			
				9	16		
					7	14	
						5	12
10							3

Now suppose that q is odd, and let the cyclic ordering of the edges of C_{2q} be given by the matrix $\widehat{A}$ in Figure 6.

1	$q+2$											
	3	$q+4$										
		5	$q+6$									
			7	$q+8$								
				$\ddots$	$\ddots$							
					$q-1$	$2q$						
						q	$2q-1$					
							$q-2$	$2q-3$				
								$\ddots$	$\ddots$			
									8	$q+7$		
										6	$q+5$	
											4	$q+3$
$q+1$												2

Figure 6

The difference of two positive integers in the same row and column of $\widehat{A}$ is $\pm(q-1)$ or q modulo $2q$ implying that every (cyclical) set of $q-1$ consecutive positive integers lies in different rows and columns. Thus

$$\mathrm{cms}(C_{2q}) = \mathrm{ms}(C_{2q}) = q - 1 \quad (q \text{ equal } 2 \text{ modulo } 4).$$

The matrix $\widehat{A}$ when $q = 6$ is

$$\begin{bmatrix} 1 & 8 & & & & \\ & 3 & 10 & & & \\ & & 5 & 12 & & \\ & & & 6 & 11 & \\ & & & & 4 & 9 \\ 7 & & & & & 2 \end{bmatrix}.$$

In summary, we have

$$\mathrm{cms}(C_n) = \mathrm{ms}(C_n) = \left\lfloor \frac{n-1}{2} \right\rfloor \quad (n \geq 3).$$

Now consider a path P_n of n vertices. First assume that $n = 2q$. Then a $q \times q$ biadjacency matrix A of P_{2q} is obtained from the biadjacency matrix of the cycle C_{2q} by replacing the 1 in its $(q, 1)$-position with a zero. It is easy to obtain an ordering of the edges of P_{2q} (the 1s in A) that shows that $\mathrm{cms}(P_{2q}) = \mathrm{ms}(P_{2q}) = q - 1$. For example, if $q = 5$, the following ordering works and generalizes in the obvious way:

$$\widehat{A} = \begin{bmatrix} 4 & 8 & & & \\ & 3 & 7 & & \\ & & 2 & 6 & \\ & & & 1 & 5 \\ & & & & 9 \end{bmatrix}.$$

Now assume that $n = 2q + 1$ is odd. Then a $q \times (q + 1)$ biadjacency matrix A of P_{2q+1} is obtained from the biadjacency matrix of the cycle C_{2q+1} by deleting its last row. It is easy to show that $\mathrm{ms}(P_{2q+1}) = q$. For example, if $q = 5$, the following ordering works and generalizes in the obvious way:

$$\widehat{A} = \begin{bmatrix} 5 & 10 & & & & \\ & 4 & 9 & & & \\ & & 3 & 8 & & \\ & & & 2 & 7 & \\ & & & & 1 & 6 \end{bmatrix}.$$

Note that this construction does not give $\mathrm{cms}(P_{2q+1}) = q$. In fact, $\mathrm{cms}(P_{2q+1}) = q-1$, as we now argue by contradiction. Suppose there were a cyclic ordering of the $2q$ edges of P_{2q+1} such that every cyclic set of q edges form a matching of P_{2q+1}. Since

we have a cyclic ordering, we may assume that the edge $[1]$ is one of the pendent edges of P_{2q+1}. But then, no matter which matching is determined by the edges $[1], [2], \ldots, [q]$, all of the other edges meet a vertex of one of the edges $[2], \ldots, [q]$, a contradiction.

In summary, we have

$$
\begin{array}{rcll}
\mathrm{cms}(P_n) = \mathrm{ms}(P_n) & = & \frac{n-2}{2} & \text{if } n \text{ is even,} \\
\mathrm{ms}(P_n) & = & \frac{n-1}{2} & \text{if } n \text{ is odd,} \\
\mathrm{cms}(P_n) & = & \frac{n-3}{2} & \text{if } n \text{ is odd.}
\end{array}
$$

The (cyclic) matching sequencibility has been computed for complete graphs, complete bipartite graphs, cycles, and paths. It may be of interest to compute it for other important classes of graphs, but such computations may be very difficult and they will in general depend on the particular graph in the class and thus be of less interest. For a tree, the (cyclic) matching sequencibility may differ considerably from the *matching number* (the maximum number of edges in a matching). For example, in Figure 7 there is a tree T of order 18 with a perfect matching (matching number equal to 9) and an ordering of the edges showing that $\mathrm{ms}(T) \geq 2$. It is not hard to see that $\mathrm{cms}(T) = 1$ and $\mathrm{ms}(T) = 2$. If one takes any graph G of order n and adjoins $n + 1$ (respectively, $n + 2$) pendent edges at one of its vertices to get a graph H of order $2n + 1$ (respectively, $2n + 2$), then $\mathrm{ms}(G) = 1$ (respectively, $\mathrm{cms}(H) = 1$), since in any ordering (respectively, cyclic ordering) of the edges of H, some two of these pendent edges would have to be consecutive.
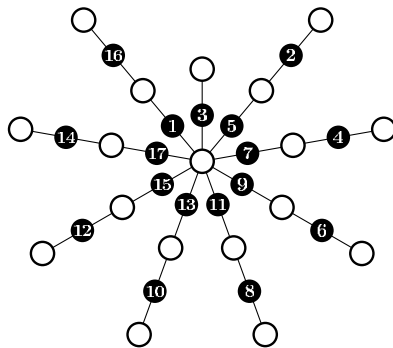

Figure 7

Another class of graphs of potential interest are the k-regular bipartite graphs $G \subseteq K_{n,n}$. For example, consider the 3-regular bipartite graph $G \subseteq K_{n,n}$ whose biadjacency matrix is $P^{-1} + I + P$ (equivalently, $I + P + P^2$) where P is as before the full cycle permutation matrix of order n. Then $\mathrm{cms}(G) = \mathrm{ms}(G) = n - 1$. We

illustrate this in terms of the biadjacency matrix A and the labeling given by $\widehat{A}$ for the n odd and n even cases with $n = 7$ and 8:

$$\widehat{A} = \begin{bmatrix} 1 & 16 & & & & & 7 \\ 15 & 2 & 9 & & & & \\ & 10 & 3 & 18 & & & \\ & & 17 & 4 & 11 & & \\ & & & 12 & 5 & 20 & \\ & & & & 19 & 6 & 13 \\ 8 & & & & & 14 & 21 \end{bmatrix}, \quad \begin{bmatrix} 1 & 18 & & & & & & 8 \\ 17 & 2 & 10 & & & & & \\ & 11 & 3 & 20 & & & & \\ & & 19 & 4 & 12 & & & \\ & & & 13 & 5 & 22 & & \\ & & & & 21 & 6 & 14 & \\ & & & & & 15 & 7 & 24 \\ 9 & & & & & & 23 & 16 \end{bmatrix}.$$

We conclude with three additional questions applying to all graphs that appear to be very difficult.

Question 1: Given a graph G with matching number p, is there a positive integer k such that $\mathrm{ms}(kG) = p$ $(\mathrm{cms}(kG) = p)$?

Question 2: For a graph G, we have $\mathrm{ms}(G) \geq \mathrm{cms}(G)$. How large can $\mathrm{ms}(G) - \mathrm{cms}(G)$ be? Is $\mathrm{cms}(G) \geq \mathrm{ms}(G) - 1$?

Question 3: Given a graph G, is $\mathrm{cms}(2G) = \mathrm{ms}(G)$?

References

[1] B. Alspach, The wonderful Walecki construction, *Bull. Inst. Combin. Applic.* 52 (2008), 7–20.