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CYCLIC ORDERED GROUPS AND MV-ALGEBRAS

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In the forties and fifties two—at the moment—unrelated concepts derived from that of an ordered group appeared. The notion of cyclic-ordered group (c-group) (see [9], [10], [13] and [14]) and that of MV-algebra (see [4] and [5]). The first one appeared as a way of generalizing the notion of totally ordered groups. That notion was further extended to that of partially cyclically ordered groups. The notion of MV-algebras resulted from a succesfull attempt of giving an algebraic structure to the infinite-valued Lukasievicz propositional logics. In the last decade, that theory was fruitfully linked with that of a class of $\mathbf{C}^*$-algebras (see [8]). The objective of this work is to show that suitable subclasses of that notions can be linked by the way of a covariant functor.

1. DEFINITIONS AND FIRST FACTS

A cyclically ordered group (c-group) is a system $\langle G,+,-,0,T\rangle$ where $\langle G,+,-,0\rangle$ is a group (not necessarilly commutative) and T is a ternary relation verifying the following properties:

C1. $\forall abc$ (if $a \neq b \neq c \neq a$ then exactly one of $T(a,b,c)$ and $T(a,c,b)$ holds);

C2. $\forall abc\ \big(T(a,b,c) \Longrightarrow a \neq b \neq c \neq a\big)$;

C3. $\forall abc\ \big(T(a,b,c) \Longrightarrow T(c,a,b)\big)$;

C4. $\forall abcd\ \big(T(b,c,a)\ \&\ T(c,d,a) \Longrightarrow T(b,d,a)\big)$;

C5. $\forall abcd\ \big(T(a,b,c) \Longrightarrow T(d+a,d+b,d+c)\ \&\ T(a+d,b+d,c+d)\big)$.

A fundamental result of Rieger (see [9]) says that any such a group is isomorphic to a quotient of a totally ordered group (o-group) by the subgroup generated by a strong unit (a cofinal element in its centre). In that case, if $G = \langle G,+,-,0,u,\leqslant\rangle$ is an o-group with strong unit u, the quotient group $G_u = G/\langle u\rangle$ can be endowed with a cyclic order by defining $T(a,b,c)$ if and only if, for the only representatives a, b, c such that $0 \leqslant a,b,c < u$, either $a<b<c$ or $b<c<a$ or $c<a<b$ holds.

The notion of c-group generalizes that of totally ordered groups (o-groups) in the sense that for a c-group with the property: for all $a \in G$, $T(-a, 0, a)$ implies, for all $n \in \mathbf{N}$, $T(-na, 0, na)$ a total order (compatible with the group operation) can be defined by $0 < a$ if and only if $T(-a, 0, a)$. Conversely, an o-group can be endowed with a c-group structure by defining $T(a, b, c)$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$.

A partially cyclically ordered group (pco-group) is a system $\langle G, +, -, 0, T\rangle$ where the axioms C3, C4, C5 and

C1p. $\forall abc(T(a, b, c) \Longrightarrow \neg T(a, c, b))$;

C6. $\forall abc(T(a, b, c) \Longrightarrow T(-c, -b, -a))$ hold.

This last axiom is consequence of axioms C1 ...C5 and C2 is consequence of C1p and C3.

Observe that, Rieger's theorem also holds in this case by replacing the o-group by a partially ordered group (po-grup) (see [13] or [14]).

An MV-algebra (see [4], [5] and [8]) is a system $\langle A, \oplus, *, \neg, 0, 1\rangle$ which satisfies the following universal identities:

m_1	$x \oplus (y \oplus z) = (x \oplus y) \oplus z$
m_2	$x \oplus 0 = x$
m_3	$x \oplus y = y \oplus x$
m_4	$x \oplus 1 = 1$
m_5	$\neg\neg x = x$
m_6	$\neg 0 = 1$
m_7	$x \oplus \neg x = 1$
m_8	$\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$
m_9	$x * y = \neg(\neg x \oplus \neg y)$

By defining $x \vee y := (x * \neg y) \oplus y$ and, by duality, $x \wedge y := \neg(\neg x \vee \neg y)$ we have that $\langle A, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice.

Another approach for this structures is that of Wajsberg algebras (W-algebras) (see [6] and [11]). Such an algebra is a system $\langle A, \rightarrow, \neg, 0, 1\rangle$ satisfying the following universal identities:

$$\begin{array}{ll}
\text{W1.} & (x \to y) \to ((y \to z) \to (x \to z)) = 1; \\
\text{W2.} & (x \to y) \to y = (y \to x) \to x; \\
\text{W3.} & (\neg x \to \neg y) \to (y \to x) = 1; \\
\text{W4.} & 1 \to x = x; \\
\text{W5.} & x \to 0 = \neg x; \\
\text{W6.} & \neg 1 = 0; \\
\text{W7.} & \neg 0 = 1.
\end{array}$$

By defining $x \vee y := (x \to y) \to y$ and $x \wedge y := \neg(\neg x \vee \neg y)$ $\langle A, \vee, \wedge, 0, 1\rangle$ results also a bounded distributive lattice.

In [6] it is proved that a W-algebra can be thought of as an MV-algebra (and viceversa) by identifying the respective 0,1 and $\neg$ and defining:

$$a \to b := \neg a \oplus b \qquad \text{and} \qquad a \oplus b := \neg a \to b;$$

(recall that the operation $*$ of the MV-algebra can be defined in terms of $\oplus$ and $\neg$).

In [4] it is proved that any MV-algebra A can be obtained from an abelian lattice-ordered group (l-group) with strong unit u $G = \langle G, \vee, \wedge, +, -, 0, u\rangle$ by defining:

$$A = [0, u] = \{a \mid 0 \leqslant a \leqslant u\}; \quad a \oplus b = (a + b) \wedge u; \quad \neg a = u - a \text{ and } 1 = u.$$

Since any MV-algebra derives from an abelian l-group, in the sequel group will stand for abelian group, homomorphism and subgroup for homomorphism and subgroup for the respective structures (o-groups, c-groups, pco-groups, l-groups, MV-algebras).

2. Lattice pco-groups

For any pco-group G, a partial order be defined by

$$(*) \qquad a \leqslant b \quad \text{if and only if} \quad a = b \quad \text{or} \quad T(0, a, b) \quad \text{or} \quad a = 0.$$

This order makes every element "positive". Observe that, in general, $\leqslant$ is not compatible with the group operation, for example, by setting $G = \mathbb{Z}/3\mathbb{Z}$ with its natural cyclical order, the total order $(*)$ induced is given by the set of pairs $\{(0,0), (1,1), (2,2), (0,1), (0,2), (1,2)\}$ which is obviously non-compatible, since $1 \leqslant 2$ holds but $2 = 1 + 1 \leqslant 2 + 1 = 0$ does not hold.

We say that a group homomorphism $f: G \to H$ between pco-groups is a pco-homomorphism if, for $a, b, c \in G$ such that $T(a, b, c)$, if $f(a) \neq f(b) \neq f(c) \neq f(a)$ then $T(f(a), f(b), f(c))$.

Observe that a pco-homomorphism is also a homomorphism for the order given in $(*)$.

Definition 2.1. A pco-group G will be called a lattice-cyclical-group (and denoted lc-group), if, for the order defined in (*) the structure $\langle G, 0, \leqslant \rangle$ admits a distributive lattice structure with first element.

Lemma 2.2. *Let G be an lc-group, $a, b \in G$. If $a \leqslant a+b$ ($b \leqslant a+b$) then $b \leqslant a+b$ ($a \leqslant a+b$), implying $a \vee b \leqslant a+b$.*

Proof. Suppose $0 < a < a+b$ (the other cases are immediate). Then we have $T(0, a, a+b)$, which, adding $-(a+b)$ to each term, implies $T(-(a+b), -b, 0)$ which, by axiom C6, is equivalent to $T(0, b, a+b)$, proving our claim. □

Definition 2.3. Let G be an lc-group and H a subgroup.

(i) H is called an lc-ideal if it is convex for the order $\leqslant$ (that is, for all $x \in H$, $z \in G$, $z \leqslant x$ implies $z \in H$), and is an l-subgroup (that is, for $x, y \in H$, $x \vee y \in H$).

(ii) H is called a pc-subgroup if it is convex for the relation T (that is, for $x, y \in H$ and $z \in G$, $T(x, z, y)$ implies $z \in G$).

Observe that the lc-ideals (pc-subgroups) are the kernels of lc(pc)-homomorphisms. Moreover, the lc-ideals are lattice-ideals for the structure $\langle G, 0, \vee, \wedge \rangle$. Observe also that for cyclically ordered groups, the T-convex subgroups are always trivial.

Lemma 2.4. *Let G be an lc-group and H a subgroup. H is T-convex if and only if it is $\leqslant$-convex. So, any pc-subgroup preserving the lattice operations is also an lc-ideal.*

Proof. Let H be T-convex, $a \in H$, $b \in G$ such that $0 \leqslant b \leqslant a$. If $b = 0$ or $b = a$, it is immediate that $b \in H$. So we can write $T(0, b, a)$, implying, by T-convexity, that $b \in H$.

For the converse, if H is $\leqslant$-convex, $a, c \in H$, $b \in G$ such that $T(a, b, c)$. By axiom C5 we have $T(0, b-a, c-a)$. Since H is $\leqslant$-convex, we conclude that $b - a \in H$ and then $b \in H$. □

So, without abuse of notation, we can speak about convex subgroups.

Lemma 2.5. *Let G be an lc-group, $H \subseteq G$ an lc-ideal. H is prime if and only if the quotient G/H is cyclically ordered.*

Proof. By a result on distributive lattices (see [1, III.3]) we have that the lattice $\langle G/H, 0, \vee, \wedge \rangle \simeq \langle G, 0, \vee, \wedge \rangle / H$ is totally ordered if and only if H is prime as a lattice ideal. Since the notion of primeness is a set theoretic one, H is prime as lattice ideal if and only if it is so as lc-ideal. It is immediate to verify that the induced order $\leqslant$ on a pco-group is total if and only if the group is cyclically ordered. □

As in the case of l-groups, we can define the notions of orthogonality, projectability and weak unit:

Definitions 2.6. Let G be an lc-group, $g, h \in G$, A, B subsets of G.

(i) g and h are orthogonal, $g \perp h$, if $g \wedge h = 0$.

(ii) The polar of A, $A^{\perp} = \{x \mid \forall a(a \in A \Rightarrow x \perp a)\}$. B is called a polar if $B = A^{\perp}$ for some A. If $A = \{g\}$ we shall write $g^{\perp}$ in place of $\{g\}^{\perp}$.

(iii) The double polar of A, $A^{\perp\perp} = \{x \mid \forall y(y \in A^{\perp} \Rightarrow x \perp y)\}$. Observe that B is a double polar if and only if it is a polar.

(iv) G is called projectable if one can define a binary operation pr on G, compatible for the left argument with the group operations, such that, $h' = \mathrm{pr}(g, h)$ implies $h' \in h^{\perp}$ and $g - h' \in h^{\perp\perp}$.

(v) $u \in G$ is called a weak unit if, for all $g \in G$, $g \perp u$ implies $g = 0$.

Lemma 2.7. *Let G be a projectable lc-group. Its polars are lc-ideals.*

Proof. Let $g, h \in G$, A a subset of G. Consider a generic $a \in A$. By distributivity, it is immediate that $(g \vee h) \wedge a = (g \wedge a) \vee (h \wedge a)$. Since $g \leqslant h$ implies $g \wedge a \leqslant h \wedge a$, we have that $h \in a^{\perp}$ implies $g \in a^{\perp}$. Since $A^{\perp} = \bigcup\{a^{\perp} \mid a \in A\}$, we conclude that $A^{\perp}$ is a lattice-ideal. Suppose $g \perp a$ and $h \perp a$. By projectability, observe that $g = \mathrm{pr}(g, a)$ and $h = \mathrm{pr}(h, a)$. Since pr is compatible at left with the sum and the inverse, we have that $\mathrm{pr}(g + h, a) = g + h$ and $\mathrm{pr}(-g, a) = -g$, implying $(g + h) \perp a$ and $-g \perp a$. So we can conclude that $A^{\perp}$ is an lc-ideal. □

Lemma 2.8. *Let G be a projectable lc-group, $h, h_1, h_2, h_3, h_4 \in G$ such that $h_1, h_3 \in h^{\perp}$; $h_2, h_4 \in h^{\perp\perp}$ and $h_1 + h_2 = h_3 + h_4$ then $h_1 = h_3$ and $h_2 = h_4$.*

Proof. We have $h_1 + h_2 = h_3 + h_4$ implies $h_1 - h_3 = h_4 - h_2$. Since the polars are lc-ideals, we have that the first member belongs to $h^{\perp}$ and the second to $h^{\perp\perp}$, implying that both equal zero. □

From the above proved lemma, we conclude that the decomposition in terms of $h^{\perp}$ and $h^{\perp\perp}$ given by $\mathrm{pr}(\ , h)$ is the only one possible and, since $\mathrm{pr}\big(\mathrm{pr}(g, h), h\big) = \mathrm{pr}(g, h)$ it can be well considered a projection.

We recall (see [3; § 8.1]) that given a language L, an L-structure G and a family $(L_i)_{i \in I}$ of L-structures, G is a Boolean product of the family $(L_i)_{i \in I}$ (denoted by $G \in \Gamma\big(I, (L_i)_{i \in I}\big)$) if and only if:

(i) G is a subdirect product of the family $(L_i)_{i\in I}$ and

(ii) I can be endowed with a Boolean space topology such that:

(α) For any atomic L-formula $\varphi(x_1,\dots,x_n)$ and $g_1,\dots,g_n \in G$, the set $\{i \mid L_i \models \varphi[g_1(i),\dots,g_n(i)]\}$ (denoted by $[\![\varphi[g_1,\dots,g_n]]\!]$) is clopen;

(β) For $g,h \in G$ and J a clopen set of I, there exists the element of G given by $g \restriction J \cup h \restriction I \setminus J$ (patchwork property).

Let $(C_i)_{i\in I}$ be a family of c-groups and G a subgroup of $\prod C_i$. G will be endowed with a pco structure by considering the product ternary relation $T = \prod T_i$. That is $T(a,b,c)$ if and only for all $i \in I$ $T(a_i,b_i,c_i)$ holds.

The following proposition is analogous to a result of Weispfenning on l-groups (see [12]):

Proposition 2.9. *An lc-group G is isomorphic to a Boolean product (in the language $\langle +,-,0,T,\vee,\wedge\rangle$) of (non-trivial) c-groups if and only if it is projectable and has a weak unit.*

Proof. Let $G \in \Gamma(I,(C_i)_{i\in I})$ where $(C_i)_{i\in I}$ is a family of non-trivial c-groups. For each $i \in I$ there exists $h_i \in C_i$ such that $h_i \neq 0$. Since G is a subdirect product, there exist a family $(h'_i)_{i\in I} \subseteq G$ such that, for each $i \in I$, $h'_i(i) = h_i$. By property (ii-α) above, for each $i \in I$, the set $[\![h'_i \neq 0]\!]$ is clopen. By compacity of I, a finite subset J of I can be found such that the family $\{[\![h'_i \neq 0]\!] \mid i \in J\}$ covers I. By property (ii-β), that family can be considered disjoint. Now, applying $|J|$ times the same property, an element $h \in G$ such that $h \restriction [\![h'_i \neq 0]\!] = h_{i|[\![h'_i\neq 0]\!]}$ $(i \in J)$ can be found. (This line of argumentation on Boolean products is standard and will not be repeated in the following proofs.) We shall see that h is, indeed, a weak unit. For, suppose $g \in G$ and $g \wedge h = 0$. Since G is a subdirect product and $x \wedge y = 0$ is an atomic formula, for each $i \in I$, $g(i) \wedge h(i) = 0$ holds. But, for each c-group C_i, $h(i)$ is different from 0, implying that $g(i) = 0$ for all i and then $g = 0$.

For the projectability, let $g,h \in G$. Consider the clopen subset of I $J = [\![h \neq 0]\!]$. By property (ii-β) call h'' the restriction of g to J and h' its restriction to $I \setminus J$. It is immediate to verify (since G is a subdirect product) that $g = h' + h''$ and $h' = \mathrm{pr}(g,h)$.

For the converse. Let G be a projectable lc-group with weak unit u. We consider the Boolean algebra $B(G,u)$ with underlying set $\{\mathrm{pr}(u,g) \mid g \in G\}$ and operations $\mathrm{pr}(u,g) \vee \mathrm{pr}(u,h) = \mathrm{pr}(u, g\wedge h)$; $\neg\,\mathrm{pr}(u,g) = u - \mathrm{pr}(u,g) = \mathrm{pr}\,(u, \mathrm{pr}(u,g))$; $0_B = \mathrm{pr}(u,u) = 0$ and $1 = \mathrm{pr}(u,0) = u$. It is easy to verify that, if u,u' are weak units, we have the isomorphism $B(G,u) \simeq B(G,u')$. So we can forget the weak unit and write $B(G)$ for the Boolean algebra of the group. Observe that polars of G and ideals of $B(G)$ are in a bijective correspondence: If A is a polar of G, $A \cap B(G)$ is an ideal of $B(G)$. If J is an ideal of $B(G)$, $J^G = \{g \in G \mid u - \mathrm{pr}(u,g) \in J\}$ is a polar of G. Both constructions are each other inverses.

Let $I = \mathrm{Sp}(B(G))$ the space of prime ideals of $B(G)$. By the above remark and Lemma 2.7, we can identify it as a subspace of the space of prime lc-ideals of G. That set of lc-ideals distinguishes points: In particular, if $g \in G$, $g \neq 0$, there exists a prime ideal P of $B(G)$ such that $u - \mathrm{pr}(u, g) \notin P$. Then $g/p^G \neq 0$. So G can be represented as a subdirect product of the family $(C_i)_{i \in I}$ of lc-groups given by the quotients by the elements of I. Since, each of those lc-ideals is prime, by Lemma 2.5, each C_i results cyclically ordered for the quotient of the relation T.

Finally we show that G (considered as a subdirect product) has properties (ii-α) and (ii-β) of the Boolean product definition. Any atomic formula $\varphi(\overline{x})$ is of the form or $T(t_1(\overline{x}), t_2(\overline{x}), t_3(\overline{x}))$ or $t_1(\overline{x}) = t_2(\overline{x})$ for t_1, t_2, t_3 terms in the group language.

For the sake of simplicity, we can suppose that the terms are just variables. We have, for a c-group $T(x_1, x_2, x_3) \Leftrightarrow T(0, x_2 - x_1, x_3 - x_1) \Leftrightarrow 0 < x_2 - x_1 < x_3 - x_1 \Leftrightarrow (x_2 - x_1) \vee (x_3 - x_1) = x_3 - x_1 \;\&\; x_2 - x_1 \neq 0 \;\&\; x_3 - x_2 \neq 0$. Let be now $g_1, g_2, g_3 \in G$, call $b = \neg\,\mathrm{pr}(u, g_2 - g_1)$, $a = \mathrm{pr}\left(u, g_3 - g_1 - \left((g_2 - g_1) \vee (g_3 - g_1)\right)\right)$ and $c = \neg\,\mathrm{pr}(u, g_3 - g_2)$. Now, by the above considerations about the definition of T on a subgroup of a product of c-groups, the element $a \wedge b \wedge c$ of the Boolean algebra $B(G)$ corresponds to $[\![T(g_1, g_2, g_3)]\!]$. And since the elements of $B(G)$ are in correspondence with the clopen sets of $\mathrm{Sp}(B(G))$, we are done. For the formula $x_1 = x_2$, and $g_1, g_2 \in G$, it suffices to take $a = \mathrm{pr}(u, g_1 - g_2)$, proving property (ii-$\alpha$).

Property (ii-β) results from projectability. Let $g, h \in G$ and J a clopen set of I, there exists then $c_J \in G$ such that $c_J = \mathrm{pr}(u, u - c_J) = \neg\,\mathrm{pr}(u, c_J)$ and that element "corresponds" to J. So, we have the identity $g \restriction J \cup h \restriction I \setminus J = \mathrm{pr}(g, u - c_J) + \mathrm{pr}(h, c_J)$. □

3. The standard construction

We recall the result of V. Weispfenning (see [12]), which states that an l-group is isomorphic to a Boolean product of totally ordered groups if and only if it is projectable and has a weak unit.

Let G be a projectable l-group and $u \in G$ a strong unit. Define the l-subgroup $H(u)$ generated by all the elements of the form $u \restriction g^{\perp}$ (with g ranging by all the elements of G). Consider the quotient group $G_u = G/H(u)$.

Proposition 3.1. *The group G_u admits a natural lc-structure.*

Proof. By the above stated observation, we shall consider $G \in \Gamma(I, (L_i)_{i \in I})$ for some family $(L_i)_{i \in I}$ of totally ordered groups. First, observe that, for any $g_u \in G_u$ there exists only one $a \in [0, u) = \{h \in G \mid 0 \leqslant h < u\}$ such that $a_u = g_u$: Let be $g \in G$. Since u is a strong unit, we have that there exists $n \in \mathbf{N}$ such that $nu > |g|$. For $m \in \mathbf{Z}$ such that $-n \leqslant m < n$, call I_m the clopen subset of I given by

$[\![mu \leqslant g < (m+1)u]\!]$. Calling g_m the restriction of g to I_m, we have that it has a representative in the interval $[0, u_m)$. Now, by the patchwork property, we can patch all those representatives and obtain an element $a \in [0, u)$ such that $a_u = g_u$. It is immediate that any two of the elements in the interval are not congruent modulo $H(u)$.

Now, for $a_u, b_u, c_u \in G_u$, consider the representatives $a, b, c \in [0, u)$. We shall define $T(a_u, b_u, c_u)$ if and only if

$$I = [\![a < b < c \text{ or } b < c < a \text{ or } c < a < b]\!].$$

The proof that this defines a partial cyclic order is analogous to that for the cyclic order case (see [10]).

Call $\leqslant_u$ the order induced by T. It is immediate to verify that $a_u \leqslant_u b_u$ if and only if $a \leqslant b$ for a, b representatives in $[0, u)$. Since for this order that interval is a distributive lattice with first element, we can conclude that its lattice structure is copied, isomorphically on G_u. □

The Boolean product characterization allows us to prove the converse.

Proposition 3.2. *Let G be a projectable lc-group with weak unit. There exists an l-group G' with a strong unit u such that $G \simeq G'$ in the above sense.*

Proof. We can suppose $G \in \Gamma(I, (C_i)_{i\in I})$ for some family $(C_i)_{i\in I}$ of c-groups. By Rieger's theorem, there exists a family $(L_i, u_i)_{i\in I}$ of o-groups with strong units such that for each $i \in I$, $C_i \simeq (L_i)/\langle u_i\rangle$. Consider now the direct product $\prod L_i$ and identify the elements of G with the elements in the product of intervals $\prod [0, u_i)$. Now call G' the l-group spanned by G and $(u_i)_{i\in I}$ in $\prod L_i$. By construction, it results that $G' \in \Gamma(I, (L_i)_{i\in I})$ and it is immediate to prove that, setting $u = (u_i)_{i\in I}$, $G \simeq G'_u$. □

4. The functorial equivalence

In the sequel we shall restrict ourselves to projectable MV-algebras, which can be defined analogously to the case of lc(l)-groups. In particular, it holds that a projectable MV-algebra is isomorphic to an element of $\Gamma\big(I, (L_i)\big)_{i\in I}$ for a family $(L_i)_{i\in I}$ of totally ordered MV-algebras. (This result is analogous of that of Weispfenning on l-groups and can be found—implicitly—in [11]).

In an MV-algebra, an element a is called boolean if $a \perp \neg a$.

Let $A = \langle A, \oplus, *, \neg, 0, 1\rangle$ be an MV-algebra and consider the equivalence relation $\sim$ given by:

$a \sim b$ if and only if there exist boolean elements a' and b' such that $a \oplus a' = b \oplus b'$, $a \perp a'$, $b \perp b'$ and $a' \perp b'$. By considering A as a boolean product over a space I, this corresponds to the identity $I = [\![a = b]\!] \cup [\![a = 0 \ \&\ b = 1]\!] \cup [\![b = 0 \ \&\ a = 1]\!]$. We show that $\sim$ is, indeed, an equivalence relation:

- By taking $a' = 0$, we prove that $a \sim a$.
- The simmetry results from the definition.
- Let be $a \sim b \sim c$. We shall use the boolean product characterization of the relation $\sim$:

$$I_1 = [\![a = c]\!] = ([\![a = b]\!] \cap [\![b = c]\!]) \cup [\![a = 0 \ \&\ c = 0]\!] \cup [\![a = 1 \ \&\ c = 1]\!];$$
$$I_2 = [\![a = 0 \ \&\ c = 1]\!] = ([\![a = b]\!] \cap [\![b = 0 \ \&\ c = 1]\!]) \cup ([\![c = b]\!] \cap [\![b = 1 \ \&\ a = 0]\!]);$$
$$I_3 = [\![a = 1 \ \&\ c = 0]\!] = ([\![a = b]\!] \cap [\![b = 1 \ \&\ c = 0]\!]) \cup ([\![c = b]\!] \cap x[\![b = 0 \ \&\ a = 1]\!]).$$

A simple set-theoretic manipulation proves that $I = I_1 \cup I_2 \cup I_3$ and then $a \sim c$.

We define the group operations in $G = A/\!\sim$ by

$$-(a/\!\sim) := \neg a/\!\sim.$$

Given $a/\!\sim, b/\!\sim \in G$ consider the clopen set $J = [\![a \oplus b < 1]\!]$ and define

$$(a/\!\sim) + (b/\!\sim) := \big((a \oplus b)\big|J \cup (a * b)\big|I \setminus J\big)/\!\sim.$$

To verify that those operations are well-defined, since we are dealing with subdirect products, it suffices to consider the totally ordered case:

For that case we have $a \sim b$ if and only if $a = b$ or ($a = 0$ and $b = 1$) or ($a = 1$ and $b = 0$). For the difference: $\neg 0/\!\sim\ = 1/\!\sim\ = 0/\!\sim\ = \neg 1/\!\sim$. For the sum, it suffices to consider the case $a/\!\sim\ = 0/\!\sim$ and $0 < b < 1$. So we have $0/\!\sim + b/\!\sim\ = (0 \oplus b)/\!\sim\ = b/\!\sim\ = (1 * b)/\!\sim\ = 1/\!\sim + b/\!\sim$. We show that $\langle G, +, -, 0\rangle$ is an abelian group:

Recall the Theorem 16 in [6] which implies that the variety of MV-algebras is generated by the MV-algebra $\mathbf{Q}[0,1]$ with underlying set $\{x \in \mathbf{Q} \mid 0 \leqslant x \leqslant 1\}$ and operations $x \oplus y = 1 \wedge (x + y)$ and $\neg x = 1 - x$. So any equation is true in the variety if and only if it holds in $\mathbf{Q}[0,1]$. We shall consider then $A = \mathbf{Q}[0,1]$.

- The commutativity results from that of $\oplus$ and $*$;
- $a/\!\sim + 0/\!\sim\ = (a \oplus 0)/\!\sim\ = a/\!\sim$;
- $-a/\!\sim + \big(-(a/\!\sim)\big) = a/\!\sim + \neg a/\!\sim\ = (a * \neg a)/\!\sim\ = 0/\!\sim$ because $a \oplus \neg a = 1$;
- For the associativity, let $a/\!\sim, b/\!\sim, c/\!\sim \in G$:

Case $(a \oplus b) \oplus c < 1$: Results from the associativity of $\oplus$;

Case $a \oplus b = 1$ and $(a * b) \oplus c = 1$: Since $a * b \leqslant b$, we have $b \oplus c = 1$ and then

$$(a/\!\sim + b/\!\sim) + c/\!\sim\ = (a * b) * c. \tag{1}$$

$a\oplus(b*c) = 1\wedge(a+(b*c)) = 1\wedge(a+\neg(\neg b\oplus\neg c)) = 1\wedge(a+(1-(1\wedge(1-b+(1-c))))) = 1\wedge(a+(1-(1\wedge(2-(b+c))))) = 1\wedge(a+(1-(2-(b+c)))) = 1\wedge(a+b+c-1) = (a*b)\oplus c$ because $a*b = a+b-1$. And, by hipothesis, $(a*b)\oplus c = 1$. So we have $a/\sim + (b/\sim + c/\sim) = (a*b)*c$ which coincides with (1).

Case $a\oplus b = 1$, $(a*b)\oplus c < 1$ and $b\oplus c < 1$:

$$\begin{aligned}(2)\quad &(a/\sim + b/\sim) + c/\sim = (a*b)\oplus c = 1\wedge(a*b+c) = 1\wedge(\neg(\neg a\oplus\neg b)+c) = \\ &= 1\wedge(1-(1\wedge(1-a+(1-b)))+c) = 1\wedge(1-(1\wedge(2-(a+b)))+c) = \\ &= 1\wedge(1-(2-(a+b))+c) = 1\wedge(a+b+c-1).\end{aligned}$$

Since $a\oplus(b\oplus c) \geqslant a\oplus b = 1$, we have $a/\sim + (b/\sim + c/\sim) = a*(b\oplus c)$. An analogous treatment yields $a*(b\oplus c) = (2)$.

The rest of the cases are treated in a similar way, proving the associativy.

Now, for the relation T, given $a/\sim, b/\sim, c/\sim \in G$, define the following clopen sets:

$$\begin{aligned} I_1 &= [\![(a<b<c) \;\&\; (a\neq 0 \text{ or } c\neq 1)]\!], \\ I_2 &= [\![(b<c<a) \;\&\; (b\neq 0 \text{ or } a\neq 1)]\!], \\ I_3 &= [\![(c<a<b) \;\&\; (c\neq 0 \text{ or } b\neq 1)]\!]. \end{aligned}$$

Define a pc-order by $T(a/\sim, b/\sim, c/\sim)$ if and only if $I = \bigcup_{j=1}^{3} I_j$. It is immediate that T satisfies properties C1p, C3, C4, C5 and C6. The good definition results from the second condition in each I_j. Since the order $\leqslant_c$ defined on G by $g \leqslant_c h$ if and only if $T(0,g,h)$ or $g=0$ or $g=h$ coincides with the order $\leqslant$ of A (modulo $\sim$), we have that it induces a lattice structure.

For the compatibility of $+$ and T it also suffices to consider the totally ordered case: Let be $a,b,c,d\in A$ such that $a<b<c<1$ and $d<1$.

– If $c\oplus d<1$ we have $a\oplus d < b\oplus d < c\oplus d < 1$;

– If $a\oplus d = b\oplus d = c\oplus d = 1$, we have $a*d < b*d < c*d$;

– If $a\oplus d, b\oplus d < 1$ and $c\oplus d = 1$ we have $c*d < d \leqslant a\oplus d < b\oplus d$;

– The case $a\oplus d < 1$ and $b\oplus d, c\oplus d = 1$ is analogous.

If $f\colon A\to B$ is an MV-homomorphism, it is immediate to verify that $f/\sim$ is well-defined and then, an lc-group homomorphism.

Reciprocally, let $G = \langle G,+,-,0,u,T\rangle$ be a projectable lc-group with weak unit. We can identify G with an element of $\Gamma(I,(L_i)_{i\in I})$ for some family $(L_i)_i \in I$ of c-groups, where the Boolean space I is the one constructed in the second part of the proof of Proposition 2.9. The Boolean algebra $B(I)$ of clopen sets of I (considered as a set algebra) can be also identified with the algebra of supports of elements of G.

Define $A = \{(g,\alpha)\in G\times B(I) \mid \mathrm{supp}(g)\cap\alpha = \emptyset\}$.

We define on A the MV operations:

The 0 of the MV-algebra will be the element $(0, \emptyset)$ and the 1 the element $(0, I)$.

Let $(g, \alpha) \in A$, call $\beta = I \setminus \mathrm{supp}(g)$. Define $\neg(g, \alpha) = \big(-g, (I \setminus \alpha) \cap \beta\big)$.

Given $(g, \alpha), (h, \beta) \in A$, consider the clopen set $\gamma = I \setminus (\alpha \cup \beta)$ and the elements of G $g' = g|\gamma$ and $h' = h|\gamma$. Call δ the clopen set $\gamma \cap \big([\![T(0, g', g' + h')]\!] \cup [\![g' = 0]\!] \cup [\![h' = 0]\!]\big)$ which coincides with $\gamma \cap [\![g' \leqslant g' + h']\!]$. (Observe that Lemma 2.2 implies $T(0, g', g' + h')$ if and only if $T(0, h', g' + h')$). And finally $\eta = [\![\neg T(0, g', g' + h')]\!]$. Now define:

$$(g, \alpha) \oplus (h, \beta) = \big((g' + h')|\delta, \alpha \cup \beta \cup \eta\big).$$

The operation $*$ is defined in terms of $\oplus$ and $\neg$.

We shall proof that $A = \langle A, \oplus, *, \neg, 0, 1\rangle$ is in effect an MV-algebra.

m_1: Let $(g, \alpha), (h, \beta), (k, \gamma) \in A$.

By setting

$$\delta = I \setminus \alpha \cup \beta \cup \gamma, \quad g' = g|\delta, \quad h' = h|\delta, \quad k' = k|\delta,$$
$$\varepsilon = [\![g' \leqslant g' + h' \leqslant g' + h' + k']\!], \quad \eta = \varepsilon \cap \delta$$

and

$$\kappa = \neg[\![g' \leqslant g' + h' \leqslant g' + h' + k']\!],$$

we have that $\big((g, \alpha) \oplus (h, \beta)\big) \oplus (k, \gamma) = (g, \alpha) \oplus (h, \beta) \oplus (k, \gamma) = \big((g' + h' + k')|\eta, \alpha \cup \beta \cup \gamma \cup \kappa\big)$, implying the associativity.

m_5: Let $(g, \alpha) \in A$, $\beta = I \setminus \mathrm{supp}(g)$, then $\neg(g, \alpha) = \big(-g, (I \setminus \alpha) \cap \beta\big)$. Since $\mathrm{supp}(-g) = \mathrm{supp}(g)$, we have $\neg\neg(g, \alpha) = \big(g, I \setminus \big((I \setminus \alpha) \cap \beta\big) \cap \beta\big) = (g, \alpha)$ because $\alpha \subseteq \beta$.

m_8: We shall prove that $\neg(\neg x \oplus y) \oplus y = x \vee y$, proving then the equation $\neg(\neg x \oplus y) \oplus y = \neg(x \oplus \neg y) \oplus x$. Let $(g, \alpha), (h, \beta) \in A$. Using the Boolean product characterization, we have $\neg(\neg x \oplus y) \oplus y = x \vee y$ if and only if, for each $i \in I$,

$$\big(\neg(\neg x \oplus y) \oplus y\big)(i) = \begin{cases} x(i) & \text{if } y(i) \leqslant x(i); \\ y(i) & \text{if } x(i) \leqslant y(i). \end{cases},$$

which translated to the elements of A results:

$\big(\neg\big(\neg(g, \alpha) \oplus (h, \beta)\big) \oplus (h, \beta)\big)(i) =$

$(g, \alpha)(i)$ if $T\big(0, h(i), g(i)\big)$ or $\big(g(i) \neq 0$ and $h(i) = g(i)\big)$ or $\big(g(i) = 0$ and $\alpha(i) = 1\big)$ or $h(i) = \beta(i) = 0$;

$(h, \beta)(i)$ if $T\big(0, g(i), h(i)\big)$ or $\big(h(i) \neq 0$ and $g(i) = h(i)\big)$ or $\big(h(i) = 0$ and $\beta(i) = 1\big)$ or $g(i) = \alpha(i) = 0$.

Case $g(i) = \alpha(i) = 0$:

$\neg(g, \alpha)(i) = (0, 1)$ and then $\big(\neg\big(\neg(g, \alpha) \oplus (h, \beta)\big) \oplus (h, \beta)\big)(i) = \big(\neg\big((0, 1) \oplus (h, \beta)\big) \oplus (h, \beta)\big)(i) = \big((0, 0) \oplus (h, \beta)\big)(i) = (h, \beta)(i)$.

Case $g(i) = 0$, $\alpha(i) = 1$:
$\neg(g,\alpha)(i) = (0,0)$ and then $\big(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \big(\neg((0,0)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \big(\neg(h,\beta)\oplus(h,\beta)\big)(i) = (0,1) = (g,\alpha)(i)$.

Case $h(i) = \beta(i) = 0$:
$\big(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \big(\neg(\neg(g,\alpha)\oplus(0,0))\oplus(0,0)\big)(i) = \neg\neg(g,\alpha)(i) = (g,\alpha)(i)$.

Case $h(i) = 0$, $\beta(i) = 1$:
$\big(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \big(\neg(\neg(g,\alpha)\oplus(0,1))\oplus(0,1)\big)(i) = (0,1) = (h,\beta)(i)$.

Case $T(0,g(i),h(i))$, that is $0 < g(i) < h(i)$ and $\alpha(i) = \beta(i) = 0$:
that implies $\neg(g,\alpha)(i) = (-g,0)(i) > (-h,0)(i) = \neg(h,\beta)(i)$, and then

$$\neg(g,\alpha)(i)\oplus(h,\beta)(i) = (0,1),$$

concluding that

$$\big(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \neg(0,1)\oplus(h,\beta)(i) = (h,\beta)(i).$$

Case $T(0,h(i),g(i))$, that is $0 < h(i) < g(i)$ and $\alpha(i) = \beta(i) = 0$:
Since $\neg(g,\alpha)(i) < \neg(h,\beta)(i)$, we have $\neg(g,\alpha)(i)\oplus(h,\beta)(i) < (0,1)$, implying $\neg(g,\alpha)(i)\oplus(h,\beta)(i) = (-g+h,0)(i)$. Then $\big(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \big(\neg(-g+h,0)\big)\oplus(h,0)(i) = \big((g-h,0)(i)\big)\oplus(h,0)(i)$ which is equal to $(g,0)(i)$ because we have $T(0,g(i)-h(i),g(i))$.

Case $g(i) = h(i) \neq 0 = \alpha(i) = \beta(i)$:
We have $\neg(g,\alpha)(i) = \neg(h,\beta)(i)$.
So $\big(\neg(\neg(g,\alpha)\oplus(h,\beta))\oplus(h,\beta)\big)(i) = \big(\neg(0,1)\oplus(h,\beta)\big)(i) = \big((0,0)\oplus(h,\beta)\big)(i)$ which equals to $(h,\beta)(i)$.

m_2, m_3, m_4, m_6 and m_7 are immediate and m_9 can be considered a definition.

If $f\colon G \to H$ is an lc-homomorphism, observe that f induces a Boolean algebra homomorphism $B(f) = B(G) \to B(H)$, where $B(G)$ and $B(H)$ are the respective Boolean algebras of supports: Define $B(f)\big(\mathrm{supp}(g)\big) = \mathrm{supp}\,\big(f(g)\big)$. The good definition results from the fact that f maps weak units on weak units and preserves the lattice operations: So, let $g, g' \in G$ such that $\mathrm{supp}(g) = \mathrm{supp}(g')$. Let u be a weak unit in G. The element $g'' = \mathrm{pr}(u,g')$ is orthogonal to both g and g', and both $g+g''$ and $g'+g''$ are weak units. So since $\mathrm{supp}\,\big(f(g)+f(g'')\big) = \mathrm{supp}\,\big(f(g')+f(g'')\big) = I'$ (where I' is the Boolean space of H) and $f(g') \perp f(g'')$ we have that $\mathrm{supp}\,\big(f(g')\big) \subseteq \mathrm{supp}\,\big(f(g)\big)$. The proof of the other inclusion is analogous.

Now, if A and B are the respective MV-algebras constructed from G and H respectively, as above, define $\tilde{f}\colon A \to B$ by $\tilde{f}\big((g,\alpha)\big) = \big(f(g), B(f)(\alpha)\big)$. We shall proof that it is an MV-homomorphism: Let $(g,\alpha),(h,\beta)\in A$, call $\alpha' = I\setminus \operatorname{supp}(g)$ (where I is the Boolean space of G). Then

$$\begin{aligned}\tilde{f}(\neg(g,\alpha)) &= \tilde{f}(-g,(I\setminus\alpha)\cap\alpha') = (f(-g), B(f)((I\setminus\alpha)\cap\alpha'))\\ &= \big(-f(g), (B(f)(I)\setminus B(f)(\alpha))\cap B(f)(\alpha')\big)\\ &= \big(-f(g), (I'\setminus B(f)(\alpha))\cap B(f)(\alpha')\big).\end{aligned}$$

By calling $\alpha'' = I'\setminus \operatorname{supp}\big(f(g)\big)$, we have also

$$\neg\tilde{f}((g,\alpha)) = \big(-f(g), (I'\setminus B(f)(\alpha))\cap\alpha''\big).$$

Since $\alpha'' = B(f)(\alpha')$ we have that $\tilde{f}$ preserves the operation $\neg$.

For $\oplus$, call $\gamma = I\setminus(\alpha\cup\beta)$, $g' = g\big|\gamma$, $h' = h\big|\gamma$, $\delta = \gamma\cap[\![g'\leqslant g'+h']\!]$ and $\eta = \neg[\![g'\leqslant g'+h']\!]$. We have

$$\begin{aligned}(g,\alpha)\oplus(h,\beta) &= \big((g'+h')\big|\delta, \alpha\cup\beta\cup\eta\big),\\ \tilde{f}((g,\alpha)\oplus(h,\beta)) &= \big(f((g'+h')\big|\delta), B(f)(\alpha\cup\beta\cup\eta)\big)\\ &= \big(f(g'\big|\delta)+f(h'\big|\delta), B(f)(\alpha\cup\beta\cup\eta)\big)\end{aligned}$$

By the other side, calling $\mu = B(f)(\alpha)$, $\nu = B(f)(\beta)$, $\sigma = I'\setminus(\mu\cup\nu) = B(f)(\gamma)$, $\upsilon = \neg[\![f(g')\leqslant f(g')+f(h')]\!]$ (because f preserves the relation T), $g'' = f(g)\big|\sigma$, $h'' = f(h)\big|\sigma$, and $\tau = \sigma\cap[\![g''\leqslant g''+h'']\!]$, we have

$$\tilde{f}((g,\alpha))\oplus\tilde{f}((h,\beta)) = (f(g),\mu)\oplus(f(h),\nu) = \big((f(g)+f(h))\big|\tau, \mu\cup\nu\cup\upsilon\big).$$

Since, for each $i\in I$, $g'(i)\leqslant g'(i)+h'(i)$ if and only if $f(g')(i)\leqslant f(g')(i)+f(h')(i)$ because of axiom C1 and the fact that f is an lc-homomorphism, we have that $\upsilon = B(f)(\eta)$, proving $\tilde{f}((g,\alpha)\oplus(h,\beta)) = \tilde{f}((g,\alpha))\oplus\tilde{f}((h,\beta))$.

Finally we show that the compositions of both functors are the identity:

Call LC and MV, the categories of projectable lc-groups with weak unit and projectable MV-algebras, respectively, $\Psi\colon \mathrm{MV}\to\mathrm{LC}$ and $\Phi\colon \mathrm{LC}\to\mathrm{MV}$ the above constructed functors.

Let $G\in\mathrm{LC}$, $\Phi(G) = \{(g,\alpha)\in G\times B(I)\mid \operatorname{supp}(g)\cap\alpha = \emptyset\}$ (as a set) and $\Psi\big(\Phi(G)\big) = \Phi(G)/\!\sim$ (as a set). Observe that $a = (g,\alpha)\sim(h,\beta) = b$ if and only if $g = h$: by taking $a' = (0,\beta\setminus\alpha)$ and $b' = (0,\alpha\setminus\beta)$, we have $a\oplus a' = b\oplus b'$, $a'\perp b'$, $a\perp a'$ and $b\perp b'$, implying $(g,\alpha)\sim(g,\beta)$. Suppose now $g\neq h$, then the set $[\![a=b]\!]\cup[\![a=0\ \&\ b=1]\!]\cup[\![a=1\ \&\ b=0]\!]$ is strictly contained in I, implying that (g,α) is not equivalent to (h,β). Now, for the operations, it is immediate for 0 and $-$. Let $g,h\in G$, we can choose, for their images in $\Phi(G)$, the elements $(g,\emptyset)$ and $(h,\emptyset)$ respectively. By calling $J = [\![(g,\emptyset)\oplus(h,\emptyset)<1]\!]$, we have, in $\Psi(\Phi(G))$, $g+h = (((g,\emptyset)\oplus(h,\emptyset))|J \cup ((g,\emptyset)*(h,\emptyset))|I\setminus J)/\sim$. Observe that $J = [\![g \leqslant g+h]\!]$ and then $(g,\emptyset)\oplus(h,\emptyset) = ((g+h)|J, I\setminus J)$. So, it holds $g+h = (g+h)|J\cup(((g,\emptyset)*(h,\emptyset))|I\setminus J)/\sim = (g+h)|J\cup(\neg(\neg(g,\emptyset)\oplus\neg(h,\emptyset))|I\setminus J)/\sim = (g+h)|J\cup(\neg((-g,\emptyset)\oplus(-h,\emptyset))|I\setminus J)/\sim = (g+h)|J\cup(\neg((-g-h,\emptyset))_{I\setminus J})/\sim$ because $[\![-g \leqslant -g-h]\!] = I\setminus J$. So, we can conclude (in $\Psi(\Phi(G))$), $g+h = (g+h)|J\cup(-(-g,h))|I\setminus J = g+h$ (in G). We have, proved, then, that $\Psi\circ\Phi = \mathrm{Id}_G$.

For the converse, let $A \in$ MV. In $\Psi(A)$ the elements of A which coincide modulo a Boolean element are identified. Let $a\in A$. By setting $\alpha = [\![a=1]\!]$, we have that, in $\Phi\circ\Psi(A)$ the element $(a/\sim,\alpha)$ corresponds to a (in A). So, it is immediate to verify that the application $a \to (a/\sim,\alpha)$ gives a bijection between A and $\Phi\circ\Psi(A)$ preserving the 0 and 1. For the negation, $[\![\neg a = 1]\!] = [\![a=0]\!] = I\setminus(\alpha\cup\mathrm{supp}(a/\sim))$ and call $\beta = I\setminus\mathrm{supp}(a/\sim)$. We have then $\neg(a/\sim,\alpha) = (-(a/\sim),(I\setminus\alpha)\cap\beta) = (\neg a/\sim, I\setminus(\alpha\cup\mathrm{supp}(a/\sim)))$ proving that the above defined map preserves also the negation.

Finally, for the MV sum, let $a,b\in A$, $\alpha = [\![a=1]\!]$ and $\beta = [\![b=1]\!]$. Define $\gamma = I\setminus(\alpha\cup\beta)$, $(a/\sim)' = (a/\sim)|\gamma = (a|\gamma)/\sim$, $(b/\sim)' = (b/\sim)|\gamma = (b|\gamma)/\sim$, $\delta = \gamma\cap[\![(a/\sim)' \leqslant (a/\sim)'+(b/\sim)']\!]$ and $\eta = \neg[\![(a/\sim)' \leqslant (a/\sim)'+(b/\sim)']\!]$. So, we can write $(a/\sim,\alpha)\oplus(b/\sim,\beta) = (((a/\sim)'+(b/\sim)')|\delta, \alpha\cup\beta\cup\eta)$. Call now $J = [\![a|\gamma\oplus b|\gamma < 1]\!]$. We have then $(a/\sim)'+(b/\sim)' = (a\oplus b)|J\cap\gamma\cup(a*b)|(I\setminus J)\cap\gamma$, which implies $(a/\sim,\alpha)\oplus(b/\sim,\beta) = (a\oplus b)|J\cap\delta\cup(a*b)|(I\setminus J)\cap\delta\cup\alpha\cup\beta\cup\eta$. It is easy to verify that $J=\delta$, implying $(a/\sim,\alpha)\oplus(b/\sim,\beta) = (a\oplus b)|\delta\cup\alpha\cup\beta\cup\eta = a\oplus b$ because $\alpha\cup\beta\cup\eta = [\![a\oplus b]\!] = 1$.

So we can state the

Theorem 4.1. *The categories LC and MV are equivalent.*

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