Cyclical Properties of Baxter-King Filtered Time Series

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January 15, 2002

Abstract

This note demonstrates that the Baxter-King (1999) filter, and in general any band-pass filter, does not isolate the cycle in an unobserved components model with a stochastic trend. The first difference of the trend passes through the filter, and as a result, the spectral properties of the filtered series depend on the trend in the unfiltered series. It is demonstrated that for post-war U.S. Real GDP, the spectral properties of the BK filtered series are primarily to due to the stochastic trend in output.

JEL Codes: C32, E32

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I thank two anonymous referees, Luca Benati, Tim Cogley, Clive Granger, James Hamilton, Andrew Harvey, James Morley, David Papell, James Stock, Mark Wohar, and especially Charles Nelson, for helpful comments and discussions.

1. Introduction

In a recent paper, Baxter and King (1999) propose an approximate band-pass filter to extract the business cycle component from a time series with deterministic or stochastic trends. This note analyzes the relationship between the output from the Baxter-King (BK) filter and the cycle in an unobserved components model, where the cycle is the stationary deviation from a stochastic trend. I demonstrate that the BK filter, and more generally any band-pass filter, does not isolate the cycle when the trend component of the series to be filtered is integrated. The first difference of the trend passes through the filter, and as a result, the spectral properties of the BK filtered series depend on the trend in the unfiltered series.

Other studies of filtering nonstationary processes include the work by Harvey and Jaeger (1993) and Cogley and Nason (1995) on the Hodrick-Prescott (1980/1997) filter. Osborn (1995) studies simple moving average (*i.e.* non band-pass) filtering of integrated processes, and Benati (2001) analyzes the properties of time series generated from macroeconomic models and passed through the BK filter.

This note is organized as follows. Section 2 discusses alternative definitions of the business cycle. I define the business cycle as stationary deviations from a stochastic trend. In Section 3, I demonstrate that the spectrum of a BK filtered unobserved components model is comprised of three components: one due to the stochastic trend, one due to the cycle, and a covariance term. In addition, the BK filter assigns a higher weight to the trend component than it does the cyclical component in determining the spectral power of the filtered series. Section 4 summarizes and offers concluding remarks.

2. Defining the Business Cycle

Hodrick and Prescott (1997/1980), Baxter and King (1999), and others define the business cycle as the stationary component that remains after (the log of) output is passed through an ideal band-pass filter. This definition relies on frequency components of the data, and in the case of the BK filter, the frequency band is 1.5 to 8 years per cycle.

An alternative definition of the business cycle relies on an unobserved components (UC) view of output. In this case, (the log of) output is the sum of an unobserved trend and cycle:

$$y_t = t_t + c_t ,$$

where t_t is the nonstationary trend and c_t is the stationary cycle around this trend. The UC model has a rich history in econometrics, and has been analyzed by Harvey (1985), Watson (1986), Clark (1987), Harvey and Jaeger (1993), and Morley, Nelson, and Zivot (2002) among others. See Cogley (2001) for additional discussion on alternative definitions of the business cycle.

In this note, I adopt the UC definition of the business cycle. The purpose of this study is to ascertain whether or not the BK filter can isolate the cyclical component of an integrated series, where the cycle is defined as stationary deviations from trend. In other words, how closely does the output from passing y_t through the BK filter resemble c_t over a particular frequency band?

3. Cyclical Properties of Baxter-King Filtered Time Series

3.1 The Baxter-King Filter

The BK filter is an approximation to an ideal band-pass filter. This ideal filter has the following 2-sided infinite moving average representation:

$$a(L) = \sum_{k=-\infty}^{\infty} a_k L^k$$

where symmetry $(a_k = a_{-k})$ is imposed so that the filter does not induce a phase shift. The transfer function of a filter determines the extent to which periodic components of the filtered series are related to periodic components of the underlying (unfiltered) series. The BK filter is designed to pass through the stationary component of output whose periodicity ranges from 1.5 to 8 years per cycle. For stationary time series, the transfer function of this ideal filter takes the form:[1]

$$a(w) = \begin{cases} 1 & \text{if } p/16 \le |w| \le p/3 \\ 0 & \text{otherwise} \end{cases} .$$

This ideal filter is not feasible since it requires an infinite amount of data. Baxter and King employ the following truncated version of the ideal filter, which is the optimal approximation:

[1] At quarterly frequencies, the desired band is 6 to 32 quarters per cycle. Since $\omega = 2\pi/P$, this translates into a frequency band of $p/16 \le |w| \le p/3$.

$$a_K(L) = \sum_{k=-K}^{K} a_k L^k \ .$$

This approximate band-pass filter, with corresponding transfer function $a_K(w)$, sacrifices 2K data points.

The BK filter is designed so that it renders trending series stationary. This is achieved by constraining the frequency response of the filter to be zero at zero frequency. The BK filter may thus be factored as:

$$\begin{aligned} a_K(L) &= -(1-L)(1-L^{-1})y_{K-1}(L) \\ &= L^{-1}(1-L)^2 y_{K-1}(L) \end{aligned}$$

where

$$y_{K-1}(L) = \sum_{h=-(K-1)}^{K-1} y_h L^h$$

and the coefficients of $y_{K-1}(L)$ are given by

$$y_{|h|} = \sum_{j=|h|+1}^{K} (j-|h|)a_j \ .$$

Since the BK filter contains two differencing operators, it removes linear and quadratic time trends, and up to two unit roots.

3.2 Properties of BK Filtered Series When the Trend is Integrated

Consider the following unobserved components model:

$$y_t = t_t + c_t$$

$$t_t = m + t_{t-1} + h_t$$

where c_t and h_t are both stationary processes. It is instructive to factor the BK filter as follows:

$$a_K(L) = (1-L)b_K(L)$$

where $b_K(L) = -(1-L^{-1})y_{K-1}(L)$.

Define x_t as the output from passing y_t through the BK filter:

$$x_t \equiv a_K(L)y_t \ .$$

For the above UC model, the BK filtered series is:

$$x_t = a_K(L)t_t + a_K(L)c_t = b_K(L)h_t + a_K(L)c_t \ .$$

The appearance of $a_K(L)c_t$ in the above expression demonstrates that the BK filter approximately passes the UC cycle through at business cycle frequencies. However, the BK filter also allows the first difference of the trend, h_t, to pass through. In addition, the periodic components of h_t and c_t are passed through with different weights. When the BK filter is applied to an integrated process, it differences the series, rendering it stationary, and then filters the resulting stationary series. Transforming the series "uses up" one of the difference operators in $a_K(L)$, and the asymmetric filter $b_K(L)$ is then applied to the first difference of the trend. Therefore, while the BK filter removes unit roots, it does not remove stochastic trends.

The spectrum of the BK filtered series is:

$$f_x(w) = \left|b_K(w)\right|^2 f_h(w) + \left|a_K(w)\right|^2 f_c(w) + 2\left|a_K(w)\right|^2 \operatorname{Re}\left[\frac{f_{hc}(w)}{\Delta(w)}\right],$$

where $b_K(L)$ is the transfer function of $b_K(L)$, $f_h(w)$ and $f_c(w)$ are the spectra of h_t and c_t respectively, $f_{hc}(w)$ is the cross-spectrum between h_t and c_t, and $\Delta(w)$ is the transfer function of the difference operator. The last term in the above expression can be further simplified as:

$$2\left|a_K(w)\right|^2 \operatorname{Re}\left[\frac{f_{hc}(w)}{\Delta(w)}\right] = \left|a_K(w)\right|^2 co_{hc}(w) - \frac{\sin(w)}{1-\cos(w)}\left|a_K(w)\right|^2 q_{hc}(w),$$

where $co_{hc}(w)$ and $q_{hc}(w)$ are the co-spectra and quadrature spectra between h_t and c_t respectively.

The periodicity displayed by the BK filtered UC model thus arises from three sources: the first difference of the stochastic trend, the UC cycle, and their covariance. However, in most of the existing literature, such as Harvey (1985) and Clark (1987), the trend and cycle are assumed to be uncorrelated. In this case, the term involving the cross-spectrum vanishes.

The extent to which the periodic behavior present in x_t reflects c_t and h_t is determined by $\left|a_K(L)\right|^2$ (the squared gain of $a_K(L)$) and $\left|b_K(w)\right|^2$ (the squared gain of $b_K(L)$) respectively. To quantify the influence of the stochastic trend and the UC cycle in determining the spectral power of a BK filtered unobserved components model, Figure

1 plots $|a_K(w)|^2$ (the solid line) and $|b_K(w)|^2$ (the dashed line), for various values of K. In addition, the frequency band of 1.5 to 8 years per cycle is shaded.

We can clearly see the extent to which the behavior present in x_t depends on the underlying stochastic trend and the UC cycle. For each value of K, the BK filter ascribes a higher frequency response to h_t than it does to c_t. Furthermore, the contribution of h_t to the spectral power of x_t increases as more data is sacrificed. Indeed, when 20 years of quarterly data are sacrificed (K=40), $|a_K(w)|^2$ is dwarfed by $|b_K(w)|^2$. Therefore, when the trend component of the underlying series is integrated, and uncorrelated with the UC cycle, the BK filter will overstate the importance of transitory dynamics at business cycle frequencies.

We note that when the trend and cycle are correlated, it is possible for the BK filter to understate transitory variation at business cycle frequencies. This will occur if

$$|b_K(w)|^2 f_h(w) + 2|a_K(w)|^2 \operatorname{Re}\left[\frac{f_{hc}(w)}{\Delta(w)}\right] < 0.$$

Whether or not this condition holds will depend on the properties of $f_h(w)$ and $f_{hc}(w)$, as well as $|a_K(L)|^2$ and $|b_K(w)|^2$.

3.3 BK Filtered Post-War Quarterly U.S. Real GDP

As an illustration of the potential for the BK filter to overstate transitory dynamics in practice, we employ a parameterization taken from Morley, Nelson, and Zivot (2002). They estimate the following stochastic trend plus cycle model for post-war quarterly U.S. Real GDP, 1947.1-1998.2:

$$y_t = t_t + c_t$$

$$t_t = 0.82 + t_{t-1} + h_t \qquad s_h = 1.24$$

$$(1 - 1.34L + 0.71L^2)c_t = e_t \qquad s_e = 0.75$$

with $E(h_t e_t) = s_{he} = -0.84$. This parameterization is particularly convenient since c_t has 84% of its spectral power in the BK band.

Figure 2 plots the three spectral components of x_t, the BK filtered series, from this data generating process. $\left|a_K(w)\right|^2 f_c(w)$ is represented by the solid line, $\left|b_K(w)\right|^2 f_h(w)$ by the dashed line, and $2\left|a_K(w)\right|^2 \mathrm{Re}\left[\frac{f_{hc}(w)}{\Delta(w)}\right]$ by the solid and dashed line. Again, the contribution of the first difference of the trend in determining the spectral power of the BK filtered series increases with K. Also, since the trend and cycle are negatively correlated, the cross-spectral term is negative. This demonstrates the potential to mitigate the presence of the term involving the stochastic trend. Notice however, that in this case, $2\left|a_K(w)\right|^2 \mathrm{Re}\left[\frac{f_{hc}(w)}{\Delta(w)}\right]$ and $\left|a_K(w)\right|^2 f_c(w)$ nearly cancel each other, so that the spectral power of the BK filtered UC model *is almost entirely determined by the first difference of the stochastic trend*.

4. Summary and Concluding Remarks

This note analyzes the relationship between the output of the Baxter-King filter and the cycle in an unobserved components model. I demonstrate that the Baxter-King filter does not isolate the cycle, but rather passes the first difference of the trend through to the filtered series. Furthermore, the weight that the BK filter assigns to the first difference of the trend in determining the spectral power of the BK filtered series is much higher than the weight it assigns to the UC cycle. This illustrates the potential for the BK filter to overstate the importance of transitory dynamics at business cycle frequencies. An empirical example using post-war quarterly U.S. Real GDP demonstrates the importance of this phenomenon in practice.

The analysis presented in this note is not specific to the Baxter-King filter, but indeed applies to all band-pass filters. The simple act of differencing does not remove a stochastic trend, it merely renders it stationary. Therefore, while band-pass filtering can render an integrated series stationary, the properties of the filtered series will depend on the trend in the unfiltered series.

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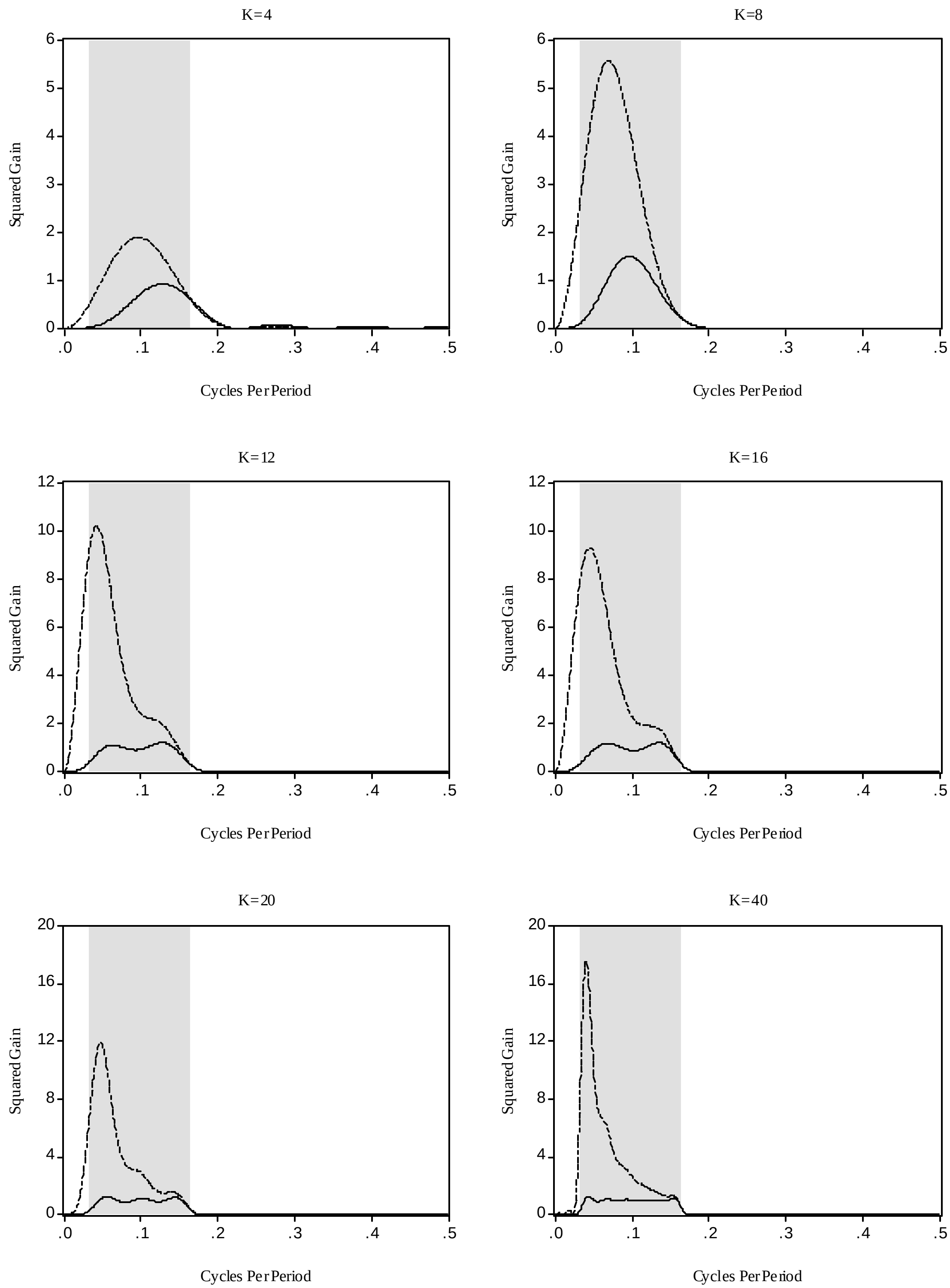
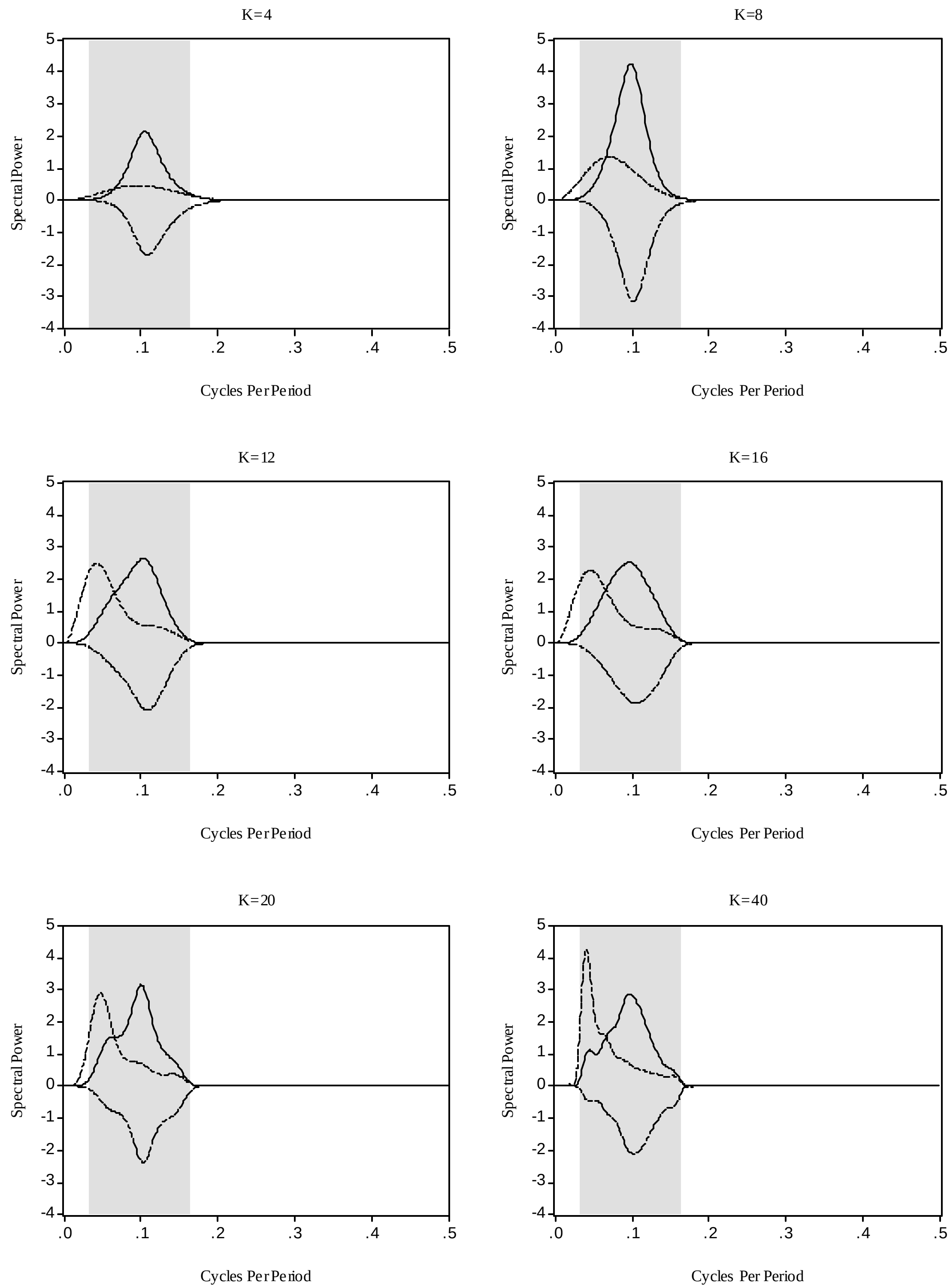
Figure 1. $|a_K(L)|^2$ **and** $|b_K(L)|^2$



$|a_K(L)|^2$ is represented by the solid line, and $|b_K(L)|^2$ by the dashed line.

Figure 2. Spectral Components of BK Filtered U.S. Real GDP: 1947.1 – 1998.2



$|a_K(L)|^2 f_c(w)$ is represented by the solid line, $|b_K(L)|^2 f_h(w)$ by the dashed line, and $2|a_K(w)|^2 \,\mathrm{Re}\left[\frac{f_{hc}(w)}{\Delta(w)}\right]$ by the solid and dashed line.