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CYCLOTOMY OF ORDER 15 OVER GF (p^2), p = 4, 11(MOD 15)

CHRISTIAN FRIESEN
Department of Mathematics and Statistics
University of New Brunswick
Fredericton, New Brunswick, Canada E3B 5A3

JOSEPH B. MUSKAT
Department of Mathematics and Computer Science
Bar-Ilan University
Ramat-Gan, Israel

BLAIR K. SPEARMAN
Department of Mathematics
College of New Caledonia
Prince George, British Columbia, Canada V2N 1P8

KENNETH S. WILLIAMS
Department of Mathematics and Statistics
Carleton University, Ottawa, Ontario, Canada K1S 5B6

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ABSTRACT. For primes $p \equiv 4, 11 \pmod{15}$ explicit formulae are obtained for the cyclotomic numbers of order 15 over $GF(p^2)$.

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1. INTRODUCTION.

Let $e \geq 2$ and $\ell \geq 1$ be integers and let p be an odd prime such that e divides $p^\ell - 1$. We set $q = p^\ell$ and denote the finite field with q elements by $GF(q)$. The positive integer f is defined by $q = ef + 1$. We fix once and for all a generator γ of the multiplicative group $GF(q)^* = GF(q) - \{0\}$, and set $g = \gamma^{1+p+ \ldots +p^{\ell-1}}$ so that g is a primitive root modulo p. For $\alpha \in GF(q)^*$ the index of α with respect to γ, denoted by $\operatorname{ind}_\gamma \alpha$, is the unique integer m such that $\alpha = \gamma^m$, $0 \leq m \leq q - 2$, and for $a \in GF(p)^*$ the index of a with respect to g, denoted by $\operatorname{ind}_g a$, is the unique integer n such that $a \equiv g^n \pmod{p}$, $0 \leq n \leq p - 2$. These indices are related by the following congruence: for $a \in GF(p)^*$ we have

$$\operatorname{ind}_\gamma a \equiv (1 + p + \ldots + p^{\ell-1}) \operatorname{ind}_g a \pmod{p^\ell - 1}. \tag{1.1}$$

The congruence (1.1) will be used many times throughout the paper.

The number of solutions $\alpha \in GF(q)^+ = GF(q)^* - \{1\}$ of the pair of congruences

$$\operatorname{ind}_\gamma (\alpha - 1) \equiv h \pmod{e},$$
$$\operatorname{ind}_\gamma \alpha \equiv k \pmod{e}, \tag{1.2}$$

is denoted by $(h, k)_e$, where h and k are integers such that $0 \le h \le e - 1$, $0 \le k \le e - 1$. The numbers $(h, k)_e$ are called the cyclotomic numbers of order e over GF(q), and they depend on p, ℓ, e and γ. It is a central problem in the theory of cyclotomy to obtain precise formulae for these numbers. This has been done for a number of values of e and ℓ—for small values of e these formulae are given in [11] and references to further results are given, for example, in [9].

The purpose of this paper is to give the first complete determination of the cyclotomic numbers $(h, k)_e$ in the case $e = 15$, $\ell = 2$, $p \equiv 4, 11 \pmod{15}$ (so that $q = p^2 \equiv 1 \pmod{15}$). The two basic properties of cyclotomic numbers (6.1), (6.2) show that the $e^2 = 225$ cyclotomic numbers take on at most 46 values. As we have assumed that $p \not\equiv 1 \pmod{15}$ the additional property $(ph, pk)_{15} = (h, k)_{15}$ reduces the maximum number of possible different values to 28 in the case $p \equiv 4 \pmod{15}$ and 29 in the case $p \equiv 11 \pmod{15}$. It should be remarked that in the remaining case for which $p^2 \equiv 1 \pmod{15}$, namely, $p \equiv 14 \pmod{15}$, the phenomenom of uniform cyclotomy occurs (see [2: Theorem 1]) and there are just 3 different cyclotomic numbers.

In the case $p \equiv 4 \pmod{15}$, the 28 cyclotomic numbers are expressed (see Tables 13a, 13b, 13c) as linear combinations of p^2, p, 1, A^2, AB, B^2, AT, BT, AU, BU, $T^2 - 15U^2$, TU, where the integers A, B, T, U are defined in Theorem 1 and satisfy

$$p = A^2 - AB + B^2 , \quad A \equiv -1 \pmod{3} , \quad B \equiv 0 \pmod{3}, \tag{1.3}$$

$$p = T^2 + 15U^2 , \quad T \equiv -1 \pmod{3} . \tag{1.4}$$

It should be noted that (1.3) alone does not distinguish between the two solutions (A, B) and (A - B, -B), while (1.4) determines T uniquely but determines U only up to sign.

For the case $p \equiv 11 \pmod{15}$, the 29 cyclotomic numbers are expressed (see Tables 14a - 14g) as linear combinations of p^2, p, 1, XU, XV, U^2, UV, UW, V^2, VW, W^2, where the integers X, U, V, W are defined in Theorem 2 and satisfy

$$\begin{cases} p = X^2 + 5U^2 + 5V^2 + 5W^2, \quad X \equiv -1 \pmod{5} , \\ XW = V^2 - UV - U^2 . \end{cases} \tag{1.5}$$

Again, it should be noted that (1.5) alone does not distinguish between the four solutions (X, U, V, W) , (X, V, -U, -W) , (X, -U, -V, W) , (X, -V, U, -W) (see [10]).

As an application of the formulae for the cyclotomic numbers in the case $p \equiv 4 \pmod{15}$, we prove the following theorem in §7.

THEOREM 8. Let $p \equiv 4 \pmod{15}$ be a prime. Let γ be a generator of $GF(p^2)^*$. Set $g = \gamma^{p+1}$, so that g is a primitive root (mod p). The quantity $\gamma^{(q-1)/5} - \gamma^{2(q-1)/5} - \gamma^{3(q-1)/5} + \gamma^{4(q-1)/5} \in GF(p)$ and a unique square root of 5 modulo p is given by

$$\sqrt{5} \equiv \gamma^{(q-1)/5} - \gamma^{2(q-1)/5} - \gamma^{3(q-1)/5} + \gamma^{4(q-1)/5} \pmod{p}. \tag{1.6}$$

Then we have

$$\operatorname{ind}_g \tfrac{1}{2}(1 + \sqrt{5}) \equiv -U \pmod{3}, \tag{1.7}$$

where U is given by Theorem 1.

In particular, we see that $\frac{1}{2}(1 + \sqrt{5})$ (equivalently $\frac{1}{2}(1 - \sqrt{5})$) is a cube modulo the prime $p \equiv 4 \pmod{15}$, if and only if, $U \equiv 0 \pmod{3}$, where U is given by (1.4). A sketch of a proof of this, using classfield theory, has been given by Aigner [1]. This result complements that of Emma Lehmer (see [9: Theorem 2]) for the case $p \equiv 1 \pmod{15}$.

2. BASIC PROPERTIES OF JACOBI, GAUSS AND EISENSTEIN SUMS.

With the notation of §1, we define the Jacobi sum $J_q(\beta^m, \beta^n)$ over GF(q) of order e for integers m and n by

$$J_q(\beta^m, \beta^n) = \sum_{\alpha \in GF(q)^\dagger} \beta^{m \operatorname{ind}_\gamma \alpha + n \operatorname{ind}_\gamma(1-\alpha)} , \tag{2.1}$$

where $\beta = \exp(2\pi i/e)$. These sums have the following well-known properties (see, for example, [11: Lemmas 13-16]):

$$J_q(\beta^m, \beta^n) = J_q(\beta^n, \beta^m) = (-1)^{nf} J_q(\beta^{-m-n}, \beta^n) . \tag{2.2}$$

$$J_q(\beta^m, \beta^n)\, J_q(\beta^{-m}, \beta^{-n}) = q , \text{ if } e \nmid m , e \nmid n , e \nmid m+n , \tag{2.3}$$

$$J_q(\beta^m, \beta^n) = -1 , \text{ if } e \nmid m , e \mid n , \text{ or } e \mid m , e \nmid n , \tag{2.4}$$

$$J_q(\beta^m, \beta^n) = q - 2 , \text{ if } e \mid m , e \mid n , \tag{2.5}$$

$$J_q(\beta^m, \beta^n) = (-1)^{nf+1} , \text{ if } e \nmid m , e \nmid n , e \mid m+n , \tag{2.6}$$

$$J_q(\beta^{pm}, \beta^{pn}) = J_q(\beta^m, \beta^n) . \tag{2.7}$$

If (j, e) = 1, the mapping which sends β to β^j is an automorphism of the cyclotomic field $Q(\beta)$, which we shall denote by σ_j. We have

$$\sigma_j(J_q(\beta^m, \beta^n)) = J_q(\beta^{jm}, \beta^{jn}) . \tag{2.8}$$

Closely related to the Jacobi sum is the Gauss sum $G_q(\beta^m)$, defined for any integer m by

$$G_q(\beta^m) = \sum_{\alpha \in GF(q)^*} \beta^{m \operatorname{ind}_\gamma \alpha} \zeta^{\operatorname{tr} \alpha} , \tag{2.9}$$

where tr α denotes the trace of α from GF(q) to GF(p), that is

$$\operatorname{tr} \alpha = \alpha + \alpha^p + \alpha^{p^2} + \ldots + \alpha^{p^{\ell-1}} ,$$

and $\zeta = \exp(2\pi i/p)$. Gauss sums have the following properties (see for example [8: pp. 3-5] and [3: Theorem 2.12])

$$G_q(\beta^m) = -1, \; e \mid m , \tag{2.10}$$

$$G_{p^2}(\beta^m) = p(-1)^{(p+1)/e'} , \text{ if } e \nmid m \text{ and } e' \mid p+1 , \tag{2.11}$$

where we have written e' for e/(e, m) ,

$$G_q(\beta^m)\, G_q(\bar{\beta}^m) = (-1)^{mf} q, \text{ if } e \nmid m , \tag{2.12}$$

$$G_q(\beta^{pm}) = G_q(\beta^m) . \tag{2.13}$$

In addition, these sums satisfy the Davenport-Hasse relations [4: eqns. (0.8) and (0.9_1)] (see, for example, [8: p. 20, Theorem 5.1] and [3: §8])

$$G_q(\beta^{m(q-1)/(p-1)}) = (-1)^{\ell-1} G^\ell(\beta^m) , \tag{2.14}$$

where the Gauss sum $G(\beta^m)$ is defined in (2.18), and for e = xy and an arbitrary integer t

$$G_q(\beta^{ty}) \prod_{k=1}^{y-1} G_q(\beta^{kx}) = \beta^{ty \operatorname{ind}_\gamma y} \prod_{k=0}^{y-1} G_q(\beta^{kx+t}) . \tag{2.15}$$

The basic relationship between Gauss and Jacobi sums is (see, for example, [3: Theorem 2.2].)

$$J_q(\beta^m, \beta^n) = \frac{G_q(\beta^m)\, G_q(\beta^n)}{G_q(\beta^{m+n})}, \text{ if } e \nmid m,\ e \nmid n,\ e \nmid m+n. \tag{2.16}$$

We also define the Jacobi sum

$$J(\beta^m, \beta^n) = \sum_{a \in GF(p)^\dagger} \beta^{m \operatorname{ind}_\gamma a + n \operatorname{ind}_\gamma (1-a)} \tag{2.17}$$

and the Gauss sum

$$G(\beta^m) = \sum_{a \in GF(p)^*} \beta^{m \operatorname{ind}_\gamma a} \zeta^a. \tag{2.18}$$

We note that if $\ell = 1$ (so that $q = p$), then $J_q(\beta^m, \beta^n) = J(\beta^m, \beta^n)$ and $G_q(\beta^m) = G(\beta^m)$. The sums $J(\beta^m, \beta^n)$ and $G(\beta^m)$ also have the properties (2.2), (2.3) with the right hand side replaced by p, (2.4), (2.5) with the right hand side replaced by $p - 2$, (2.6), (2.7), (2.8), (2.10), (2.12) with the right hand side replaced by $(-1)^{mf} p$, (2.13), and (2.16).

Next, we define the Eisenstein sums $E(\beta^m)$ over $GF(p^2)$. We set

$$E(\beta^m) = \sum_{b=0}^{p-1} \beta^{m \operatorname{ind}_\gamma (1+bx)}, \tag{2.19}$$

where x is any element of $GF(p^2)$, such that

$$GF(p^2) = \{ a + bx \mid a, b = 0, 1, 2, \ldots, p-1 \} \tag{2.20}$$

and $x^2 \in GF(p)$. (For example, we could take $x = \gamma^{(p+1)/2}$.) Clearly, $E(\beta^m)$ is independent of the choice of x as $bx (1 \le b \le p-1)$ runs twice through the square roots of the quadratic non-residues of $GF(p)$. Eisenstein sums have the properties (see, for example, [3: pp. 377-381])

$$E(\beta^m) = p, \text{ if } e \mid m, \tag{2.21}$$

$$E(\beta^m) = -(-1)^{(p+1)/e'}, \text{ if } e \nmid m \text{ and } e' \mid p+1, \tag{2.22}$$

$$E(\beta^m)\, E(\beta^{-m}) = p, \text{ if } e' \nmid p+1, \tag{2.23}$$

$$E(\beta^{pm}) = E(\beta^m), \tag{2.24}$$

$$\sigma_j(E(\beta^m)) = E(\beta^{jm}), \text{ if } (j, e) = 1, \tag{2.25}$$

where, again, we have written e' for $e/(e, m)$.

Finally, we remark that the Eisenstein sums are related to the Gauss sums as follows (see, for example, [3: Theorem 2.23]):

$$E(\beta^m) = \beta^{m \operatorname{ind}_\gamma 2}\, G_{p^2}(\beta^m)/G(\beta^m), \text{ if } e' \nmid p+1. \tag{2.26}$$

3. DETERMINATION OF EISENSTEIN SUMS OVER $GF(p^2)$ FOR $e = 15$, $p \equiv 4, 11 \pmod{15}$.

Throughout §3 and §4, we take $e = 15$, $\ell = 2$, $q = p^2$ and $p \equiv 4, 11 \pmod{15}$. We begin by determining the Eisenstein sums $E(\beta^m)$ $(m = 1, 2, \ldots, 14)$ in the case $p \equiv 4 \pmod{15}$.

THEOREM 1. Let $p \equiv 4 \pmod{15}$ be a prime. Let γ be a generator of $GF(p^2)^*$. Set $g = \gamma^{p+1}$, $\beta = \exp(2\pi i/15)$, and $\omega = \beta^5 = \exp(2\pi i/3) = \frac{1}{2}(-1 + \sqrt{-3})$. Then $\omega^{-\operatorname{ind}_g 10} E(\beta)$ is an integer of $Q(\sqrt{-15})$ of the form $T + U\sqrt{-15}$, where T and U are

rational integers, and we have

$$E(\beta) = E(\beta^4) = \overline{E(\beta^{11})} = \overline{E(\beta^{14})} = \omega^{\text{ind}_g 10}(T + U\sqrt{-15}) , \tag{3.1}$$

$$E(\beta^2) = \overline{E(\beta^7)} = E(\beta^8) = \overline{E(\beta^{13})} = \omega^{-\text{ind}_g 10}(T + U\sqrt{-15}) , \tag{3.2}$$

$$E(\beta^3) = E(\beta^6) = E(\beta^9) = E(\beta^{12}) = -1 , \tag{3.3}$$

$$E(\beta^5) = \overline{E(\beta^{10})} = -\omega^{-\text{ind}_g 2}(A + B\omega) , \tag{3.4}$$

where

$$J(\beta, \beta) = A + B\omega . \tag{3.5}$$

Moreover, A, B, T, U satisfy (1.3), (1.4), as well as the congruences given in the following chart.

$\text{ind}_g 5$ (mod 3)	A (mod 5)	B (mod 5)
0	-T	0
1	T	1
2	0	-T

(3.6)

PROOF. From (1.1) we have for $a \in GF(p)^*$

$$\text{ind}_\gamma a \equiv (1 + p)\,\text{ind}_g a \pmod{p^2 - 1} ,$$

so that

$$\text{ind}_\gamma a \equiv -\text{ind}_g a \pmod 3 \tag{3.7}$$

and

$$\text{ind}_\gamma a \equiv 0 \pmod 5 . \tag{3.8}$$

From (2.18) and (3.8) we see that

$$G(\beta^m) = G(\beta^n) \text{ , if } m \equiv n \pmod 3 . \tag{3.9}$$

We begin by proving (3.4). We have

$$\begin{aligned} E(\beta^5) &= \omega^{\text{ind}_\gamma 2}\, G_q(\beta^5)/G(\beta^5) && \text{(by (2.26))} \\ &= \omega^{-\text{ind}_g 2}\, G_q(\beta^5)/G(\beta^2) && \text{(by (3.7) and (3.9))} \\ &= -\omega^{-\text{ind}_g 2}\, \{G(\beta)\}^2/G(\beta^2) && \text{(by (2.14))} \\ &= -\omega^{-\text{ind}_g 2}\, J(\beta, \beta) && \text{(by (2.16))} \\ &= -\omega^{-\text{ind}_g 2}(A + B\omega) , \end{aligned}$$

where A and B satisfy (1.3) (see, for example, [7: Prop. 8.3.4]). Appealing to (2.25), we obtain $\overline{E(\beta^{10})} = E(\beta^5)$, which completes the proof of (3.4).

We note that (3.3) follows from (2.22), and in view of (2.24) and (2.25), to complete the proof it suffices to prove

$$E(\beta^2) = \omega^{\text{ind}_g 10}\, E(\beta) \tag{3.10}$$

and

$$E(\beta) = \omega^{\text{ind}_g 10}(T + U\sqrt{-15}) . \tag{3.11}$$

First, we prove (3.10). Taking t = 1, x = 3, y = 5 in (2.15) and appealing to

(2.12), we obtain

$$q^2 G_q(\beta^5) = \omega^{\text{ind}_\gamma 5}\, G_q(\beta)\, G_q(\beta^4)\, G_q(\beta^7)\, G_q(\beta^{10})\, G_q(\beta^{13}) .$$

Applying (2.26) and (3.9), we have

$$\omega^{\text{ind}_g 5}\, q^2 G(\beta^2)\, E(\beta^5) = \{G(\beta)\}^5\, E(\beta)\, E(\beta^4)\, E(\beta^7)\, E(\beta^{10})\, E(\beta^{13}) .$$

Multiplying both sides by $G(\beta)$ and appealing to (2.12), (2.23) and (2.24), we obtain

$$\omega^{2\,\text{ind}_g 5}\, G(\beta)^6\, E(\beta)^2 = p^2\, E(\beta^2)^2\, E(\beta^5)^2 ,$$

giving

$$\omega^{\text{ind}_g 5}\, G(\beta)^3\, E(\beta) = \pm p\, E(\beta^2)\, E(\beta^5) . \tag{3.12}$$

As

$$G(\beta)^3 = J(\beta, \beta)\, G(\beta^2)\, G(\beta) = p(A + B\omega) ,$$

we obtain from (3.4) and (3.12)

$$E(\beta^2) = \pm\omega^{\text{ind}_g 10}\, E(\beta) . \tag{3.13}$$

We now determine which sign holds in (3.13). Cubing both sides of (3.13), we obtain

$$E(\beta^2)^3 = \pm E(\beta)^3 ,$$

and so

$$E(\beta^6) \equiv \pm E(\beta^3) \pmod 3 . \tag{3.14}$$

Since $E(\beta^3) = E(\beta^6) = -1$, we must have the + sign holding in (3.14), giving (3.10).

Raising (3.10) to the 5th power and reducing modulo 5, we obtain

$$E(\beta^{10}) \equiv \{E(\beta^2)\}^5 \equiv \omega^{-\text{ind}_g 10}\, \{E(\beta)\}^5 \equiv \omega^{-\text{ind}_g 10}\, E(\beta^5) \pmod 5 .$$

Then, appealing to (3.4), we obtain

$$A + B\omega^2 \equiv \omega^{-\text{ind}_g 5}\, (A + B\omega) \pmod 5 ,$$

which gives

$$\begin{aligned} &B \equiv 0 \pmod 5, \text{ if } \text{ind}_g 5 \equiv 0 \pmod 3 , \\ &A \equiv B \pmod 5, \text{ if } \text{ind}_g 5 \equiv 1 \pmod 3 , \\ &A \equiv 0 \pmod 5, \text{ if } \text{ind}_g 5 \equiv 2 \pmod 3 . \end{aligned} \tag{3.15}$$

Next, we prove (3.11). We have

$$\sigma_2(\omega^{-\text{ind}_g 10}\, E(\beta)) = \omega^{\text{ind}_g 10}\, E(\beta^2) = \omega^{-\text{ind}_g 10}\, E(\beta) .$$

From this equation, we see that $\omega^{-\text{ind}_g 10}\, E(\beta)$ is invariant under the automorphism σ_2 so that $\omega^{-\text{ind}_g 10}\, E(\beta)$ is an integer of $Q(\sqrt{-15})$. Hence, we have

$$\omega^{-\text{ind}_g 10}\, E(\beta) = \tfrac{1}{2}(t + u\sqrt{-15}),\ t \equiv u \pmod 2 ,$$

where t and u are integers such that $4p = t^2 + 15u^2$. Clearly, t and u cannot both be odd, so $t = 2T$, $u = 2U$, and

$$\omega^{-\text{ind}_g 10}\, E(\beta) = T + U\sqrt{-15} ,$$

where $p = T^2 + 15U^2$. This proves (3.11).

We now show that $T \equiv -1 \pmod 3$. Cubing (3.11), we obtain by (3.3)

$$T \equiv T^3 \equiv E(\beta)^3 \equiv E(\beta^3) \equiv -1 \pmod 3 .$$

Finally, we prove (3.6). Raising (3.11) to the fifth power, we obtain by (3.4)

$$-\omega^{\operatorname{ind}_g 2}(A + B\omega) \equiv E(\beta^5) \equiv E(\beta)^5 \equiv \omega^{-\operatorname{ind}_g 10} T^5 \equiv \omega^{-\operatorname{ind}_g 10} T \pmod 5 ,$$

that is

$$T \equiv -\omega^{\operatorname{ind}_g 5}(A + B\omega) \pmod 5 ,$$

which together with (3.15) gives (3.6).

This completes the proof of Theorem 1.

Next we determine the Eisenstein sums $E(\beta^m)$ $(1 \le m \le 14)$ in the case $p \equiv 11 \pmod{15}$.

THEOREM 2. Let $p \equiv 11 \pmod{15}$ be a prime. Let γ be a generator of $GF(p^2)^*$. Set $g = \gamma^{p+1}$, $\beta = \exp(2\pi i/15)$, and $\theta = \beta^3 = \exp(2\pi i/5) = (5^{\frac{1}{2}} - 1 + i(10+2(5)^{\frac{1}{2}})^{\frac{1}{2}})/4$. Then

$$E(\beta) = \overline{E(\beta^4)} = E(\beta^{11}) = \overline{E(\beta^{14})} = \theta^{2\operatorname{ind}_g 2 - \operatorname{ind}_g 3} \sum_{i=0}^{4} t_i \theta^i , \tag{3.16}$$

$$E(\beta^2) = E(\beta^7) = \overline{E(\beta^8)} = \overline{E(\beta^{13})} = \theta^{-\operatorname{ind}_g 2 - 2\operatorname{ind}_g 3} \sum_{i=0}^{4} t_i \theta^{2i} , \tag{3.17}$$

$$E(\beta^3) = E(\beta^{12}) = -\sum_{i=0}^{4} t_i \theta^i , \tag{3.18}$$

$$E(\beta^5) = E(\beta^{10}) = -1 , \tag{3.19}$$

$$E(\beta^6) = \overline{E(\beta^9)} = -\sum_{i=0}^{4} t_i \theta^{2i} , \tag{3.20}$$

where the integers t_i $(0 \le i \le 4)$ satisfy

$$\sum_{i=0}^{4} t_i = -1$$

and are of the form

$$t_0 = \tfrac{1}{5}(4X - 1) , \tag{3.21}$$

$$t_1 = \tfrac{1}{5}(-1 - X) + U + V + W , \tag{3.22}$$

$$t_2 = \tfrac{1}{5}(-1 - X) + U - V - W , \tag{3.23}$$

$$t_3 = \tfrac{1}{5}(-1 - X) - U + V - W , \tag{3.24}$$

$$t_4 = \tfrac{1}{5}(-1 - X) - U - V + W , \tag{3.25}$$

where (X, U, V, W) is a solution of

$$P = X^2 + 5U^2 + 5V^2 + 5W^2 , \quad X \equiv -1 \pmod 5 , \tag{3.26}$$
$$XW = V^2 - UV - U^2 .$$

Moreover, we have

$$\sum_{i=0}^{4} t_i \theta^i = X + Ui(5 + 2(5)^{\frac{1}{2}})^{\frac{1}{2}} + Vi(5 - 2(5)^{\frac{1}{2}})^{\frac{1}{2}} + W(5)^{\frac{1}{2}}$$

and

$$\sum_{i=0}^{4} t_i\theta^{2i} = X + Vi(5 + 2(5)^{\frac{1}{2}})^{\frac{1}{2}} - Ui(5 - 2(5)^{\frac{1}{2}})^{\frac{1}{2}} - W(5)^{\frac{1}{2}} .$$

PROOF. From (1.1), we have for $a \in GF(p)^*$

$$\text{ind}_\gamma a \equiv (1 + p)\,\text{ind}_g a \pmod{p^2 - 1} ,$$

so that

$$\text{ind}_\gamma a \equiv 0 \pmod 3 \tag{3.27}$$

and

$$\text{ind}_\gamma a \equiv 2\,\text{ind}_g a \pmod 5 . \tag{3.28}$$

From (2.18) and (3.27) we see that

$$G(\beta^m) = G(\beta^n) , \text{ if } m \equiv n \pmod 5 . \tag{3.29}$$

We set

$$A = G(\beta) , \quad B = G(\beta^2) . \tag{3.30}$$

From (3.29) and (3.30) we have

$$G(\beta) = G(\beta^6) = G(\beta^{11}) = A , \tag{3.31}$$

$$G(\beta^2) = G(\beta^7) = G(\beta^{12}) = B .$$

Next appealing to (2.12), (3.29) and (3.31), we have

$$G(\beta^3) = G(\beta^8) = G(\beta^{13}) = p/B , \tag{3.32}$$

$$G(\beta^4) = G(\beta^9) = G(\beta^{14}) = p/A . \tag{3.33}$$

Also, from (2.18) and (3.27) we have

$$G(1) = G(\beta^5) = G(\beta^{10}) = -1 . \tag{3.34}$$

Next, we determine the values of the Gauss sums $G_q(\beta^i)$ $(0 \le i \le 14)$ in terms of A, B, p and g. Taking m = 4n in (2.14), we obtain

$$G_q(\beta^{3n}) = -\{G(\beta^{4n})\}^2 . \tag{3.35}$$

Thus, taking n = 1, 2, 3, 4 and appealing to (3.31), (3.32), (3.33), we obtain

$$G_q(\beta^3) = -p^2/A^2 , \tag{3.36}$$

$$G_q(\beta^6) = -p^2/B^2 , \tag{3.37}$$

$$G_q(\beta^9) = -B^2 , \tag{3.38}$$

$$G_q(\beta^{12}) = -A^2 . \tag{3.39}$$

Next, taking m = 5, 10 in (2.11), we obtain

$$G_q(\beta^5) = G_q(\beta^{10}) = p . \tag{3.40}$$

Also, taking m = 0 in (2.10), we have

$$G_q(1) = -1 . \tag{3.41}$$

Further, from (2.13), we have

$$G_q(\beta^{11m}) = G_q(\beta^m) . \tag{3.42}$$

From the Davenport-Hasse relation (2.15) with x = 5, y = 3, t = 1, we obtain (appealing to (3.40))

$$G_q(\beta^3)\,p^2 = \beta^{3\,\text{ind}_\gamma 3}\, G_q(\beta)\, G_q(\beta^6)\, G_q(\beta^{11}) , \tag{3.43}$$

and with x = 5, y = 3, t = 2, we have

$$G_q(\beta^6)\,p^2 = \beta^{6\,\text{ind}_\gamma 3}\, G_q(\beta^2)\, G_q(\beta^7)\, G_q(\beta^{12}) . \tag{3.44}$$

Appealing to (3.36), (3.37), (3.42), with m = 1, the equation (3.43) becomes

$$G_q(\beta)^2 = \theta^{-2\,\text{ind}_g 3}\, p^2\, B^2/A^2 , \tag{3.45}$$

and appealing to (3.37), (3.39), (3.42) with m = 2, the equation (3.44) becomes

$$G_q(\beta^2)^2 = \theta^{-4 \operatorname{ind}_g 3}\, p^4/A^2B^2 \quad . \tag{3.46}$$

Taking square roots in (3.45) and (3.46), we have

$$G_q(\beta) = \pm\theta^{-\operatorname{ind}_g 3}\, p\, B/A \tag{3.47}$$

$$G_q(\beta^2) = \pm\theta^{-2 \operatorname{ind}_g 3}\, p^2/A\, B \quad . \tag{3.48}$$

Raising both sides of (3.47) and (3.48) to the fifth power and reducing modulo 5, we see that the + signs must hold in both equations; that is,

$$G_q(\beta) = \theta^{-\operatorname{ind}_g 3}\, p\, B/A \quad , \tag{3.49}$$

$$G_q(\beta^2) = \theta^{-2 \operatorname{ind}_g 3}\, p^2/A\, B \quad . \tag{3.50}$$

The values of the remaining $G_q(\beta^i)$ follow immediately from (3.49) and (3.50) as

$$G_q(\beta) = \overline{G_q(\beta^4)} = G_q(\beta^{11}) = \overline{G_q(\beta^{14})} \tag{3.51}$$

$$G_q(\beta^2) = G_q(\beta^7) = \overline{G_q(\beta^8)} = \overline{G_q(\beta^{14})} \quad . \tag{3.52}$$

We now appeal to (2.22), (2.26) and the above formulae for the $G(\beta^i)$ and $G_q(\beta^i)$ to obtain

$$E(\beta) = \overline{E(\beta^4)} = E(\beta^{11}) = \overline{E(\beta^{14})} = \theta^{-\operatorname{ind}_g 6}\, p\, B\, /\, A^2 \quad , \tag{3.53}$$

$$E(\beta^2) = E(\beta^7) = \overline{E(\beta^8)} = \overline{E(\beta^{13})} = \theta^{-2 \operatorname{ind}_g 6}\, p^2\, /\, A\, B^2 \quad , \tag{3.54}$$

$$E(\beta^3) = \overline{E(\beta^{12})} = -\,\theta^{2 \operatorname{ind}_g 2}\, p\, B\, /\, A^2 \quad , \tag{3.55}$$

$$E(\beta^5) = E(\beta^{10}) = -\,1 \quad , \tag{3.56}$$

$$E(\beta^6) = \overline{E(\beta^9)} = -\,\theta^{-\operatorname{ind}_g 2}\, p^2/\, A\, B^2 \quad . \tag{3.57}$$

We now examine $E(\beta^3)$. We have

$$\begin{aligned}
E(\beta^3) &= -\,\theta^{2 \operatorname{ind}_g 2}\, p\, B\, /\, A^2 \\
&= -\,\theta^{2 \operatorname{ind}_g 2} G(\beta^4)^2 \,/\, G(\beta^3) \\
&= -\,\theta^{2 \operatorname{ind}_g 2} J(\beta^4, \beta^4) \qquad \text{(by (2.16) and (3.29))} \\
&= -\,\theta^{2 \operatorname{ind}_g 2} \sum_{a=2}^{p-1} \beta^{4 \operatorname{ind}_\gamma a + 4 \operatorname{ind}_\gamma (1-a)} \\
&= -\,\theta^{2 \operatorname{ind}_g 2} \sum_{a=2}^{p-1} \theta^{\operatorname{ind}_g a + \operatorname{ind}_g (1-a)} \\
&= -\,\theta^{2 \operatorname{ind}_g 2} R_5(1, 1) \qquad \text{(in the notation of [5: eqn. (3.1)])} \\
&= -\,\theta^{2 \operatorname{ind}_g 2} \sum_{i=0}^{4} B_5(i, 1)\, \theta^i \qquad \text{[5: p. 345]}
\end{aligned}$$

$$= -\sum_{j=0}^{4} B_5(j - 2\,\mathrm{ind}_g 2,\ 1)\ \theta^j$$

$$= -\sum_{j=0}^{4} \{ B_5(j - 2\,\mathrm{ind}_g 2,\ 1) - \tfrac{1}{5}(p-1)\}\ \theta^j$$

$$= -\sum_{j=0}^{4} t_j\ \theta^j \ ,$$

where (see [5: eqn. (9.5)]) for $0 \le j \le 4$

$$t_j = B_5(j - 2\,\mathrm{ind}_g 2,\ 1) - \tfrac{1}{5}(p-1) \ . \tag{3.58}$$

From [5: eqn. (9.6)], we have

$$\sum_{j=0}^{4} t_j = -1 \ , \tag{3.59}$$

and [5: Theorem 8]

$$\begin{gathered} 5t_0 + 1 = 4X,\ t_1 + t_2 - t_3 - t_4 = 4U \ , \\ t_1 - t_2 + t_3 - t_4 = 4V,\ t_1 - t_2 - t_3 + t_4 = 4W, \end{gathered} \tag{3.60}$$

where X, U, V, W are integers satisfying (3.26). Solving (3.59) and (3.60) for t_0, t_1, t_2, t_3, t_4, we obtain (3.21) - (3.25). This completes the proof of (3.18).

From (3.53) and (3.55), we have

$$E(\beta) = -\theta^{2\,\mathrm{ind}_g 2 - \mathrm{ind}_g 3}\ E(\beta^3) \ ,$$

which together with (3.18), gives (3.16).

Applying the automorphism σ_2 to (3.16), we obtain (3.17), and applying the same automorphism to (3.18), we obtain (3.20).

Finally, we note that

$$\begin{aligned} \sum_{i=0}^{4} t_i\theta^i &= \sum_{i=1}^{4} (t_i - t_0)\ \theta^i \\ &= (-X + U + V + W)\theta + (-X + U - V - W)\theta^2 + (-X - U + V - W)\theta^3 + (-X-U-V+W)\theta^4 \\ &= X(-\theta-\theta^2-\theta^3-\theta^4) + U(\theta + \theta^2 - \theta^3 - \theta^4)+V(\theta-\theta^2 + \theta^3 - \theta^4)+W(\theta - \theta^2-\theta^3 + \theta^4), \end{aligned}$$

that is

$$\sum_{i=0}^{4} t_i\theta^i = X + Ui(5 + 2(5)^{\frac{1}{2}})^{\frac{1}{2}} + Vi(5 - 2(5)^{\frac{1}{2}})^{\frac{1}{2}} + W(5)^{\frac{1}{2}}, \tag{3.61}$$

as required. Applying σ_2 to (3.61), we obtain

$$\sum_{i=0}^{4} t_i\theta^{2i} = X + Vi(5 + 2(5)^{\frac{1}{2}})^{\frac{1}{2}} - Ui(5 - 2(5)^{\frac{1}{2}})^{\frac{1}{2}} - W(5)^{\frac{1}{2}} \ , \tag{3.62}$$

as

$$\begin{aligned} \sigma_2(i(5 + 2(5)^{\frac{1}{2}})^{\frac{1}{2}}) &= \sigma_2(\theta + \theta^2 - \theta^3 - \theta^4) = \theta^2 + \theta^4 - \theta^6 - \theta^8 \\ &= -\theta + \theta^2 - \theta^3 + \theta^4 = -i(5 - 2(5)^{\frac{1}{2}})^{\frac{1}{2}} \ , \\ \sigma_2(i(5 - 2(5)^{\frac{1}{2}})^{\frac{1}{2}}) &= i(5 + 2(5)^{\frac{1}{2}})^{\frac{1}{2}} \ , \end{aligned}$$

and

$$\sigma_2((5)^{\frac{1}{2}}) = -(5)^{\frac{1}{2}} \ .$$

This completes the proof of Theorem 2.

4. DETERMINATION OF JACOBI SUMS FOR e = 15 AND p ≡ 4, 11 (mod 15).

The Jacobi sums of order 15 over GF(p^2) are conjugate to $J_q(\beta^m, \beta)$ $(1 \le m \le 5)$, $J_q(\beta^3, \beta^3)$ and $J_q(\beta^5, \beta^5)$. We first evaluate these for p ≡ 4 (mod 15). We prove

THEOREM 3. For p ≡ 4 (mod 15), with the notation of Theorem 1, we have

$$J_q(\beta, \beta) = (A + B\omega)\,(T + U\sqrt{-15}) \quad , \tag{4.1}$$

$$J_q(\beta^2, \beta) = (T + U\sqrt{-15})^2 \quad , \tag{4.2}$$

$$J_q(\beta^3, \beta) = p \quad , \tag{4.3}$$

$$J_q(\beta^4, \beta) = -\,\omega^{-\mathrm{ind}_g 5}\,(T + U\sqrt{-15})^2 \quad , \tag{4.4}$$

$$J_q(\beta^5, \beta) = -\,\omega^{\mathrm{ind}_g 5}\,(A + B\omega)(T + U\sqrt{-15}) \quad , \tag{4.5}$$

$$J_q(\beta^3, \beta^3) = p \quad , \tag{4.6}$$

$$J_q(\beta^5, \beta^5) = -\,(A + B\omega)^2 \quad . \tag{4.7}$$

PROOF. From Theorem 1, we have

$$G_q(\beta) = G_q(\beta^4) = \overline{G_q(\beta^{11})} = \overline{G_q(\beta^{14})} = \omega^{\mathrm{ind}_g 5}\,(T + U\sqrt{-15})\,G(\beta) \quad , \tag{4.8}$$

$$G_q(\beta^2) = \overline{G_q(\beta^7)} = G_q(\beta^8) = \overline{G_q(\beta^{13})} = \omega^{-\mathrm{ind}_g 5}\,(T + U\sqrt{-15})\,G(\beta^2) \quad , \tag{4.9}$$

$$G_q(\beta^3) = G_q(\beta^6) = G_q(\beta^9) = G_q(\beta^{12}) = p \quad , \tag{4.10}$$

$$G_q(\beta^5) = \overline{G_q(\beta^{10})} = -\,G(\beta)^2 \quad , \tag{4.11}$$

and

$$G(\beta)^3 = p(A + B\omega) \quad , \quad G(\beta^2)^3 = p(A + B\omega^2) \quad , \quad G(\beta)\,G(\beta^2) = p \quad . \tag{4.12}$$

The values of the Jacobi sums (4.1) - (4.7) follow from (4.8) - (4.12) and (2.16). This completes the proof of Theorem 3.

Next, we evaluate the required Jacobi sums in the case p ≡ 11 (mod 15). We prove

THEOREM 4. For p ≡ 11 (mod 15), with the notation of Theorem 2, we have

$$J_q(\beta, \beta) = \theta^{2\,\mathrm{ind}_g 2}\,\{\sum_{i=0}^{4} t_i\theta^i\}\,\{\sum_{j=0}^{4} t_j\theta^{3j}\} \quad , \tag{4.13}$$

$$J_q(\beta^2, \beta) = -\,\theta^{2\,\mathrm{ind}_g 3}\,p \quad , \tag{4.14}$$

$$J_q(\beta^3, \beta) = -\,\theta^{\mathrm{ind}_g 2 - 2\,\mathrm{ind}_g 3}\,\{\sum_{i=0}^{4} t_i\theta^i\}^2 \quad , \tag{4.15}$$

$$J_q(\beta^4, \beta) = p \quad , \tag{4.16}$$

$$J_q(\beta^5, \beta) = -\,\theta^{2\,\mathrm{ind}_g 2 - \mathrm{ind}_g 3}\,\{\sum_{i=0}^{4} t_i\theta^i\}\{\sum_{j=0}^{4} t_j\theta^{3j}\} \tag{4.17}$$

$$J_q(\beta^3, \beta^3) = -\,\theta^{\mathrm{ind}_g 2}\,\{\sum_{i=0}^{4} t_i\theta^i\}^2 \quad , \tag{4.18}$$

$$J_q(\beta^5, \beta^5) = p \quad . \tag{4.19}$$

PROOF. Theorem 4 follows from (2.16), (3.36) - (3.40), (3.49) - (3.57) and Theorem 2.

5. DETERMINATION OF DICKSON-HURWITZ SUMS FOR $e = 15$ AND $p \equiv 4, 11 \pmod{15}$.

For any integers i and v, the Dickson-Hurwitz sum $B_e(i, v)$ is defined by

$$B_e(i, v) = \sum_{h=0}^{e-1} (h, i - vh)_e \quad , \tag{5.1}$$

where the cyclotomic number $(h, k)_e$ was defined in (1.2).

Jacobi sums and Dickson-Hurwitz sums are related by the formula

$$J_q(\beta^{nv}, \beta^n) = (-1)^{nvf} \sum_{i=0}^{e-1} B_e(i, v) \beta^{ni} \quad . \tag{5.2}$$

The Dickson-Hurwitz sums have the well-known properties

$$B_e(i, v) = B_e(i, e - v - 1) \tag{5.3}$$

and

$$B_e(i, 0) = B_e(i, e - 1) = \begin{cases} f - 1 \ , & \text{if } e \mid i \ , \\ f \ , & \text{if } e \nmid i \ . \end{cases} \tag{5.4}$$

Taking $n = 0$ in (5.2), we obtain

$$\sum_{i=0}^{e-1} B_e(i, v) = q - 2 \quad . \tag{5.5}$$

In addition, Whiteman [12: Lemma 1] showed in the case $\ell = 1$ that if $(v, e) = 1$

$$B_e(i, v) = B_e(iv^{-1}, v^{-1}) \ , \text{ where } vv^{-1} \equiv 1 \pmod{e} \quad . \tag{5.6}$$

It is very easily checked that, in fact, (5.6) holds for $\ell \geq 1$.

We next establish the relation

$$(h, k)_e = (ph, pk)_e \quad , \tag{5.7}$$

which will be needed in §6 as well as to establish property (5.8) below. Defining r to be the least positive integer, such that $p^r \equiv 1 \pmod{e}$, we have

$$(ph, pk)_e = \sum_{\substack{\alpha \in GF(q)^\dagger \\ \operatorname{ind}_\gamma(\alpha - 1) \equiv ph \pmod{e} \\ \operatorname{ind}_\gamma \alpha \equiv pk \pmod{e}}} 1 = \sum_{\substack{\alpha \in GF(q)^\dagger \\ p^{r-1}\operatorname{ind}_\gamma(\alpha - 1) \equiv h \pmod{e} \\ p^{r-1}\operatorname{ing}_\gamma \alpha \equiv k \pmod{e}}} 1$$

$$= \sum_{\substack{\alpha \in GF(q)^\dagger \\ \operatorname{ind}_\gamma(\alpha-1)^{p^{r-1}} \equiv h \pmod{e} \\ \operatorname{ind}_\gamma \alpha^{p^{r-1}} \equiv k \pmod{e}}} 1 = \sum_{\substack{\alpha \in GF(q)^\dagger \\ \operatorname{ind}_\gamma(\alpha^{p^{r-1}} - 1) \equiv h \pmod{e} \\ \operatorname{ind}_\gamma \alpha^{p^{r-1}} \equiv k \pmod{e}}} 1$$

$$= \sum_{\substack{\alpha_1 \in GF(q)^\dagger \\ \operatorname{ind}_\gamma(\alpha_1 - 1) \equiv h \pmod{e} \\ \operatorname{ind}_\gamma \alpha_1 \equiv k \pmod{e}}} 1 = (h, k)_e \ .$$

From (5.7) we see that

$$B_e(pi, v) = B_e(i, v) \quad , \tag{5.8}$$

as

$$B_e(pi, v) = \sum_{h=0}^{e-1} (h, pi - vh)_e$$

$$= \sum_{k=0}^{e-1} (pk, pi - vpk)_e$$

$$= \sum_{k=0}^{e-1} (k, i - vk)_e \ .$$

From (5.3), (5.4), (5.6), (5.8), we see that for $e = 15$, $q = p^2$, the values of all the $e^2 = 225$ Dickson-Hurwitz sums $B_{15}(i, v)(i, v = 0, 1, \ldots, 14)$ are known once the values of the 45 sums $B_{15}(i, v)(i = 0, 1, 2, 3, 5, 6, 7, 10, 11; v = 1, 2, 3, 4, 5)$ have been determined in the case $p \equiv 4 \pmod{15}$, and the values of the 50 sums $B_{15}(i, v)$ $(i = 0, 1, 2, 3, 4, 5, 6, 8, 9, 12; v = 1, 2, 3, 4, 5)$ have been determined in the case $p \equiv 11 \pmod{15}$. This is clear in view of the following: for each i

$$\begin{aligned} B_{15}(i, 0) = B_{15}(i, 14) &= \begin{cases} f - 1, & \text{if } 15 \mid i \\ f, & \text{if } 15 \nmid i \end{cases} , \\ B_{15}(i, v) &= B_{15}(i, 14 - v) \quad (8 \le v \le 14) , \\ B_{15}(i, 7) &= B_{15}(13i, 13) = B_{15}(13i, 1) , \\ B_{15}(i, 6) &= B_{15}(i, 8) = B_{15}(2i, 2) , \end{aligned} \tag{5.9}$$

and for each v

$$B_{15}(i, v) = \begin{cases} B_{15}(4i, v), & \text{if } p \equiv 4 \pmod{15} , \\ B_{15}(11i, v), & \text{if } p \equiv 11 \pmod{15} . \end{cases} \tag{5.10}$$

We begin by determining the system of linear equations which have the nine Dickson-Hurwitz sums $B_{15}(i, v)$ $(i = 0, 1, 2, 3, 5, 6, 7, 10, 11)$ as solutions when $p \equiv 4 \pmod{15}$.

THEOREM 5. For $p \equiv 4 \pmod{15}$ and for each integer v, $1 \le v \le 6$, the nine Dickson-Hurwitz sums $B_{15}(i, v)$ $(i = 0, 1, 2, 3, 5, 6, 7, 10, 11)$ are the solutions of the matrix equation

$$M\, B(v) = C(v) \ , \tag{5.11}$$

where the 9 x 9 coefficient matrix of determinant $-3375 = -3^3 \cdot 5^3$

$$M = \begin{bmatrix} 1 & 2 & 2 & 2 & 1 & 2 & 2 & 1 & 2 \\ 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & -1 \\ 0 & 1 & -1 & -1 & 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & -2 & 2 & -1 & 2 & 0 & 0 & -2 \\ 0 & 2 & -2 & 0 & -1 & 0 & 2 & 1 & -2 \\ 1 & 0 & -1 & 0 & 0 & -1 & 1 & -1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 & 1 & 0 & 1 & -1 & 0 \end{bmatrix} \tag{5.12}$$

is independent of v,

$$B(v) = \begin{bmatrix} B_{15}(0, v) \\ B_{15}(1, v) \\ B_{15}(2, v) \\ B_{15}(3, v) \\ B_{15}(5, v) \\ B_{15}(6, v) \\ B_{15}(7, v) \\ B_{15}(10, v) \\ B_{15}(11, v) \end{bmatrix}, \qquad C(v) = \begin{bmatrix} C_1(v) \\ C_2(v) \\ C_3(v) \\ C_4(v) \\ C_5(v) \\ C_6(v) \\ C_7(v) \\ C_8(v) \\ C_9(v) \end{bmatrix}. \tag{5.13}$$

The values of the $C_i(v)$ are given by

$$C_1(v) = p^2 - 2, \tag{5.14}$$

$$C_2(v) = \begin{cases} p, & \text{if } v = 1, 2, 3, 6, \\ -1, & \text{if } v = 4, 5, \end{cases} \tag{5.15}$$

$$C_3(v) = 0, \tag{5.16}$$

$$C_4(v) = \begin{cases} -A^2 + B^2, & \text{if } v = 1, 4, \\ -1, & \text{if } v = 2, 3, 5, 6, \end{cases} \tag{5.17}$$

$$C_5(v) = \begin{cases} -2AB + B^2, & \text{if } v = 1, 4, \\ 0, & \text{if } v = 2, 3, 5, 6, \end{cases} \tag{5.18}$$

$$C_6(1) = AT - 3AU, \; C_7(1) = 4AU - 2BU, \tag{5.19}$$
$$C_8(1) = 2AU + 2BU, \; C_9(1) = BT - 2AU + BU,$$

$$C_6(2) = (T^2 - 15U^2) - 6TU, \; C_7(2) = 8TU, \tag{5.20}$$
$$C_8(2) = 4TU, \; C_9(2) = -4TU,$$

$$C_6(3) = p, \; C_7(3) = 0, \; C_8(3) = 0, \; C_9(3) = 0, \tag{5.21}$$

$$C_6(4) = -(T^2 - 15U^2) + 6TU, \; C_7(4) = -8TU,$$
$$C_8(4) = -4TU, \; C_9(4) = 4TU, \text{ if } \operatorname{ind}_g 5 \equiv 0 \pmod 3,$$
$$C_6(4) = (T^2 - 15U^2) - 6TU, \; C_7(4) = 4TU, \tag{5.22}$$
$$C_8(4) = 8TU, \; C_9(4) = (T^2 - 15U^2) - 2TU, \text{ if } \operatorname{ind}_g 5 \equiv 1 \pmod 3,$$
$$C_6(4) = 0, \; C_7(4) = 4TU, \; C_8(4) = -4TU, \; C_8(4) = -(T^2 - 15U^2) - 2TU,$$
$$\text{if } \operatorname{ind}_g 5 \equiv 2 \pmod 3,$$

$$C_6(5) = -AT + 3AU, \; C_7(5) = -4AU + 2BU,$$
$$C_8(5) = -2AU - 2BU, \; C_9(5) = -BT + 2AU - BU, \text{ if } \operatorname{ind}_g 5 \equiv 0 \pmod 3,$$
$$C_6(5) = BT - 3BU, \; C_7(5) = 2AU + 2BU, \; C_8(5) = -2AU + 4BU,$$
$$C_9(5) = -AT + BT - AU - BU, \text{ if } \operatorname{ind}_g 5 \equiv 1 \pmod 3, \tag{5.23}$$
$$C_6(5) = AT - 3AU - BT + 3BU, \; C_7(5) = 2AU - 4BU,$$
$$C_8(5) = 4AU - 2BU, \; C_9(5) = AT - AU + 2BU, \text{ if } \operatorname{ind}_g 5 \equiv 2 \pmod 3,$$

$$C_6(6) = (T^2 - 15U^2) - 6TU, \; C_7(6) = 8TU, \tag{5.24}$$
$$C_8(6) = 4TU, \; C_9(6) = -4TU.$$

We remark that the value $v = 6$ has also been included in the statement of Theorem 5 as the solution of (5.11) yields an additional property of the Dickson-

Hurwitz sums of order 15, namely,

$$B_{15}(i, 2) = B_{15}(i, 6) , \tag{5.25}$$

so that by (5.3), (5.6) and (5.25), we obtain

$$B_{15}(2i, 2) = B_{15}(i, 2) . \tag{5.26}$$

PROOF. The first row in the matrix equation (5.11) follows from (5.5) and (5.10).

The second and third rows are derived as follows. From (5.2) (with n = 3, e = 15), we have

$$J_q(\beta^{3v},\beta^3) = \sum_{i=0}^{14} B_{15}(i, v)\beta^{3i} = \sum_{j=0}^{4} S(j, v)\beta^{3j}, \tag{5.27}$$

where

$$S(j, v) = \sum_{i=0}^{2} B_{15}(j + 5i, v) . \tag{5.28}$$

Appealing to (5.10) we deduct that

$$S(4, v) = S(1, v), S(3, v) = S(2, v) . \tag{5.29}$$

Thus expressing (5.27) in terms of the basis $\{1, \theta, \theta^2, \theta^3\}$ for $Q(\theta)$ we obtain

$$J_q(\beta^{3v}, \beta^3) = (S(0, v) - S(1, v)) + (S(2, v) - S(1, v))(\theta^2 + \theta^3) . \tag{5.30}$$

From (2.2), (2.4), (2.6) and Theorem 3 we have

$$J_q(\beta^{3v},\beta^3) = \begin{cases} p & , \text{ if } v = 1, 2, 3, 6 \\ -1 & , \text{ if } v = 4, 5 \end{cases} , \tag{5.31}$$

and so

$$S(0, v) - S(1, v) = C_2(v) , \quad S(2, v) - S(1, v) = C_3(v) ,$$

which are the second and third rows of (5.11). The fourth and fifth rows of (5.11) are obtained as follows. From (5.2) (with n = 5, e = 15) we have

$$J_q(\beta^{5v},\beta^5) = \sum_{i=0}^{14} B_{15}(i, v)\beta^{5i} = \sum_{j=0}^{2} T(j, v)\beta^{5j}, \tag{5.32}$$

that is

$$J_q(\beta^{5v}, \beta^5) = (T(0, v) - T(2, v)) + (T(1, v) - T(2, v))\omega ,$$

where

$$T(j, v) = \sum_{i=0}^{4} B_{15}(j + 3i, v) . \tag{5.33}$$

From (2.4), (2.6) and Theorem 3 we have

$$J_q(\beta^{5v},\beta^5) = \begin{cases} -(A + B\omega)^2 & , \text{ if } v = 1, 4 \\ -1 & , \text{ if } v = 2, 3, 5, 6 \end{cases} . \tag{5.34}$$

From (5.32) and (5.34) we derive

$$T(0, v) - T(2, v) = C_4(v), \quad T(1, v) - T(2, v) = C_5(v) , \tag{5.35}$$

which are the required fourth and fifth equations.

The sixth, seventh, eighth, ninth rows of (5.11) are obtained by equating the coefficients of 1, β, β^2 and β^5 respectively in

$$J_q(\beta^v, \beta) = \sum_{i=0}^{14} B_{15}(i, v)\beta^i \tag{5.36}$$

after representing both sides in terms of a basis $\{1, \beta, \ldots, \beta^7\}$ of $Q(\beta)$ with the values of the Jacobi sums $J_q(\beta^v, \beta)(v = 1, 2, \ldots, 6)$ given by Theorem 3. The right-hand side of (5.36) in terms of the basis $\{1, \beta, \ldots, \beta^7\}$ is $\sum_{i=0}^{7} W(i, v)\beta^i$ where

$$\begin{aligned} W(0,v) &= B_{15}(0,v) - B_{15}(2,v) - B_{15}(6,v) + B_{15}(7,v) - B_{15}(10,v) + B_{15}(11,v), \\ W(1,v) &= W(4,v) = B_{15}(1,v) + B_{15}(2,v) - B_{15}(7,v) - B_{15}(11,v), \\ W(2,v) &= -W(3,v) = W(7,v) = B_{15}(2,v) - B_{15}(3,v) + B_{15}(6,v) - B_{15}(11,v), \\ W(5,v) &= -B_{15}(2,v) + B_{15}(5,v) + B_{15}(7,v) - B_{15}(10,v), \\ W(6,v) &= 0. \end{aligned} \tag{5.37}$$

When $v = 1$ we have by Theorem 3

$$\begin{aligned} J_q(\beta, \beta) &= (A + B\beta^5)(T + U(-3 + 4\beta + 2\beta^2 - 2\beta^3 + 4\beta^4 - 2\beta^5 + 2\beta^7)) \\ &= (AT - 3AU) + (4AU - 2BU)\beta + (2AU + 2BU)\beta^2 \\ &\quad + (-2AU - 2BU)\beta^3 + (4AU - 2BU)\beta^4 \\ &\quad + (BT - 2AU + BU)\beta^5 + (2AU - 2BU)\beta^7, \end{aligned}$$

from which (5.19) follows. When $v = 2$ we have

$$\begin{aligned} J_q(\beta^2, \beta) &= (T + U(-15)^{\frac{1}{2}})^2 \\ &= ((T^2 - 15U^2) - 6TU) + 8TU\beta + 4TU\beta^2 - 4TU\beta^3 + 8TU\beta^4 - 4TU\beta^5 + 4TU\beta^7, \end{aligned}$$

from which (5.20) follows. The remaining v can be treated similarly. It should be noted that for $v = 4$ and 5 there are three cases to be considered depending upon $\operatorname{ind}_g 5 \pmod 3$.

This completes the proof of Theorem 5.

We next determine the system of linear equations which have the ten Dickson-Hurwitz sums $B_{15}(i, v)(i = 0, 1, 2, 3, 4, 5, 6, 8, 9, 12)$ as solutions of the matrix equation

$$MB(v) = C(v), \tag{5.38}$$

where the 10x10 coefficient matrix of determinant $-6075 = -3^5 \cdot 5^2$

$$M = \begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 2 & 1 & 2 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 & 1 \\ 1 & 0 & 0 & 0 & -2 & 2 & 0 & 0 & -1 & 0 \\ 0 & 2 & 0 & 0 & -2 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 2 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \tag{5.39}$$

is independent of v,

$$B(v) = \begin{bmatrix} B_{15}(0, v) \\ B_{15}(1, v) \\ B_{15}(2, v) \\ B_{15}(3, v) \\ B_{15}(4, v) \\ B_{15}(5, v) \\ B_{15}(6, v) \\ B_{15}(8, v) \\ B_{15}(9, v) \\ B_{15}(12, v) \end{bmatrix}, \qquad C(v) = \begin{bmatrix} C_1(v) \\ C_2(v) \\ C_3(v) \\ C_4(v) \\ C_5(v) \\ C_6(v) \\ C_7(v) \\ C_8(v) \\ C_9(v) \\ C_{10}(v) \end{bmatrix}. \tag{5.40}$$

The values of the $C_i(v)$ are given by

$$C_1(v) = p^2 - 2, \tag{5.41}$$

$$C_2(v) = \begin{cases} p, & \text{if } v = 1, 4, \\ -1, & \text{if } v = 2, 3, 5, \end{cases} \tag{5.42}$$

$$C_i(1) = C_i(3) = Q(1 + \text{ind}_g 2) - Q(3 - i + \text{ind}_g 2), \quad i = 3, 4, 5, 6, \tag{5.43}$$

$$C_i(2) = Q(3 + \text{ind}_g 2) - Q(4 + 2i + \text{ind}_g 2), \quad i = 3, 4, 5, 6, \tag{5.44}$$

$$C_3(4) = C_3(5) = -1, \; C_j(4) = C_j(5) = 0, \; j = 4, 5, 6, \tag{5.45}$$

$$C_i(1) = R(2i + 1 + \text{ind}_g 2) - R(3 + \text{ind}_g 2), \; i = 7, 8, 9, 10, \tag{5.46}$$

$$C_i(2) = \begin{cases} p, \text{ if } \text{ind}_g 3 \equiv 2 \pmod 5, \\ -p, \text{ if } \text{ind}_g 3 \equiv 3i - 1 \pmod 5, \; i = 7, 8, 9, 10 \\ 0, \text{ otherwise}, \end{cases} \tag{5.47}$$

$$C_i(3) = Q(1 + \text{ind}_g 2 + 3\text{ind}_g 3) - Q(2 - i + \text{ind}_g 2 + 3\text{ind}_g 3), \; i = 7, 8, 9, 10, \tag{5.48}$$

$$C_7(4) = p, \; C_j(4) = 0, \; j = 8, 9, 10, \tag{5.49}$$

$$C_i(5) = R(3 + \text{ind}_g 2 + 2\text{ind}_g 3) - R(2i + 1 + \text{ind}_g 2 + 2\text{ind}_g 3), \; i = 7, 8, 9, 10, \tag{5.50}$$

where (the t_i are specified in Theorem 2)

$$\begin{aligned} Q(0) &= t_0^2 + 2t_1t_4 + 2t_2t_3, \\ Q(1) &= t_2^2 + 2t_0t_4 + 2t_1t_3, \\ Q(2) &= t_4^2 + 2t_0t_3 + 2t_1t_2, \\ Q(3) &= t_1^2 + 2t_0t_2 + 2t_3t_4, \\ Q(4) &= t_3^2 + 2t_0t_1 + 2t_2t_4, \end{aligned} \tag{5.51}$$

$$\begin{aligned} R(0) &= t_0^2 + t_1t_2 + t_1t_3 + t_2t_4 + t_3t_4, \\ R(1) &= t_2^2 + t_0t_1 + t_0t_3 + t_1t_4 + t_3t_4, \\ R(2) &= t_4^2 + t_0t_1 + t_0t_2 + t_1t_3 + t_2t_3, \\ R(3) &= t_1^2 + t_0t_3 + t_0t_4 + t_2t_3 + t_2t_4, \\ R(4) &= t_3^2 + t_0t_2 + t_0t_4 + t_1t_2 + t_1t_4, \end{aligned} \tag{5.52}$$

and

$$Q(n) = Q(n + 5) \ , \ R(n) = R(n + 5)$$

for all integers n.

PROOF. The first row in the matrix equation (5.38) follows from (5.5) and (5.10).

The second row is derived as follows. From (5.2) (with $n = 5$, $e = 15$) we have

$$\begin{aligned} J_q(\beta^{5v}, \beta^5) &= \sum_{i=0}^{14} B_{15}(i, v)\beta^{5i} \\ &= \sum_{j=0}^{2} \Big(\sum_{i=0}^{4} B_{15}(3i + j, v)\Big)\omega^j \\ &= \sum_{j=0}^{1} \Big(\sum_{i=0}^{4} (B_{15}(3i + j, v)) - B_{15}(3i + 2, v))\Big)\omega^j \\ &= B_{15}(0, v) - B_{15}(1, v) - B_{15}(2, v) + B_{15}(3, v) - B_{15}(4, v) \\ &\quad - B_{15}(5, v) + B_{15}(6, v) - B_{15}(8, v) + B_{15}(9, v) + B_{15}12, v) \ . \end{aligned}$$

The values of $C_2(v)$ now follow from (4.19), (2.4) and (2.6).

The third, fourth, fifth, and sixth rows are derived as follows. From (5.2) (with $n = 3$, $e = 15$), we have

$$J_q(\beta^{3v}, \beta^3) = \sum_{i=0}^{14} B_{15}(i, v)\beta^{3i} = \sum_{j=0}^{14} S(j, v)\theta^j \ ,$$

where

$$S(j, v) = \sum_{i=0}^{2} B_{15}(j + 5i, v) \ .$$

Hence, as $\theta^4 = -1 - \theta - \theta^2 - \theta^3$, we have

$$J_q(\beta^{3v}, \beta^3) = \sum_{j=0}^{3} T(j, v)\theta^j \ , \tag{5.53}$$

where

$$T(j, v) = S(j, v) - S(4, v) \ .$$

Appealing to (5.10) we obtain

$$\begin{aligned} T(0, v) &= B_{15}(0, v) - 2B_{15}(4, v) + 2B_{15}(5, v) - B_{15}(9, v) \ , \\ T(1, v) &= 2B_{15}(1, v) - 2B_{15}(4, v) + B_{15}(6, v) - B_{15}(9, v) \ , \\ T(2, v) &= 2B_{15}(2, v) - 2B_{15}(4, v) - B_{15}(9, v) + B_{15}(12, v) \ , \\ T(3, v) &= B_{15}(3, v) - 2B_{15}(4, v) + 2B_{15}(8, v) - B_{15}(9, v) \ . \end{aligned}$$

The values of $C_3(v)$, $C_4(v)$, $C_5(v)$, and $C_6(v)$ follow from (4.18), (2.2), (2.4), (2.6), and (2.7), by equating coefficients of 1, θ, θ^2 and θ^3 in (5.53).

The seventh, eighth, ninth, tenth rows of (5.38) are obtained by equating the coefficients of 1, θ, θ^2 and θ^3 respectively in

$$\begin{aligned} J(\beta^v, \beta) &= (B_{15}(0, v) + B_{15}(2, v) - B_{15}(5, v) - B_{15}(12, v)) \\ &\quad + (B_{15}(2, v) + B_{15}(3, v) - B_{15}(8, v) - B_{15}(12, v))\theta \\ &\quad + (-B_{15}(1, v) + B_{15}(2, v) + B_{15}(6, v) - B_{15}(12, v))\theta^2 \\ &\quad + (B_{15}(2, v) - B_{15}(4, v) + B_{15}(9, v) - B_{15}(12, v))\theta^3, \end{aligned}$$

which was obtained from (5.2) (with $e = 5$, $n = 1$) by expressing the right-hand side in terms of a basis $\{1, \beta, \ldots, \beta^7\}$ of $Q(\beta)$ and then using (5.10) to express it in

terms of $\{1, \theta, \theta^2, \theta^3\}$. The values of the Jacobi sums $J_q(\beta^v, \beta)(v = 1, 2, \ldots, 6)$ are given by Theorem 4.

This completes the proof of Theorem 6.

Next we use Theorems 5 and 6 to determine the values of the Dickson-Hurwitz sums. We treat the case $p \equiv 4 \pmod{15}$ first. Inverting the matrix M given in (5.12), we obtain from (5.11)

$$B(v) = \frac{1}{15}NC(v) , \tag{5.54}$$

where

$$N = \begin{bmatrix} 1 & 4 & 2 & 2 & -1 & 8 & 2 & 4 & -4 \\ 1 & -1 & 2 & -1 & 2 & 1 & 6 & -2 & 1 \\ 1 & -1 & -3 & -1 & -1 & 1 & 2 & 3 & -2 \\ 1 & -1 & -3 & 2 & -1 & -2 & 2 & -6 & 1 \\ 1 & 4 & 2 & -1 & -1 & -4 & 2 & -2 & 8 \\ 1 & -1 & 2 & 2 & -1 & -2 & -3 & 4 & 1 \\ 1 & -1 & -3 & -1 & 2 & 1 & -4 & 3 & 1 \\ 1 & 4 & 2 & -1 & 2 & -4 & -4 & -2 & -4 \\ 1 & -1 & 2 & -1 & -1 & 1 & -3 & -2 & -2 \end{bmatrix} . \tag{5.55}$$

Substituting the values for $C_1(v), \ldots, C_9(v)$ specified in Theorem 5 into (5.54) we obtain the following tables of values of $B_{15}(i, v)$ $(i = 0, 1, 2, 3, 5, 6, 7, 10, 11;$ $v = 1, 2, 3, 4, 5)$.

Table 1: Dickson-Hurwitz sums $B_{15}(i, 1)$, $p \equiv 4 \pmod{15}$

	p^2	p	1	A^2	AB	B^2	AT	BT	AU	BU
$15\,B_{15}(0,1)$:	1	4	-2	-2	2	1	8	-4		
$15\,B_{15}(1,1)$:	1	-1	-2	1	-4	1	1	1	15	-15
$15\,B_{15}(2,1)$:	1	-1	-2	1	2	-2	1	-2	15	
$15\,B_{15}(3,1)$:	1	-1	-2	-2	2	1	-2	1		-15
$15\,B_{15}(5,1)$:	1	4	-2	1	2	-2	-4	8		
$15\,B_{15}(6,1)$:	1	-1	-2	-2	2	1	-2	1		15
$15\,B_{15}(7,1)$:	1	-1	-2	1	-4	1	1	1	-15	15
$15\,B_{15}(10,1)$:	1	4	-2	1	-4	1	-4	-4		
$15\,B_{15}(11,1)$:	1	-1	-2	1	2	-2	1	-2	-15	

Table 2: Dickson-Hurwitz sums $B_{15}(i, 2)$, $p \equiv 4 \pmod{15}$

	p^2	p	1	T^2	U^2	TU
$15\,B_{15}(0,2)$:	1	4	-4	8	-120	
$15\,B_{15}(1,2)$:	1	-1	-1	1	-15	30
$15\,B_{15}(2,2)$:	1	-1	-1	1	-15	30
$15\,B_{15}(3,2)$:	1	-1	-4	-2	30	
$15\,B_{15}(5,2)$:	1	4	-1	-4	60	
$15\,B_{15}(6,2)$:	1	-1	-4	-2	30	
$15\,B_{15}(7,2)$:	1	-1	-1	1	-15	-30
$15\,B_{15}(10,2)$:	1	4	-1	-4	60	
$15\,B_{15}(11,2)$:	1	-1	-1	1	-15	-30

Table 3: Dickson-Hurwitz sums $B_{15}(i, 3)$, $p \equiv 4 \pmod{15}$

	p^2	p	1
$15\ B_{15}(0,3)$:	1	12	-4
$15\ B_{15}(1,3)$:	1		-1
$15\ B_{15}(2,3)$:	1		-1
$15\ B_{15}(3,3)$:	1	-3	-4
$15\ B_{15}(5,3)$:	1		-1
$15\ B_{15}(6,3)$:	1	-3	-4
$15\ B_{15}(7,3)$:	1		-1
$15\ B_{15}(10,3)$:	1		-1
$15\ B_{15}(11,3)$:	1		-1

Table 4a: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 0 \pmod 3$

	p^2	1	A^2	AB	B^2	T^2	TU	U^2
$15\ B_{15}(0,4)$:	1	-6	-2	2	1	-8		120
$15\ B_{15}(1,4)$:	1	-1	1	-4	1	-1	-30	15
$15\ B_{15}(2,4)$:	1	-1	1	2	-2	-1	-30	15
$15\ B_{15}(3,4)$:	1	-1	-2	2	1	2		-30
$15\ B_{15}(5,4)$:	1	-6	1	2	-2	4		-60
$15\ B_{15}(6,4)$:	1	-1	-2	2	1	2		-30
$15\ B_{15}(7,4)$:	1	-1	1	-4	1	-1	30	15
$15\ B_{15}(10,4)$:	1	-6	1	-4	1	4		-60
$15\ B_{15}(11,4)$:	1	-1	1	2	-2	-1	30	15

Table 4b: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 1 \pmod 3$

	p^2	1	A^2	AB	B^2	T^2	TU	U^2
$15\ B_{15}(0,4)$:	1	-6	-2	2	1	4		-60
$15\ B_{15}(1,4)$:	1	-1	1	-4	1	2		-30
$15\ B_{15}(2,4)$:	1	-1	1	2	-2	-1	30	15
$15\ B_{15}(3,4)$:	1	-1	-2	2	1	-1	-30	15
$15\ B_{15}(5.4)$:	1	-6	1	2	-2	4		-60
$15\ B_{15}(6,4)$:	1	-1	-2	2	1	-1	30	15
$15\ B_{15}(7,4)$:	1	-1	1	-4	1	2		-30
$15\ B_{15}(10,4)$:	1	-6	1	-4	1	-8		120
$15\ B_{15}(11,4)$:	1	-1	1	2	-2	-1	-30	15

Table 4c: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 2 \pmod 3$

	p^2	1	A^2	AB	B^2	T^2	TU	U^2
15 $B_{15}(0,4)$:	1	-6	-2	2	1	4		-60
15 $B_{15}(1,4)$:	1	-1	1	-4	1	-1	30	15
15 $B_{15}(2,4)$:	1	-1	1	2	-2	2		-30
15 $B_{15}(3,4)$:	1	-1	-2	2	1	-1	30	15
15 $B_{15}(5,4)$:	1	-6	1	2	-2	-8		120
15 $B_{15}(6,4)$:	1	-1	-2	2	1	-1	-30	15
15 $B_{15}(7,4)$:	1	-1	1	-4	1	-1	-30	15
15 $B_{15}(10,4)$:	1	-6	1	-4	1	4		-60
15 $B_{15}(11,4)$:	1	-1	1	2	-2	2		-30

Table 5a: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 0 \pmod 3$

	p^2	1	AT	BT	AU	BU
15 $B_{15}(0,5)$:	1	-8	-8	4		
15 $B_{15}(1,5)$:	1		-1	-1	-15	15
15 $B_{15}(2,5)$:	1		-1	2	-15	
15 $B_{15}(3,5)$:	1	-3	2	-1		15
15 $B_{15}(5,5)$:	1	-5	4	-8		
15 $B_{15}(6,5)$:	1	-3	2	-1		-15
15 $B_{15}(7,5)$:	1		-1	-1	15	-15
15 $B_{15}(10,5)$:	1	-5	4	4		
15 $B_{15}(11,5)$:	1		-1	2	15	

Table 5b: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 1 \pmod 3$

	p^2	1	AT	BT	AU	BU
15 $B_{15}(0,5)$:	1	-8	4	4		
15 $B_{15}(1,5)$:	1		-1	2	15	
15 $B_{15}(2,5)$:	1		2	-1		15
15 $B_{15}(3,5)$:	1	-3	-1	-1	15	-15
15 $B_{15}(5,5)$:	1	-5	-8	4		
15 $B_{15}(6,5)$:	1	-3	-1	-1	-15	15
15 $B_{15}(7,5)$:	1		-1	2	-15	
15 $B_{15}(10,5)$:	1	-5	4	-8		
15 $B_{15}(11,5)$:	1		2	-1		-15

Table 5c: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 2 \pmod 3$

	p^2	1	AT	BT	AU	BU
15 $B_{15}(0,5)$:	1	-8	4	-8		
15 $B_{15}(1,5)$:	1		2	-1		-15
15 $B_{15}(2,5)$:	1		-1	-1	15	-15
15 $B_{15}(3,5)$:	1	-3	-1	2	-15	
15 $B_{15}(5,5)$:	1	-5	4	4		
15 $B_{15}(6,5)$:	1	-3	-1	2	15	
15 $B_{15}(7,5)$:	1		2	-1		15
15 $B_{15}(10,5)$:	1	-5	-8	4		
15 $B_{15}(11,5)$:	1		-1	-1	-15	15

Next, we treat the case $p \equiv 11 \pmod{15}$. Inverting the matrix M given in (5.39), we obtain from (5.38)

$$B(v) = \frac{1}{15} N C(v) , \tag{5.56}$$

where

$$N = \begin{bmatrix} 1 & 2 & 4 & -1 & -1 & -1 & 8 & -2 & -2 & -2 \\ 1 & -1 & -1 & 4 & -1 & -1 & 1 & 1 & -4 & 1 \\ 1 & -1 & -1 & -1 & 4 & -1 & 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -1 & -1 & 4 & -2 & 8 & -2 & -2 \\ 1 & -1 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & -4 \\ 1 & -1 & 4 & -1 & -1 & -1 & -4 & 1 & 1 & 1 \\ 1 & 2 & -1 & 4 & -1 & -1 & -2 & -2 & 8 & -2 \\ 1 & -1 & -1 & -1 & -1 & 4 & 1 & -4 & 1 & 1 \\ 1 & 2 & -1 & -1 & -1 & -1 & -2 & -2 & -2 & 8 \\ 1 & 2 & -1 & -1 & 4 & -1 & -2 & -2 & -2 & -2 \end{bmatrix} . \tag{5.57}$$

Substituting the values of the $C_i(v)$ given in Theorem 6 into (5.56), we obtain the Dickson-Hurwitz sums in terms of the Q(i) and R(i) ($0 \le i \le 4$). There will be 61 tables as follows:

v	no. of tables
1	5 (depending upon $\text{ind}_g 2 \pmod 5$)
2	25 (depending upon $\text{ind}_g 2$, $\text{ind}_g 3 \pmod 5$)
3	25 (depending upon $\text{ind}_g 2$, $\text{ind}_g 3 \pmod 5$)
4	1
5	5 (depending upon $(\text{ind}_g 2 + 2\ \text{ind}_g 3) \pmod 5$)

As we cannot display them all here, we just give one table for each value of v.

Table 6: Dickson-Hurwitz sums $B_{15}(i,1)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 0 \pmod 5$

	p^2	p	1	Q(0)	Q(1)	Q(2)	Q(3)	Q(4)	R(0)	R(1)	R(2)	R(3)	R(4)
15 $B_{15}(0,1)$:	1	2	-2	-4	1	1	1	1	8	-2	-2	-2	-2
15 $B_{15}(1,1)$:	1	-1	-2	1	1	1	1	-4	1	1	1	1	-4
15 $B_{15}(2,1)$:	1	-1	-2	1	1	1	-4	1	1	1	1	-4	1
15 $B_{15}(3,1)$:	1	2	-2	1	1	-4	1	1	-2	-2	8	-2	-2
15 $B_{15}(4,1)$:	1	-1	-2	1	-4	1	1	1	1	-4	1	1	1
15 $B_{15}(5,1)$:	1	-1	-2	-4	1	1	1	1	-4	1	1	1	1
15 $B_{15}(6,1)$:	1	2	-2	1	1	1	1	-4	-2	-2	-2	-2	8
15 $B_{15}(8,1)$:	1	-1	-2	1	1	-4	1	1	1	1	-4	1	1
15 $B_{15}(9,1)$:	1	2	-2	1	-4	1	1	1	-2	8	-2	-2	-2
15 $B_{15}(12,1)$:	1	2	-2	1	1	1	-4	1	-2	-2	-2	8	-2

Table 7: Dickson-Hurwitz sums $B_{15}(i,2)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 1 \pmod 5$, $\text{ind}_g 3 \equiv 2 \pmod 5$.

	p^2	p	1	Q(0)	Q(1)	Q(2)	Q(3)	Q(4)
15 $B_{15}(0,2)$:	1	2	-4	1	-4	1	1	1
15 $B_{15}(1,2)$:	1	-1	-1	1	1	1	-4	1
15 $B_{15}(2,2)$:	1	4	-1	-4	1	1	1	1
15 $B_{15}(3,2)$:	1	2	-4	1	1	-4	1	1
15 $B_{15}(4,2)$:	1	-1	-1	1	1	1	1	-4
15 $B_{15}(5,2)$:	1	-1	-1	1	-4	1	1	1
15 $B_{15}(6,2)$:	1	2	-4	1	1	1	-4	1
15 $B_{15}(8,2)$:	1	-1	-1	1	1	-4	1	1
15 $B_{15}(9,2)$	1	2	-4	1	1	1	1	-4
15 $B_{15}(12,2)$:	1	-8	-4	-4	1	1	1	1

Table 8: Dickson-Hurwitz sums $B_{15}(i,3)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 0 \pmod 5$, $\text{ind}_g 3 \equiv 2 \pmod 5$.

	p^2	p	1	Q(0)	Q(1)	Q(2)	Q(3)	Q(4)
$15\ B_{15}(0,3)$:	1	0	-4	-2	3	3	-7	3
$15\ B_{15}(1,3)$:	1	0	-1	0	5	0	0	-5
$15\ B_{15}(2,3)$:	1	0	-1	0	0	0	-5	5
$15\ B_{15}(3,3)$:	1	0	-4	3	3	-12	3	3
$15\ B_{15}(4,3)$:	1	0	-1	5	-5	0	0	0
$15\ B_{15}(5,3)$:	1	0	-1	-5	0	0	5	0
$15\ B_{15}(6,3)$:	1	0	-4	3	-7	3	3	-2
$15\ B_{15}(8,3)$:	1	0	-1	0	0	0	0	0
$15\ B_{15}(9,3)$:	1	0	-4	-7	-2	3	3	3
$15\ B_{15}(12,3)$:	1	0	-4	3	3	3	-2	-7

Table 9: Dickson-Hurwitz sums $B_{15}(i,4)$, $p \equiv 11 \pmod{15}$

	p^2	p	1
$15\ B_{15}(0,4)$:	1	10	-6
$15\ B_{15}(1,4)$:	1	0	-1
$15\ B_{15}(2,4)$:	1	0	-1
$15\ B_{15}(3,4)$:	1	0	-1
$15\ B_{15}(4,4)$:	1	0	-1
$15\ B_{15}(5,4)$:	1	-5	-6
$15\ B_{15}(6,4)$:	1	0	-1
$15\ B_{15}(8,4)$:	1	0	-1
$15\ B_{15}(9,4)$:	1	0	-1
$15\ B_{15}(12,4)$:	1	0	-1

Table 10: Dickson-Hurwitz sums $B_{15}(i,5)$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 + 2\,\text{ind}_g 3 \equiv 4 \pmod 5$

	p^2	p	1	R(0)	R(1)	R(2)	R(3)	R(4)
15 $B_{15}(0,5)$:	1	0	-8	2	2	2	2	-8
15 $B_{15}(1,5)$:	1	0	0	-1	-1	-1	4	-1
15 $B_{15}(2,5)$:	1	0	0	-1	-1	4	-1	-1
15 $B_{15}(3,5)$:	1	0	-3	2	-8	2	2	2
15 $B_{15}(4,5)$:	1	0	0	4	-1	-1	-1	-1
15 $B_{15}(5,5)$:	1	0	-5	-1	-1	-1	-1	4
15 $B_{15}(6,5)$:	1	0	-3	2	2	2	-8	2
15 $B_{15}(8,5)$:	1	0	0	-1	4	-1	-1	-1
15 $B_{15}(9,5)$:	1	0	-3	-8	2	2	2	2
15 $B_{15}(12,5)$:	1	0	-3	2	2	-8	2	2

6. EVALUATION OF CYCLOTOMIC NUMBERS $(h,k)_{15}$ OVER GF(p^2).

The cyclotomic numbers have the following well-known properties (see, for example, [11: p. 25])

$$(h,k)_e = (e-h,k-h)_e \,, \tag{6.1}$$

$$(h,k)_e = \begin{cases} (k,h)_e & \text{, if } f \text{ is even,} \\ (k + \tfrac{1}{2}e,\ h + \tfrac{1}{2}e)_e & \text{, if } f \text{ is odd,} \end{cases} \tag{6.2}$$

as well as the property

$$(h,\ k)_e = (ph,\ pk)_e$$

noted in (5.7). Taking $e = 15$, $\ell = 2$, $q = p^2$, $p \equiv 4, 11 \pmod{15}$, and appealing to (5.7), (6.1) and (6.2), we see that each of the $e^2 = 225$ cyclotomic numbers $(h,\ k)_{15}$ is equal to one of the 28 cyclotomic numbers

$$\begin{aligned} &(0,\ k)_{15} \,, && k = 0,\ 1,\ 2,\ 3,\ 5,\ 6,\ 7,\ 10,\ 11 \,, \\ &(1,\ k)_{15} \,, && k = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 12,\ 13 \,, \\ &(2,\ k)_{15} \,, && k = 5,\ 7,\ 8,\ 9,\ 12 \,, \\ &(3,\ k)_{15} \,, && k = 6,\ 9 \,, \\ &(5,\ 10)_{15} \,, \end{aligned} \tag{6.3}$$

when $p \equiv 4 \pmod{15}$, and to one of the 29 cyclotomic numbers

$$\begin{aligned} &(0,\ k)_{15} \,, && k = 0,\ 1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 8,\ 9,\ 12 \,, \\ &(1,\ k)_{15} \,, && k = 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 13 \,, \\ &(2,\ k)_{15} \,, && k = 5,\ 7,\ 8,\ 10,\ 12 \,, \\ &(3,\ k)_{15} \,, && k = 6,\ 9 \,, \\ &(5,\ 10)_{15} \,, \end{aligned} \tag{6.4}$$

when $p \equiv 11 \pmod{15}$.

These relationships are exhibited in Tables 11 and 12, in which $(h, k)_{15}$ is in row $h(0 \le h \le 14)$ and column $k(0 \le k \le 14)$, and we have written A for 10, B for 11, C for 12, and D for 13.

Table 11: Relationships between cyclotomic numbers $(h,k)_{15}$ over $GF(p^2), p \equiv 4 \pmod{15}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0,0	0,1	0,2	0,3	0,1	0,5	0,6	0,7	0,2	0,6	0,A	0,B	0,3	0,7	0,B
1	0,1	0,B	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,A	1,5	1,C	1.D	1,2
2	0,2	1,2	0,7	1,D	1,8	2,5	1,7	2,7	2,8	2,9	2,7	1,9	2,C	1,8	1,3
3	0,3	1,3	1,D	0,3	1,C	2,C	3,6	1,3	2,C	3,9	2,5	1,D	3,6	2,5	1,4
4	0,1	1,4	1,8	1,C	0,B	1,5	1.9	1,D	1,2	1,6	1,A	1,2	1,3	1,7	1,5
5	0,5	1,5	2,5	2,C	1,5	0,A	1,A	2,7	2,5	1,A	5,A	1,6	2,C	2.7	1,6
6	0,6	1,6	1,7	3,6	1,9	1,A	0,6	1,9	2,9	3,9	1,6	1,A	3,9	2,8	1,7
7	0,7	1,7	2,7	1,3	1,D	2,7	1,9	0,2	1,8	2,8	2,C	1,2	2,5	2,9	1,8
8	0,2	1,8	2,8	2,C	1,2	2,5	2,9	1,8	0,7	1,7	2,7	1,3	1,D	2,7	1,9
9	0,6	1,9	2,9	3,9	1,6	1,A	3,9	2,8	1,7	0,6	1,6	1,7	3,6	1,9	1,A
10	0,A	1,A	2,7	2,5	1,A	5,A	1,6	2,C	2,7	1,6	0,5	1,5	2,5	2,C	1,5
11	0,B	1,5	1,9	1,D	1,2	1,6	1,A	1,2	1,3	1,7	1,5	0,1	1,4	1,8	1,C
12	0,3	1,C	2,C	3,6	1,3	2,C	3,9	2,5	1,D	3,6	2,5	1,4	0,3	1,3	1,D
13	0,7	1,D	1,8	2,5	1,7	2,7	2,8	2,9	2,7	1,9	2,C	1,8	1,3	0,2	1,2
14	0,B	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,A	1,5	1,C	1,D	1,2	0,1

Table 12: Relationships between cyclotomic numbers $(h,k)_{15}$ over $GF(p^2), p \equiv 11 \pmod{15}$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14
0	0,0	0,1	0,2	0,3	0,4	0,5	0,6	0,2	0,8	0,9	0,5	0,1	0,C	0,8	0,4
1	0,1	0,4	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,A	1,B	1,4	1,D	1,2
2	0,2	1,2	0,8	1,D	1,8	2,5	1,9	2,7	2,8	2,8	2,A	1,7	2,C	1,8	1,3
3	0,3	1,3	1,D	0,C	1,4	2,C	3,6	1,D	2,5	3,9	2,C	1,3	3,6	2,5	1,4
4	0,4	1,4	1,8	1,4	0,1	1,B	1,7	1,3	1,2	1,A	1,6	1,2	1,D	1,9	1,5
5	0,5	1,5	2,5	2,C	1,B	0,5	1,A	2,A	2,C	1,6	5,A	1,A	2,5	2,7	1,6
6	0,6	1,6	1,9	3,6	1,7	1,A	0,9	1,9	2,8	3,9	1,A	1,6	3,9	2,8	1,7
7	0,2	1,7	2,7	1,D	1,3	2,A	1,9	0,8	1,8	2,8	2,5	1,2	2,C	2,8	1,8
8	0,8	1,8	2,8	2,5	1,2	2,C	2,8	1,8	0,2	1,7	2,7	1,D	1,3	2,A	1,9
9	0,9	1,9	2,8	3,9	1,A	1,6	3,9	2,8	1,7	0,6	1,6	1,9	3,6	1,7	1,A
10	0,5	1,A	2,A	2,C	1,6	5,A	1,A	2,5	2,7	1,6	0,5	1,5	2,5	2,C	1,B
11	0,1	1,B	1,7	1,3	1,2	1,A	1,6	1,2	1,D	1,9	1,5	0,4	1,4	1,8	1,4
12	0,C	1,4	2,C	3,6	1,D	2,5	3,9	2,C	1,3	3,6	2,5	1,4	0,3	1,3	1,D
13	0,8	1,D	1,8	2,5	1,9	2,7	2,8	2,8	2,A	1,7	2,C	1,8	1,3	0,2	1,2
14	0,4	1,2	1,3	1,4	1,5	1,6	1,7	1,8	1,9	1,A	1,B	1,4	1,D	1,2	0,1

In order to determine explicit formulae for the cyclotomic numbers in (6.3) and (6.4), we express each cyclotomic number $(h, k)_{15}$ in terms of the Dickson-Hurwitz sums $B_{15}(i, v)(i, v = 0, 1, \ldots, 14)$ and then appeal to the formulae for the $B_{15}(i, v)$ derived in §5.

Whiteman gave the following formula, which expresses each cyclotomic number of order e, where e is an odd prime, as a linear combination of certain Dickson-Hurwitz sums of order e [12: Theorem 1]. We give the formula in the context of GF(q), as Whiteman's proof for GF(p) goes over without difficulty to this case. We have, for e an odd prime and r, s = 0, 1, ..., e-1,

$$e^2(r, s)_e = (q - 1)(1 - e) + e(1 - \delta_r) + e \sum_{v=0}^{e-1} B_e(vr + s, v) \tag{6.5}$$

where

$$\delta_r = \begin{cases} 1 , & \text{if } r \equiv 0 \pmod{e} , \\ 0 , & \text{if } r \not\equiv 0 \pmod{e} . \end{cases} \tag{6.6}$$

In Theorem 7, we give the corresponding formula to (6.5), where e is the product of two distinct odd primes x and y. The formula will be used with e = 15, x = 3, y = 5, $q = p^2$, $p \equiv 4, 11 \pmod{15}$.

We use the well-known identity

$$B_x(i, v) = \sum_{j=0}^{y-1} B_e(i + jx, v) , \quad e = xy , \tag{6.7}$$

to simplify formulae appearing in the proof of Theorem 7, often in the form

$$\sum_{j=1}^{y-1} B_e(i + jx, v) = B_x(i, v) - B_e(i, v) . \tag{6.8}$$

We also use a special case of the Ramanujan sum [6: pp. 237-238]

$$\sum_{n=1}^{e-1}{}' \beta^{nk} = \begin{cases} 1 , & \text{if } x \nmid k, y \nmid k \\ -(x-1) , & \text{if } x \mid k, y \nmid k \\ -(y-1) , & \text{if } x \nmid k, y \mid k \\ (x-1)(y-1) , & \text{if } x \mid k, y \mid k \end{cases} \tag{6.9}$$

where the prime (') indicates that the summation variable is restricted to values coprime with e. We prove

THEOREM 7. Let e = xy be the product of two distinct odd primes x and y. Then for r, s = 0, 1, 2, ..., e-1 we have

$$e^2(r, s)_e = q + 1 - e\,(\delta_r + \delta_s + \delta_{r-s})$$

$$+ \sum_{\substack{v=1 \\ (v,e)>1}}^{e-2} [(e-1)B_e(vr+s,v)-(x-1)B_x(vr+s,v)-(y-1)B_y(vr+s,v)+ \sum_{u=1}^{e-1}{}' B_e(vr+s+u,v)]$$

$$+ \sum_{v=1}^{e-2} [(e-1)B_e(vs+r,v)-(x-1)B_x(vs+r,v)-(y-1)B_y(vs+r,v)+ \sum_{u=1}^{e-1}{}' B_e(vs+r+u,v)]$$

$$+ y \sum_{w=1}^{y-2} B_y(wr+s,w)-(y-2)(q-2)+ x \sum_{w=1}^{x-2} B_x(wr+s,w)-(x-2)(q-2)$$

$$+ (e-1)(B_e(rax + sby,ax) + B_e(sax + rby,ax))$$

$$+ \sum_{j=1}^{e-1}{}' (B_e(rax + sby + j,ax) + B_e(sax + rby + j, ax))$$

$$- 4(q-1) + 2f(x+y) + (x-1)(\delta_{sy} + \delta_{ry}) + (y-1)(\delta_{rx} + \delta_{sx}) ,$$ where a and b are integers such that

$$ax + by = e - 1 . \tag{6.10}$$

PROOF. It is well-known that the cyclotomic numbers $(r,s)_e$ are related to the Jacobi sums $J_q(\beta^m, \beta^n)$ by the formula

$$e^2(r, s)_e = \sum_{m=0}^{e-1} \sum_{n=0}^{e-1} J_q(\beta^m, \beta^n)\beta^{-mr-ns} . \tag{6.11}$$

Splitting the double sum on the right-hand side into five sums according as $m = n = 0$; $m = 0$, $n \neq 0$; $m \neq 0$, $n = 0$; $m \neq 0$, $n \neq 0$, $m + n = e$; $m \neq 0$, $n \neq 0$, $m + n \neq e$; we obtain, appealing to (2.4),

$$e^2(r,s)_e = q+1 - e(\delta_r + \delta_s + \delta_{r-s}) + \sum_{\substack{m,n=1 \\ m+n\neq e}}^{e-1} J_q(\beta^m, \beta^n)\beta^{-mr-ns}. \tag{6.12}$$

The double sum on the right-hand side of (6.12) will be broken up into six sums S_i as follows:

$$\begin{aligned}
S_1 &: m + n \neq e , (m, e) > 1 , (n, e) = 1 , \\
S_2 &: m + n \neq e , (m, e) = 1 , \\
S_3 &: m + n \neq e , (m, e) = (n, e) = x , \\
S_4 &: m + n \neq e , (m, e) = (n, e) = y , \\
S_5 &: (m, e) = x , (n, e) = y , \\
S_6 &: (m, e) = y , (n, e) = x .
\end{aligned}$$

First, we evaluate S_1. We have

$$\begin{aligned}
S_1 &= \sum_{\substack{m=1 \\ (m,e)>1 \\ m+n\neq e}}^{e-1} \sum_{n=1}^{e-1}{}' J_q(\beta^m,\beta^n)\beta^{-mr-ns} \\
&= \sum_{n=1}^{e-1}{}' \sum_{\substack{v=1 \\ (v,e)>1}}^{e-2} J_q(\beta^{nv}, \beta^n)\beta^{-n(vr+s)} \\
&= \sum_{n=1}^{e-1}{}' \sum_{\substack{v=1 \\ (v,e)>1}}^{e-2} \sum_{i=0}^{e-1} B_e(i,v)\beta^{n(i-vr-s)} \qquad \text{(by (5.2))} \\
&= \sum_{\substack{v=1 \\ (v,e)>1}}^{e-2} \sum_{i=0}^{e-1} B_e(i,v) \sum_{n=1}^{e-1}{}' \beta^{n(i-vr-s)} .
\end{aligned}$$

Appealing to (6.9), we obtain

$$\begin{aligned}
S_1 = \sum_{\substack{v=1 \\ (v,e)>1}}^{e-2} [&(x-1)(y-1)B_e(vr+s,v)-(x-1)\sum_{u=1}^{y-1} B_e(vr+s+xu,v) \\
&- (y-1)\sum_{u=1}^{x-1} B_e(vr+s+yu,v) + \sum_{u=1}^{e-1}{}' B_e(vr+s+u,v)],
\end{aligned}$$

that is

$$S_1 = \sum_{\substack{v=1 \\ (v,e)>1}}^{e-2} [(e-1)B_e(vr+s,v)-(x-1)B_x(vr+s,v)-(y-1)B_y(vr+s,v)+ \sum_{u=1}^{e-1}{}' B_e(vr+s+u,v)]. \tag{6.13}$$

Similarly, we obtain

$$S_2 = \sum_{v=1}^{e-2} [(e-1)B_e(vs+r,v)-(x-1)B_x(vs+r,v)-(y-1)B_y(vs+r,v)+ \sum_{u=1}^{e-1}{}' B_e(vs+r+u,v)]. \quad (6.14)$$

Next, we evaluate S_3. We have

$$S_3 = \sum_{\substack{t=1 \\ t+u\neq y}}^{y-1} \sum_{u=1}^{y-1} J_q(\beta^{xt}, \beta^{xu})\beta^{-x(tr+us)}$$

$$= \sum_{w=1}^{y-2} \sum_{u=1}^{y-1} J_q(\beta^{xuw}, \beta^{xu})\beta^{-xu(wr+s)}$$

$$= \sum_{w=1}^{y-2} \sum_{u=1}^{y-1} \sum_{i=0}^{e-1} B_e(i, w)\beta^{xu(i-wr-s)} \quad \text{(by (5.2))}$$

$$= \sum_{w=1}^{y-2} \sum_{j=0}^{e-1} B_e(j + wr + s,w) \sum_{u=1}^{y-1} \beta^{xuj}$$

$$= (y - 1) \sum_{w=1}^{y-2} \sum_{t=0}^{x-1} B_e(yt + wr + s,w) - \sum_{w=1}^{y-1} \sum_{\substack{j=0 \\ y\nmid j}}^{e-1} B_e(j + wr + s,w)$$

$$= y \sum_{w=1}^{y-2} \sum_{t=0}^{x-1} B_e(yt + wr + s,w) - \sum_{w=1}^{y-1} \sum_{j=0}^{e-1} B_e(j + wr + s,w)$$

$$= y \sum_{w=1}^{y-2} B_y(wr + s,w) - \sum_{w=1}^{y-2} \sum_{j=0}^{e-1} B_e(j, w) \quad \text{(by (6.7))},$$

that is

$$S_3 = y \sum_{w=1}^{y-2} B_y(wr + s,w) - (y - 2)(q - 2) \quad \text{(by(5.5))}. \quad (6.15)$$

Similarly we have

$$S_4 = x \sum_{w=1}^{x-2} B_x(wr + s,w) - (x - 2)(q - 2) . \quad (6.16)$$

Next we evaluate S_5. We have

$$S_5 = \sum_{\substack{m=1 \\ x|m}}^{e-1} \sum_{\substack{n=1 \\ y|n}}^{e-1} J_q(\beta^m, \beta^n)\, \beta^{-mr-ns}$$

$$= \sum_{t=1}^{x-1} \sum_{u=1}^{y-1} J_q(\beta^{xt}, \beta^{yu})\beta^{-rxt-syu} .$$

Now we replace t by iax(mod y) and u by jby(mod x) to obtain

$$S_5 = \sum_{i=1}^{y-1} \sum_{j=1}^{x-1} J_q(\beta^{iax^2}, \beta^{jby^2})\beta^{-riax^2-sjby^2}$$

$$= \sum_{i=1}^{y-1} \sum_{j=1}^{x-1} \sigma_{ix+jy}(J_q(\beta^{ax}, \beta^{by})\beta^{-rax-sby})$$

$$= \sum_{k=1}^{e-1}{}' \sigma_k(J_q(\beta^{ax}, \beta^{by})\beta^{-rax-sby})$$

$$= \mathrm{tr}(J_q(\beta^{ax}, \beta^{by})\beta^{-rax-sby})$$

$$= \mathrm{tr}(J_q(\beta^{ax}, \beta)\beta^{-rax-sby}) \quad \text{(by (2.2) and (6.10))}$$

$$= \mathrm{tr}\Big(\sum_{i=0}^{e-1} B_e(i, ax)\beta^{i-rax-sby}\Big)$$

$$= \mathrm{tr}\Big(\sum_{j=0}^{e-1} B_e(rax + sby + j, ax)\beta^{j}\Big)$$

$$= \sum_{j=0}^{e-1} B_e(rax + sby + j, ax) \sum_{i=1}^{e-1}{}' \beta^{ij}$$

$$= (x - 1)(y - 1)B_e(rax + sby, ax) - (x - 1)\sum_{t=1}^{y-1} B_e(rax + sby + xt, ax)$$

$$- (y - 1)\sum_{t=1}^{x-1} B_e(rax + sby + yt, ax) + \sum_{j=1}^{e-1}{}' B_e(rax + sby + j, ax)$$

$$= (x - 1)(y - 1)B_e(rax + sby, ax) - (x - 1)(B_x(rax + sby, ax) - B_e(rax + sby, ax))$$

$$- (y - 1)(B_y(rax + sby, ax) - B_e(rax + sby, ax)) + \sum_{j=1}^{e-1}{}' B_e(rax + sby + j, ax),$$

that is

$$S_5 = (e - 1)B_e(rax + sby, ax) - (x - 1)B_x(sby, 0) - (y - 1)B_y(rax, ax)$$

$$+ \sum_{j=1}^{e-1}{}' B_e(rax + sby + j, ax) . \tag{6.17}$$

Appealing to (5.4) we have

$$B_x(sby, 0) = fy - \delta_{sy}$$

and

$$B_y(rax, ax) = B_y(rax, e - 1 - by) = B_y(rax, e - 1) = fx - \delta_{rx}.$$

Similarly we obtain

$$S_6 = (e - 1)B_e(sax + rby, ax) - (x - 1)B_x(rby, 0) - (y - 1)B_y(sax, ax)$$

$$+ \sum_{j=1}^{e-1}{}' B_e(sax + rby + j, ax) , \tag{6.18}$$

where as above we have

$$B_x(rby, 0) = fy - \delta_{ry}$$

and

$$B_y(sax, ax) = fx - \delta_{sx}.$$

This completes the proof of Theorem 7.

Applying Theorem 7 and appealing to the tables of the Dickson-Hurwitz sums obtained in §5, we obtain formulae for the cyclotomic numbers of order 15 over $GF(p^2)$, $p \equiv 4, 11 \pmod{15}$.

In the case $p \equiv 4 \pmod{15}$ there are 3 tables depending upon the value of $\mathrm{ind}_g 5 \pmod 3$. All 3 tables are given below (see Tables 13a, 13b, 13c). The values of the cyclotomic numbers are given as linear combinations of

$$p^2,\ p,\ 1,\ A^2,\ AB,\ B^2,\ AT,\ BT,\ AU,\ BU,\ T^2\text{-}15U^2,\ TU,$$

using the following abbreviations

$$\begin{aligned} x &= -2A^2 + 2AB + B^2 , \\ y &= A^2 - 4AB + B^2 , \\ z &= A^2 + 2AB - B^2 . \end{aligned} \tag{6.19}$$

In the case $p \equiv 11 \pmod{15}$ there are 25 tables depending upon the values of $\mathrm{ind}_g 2 \pmod 5$ and $\mathrm{ind}_g 3 \pmod 5$. The values of the cyclotomic numbers are given as linear compinations of

$$p^2,\ p,\ 1,\ XU,\ XV,\ U^2,\ UV,\ UW,\ V^2,\ VW,\ W^2$$

after eliminating the $Q(i)$ and $R(i)$ $(0 \le i \le 4)$ by means of the following formulae (see (3.21) - (3.26), (5.51), 5.52)):

$$\begin{aligned}
Q(0) &= \tfrac{4}{5}p + \tfrac{1}{5} - 8U^2 - 8V^2 ,\\
Q(1) &= -\tfrac{1}{5}p + \tfrac{1}{5} - 2XU - 2XV - 2U^2 - 4UV - 6UW + 6V^2 + 2VW ,\\
Q(2) &= -\tfrac{1}{5}p + \tfrac{1}{5} - 2XU + 2XV + 6U^2 + 4UV - 2UW - 2V^2 - 6VW ,\\
Q(3) &= -\tfrac{1}{5}p + \tfrac{1}{5} + 2XU - 2XV + 6U^2 + 4UV + 2UW - 2V^2 + 6VW ,\\
Q(4) &= -\tfrac{1}{5}p + \tfrac{1}{5} + 2XU + 2XV - 2U^2 - 4UV + 6UW + 6V^2 - 2VW ,
\end{aligned} \tag{6.20}$$

and

$$\begin{aligned}
R(0) &= \tfrac{4}{5}p + \tfrac{1}{5} - 4U^2 - 4V^2 - 8W^2,\\
R(1) &= -\tfrac{1}{5}p + \tfrac{1}{5} + 2XV + 2U^2 - 4UV - 2UW + 4VW + 2W^2 ,\\
R(2) &= -\tfrac{1}{5}p + \tfrac{1}{5} + 2XU + 4UV - 4UW + 2V^2 - 2VW + 2W^2 ,\\
R(3) &= -\tfrac{1}{5}p + \tfrac{1}{5} - 2XU + 4UV + 4UW + 2V^2 + 2VW + 2W^2 ,\\
R(4) &= -\tfrac{1}{5}p + \tfrac{1}{5} - 2XV + 2U^2 - 4UV + 2UW - 4VW + 2W^2 .
\end{aligned} \tag{6.21}$$

Only 7 of these tables are presented in this paper as the remainder can be deduced from them as indicated below.

For $n = 2, 3, 4$ is it convenient to define $n^* = 8, 2, 4$ respectively. The 18 cases not covered in the given tables can be divided into 3 classes as follows:

(α) $\mathrm{ind}_g 2 \equiv 0 \pmod 5$, $\mathrm{ind}_g 3 \equiv n \pmod 5$, $n = 2, 3, 4$,

(β) $\mathrm{ind}_g 2 \equiv n \pmod 5$, $\mathrm{ind}_g 3 \equiv 0 \pmod 5$, $n = 2, 3, 4$,

(γ) $\mathrm{ind}_g 2 \equiv m \pmod 5$, $\mathrm{ind}_g 3 \equiv n \pmod 5$, $m = 1, 2, 3, 4$; $n = 2, 3, 4$,

For cases in class (α) (resp. (β), resp. (γ)) the value of $(h,k)_{15}$ may be deduced from Table 14b (resp. Table 14c, resp. Table 14d (if $m \equiv n \pmod 5$), Table 14e if $m \equiv 2n \pmod 5$), Table 14f (if $m \equiv 3n \pmod 5$), Table 14g (if $m \equiv 4n \pmod 5$)) by looking up the value of $(n^*h, n^*k)_{15}$ and replacing X, U, V, W by

$$\begin{aligned}
&X, -V, U, -W, && \text{if } n = 2,\\
&X, V, -U, -W, && \text{if } n = 3,\\
&X, -U, -V, W, && \text{if } n = 4.
\end{aligned}$$

These transformations are clear in view of the basic properties

$$(h, k)_\gamma = (m^{-1}h, m^{-1}k)_{\gamma^m} ,$$

$$\mathrm{ind}_{\gamma^m}\alpha \equiv m^{-1}\mathrm{ind}_\gamma\alpha \pmod{q-1} ,$$

$$E_{\gamma^m}(\beta^n) = E_\gamma(\beta^{m^{-1}n}) ,$$

where $(m, q-1) = 1$ and the dependence upon the generator γ or γ^m of $GF(q)^*$ is shown by writing it as a subscript.

Table 13a: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 0 \pmod 3$

	p^2	p	1	A,B	AT	BT	AU	BU	T^2-15U^2	TU
225(0, 0) =	1	36	-44	x	48	-60	0	0	60	0
225(0, 1) =	1	-3	-14	y	13	13	45	-45	2	180
225(0, 2) =	1	-3	-14	z	-17	4	15	-30	-4	120
225(0, 3) =	1	-9	-14	x	-2	-5	-60	-15	-10	60
225(0, 5) =	1	12	-14	z	-12	24	0	0	-24	0
225(0, 6) =	1	-9	-14	x	-2	-5	60	15	-10	-60
225(0, 7) =	1	-3	-14	y	13	13	-45	45	2	-180
225(0,10) =	1	12	-14	y	-12	-12	0	0	12	0
225(0,11) =	1	-3	-14	z	-17	4	-15	30	-4	-120
225(1, 2) =	1	0	1	x	7	-8	-15	-30	-7	-30
225(1, 3) =	1	-3	1	y	-2	-2	60	-60	2	-60
225(1, 4) =	1	12	1	z	-2	4	30	60	-4	60
225(1, 5) =	1	0	1	x	-8	7	-30	15	8	-60
225(1, 6) =	1	-3	1	y	-2	-2	0	0	2	0
225(1, 7) =	1	-3	1	z	-2	4	30	-30	11	-30
225(1, 8) =	1	0	1	x	7	-8	15	30	-7	30
225(1, 9) =	1	-3	1	y	-2	-2	-60	60	2	60
225(1,10) =	1	-3	1	z	13	-11	-15	-15	-4	60
225(1,12) =	1	12	1	y	-2	-2	30	-30	-13	-30
225(1,13) =	1	-3	1	z	-2	4	-30	30	11	30
225(2, 5) =	1	-3	1	y	-2	-2	0	0	2	0
225(2, 7) =	1	0	1	x	-8	7	30	-15	8	60
225(2, 8) =	1	12	1	y	-2	-2	-30	30	-13	30
225(2, 9) =	1	12	1	z	-2	4	-30	-60	-4	-60
225(2,12) =	1	-3	1	z	13	-11	15	15	-4	-60
225(3, 6) =	1	6	1	x	-2	10	-30	30	5	30
225(3, 9) =	1	6	1	x	-2	10	30	-30	5	-30
225(5,10) =	1	0	1	x	12	12	0	0	-12	0

Table 13b: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 1 \pmod 3$

	p^2	p	1	A,B	AT	BT	AU	BU	T^2-15U^2	TU
225(0, 0) =	1	36	-44	x	48	12	0	0	60	0
225(0, 1) =	1	-3	-14	y	-17	13	15	15	-4	120
225(0, 2) =	1	-3	-14	z	13	-26	45	0	2	180
225(0, 3) =	1	-9	-14	x	-2	7	60	-75	-10	-60
225(0, 5) =	1	12	-14	z	-12	24	0	0	12	0
225(0, 6) =	1	-9	-14	x	-2	7	-60	75	-10	60
225(0, 7) =	1	-3	-14	y	-17	13	-15	-15	-4	-120
225(0,10) =	1	12	-14	y	-12	-12	0	0	-24	0
225(0,11) =	1	-3	-14	z	13	-26	-45	0	2	-180
225(1, 2) =	1	0	1	x	7	1	15	-45	-7	30
225(1, 3) =	1	-3	1	y	-2	-2	30	0	11	-30
225(1, 4) =	1	12	1	z	-2	4	-30	0	-13	30
225(1, 5) =	1	0	1	x	-8	1	30	-15	8	60
225(1, 6) =	1	-3	1	y	13	-2	15	-30	-4	-60
225(1, 7) =	1	-3	1	z	-2	4	60	0	2	-60
225(1, 8) =	1	0	1	x	7	1	-15	45	-7	-30
225(1, 9) =	1	-3	1	y	-2	-2	-30	0	11	30
225(1,10) =	1	-3	1	z	-2	4	0	0	2	0
225(1,12) =	1	12	1	y	-2	-2	-30	90	-4	-60
225(1,13) =	1	-3	1	z	-2	4	-60	0	2	60
225(2, 5) =	1	-3	1	y	13	-2	-15	30	-4	60
225(2, 7) =	1	0	1	x	-8	1	-30	15	8	-60
225(2, 8) =	1	12	1	y	-2	-2	30	-90	-4	60
225(2, 9) =	1	12	1	z	-2	4	30	0	-13	-30
225(2,12) =	1	-3	1	z	-2	4	0	0	2	0
225(3, 6) =	1	6	1	x	-2	-8	30	0	5	-30
225(3, 9) =	1	6	1	x	-2	-8	-30	0	5	30
225(5,10) =	1	0	1	x	12	-24	0	0	-12	0

Table 13c: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 4 \pmod{15}$, $\text{ind}_g 5 \equiv 2 \pmod 3$

	p^2	p	1	A,B	AT	BT	AU	BU	T^2-15U^2	TU
225(0, 0) =	1	36	-44	x	-24	12	0	0	24	0
225(0, 1) =	1	-3	-14	y	13	-17	-15	-15	2	60
225(0, 2) =	1	-3	-14	z	13	4	-15	30	2	60
225(0, 3) =	1	-9	-14	x	-14	7	0	45	-16	0
225(0, 5) =	1	12	-14	z	-12	24	0	0	12	0
225(0, 6) =	1	-9	-14	x	-14	7	0	-45	-16	0
225(0, 7) =	1	-3	-14	y	13	-17	15	15	2	-60
225(0,10) =	1	12	-14	y	-12	-12	0	0	12	0
225(0,11) =	1	-3	-14	z	13	4	15	-30	2	-60
225(1, 2) =	1	0	1	x	-2	1	0	-15	2	0
225(1, 3) =	1	-3	1	y	-2	-2	0	-30	2	0
225(1, 4) =	1	12	1	z	-2	4	90	-60	-13	90
225(1, 5) =	1	0	1	x	-2	1	0	45	2	0
225(1, 6) =	1	-3	1	y	-2	13	30	-15	2	-120
225(1, 7) =	1	-3	1	z	-2	4	0	30	2	0
225(1, 8) =	1	0	1	x	-2	1	0	15	2	0
225(1, 9) =	1	-3	1	y	-2	-2	0	30	2	0
225(1,10) =	1	-3	1	z	-2	-11	-30	15	2	120
225(1,12) =	1	12	1	y	-2	-2	-90	30	-13	-90
225(1,13) =	1	-3	1	z	-2	4	0	-30	2	0
225(2, 5) =	1	-3	1	y	-2	13	-30	15	2	120
225(2, 7) =	1	0	1	x	-2	1	0	-45	2	0
225(2, 8) =	1	12	1	y	-2	-2	90	-30	-13	90
225(2, 9) =	1	12	1	z	-2	4	-90	60	-13	-90
225(2,12) =	1	-3	1	z	-2	-11	30	-15	2	-120
225(3, 6) =	1	6	1	x	16	-8	0	60	14	0
225(3, 9) =	1	6	1	x	16	-8	0	-60	14	0
225(5,10) =	1	0	1	x	48	-24	0	0	-48	0

Table 14a: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 0 \pmod 5$, $\text{ind}_g 3 \equiv 0 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	-82	-44	0	0	480	0	0	480	0	240
225(0, 1) =	1	13	-14	-20	-110	-10	160	-30	-180	60	-130
225(0, 2) =	1	13	-14	-110	20	-180	-160	-60	-10	-30	-130
225(0, 3) =	1	13	-14	50	-50	30	-100	-150	-70	150	140
225(0, 4) =	1	13	-14	20	110	-10	160	30	-180	-60	-130
225(0, 5) =	1	-37	-14	0	0	180	0	0	180	0	120
225(0, 6) =	1	13	-14	-50	-50	-70	100	-150	30	-150	140
225(0, 8) =	1	13	-14	110	-20	-180	-160	60	-10	30	-130
225(0, 9) =	1	13	-14	50	50	-70	100	150	30	150	140
225(0,12) =	1	13	-14	-50	50	30	-100	150	-70	-150	140
225(1, 2) =	1	-2	1	0	0	-80	-40	0	100	0	20
225(1, 3) =	1	-2	1	0	0	-20	100	-60	-20	-180	20
225(1, 4) =	1	-2	1	0	0	40	-40	0	220	0	20
225(1, 5) =	1	-2	1	40	-80	80	-80	0	-60	0	20
225(1, 6) =	1	-2	1	20	-40	-40	-20	60	0	-120	20
225(1, 7) =	1	-2	1	0	0	-20	-100	-180	-20	60	20
225(1, 8) =	1	-2	1	0	0	100	40	0	-80	0	20
225(1, 9) =	1	-2	1	0	0	-20	-100	180	-20	-60	20
225(1,10) =	1	-2	1	-20	40	-40	-20	-60	0	120	20
225(1,11) =	1	-2	1	-40	80	80	-80	0	-60	0	20
225(1,13) =	1	-2	1	0	0	-20	100	60	-20	180	20
225(2, 5) =	1	-2	1	40	20	0	20	-120	-40	-60	20
225(2, 7) =	1	-2	1	80	40	-60	80	0	80	0	20
225(2, 8) =	1	-2	1	0	0	220	40	0	40	0	20
225(2,10) =	1	-2	1	-80	-40	-60	80	0	80	0	20
225(2,12) =	1	-2	1	-40	-20	0	20	120	-40	60	20
225(3, 6) =	1	-2	1	0	0	-20	-200	0	-20	0	-160
225(3, 9) =	1	-2	1	0	0	-20	200	0	-20	0	-160
225(5,10) =	1	98	1	0	0	-240	0	0	-240	0	-480

Table 14b: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\mathrm{ind}_g 2 \equiv 0 \pmod 5$, $\mathrm{ind}_g 3 \equiv 1 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	68	-44	-180	60	-180	-360	180	-60	-60	-360
225(0, 1) =	1	13	-14	40	-30	-90	-120	-230	0	160	-30
225(0, 2) =	1	13	-14	50	100	-140	80	-60	-50	-30	-130
225(0, 3) =	1	-37	-14	70	10	-30	-60	30	190	290	140
225(0, 4) =	1	13	-14	0	-50	-130	240	30	-160	-160	-30
225(0, 5) =	1	-37	-14	0	0	180	0	0	180	0	120
225(0, 6) =	1	13	-14	70	-90	-30	-60	130	-210	-110	-60
225(0, 8) =	1	-37	-14	10	-20	180	0	60	130	-70	170
225(0, 9) =	1	13	-14	-30	10	170	-60	30	-10	-10	-60
225(0,12) =	1	13	-14	-130	10	70	140	30	-110	90	140
225(1, 2) =	1	-2	1	-10	20	0	40	60	70	30	-30
225(1, 3) =	1	-2	1	50	0	120	40	40	-110	-30	70
225(1, 4) =	1	-2	1	-10	20	-60	-140	0	70	-150	-30
225(1, 5) =	1	-2	1	-20	-60	60	0	-140	60	-20	120
225(1, 6) =	1	-2	1	30	40	20	-60	60	-10	-70	-30
225(1, 7) =	1	-2	1	20	-40	0	-80	120	160	60	120
225(1, 8) =	1	-2	1	-10	-30	30	-80	10	-110	30	-80
225(1, 9) =	1	-2	1	-10	20	60	-20	-120	-50	90	-30
225(1,10) =	1	-2	1	-50	0	0	120	40	-90	70	-30
225(1,11) =	1	-2	1	0	100	-40	0	60	-40	80	120
225(1,13) =	1	-2	1	-10	-30	90	160	10	70	30	-80
225(2, 5) =	1	-2	1	-40	-20	-20	-100	-120	-20	-60	20
225(2, 7) =	1	23	1	70	10	-30	60	-90	-50	-70	20
225(2, 8) =	1	-2	1	-40	30	-30	40	-50	40	0	70
225(2,10) =	1	-2	1	-40	-20	-80	-40	60	100	-120	20
225(2,12) =	1	23	1	10	-20	-180	-60	120	-110	110	-130
225(3, 6) =	1	-2	1	20	60	120	40	-20	40	-60	40
225(3, 9) =	1	-2	1	20	-40	-180	40	-120	40	-60	-60
225(5,10) =	1	23	1	90	-30	90	180	-90	30	30	-180

Table 14c: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\mathrm{ind}_g 2 \equiv 1 \pmod 5$, $\mathrm{ind}_g 3 \equiv 0 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	-7	-44	90	30	30	300	330	-270	-210	-60
225(0, 1) =	1	-17	-14	-20	50	130	-200	90	180	20	-30
225(0, 2) =	1	-7	-14	-20	0	180	0	-240	-120	-20	20
225(0, 3) =	1	13	-14	-10	30	-70	100	-70	230	-110	-60
225(0, 4) =	1	-17	-14	50	-40	40	-40	0	70	250	170
225(0, 5) =	1	-7	-14	30	0	0	120	120	-90	-90	-30
225(0, 6) =	1	13	-14	-110	-70	130	100	30	30	-10	-60
225(0, 8) =	1	13	-14	-40	90	-250	-80	-70	-40	40	-30
225(0, 9) =	1	13	-14	-10	-70	-170	-100	30	130	-110	140
225(0,12) =	1	38	-14	40	-120	-120	0	-120	-120	40	-160
225(1, 2) =	1	3	1	-40	0	-20	20	-20	-60	-60	20
225(1, 3) =	1	-2	1	-50	-40	40	60	60	-30	130	-30
225(1, 4) =	1	18	1	-10	30	10	80	70	-150	-90	20
225(1, 5) =	1	13	1	10	50	-110	-20	30	30	-10	-180
225(1, 6) =	1	13	1	-40	-10	-110	-40	30	-80	40	-130
225(1, 7) =	1	-12	1	-10	30	70	140	-110	90	-30	20
225(1, 8) =	1	-2	1	40	-10	10	0	30	60	40	-30
225(1, 9) =	1	18	1	20	-30	-170	-100	-50	-120	0	-130
225(1,10) =	1	-17	1	10	-10	70	40	-90	90	-70	120
225(1,11) =	1	-2	1	80	-40	40	80	-120	-20	40	20
225(1,13) =	1	-17	1	10	-10	10	-120	30	150	-110	120
225(2, 5) =	1	-7	1	-20	0	0	120	0	60	100	20
225(2, 7) =	1	-2	1	-40	0	-100	40	80	20	40	120
225(2, 8) =	1	13	1	70	20	40	60	120	-30	10	-30
225(2,10) =	1	-7	1	-50	-30	150	-60	30	-30	-110	20
225(2,12) =	1	-2	1	20	-30	-10	-140	-10	20	-80	-30
225(3, 6) =	1	-2	1	40	80	-20	0	-120	-120	40	-60
225(3, 9) =	1	-27	1	-10	30	130	-100	130	30	90	140
225(5,10) =	1	38	1	0	120	120	-240	-120	0	240	120

Table 14d: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\mathrm{ind}_g 2 \equiv 1 \pmod 5$, $\mathrm{ind}_g 3 \equiv 1 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	53	-44	90	30	150	-180	-390	-390	30	-60
225(0, 1) =	1	-7	-14	-40	90	50	120	-110	160	-80	70
225(0, 2) =	1	-17	-14	-20	0	20	40	-40	40	180	220
225(0, 3) =	1	-27	-14	-10	30	-150	20	10	110	-270	140
225(0, 4) =	1	-7	-14	-90	-60	20	-160	100	90	150	-30
225(0, 5) =	1	-7	-14	30	0	0	120	120	-90	-90	-30
225(0, 6) =	1	23	-14	-10	30	-250	-180	10	10	130	-260
225(0, 8) =	1	3	-14	20	-30	-90	80	130	-100	-60	-130
225(0, 9) =	1	23	-14	90	30	50	20	10	-90	30	-60
225(0,12) =	1	-2	-14	40	-120	200	-80	-40	160	-120	40
225(1, 2) =	1	-2	1	10	0	20	-40	40	-50	-30	70
225(1, 3) =	1	8	1	40	-20	-120	0	60	80	80	-80
225(1, 4) =	1	13	1	-20	0	-40	80	-80	-200	60	-80
225(1, 5) =	1	8	1	-40	0	80	120	40	40	-80	-80
225(1, 6) =	1	8	1	30	0	-100	20	-20	-90	90	-30
225(1, 7) =	1	28	1	-20	0	-160	-40	40	-80	-180	-80
225(1, 8) =	1	-7	1	40	40	0	-60	-60	20	20	-80
225(1, 9) =	1	-17	1	40	30	110	80	-50	100	0	70
225(1,10) =	1	-22	1	20	-30	110	-120	10	100	-20	70
225(1,11) =	1	-7	1	30	-90	-70	20	-110	90	-30	120
225(1,13) =	1	-7	1	10	40	120	-60	180	-10	-10	70
225(2, 5) =	1	-2	1	-20	0	80	100	-100	-20	0	-80
225(2, 7) =	1	18	1	-40	0	-60	80	40	80	120	20
225(2, 8) =	1	-7	1	-20	-50	90	60	-30	20	-40	70
225(2,10) =	1	13	1	-50	-30	-10	-20	-10	-170	-30	-80
225(2,12) =	1	3	1	-10	30	-90	-220	-110	50	-30	20
225(3, 6) =	1	8	1	40	-20	0	120	60	-40	80	40
225(3, 9) =	1	-17	1	-110	30	50	20	10	10	30	40
225(5,10) =	1	8	1	0	120	60	0	240	60	120	120

Table 14e: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 2 \pmod 5$, $\text{ind}_g 3 \equiv 1 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	23	-44	90	90	30	300	-210	330	270	-60
225(0, 1) =	1	-7	-14	-10	-60	180	200	60	-70	30	170
225(0, 2) =	1	-17	-14	-10	-60	20	40	-20	190	-210	70
225(0, 3) =	1	-7	-14	-110	-110	-70	100	-10	-170	70	-60
225(0, 4) =	1	3	-14	40	40	60	-120	-80	-100	-240	-80
225(0, 5) =	1	-7	-14	0	-30	-90	-120	90	0	120	-30
225(0, 6) =	1	-7	-14	-10	90	30	-100	-210	-70	-30	140
225(0, 8) =	1	-7	-14	-20	10	-70	-200	50	-20	100	-30
225(0, 9) =	1	18	-14	40	40	-120	0	40	80	120	40
225(0,12) =	1	43	-14	-10	90	-70	100	190	-170	-30	-260
225(1, 2) =	1	8	1	10	-30	-90	0	-30	10	10	-80
225(1, 3) =	1	3	1	-40	50	-70	-100	30	-20	-60	70
225(1, 4) =	1	-22	1	10	0	120	-60	-60	190	70	70
225(1, 5) =	1	8	1	20	0	120	-40	120	-100	60	20
225(1, 6) =	1	-12	1	-20	10	-30	-60	-50	140	0	70
225(1, 7) =	1	23	1	-50	-30	-30	-60	-90	-110	130	-80
225(1, 8) =	1	-12	1	20	50	-10	80	-30	-20	60	70
225(1, 9) =	1	-7	1	10	0	0	120	120	10	10	70
225(1,10) =	1	8	1	20	30	-90	80	30	-100	-60	-130
225(1,11) =	1	3	1	-110	10	-30	60	70	50	-90	-80
225(1,13) =	1	18	1	80	-40	-160	-40	0	-80	0	-80
225(2, 5) =	1	-17	1	20	30	170	40	70	40	60	70
225(2, 7) =	1	8	1	-20	-20	80	-80	20	40	-140	120
225(2, 8) =	1	3	1	-10	-10	50	20	90	10	-30	-80
225(2,10) =	1	-2	1	20	0	80	-80	-80	40	60	-80
225(2,12) =	1	8	1	-20	10	-10	160	-190	-80	-20	-30
225(3, 6) =	1	3	1	-10	-10	-70	-100	90	130	-30	40
225(3, 9) =	1	-22	1	40	-60	180	0	-60	-20	-80	40
225(5,10) =	1	23	1	90	-90	-150	-60	-30	-30	-90	120

Table 14f: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\mathrm{ind}_g 2 \equiv 3 \pmod 5$, $\mathrm{ind}_g 3 \equiv 1 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	-7	-44	-30	90	210	180	270	150	-90	540
225(0, 1) =	1	3	-14	20	-60	0	-80	180	160	-60	-80
225(0, 2) =	1	13	-14	-80	-10	10	80	150	-200	0	-130
225(0, 3) =	1	-37	-14	-30	-10	210	-220	-30	150	-90	-60
225(0, 4) =	1	13	-14	30	20	-60	160	-60	-130	270	-130
225(0, 5) =	1	-7	-14	0	30	-90	-120	-90	0	-120	-30
225(0, 6) =	1	-12	-14	20	-60	60	280	-180	100	60	40
225(0, 8) =	1	-7	-14	30	20	-60	-40	-180	-30	-90	-30
225(0, 9) =	1	13	-14	-30	-10	-90	-20	-30	50	-90	140
225(0,12) =	1	13	-14	70	-10	10	-220	-30	-50	210	140
225(1, 2) =	1	-2	1	-40	40	20	-40	0	0	0	20
225(1, 3) =	1	8	1	0	-70	-30	40	-150	-60	0	-30
225(1, 4) =	1	-2	1	20	40	20	140	60	-60	-120	20
225(1, 5) =	1	18	1	60	0	-120	-80	0	40	0	-80
225(1, 6) =	1	-2	1	-30	-10	30	100	-30	-70	-90	20
225(1, 7) =	1	-2	1	20	40	140	-40	-120	120	-60	20
225(1, 8) =	1	-22	1	0	-40	0	40	0	120	0	120
225(1, 9) =	1	-2	1	-10	-50	-70	-100	90	30	-30	20
225(1,10) =	1	-12	1	-10	0	0	-80	0	10	150	70
225(1,11) =	1	-2	1	0	-40	180	-80	0	-160	0	20
225(1,13) =	1	8	1	-60	50	-150	-80	-30	60	60	-30
225(2, 5) =	1	-2	1	70	-10	-50	20	90	10	-30	20
225(2, 7) =	1	38	1	0	-40	-60	80	0	-120	0	-180
225(2, 8) =	1	8	1	30	20	0	-80	60	-90	30	-30
225(2,10) =	1	-2	1	40	80	40	80	0	-20	0	20
225(2,12) =	1	-22	1	-30	-10	210	80	30	150	90	120
225(3, 6) =	1	23	1	-30	-10	-90	-20	-30	-150	-90	-60
225(3, 9) =	1	-2	1	20	40	-40	80	120	0	60	-160
225(5,10) =	1	38	1	-120	0	-240	0	0	60	0	-180

Table 14g: Cyclotomic numbers $(h,k)_{15}$, $p \equiv 11 \pmod{15}$, $\text{ind}_g 2 \equiv 4 \pmod 5$, $\text{ind}_g 3 \equiv 1 \pmod 5$

	p^2	p	1	XU	XV	U^2	UV	UW	V^2	VW	W^2
225(0, 0) =	1	53	-44	30	-270	-210	60	150	-30	-150	-60
225(0, 1) =	1	33	-14	-10	60	-140	-120	100	-250	-50	-130
225(0, 2) =	1	-7	-14	60	-30	90	-240	-30	20	60	-30
225(0, 3) =	1	-2	-14	80	80	40	160	0	-280	0	-160
225(0, 4) =	1	-37	-14	20	50	110	-120	10	300	-20	270
225(0, 5) =	1	-7	-14	-30	0	0	120	-120	-90	90	-30
225(0, 6) =	1	-27	-14	-70	30	190	60	250	170	-50	140
225(0, 8) =	1	-17	-14	-40	20	40	160	-60	20	120	20
225(0, 9) =	1	23	-14	-70	-70	-10	60	-50	-30	-50	-60
225(0,12) =	1	23	-14	30	30	-210	60	-150	170	-150	-60
225(1, 2) =	1	-2	1	30	-30	50	40	-70	-30	-10	20
225(1, 3) =	1	-7	1	-50	40	100	20	20	110	10	-30
225(1, 4) =	1	-2	1	0	-60	-40	-20	80	0	140	20
225(1, 5) =	1	18	1	-40	60	-140	0	-20	-40	40	20
225(1, 6) =	1	-7	1	-10	-40	80	0	40	30	70	-30
225(1, 7) =	1	13	1	30	30	50	220	50	-90	50	20
225(1, 8) =	1	-7	1	-50	-20	-20	20	80	-10	-110	-30
225(1, 9) =	1	13	1	-30	0	-100	-80	-40	-90	-70	-130
225(1,10) =	1	-12	1	20	0	-20	0	-80	80	-140	20
225(1,11) =	1	8	1	80	20	-40	0	-20	60	40	-180
225(1,13) =	1	-7	1	-20	-20	100	20	-160	-40	-80	120
225(2, 5) =	1	8	1	-30	0	-180	-60	60	-10	30	-30
225(2, 7) =	1	13	1	-10	50	70	-140	30	50	90	20
225(2, 8) =	1	8	1	40	10	-110	-40	-70	20	40	-30
225(2,10) =	1	-7	1	30	-30	-30	60	30	50	90	120
225(2,12) =	1	-2	1	50	-10	70	40	150	-10	-150	20
225(3, 6) =	1	8	1	-20	-20	40	-40	-100	20	100	-60
225(3, 9) =	1	-17	1	30	30	-10	-140	50	-30	50	140
225(5,10) =	1	8	1	-60	0	240	-120	-120	-120	-60	120

7. AN APPLICATION: PROOF OF THEOREM 8.

Appealing to the well-known congruence (stated here for GF(q))

$$\text{ind}_\gamma(1 - \gamma^{vf}) \equiv (q-1)/2 + \sum_{u=1}^{e-1} u(u,v)_e \pmod e, \tag{7.1}$$

see for example [9: p. 499], and taking $e = 15$, $q = p^2$, $p \equiv 4 \pmod{15}$, we obtain, as (see (1.6))

$$\tfrac{1}{2}(1 + \sqrt{5}) \equiv -\gamma^{6f}(1 - \gamma^{6f})(1 - \gamma^{3f})^{-1} \pmod p , \tag{7.2}$$

the congruence

$$\text{ind}_\gamma \tfrac{1}{2}(1 + \sqrt{5}) \equiv \sum_{u=1}^{14} u((u,6)_{15} - (u,3)_{15}) \pmod 3. \tag{7.3}$$

Then, making use of Table 11, we obtain

$$\text{ind}_\gamma \tfrac{1}{2}(1 + \sqrt{5}) \equiv (1,3)_{15} + (1,4)_{15} + (1,7)_{15} + (1,10)_{15} + (2,5)_{15} + (2,8)_{15} - (1,6)_{15} - (1,9)_{15} - (1,12)_{15} - (1,13)_{15} - (2,9)_{15} - (2,12)_{15} \pmod 3 . \tag{7.4}$$

Substituting the values given in Tables 13a, 13b, 13c for $(1,3)_{15}, \ldots, (2,12)_{15}$ into (7.4) we obtain

$$\text{ind}_\gamma \tfrac{1}{2}(1 + \sqrt{5}) \equiv \tfrac{1}{3}(T - A - B)U \pmod 3 \tag{7.5}$$

independently of the value of $\text{ind}_g 5 \pmod 3$. Reducing the expressions for $225\{(0,0)_{15} - (0,1)_{15}\}$ given by Tables 13a, 13b, 13c modulo 9, and appealing to (1.3) and (1.4), we obtain

$$T - A - B \equiv 3U^2 \pmod 9. \tag{7.6}$$

Combining (7.5) and (7.6) we get

$$\operatorname{ind}_\gamma \tfrac{1}{2}(1 + \sqrt{5}) \equiv U^3 \equiv U \pmod 3,$$

that is

$$\operatorname{ind}_g \tfrac{1}{2}(1 + \sqrt{5}) \equiv -U \pmod 3,$$

which completes the proof of Theorem 8.

We remark that Theorem 8 can also be proved similarly to the proof of the case $p \equiv 1 \pmod{15}$ [9: pp. 499-501].

8. CONCLUDING REMARKS.

During the preparation of this paper, a number of Eisenstein sums, Jacobi sums, Dickson-Hurwitz sums, cyclotomic numbers, were computed for several values of p. These computations were performed using the facilities of the Computing Laboratory of the Department of Mathematics and Statistics at Carleton University and the numerical values obtained were of considerable use in formulating and checking the results. The help of Dr. B. C. Mortimer is acknowledged.

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