Cylinder Renormalization of Siegel Disks

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We study one of the central open questions in one-dimensional renormalization theory—the conjectural universality of golden-mean Siegel disks. We present an approach to the problem based on cylinder renormalization proposed by the second author. Numerical implementation of this approach relies on the constructive measurable Riemann mapping theorem proved by the first author. Our numerical study yields convincing evidence to support the hyperbolicity conjecture in this setting.

1. INTRODUCTION

One of the central examples of universality in one-dimensional dynamics is provided by Siegel disks of quadratic polynomials. Let us consider, for instance, the mapping

$$P_\theta(z) = z^2 + e^{2\pi i\theta} z,$$

where $\theta = (\sqrt{5}+1)/2$ is the golden mean. By a classical result of Siegel, the dynamics of P_θ is linearizable near the origin. The Siegel disk of P_θ, which we will herein denote by Δ_θ, is the maximal neighborhood of zero in which a conformal change of coordinates reduces P_θ to the form $w \mapsto e^{2\pi i\theta} w$. By results of Douady, Ghys, Herman, and Shishikura, the topological disk Δ_θ extends up to the only critical point of P_θ and is bounded by a Jordan curve.

It has been observed numerically [Manton and Nauenberg 83] that the boundary of Δ_θ is asymptotically self-similar near the critical point. Moreover, the scaling factor is universal in a large class of analytic mappings with a golden-mean Siegel disk. In 1983, Widom [Widom 83] defined a renormalization procedure for P_θ that "blows up" a part of the invariant curve $\partial\Delta_\theta$ near the critical point, and conjectured that the renormalizations of P_θ converge to a fixed point. In addition, he conjectured that in a suitable functional space this fixed point is hyperbolic with one-dimensional unstable direction.

In 1986, MacKay and Persival [MacKay and Persival 87] extended the conjecture to other rotation numbers, postulating the existence of a hyperbolic renormalization

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horseshoe corresponding to Siegel disks of analytic maps, analogous to Lanford's horseshoe for critical circle maps [Lanford 87, Lanford 88].

In 1994, Stirnemann [Stirnemann 94] gave a computer-assisted proof of the existence of a renormalization fixed point with a golden-mean Siegel disk. In 1998, McMullen [McMullen 98] proved the asymptotic self-similarity of golden-mean Siegel disks in the quadratic family. He constructed a version of renormalization based on holomorphic commuting pairs of de Faria [de Faria 92, de Faria 99], and showed that the renormalizations of a quadratic polynomial with a golden Siegel disk near the critical point converge to a fixed point geometrically fast. More generally, he constructed a renormalization horseshoe for bounded-type rotation numbers, and used renormalization to show that the Hausdorff dimension of the corresponding quadratic Julia sets is strictly less than two.

Having thus attracted much attention, the hyperbolicity part of the conjecture of Widom for golden-mean Siegel disks is still open.

In [Yampolsky 02], the second author introduced a new renormalization transformation $\mathcal{R}_{\rm cyl}$, which he called the *cylinder renormalization*, and used it to prove Lanford's hyperbolicity conjecture for critical circle maps. The main advantage of $\mathcal{R}_{\rm cyl}$ over the renormalization scheme based on commuting pairs is that this operator is analytic in a Banach manifold of analytic maps of a subdomain of $\mathbb{C}/\mathbb{Z}$. It is thus a natural setting in which to study the hyperbolic properties of a fixed point. In the present paper we study the fixed point of the cylinder renormalization numerically, and empirically confirm the hyperbolicity conjecture, as well as study the dynamical properties of the fixed point.

The main numerical challenge in working with cylinder renormalization is a change of coordinates involved in its definition. It is defined implicitly, and uniformizes a dynamically defined fundamental domain to the straight cylinder $\mathbb{C}/\mathbb{Z}$. To handle it, we use the constructive measurable Riemann mapping theorem developed in [Gaidashev 07, Gaidashev and Khmelev 06] by the first author for numerically solving the Beltrami partial differential equation.

2. DEFINITION AND MAIN PROPERTIES

2.1 Some Functional Spaces

For a topological disk $W \subset \mathbb{C}$ containing 0 and 1 we will let $\mathbf{A}_W$ denote the Banach space of bounded analytic functions in W equipped with the sup norm. Let $\mathbf{C}_W$ denote the Banach subspace of $\mathbf{A}_W$ consisting of analytic mappings $h : W \to \mathbb{C}$ such that $h(0) = 0$ and $h'(1) = 0$.

In the case that the domain W is the disk $\mathbb{D}_\rho$ of radius $\rho > 1$ centered at the origin, we will set $\mathbf{A}_{\mathbb{D}_\rho} \equiv \mathbf{A}_\rho$ and $\mathbf{C}_{\mathbb{D}_\rho} \equiv \mathbf{C}_\rho$.

For each $\rho > 1$ we will also consider the collection $\mathbf{B}^1_\rho$ of analytic functions $f(z)$ defined on some neighborhood of the origin with $f(0) = 0$, equipped with the weighted l_1 norm on the coefficients of the Maclaurin series:

$$\|f\|_\rho = \sum_{n=0}^{\infty} \frac{\left|f^{(n)}(0)\right|}{n!} \rho^n.$$

We will further let $\mathbf{L}^1_\rho$ denote the subset of $\mathbf{B}^1_\rho$ consisting of maps f with the normalizing condition $f'(1) = 0$.

The proof of the following elementary statement is left to the reader:

Lemma 2.1.

(1) *Let* $f \in \mathbf{L}^1_\rho$. *Then* $\sup_{\mathbb{D}_\rho} |f(z)| \le \|f\|_\rho$*;*

(2) *Let* $f \in \mathbf{A}_{\rho'}$ *and* $\rho' > \rho$. *Then* $\|f\|_\rho \le \frac{\rho}{\rho' - \rho} \sup_{\mathbb{D}_{\rho'}} |f(z)|$.

The following corollary is an immediate consequence:

Corollary 2.2. $\mathbf{L}^1_\rho$ *is a Banach space.*

2.2 The Cylinder Renormalization Operator

The cylinder renormalization operator is defined as follows. Let $f \in \mathbf{C}_W$. Suppose that for $n \in \mathbb{N}$ there exists a simple arc l that connects a fixed point a of f^n to 0, and has the property that $f^n(l)$ is again a simple arc whose only intersection with l is at the two endpoints. Let C_f be the topological disk in $\mathbb{C} \setminus \{0\}$ bounded by l and $f^n(l)$. We say that C_f is a *fundamental crescent* if the iterate $f^{-n}|_{C_f}$ mapping $f^n(l)$ to l is defined and univalent, and the quotient of $\overline{C_f \cup f^{-n}(C_f)} \setminus \{0, a\}$ by the iterate f^n is conformally isomorphic to $\mathbb{C}/\mathbb{Z}$.

Let R_f denote the first return map of C_f, and let z denote the critical point of this map (corresponding to the orbit of 0). Let g be the map that R_f becomes under the above isomorphism, mapping z to 0, and $h = e \circ g \circ e^{-1}$, where $e(z) = \exp[-2\pi i z]$. We say that f is *cylinder renormalizable with period* n if $h \in \mathbf{C}_V$ for some V, and call h a *cylinder renormalization* of f (see Figure 1).

We summarize below the basic properties of cylinder renormalization proven in [Yampolsky 02].

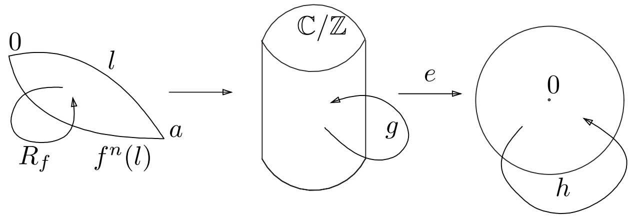

FIGURE 1. Schematics of cylinder renormalization.

Proposition 2.3. *Suppose $f \in \mathbf{C}_W$ is cylinder renormalizable and that its renormalization h_f is contained in $\mathbf{C}_V$. Let C_f denote the fundamental crescent corresponding to the renormalization. Then the following hold.*

- *Every other fundamental crescent C'_f with the same endpoints as C_f and such that $C'_f \cup C_f$ is a topological disk produces the same renormalized map h_f.*
- *There exists an open neighborhood $U(f) \subset \mathbf{C}_W$ such that every map $g \in U(f)$ is cylinder renormalizable, with a fundamental crescent C_g that can be chosen to move continuously with g.*
- *Moreover, the dependence $g \mapsto h_g$ of the cylinder renormalization on the map g is an analytic mapping $\mathbf{C}_W \to \mathbf{C}_V$.*

We now want to discuss the dynamical properties of the cylinder renormalization of maps with Siegel disks derived in [Yampolsky 06]. To simplify the exposition let us specialize to the case in which the rotation number of the Siegel disk is the golden mean $\theta = (\sqrt{5}+1)/2$. The golden mean is represented by an infinite continued fraction,

$$\theta = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \equiv 1 + [1, 1, 1, \dots].$$

As is customary, we will denote by p_n/q_n the nth convergent

$$p_n/q_n = \underbrace{[1, 1, 1, \dots, 1]}_{n}.$$

Theorem 2.4. [Yampolsky 06] *There exists a space $\mathbf{C}_U$ and an analytic mapping $\hat{f} \in \mathbf{C}_U$ that has a Siegel disk Δ_θ with rotation number θ whose boundary is a quasicircle passing through the critical point 1 such that the following hold:*

(i) *There exists a branch of cylinder renormalization with period q_k, $k \in \mathbb{N}$, which we denote by $\mathcal{R}_{\rm cyl}$, such that*
$$\mathcal{R}_{\rm cyl}\hat{f} = \hat{f}.$$

(ii) *The quadratic polynomial $P_\theta(z) = e^{2\pi i\theta}z + z^2$ is infinitely cylinder renormalizable, and*
$$\mathcal{R}^k_{\rm cyl}P_\theta \to \hat{f},$$
at a uniform geometric rate.

(iii) *The cylinder renormalization $\mathcal{R}_{\rm cyl}$ is an analytic and compact operator mapping a neighborhood of the fixed point $\hat{f}$ in $\mathbf{C}_U$ to $\mathbf{C}_U$. Its linearization $\mathcal{L}$ at $\hat{f}$ is a compact operator, with at least one eigenvalue with absolute value greater than one.*

A central open question in the study of $\mathcal{R}_{\rm cyl}$ is contained in the following conjecture:

Conjecture 2.5. *Except for the one unstable eigenvalue, the rest of the spectrum of $\mathcal{L}$ is compactly contained in the unit disk.*

Our numerical study of $\mathcal{R}_{\rm cyl}$ will begin with establishing empirically the convergence to $\hat{f}$. We will then make explicit the choice of the neighborhood U in the above theorem. Experimental evidence suggests that it can be taken as a round disk $\mathbb{D}_\rho$ for some particular value of ρ. Having numerically established this, we will then proceed to verify the conjecture experimentally.

3. CONSTRUCTION OF THE CONFORMAL ISOMORPHISM TO THE CYLINDER

The principal difficulty in both numerical and analytic study of cylinder renormalization is the inexplicit nature of the conformal isomorphism

$$\Phi : \overline{C_f \cup f^{-n}(C_f)} \setminus \{0, a\} \underset{\approx}{\longrightarrow} \mathbb{C}^*$$

of a fundamental crescent, which is a part of the definition of $\mathcal{R}_{\rm cyl}$. An analytic approach to this construction based on the measurable Riemann mapping theorem was presented by the first author in [Gaidashev 07]. It has its roots in the complex-dynamical folklore; similar arguments are found, for instance, in the work of Lyubich [Lyubich 86] and Shishikura [Shishikura 42].

In [Gaidashev 07], the first author demonstrates how this approach can be implemented constructively, with rigorous error bounds. We will give a brief outline here.

3.1 Uniformization of the Cylinder Using the Measurable Riemann Mapping Theorem

We will start with a description of our choice of a fundamental crescent C_f^n with period q_n for a map $f \in \mathbf{C}_U$ sufficiently close to $\hat{f}$.

To construct the boundary curve l_n of C_f^n, consider first the union $\tilde{l}_n$ of two parabolas $x + i(Ax^2 + Bx)$ and $(Cy^2 + Dy + E) + iy$, the first passing through points 0 and $f^{q_{n+2}+q_n}(1)$, the second through $f^{q_{n+2}+q_n}(1)$ and a repelling fixed point a_{q_n}. All parameters in these two parabolas are defined uniquely after one specifies their common tangent line at $f^{q_{n+2}+q_n}(1)$ (see equations (3–2) below). While somewhat arbitrary, this choice has the virtue of possessing a simple analytic form. It can be shown rigorously (see [Gaidashev 07]) that by modifying $\tilde{l}_n$ in sufficiently small neighborhoods of the endpoints (small enough not to influence our numerical experiments) we obtain a curve l_n that together with $f^{-q_n}(l_n)$ bounds a fundamental crescent C_f^n for $\mathcal{R}_{\rm cyl}$.

Now consider the following conformal change of coordinates for $z \in C_f^n$:

$$z = \tau(\xi) = \frac{a_{q_n}}{1 - e^{i\alpha\xi + \beta}},$$
$$\tau^{-1}(z) = \frac{1}{ia}\left[\ln\left(1 - \frac{a_{q_n}}{z}\right) - \beta\right].$$

The normalizing constant β will be chosen such that

$$\tau^{-1}(f^{q_{n+2}}(1)) = 0,$$

while a real positive α will be specified by the condition

$$|\tau^{-1}(f^{q_{n+2}+q_n}(1))| = 1.$$

The choice of this coordinate is motivated by the fact that τ^{-1} maps the interior of the fundamental crescent C_f^n conformally onto the interior of an infinite vertical closed strip $\mathcal{S}$, whose width is comparable to 1 (see Figure 2).

Next, similarly to [Shishikura 42], define a function

$$\tilde{g}_n : \mathcal{U} \equiv \{\, u + iv \in \mathbb{C} : 0 \le \operatorname{Re} w \le 1 \,\} \longrightarrow \mathcal{S}$$

by setting

$$\tilde{g}_n(u + iv) = (1 - u)\tau^{-1}(f^{-q_n}(\gamma_n(v))) + u\tau^{-1}(\gamma_n(v)),$$

where γ_n is a parameterization

$$\gamma_n : \mathbb{R} \to l_n$$

that we will specify below.

Let σ_0 be the standard conformal structure on $\mathbb{C}$, and let $\sigma = \tilde{g}_n^* \sigma_0$ be its pullback on $\mathcal{U}$. Extend this conformal structure to $\mathbb{C}$ through $\sigma \equiv (T^k)^*\sigma$ on $T^{-k}(\mathcal{U})$, where $T(w) = w + 1$ for all $k \in \mathbb{N}$.

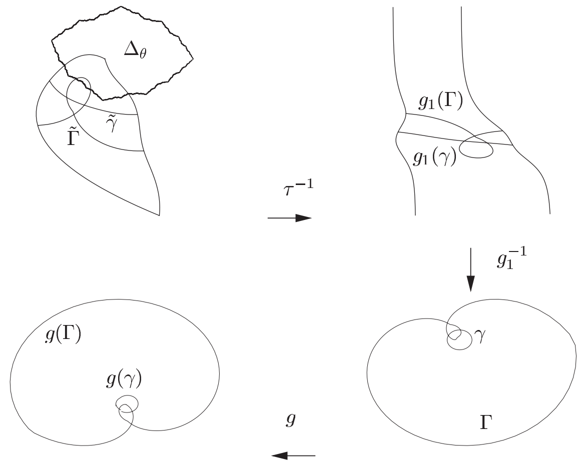

FIGURE 2. Schematics of renormalization. The contours $g(\gamma) \mapsto g(\Gamma)$ are used to find a polynomial approximation of $\mathcal{R}_{\rm cyl} f$ through the Cauchy integral formula.

Assuming that the mapping $\tilde{g}_n$ is quasiconformal, the dilatation of σ is bounded in the plane. By the measurable Riemann mapping theorem (see, for example, [Ahlfors and Bers 60]), there exists a unique quasiconformal mapping $\tilde{g} : \mathbb{C} \to \mathbb{C}$ such that $\tilde{g}^*\sigma_0 = \sigma$, normalized so that $\tilde{g}(0) = 0$ and $\tilde{g}(1) = 1$. Notice that $\tilde{g} \circ T \circ \tilde{g}^{-1}$ preserves the standard conformal structure:

$$\left(\tilde{g} \circ T \circ \tilde{g}^{-1}\right)^* \sigma_0 = (\tilde{g}^{-1})^* \circ T^* \circ \tilde{g}^*\sigma_0 = (\tilde{g}^{-1})^* \circ T^*\sigma$$
$$= (\tilde{g}^{-1})^*\sigma = (\tilde{g}^*)^{-1}\sigma = \sigma_0,$$

and therefore it is a conformal automorphism of $\mathbb{C}$. Liouville's theorem implies that this mapping is affine. By construction, it does not have any fixed points in $\mathbb{C}$, and hence is a translation. Finally, $\tilde{g} \circ T \circ \tilde{g}^{-1}(0) = 1$, and thus

$$\tilde{g} \circ T \circ \tilde{g}^{-1} \equiv T.$$

By the definition of $\tilde{g}_n$,

$$\tilde{g}_n^{-1} \circ \tau^{-1} \circ f^{q_n} \circ \tau = T \circ \tilde{g}_n^{-1}$$

on the image of $f^{-q_n}(l_n)$ by τ^{-1}. Set $\phi = \tilde{g} \circ \tilde{g}_n^{-1}$ and $\tilde{\Phi} \equiv \phi \circ \tau^{-1}$. Clearly, $\tilde{\Phi} \bmod \mathbb{Z}$ is a desired conformal isomorphism

$$\overline{C_f^n \cup f^{-q_n}(C_f^n)} \setminus \{0, a_{q_n}\} \underset{\approx}{\longrightarrow} \mathbb{C}/\mathbb{Z}.$$

Again, set $e(z) = e^{-2\pi i z}$ and $g = e \circ \tilde{g} \circ e^{-1}$. Since

$$\frac{g_{\bar{z}}(e(w))}{g_z(e(w))} = \frac{e(w)}{\overline{e(w)}} \frac{\tilde{g}_{\bar{w}}(w)}{\tilde{g}_w(w)},$$

the 1-periodic function $\tilde{g}$ is a solution of the Beltrami equation

$$\tilde{g}_{\bar{w}} = \tilde{\mu}\tilde{g}_w, \quad \tilde{\mu} = (\tilde{g}_n)_{\bar{w}}/(\tilde{g}_n)_w$$

whenever g is a solution of

$$g_{\bar{z}} = \mu g_z, \quad \mu(z) = (z/\bar{z})\tilde{\mu}(e^{-1}(z)).$$

Thus we have reduced the problem of finding

$$\Phi \equiv e \circ \tilde{\Phi} = g \circ e \circ \tilde{g}_n^{-1} \circ \tau^{-1}$$

to that of finding the properly normalized solution of the Beltrami equation

$$g_{\bar{z}} = \mu g_z, \quad \mu(z) = \frac{z}{\bar{z}} \frac{(\tilde{g}_n)_{\bar{w}}(e^{-1}(z))}{(\tilde{g}_n)_w(e^{-1}(z))} \tag{3–1}$$

on the punctured plane $\mathbb{C}^*$.

It remains to describe the choice of the parameterization of l_n in the definition of $\tilde{g}_n$. It is convenient for us to parameterize l_n using the radial coordinate in $\mathbb{C}$. For $n = 1$ and $f \in \mathbf{C}_U$ sufficiently close the empirical fixed point of the cylinder renormalization with period 1 we use the following parameterization:

$$\lambda_1(r) = \begin{cases} (x(r), Ax(r)^2 + Bx(r)), & r \leq \tilde{r}, \\ (Cy(r)^2 + Dy(r) + E, y(r)), & r > \tilde{r}, \end{cases} \tag{3–2}$$

where

$$\begin{aligned} x(r) &= \frac{\operatorname{Re} f^4(1)}{|f^4(1)|} T(r), \\ y(r) &= \operatorname{Im} f^4(1) \frac{|a_1 - f^4(1)| + |f^4(1)| - T(r)}{|a_1 - f^4(1)|} \\ &\quad + \operatorname{Im} a_1 \frac{T(r) - |f^4(1)|)}{|a_1 - f^4(1)|}, \\ T(r) &= \frac{|a_1 - f^4(1)| + |f^4(1)|}{\sqrt{r} + 1} \sqrt{r}, \end{aligned}$$

and $\tilde{r}$ is defined by $T(\tilde{r}) = |f^4(1)|$. The constants A, B, C, D, and E are fixed by the conditions $0, f^4(1), a_1 \in l_1$, together with the requirement that the slope of the common tangent line to both parabolas at the point $f^4(1)$ be equal to 1.1.

This particular choice of the parameterization is motivated by the speed of convergence of the iterative scheme in the measurable Riemann mapping theorem.

We define the following function on $\mathbb{C}^*$:

$$\begin{aligned} g_1(r, \phi) = &\left(\eta(-\phi) + \frac{\phi}{2\pi}\right) \tau^{-1}(f^{-1}(\lambda_1(r))) \\ &+ \left(1 - \eta(-\phi) - \frac{\phi}{2\pi}\right) \tau^{-1}(\lambda_1(r)), \end{aligned} \tag{3–3}$$

where $-\pi < \phi \leq \pi$ and η is the Heaviside step function (we have adopted the convention $\eta(0) = 1$). Then, according to (3–1), the Beltrami differential μ is given by

$$\mu(re^{i\phi}) = e^{2i\phi} \frac{r\partial_r g_1(r, \phi) + i\partial_\phi g_1(r, \phi)}{r\partial_r g_1(r, \phi) - i\partial_\phi g_1(r, \phi)} \tag{3–4}$$

on $\mathbb{C} \setminus (-\infty, 0]$. This is the expression that we have used to compute the Beltrami differential in our numerical studies.

3.2 A Constructive Measurable Riemann Mapping Theorem

To solve the Beltrami equation numerically, we use the constructive measurable Riemann mapping theorem (MRMT) proved by the first author in [Gaidashev 07]. Before formulating it, we need to recall two integral operators used in the classical approach to the proof of MRMT (see [Ahlfors and Bers 60]). The first is the Hilbert transform

$$T[h](z) = \frac{i}{2\pi} \lim_{\epsilon \to 0} \iint_{\mathbb{C} \setminus B(z,\epsilon)} \frac{h(\xi)}{(\xi - z)^2} \, d\bar{\xi} \wedge d\xi. \tag{3–5}$$

The second is the Cauchy transform

$$P[h](z) = \frac{i}{2\pi} \iint_{\mathbb{C}} \frac{h(\xi)}{(\xi - z)} \, d\bar{\xi} \wedge d\xi. \tag{3–6}$$

The Hilbert transform is a well-defined bounded operator on $L_p(\mathbb{C})$ for all $2 < p < \infty$. For every such p there exists a constant c_p such that the following holds (cf. [Calderón and Zygmund 56]): $\|T[h]\|_p \leq c_p \|h\|_p$ for any $h \in L_p(\mathbb{C})$, and $c_p \to 1$ as $p \to 2$.

We are now ready to state the constructive measurable Riemann mapping theorem of [Gaidashev 07]:

Theorem 3.1. *Let $\mu \in L_\infty(\bar{\mathbb{C}})$ and let an integer $p > 2$ be such that $\|\mu\|_\infty \leq K < 1$ and $Kc_p < 1$, where*

$$c_p = \cot^2(\pi/2p).$$

Assume that $\mu = \nu + \eta + \gamma$, where ν and η are compactly supported in $\mathbb{D}_R$, and $\gamma(z)$ is supported in $\bar{\mathbb{C}} \setminus \mathbb{D}_R$. Furthermore, let η be in $L_p(\mathbb{D}_R)$ and $\|\eta\|_p < \delta$ for some sufficiently small δ. Also, let $h^* \in L_p(\mathbb{C})$ and let ϵ be such that $B_p(h^*, \epsilon)$, the ball of radius ϵ around h^* in $L_p(\mathbb{C})$, contains $B_p(T[\nu(h^* + 1)], c_p\epsilon')$, with

$$\epsilon' = \delta \ \operatorname{essup}_{\mathbb{D}_R} |h^* + 1| + K\epsilon.$$

Then the solution g^μ of the Beltrami equation $g^\mu_{\bar{z}} = \mu g^\mu_z$ admits the following bound:

$$|g^\mu(z) - g^\nu_*(z)| \leq F(\epsilon', R; z, g^\nu_*(z), p, K, c_p), \tag{3–7}$$

where $g_*^\nu(z) = P[\nu(h^*+1)](z)+z$ *and* $F(\epsilon', R) = O(\epsilon', R^{-4/p})$ *is an explicit function of its arguments.*

Given the theorem, the algorithm for producing an approximate solution of the Beltrami equation is as follows. Given a μ as in the condition of the theorem, we first iterate

$$h \to T[\nu(h+1)] \tag{3–8}$$

to find a numerical approximation h_a^* to the solution of the equation $T[\mu(h^*+1)] = h^*$. After that, we compute an approximate solution as

$$g_a^\nu(z) = P[\nu(h_a^*+1)](z)+z. \tag{3–9}$$

One can obtain rigorous computer-assisted bounds on such solutions using Theorem 3.1. Such bounds have indeed been implemented in [Gaidashev 07] for a particular case of the golden-mean quadratic polynomial. However, in the present numerical work we will not require such estimates.

In the appendix, Section 7, we will discuss several numerical algorithms for the two integral transforms appearing in this scheme.

4. EMPIRICAL CONVERGENCE TO A FIXED POINT

An appropriate choice of the domains of analyticity for the renormalized functions is central to a successful numerical implementation of cylinder renormalization. Our numerical approximation to the renormalization fixed point is a finite-degree truncation of a function analytic in $\mathbb{D}_3$ (see Section 5 for a detailed explanation of this choice of the domain). However, for the purposes of obtaining bounds on higher-order terms, we will consider a smaller domain of analyticity, a disk of radius $\rho = 2.266$. Thus the cylinder renormalization will be a priori an analytic operator in a neighborhood of the fixed point in the Banach space $\mathbf{L}^1_\rho$ with $\rho = 2.266$.[1]

Given an $f \in \mathbf{L}^1_\rho$, a numerical approximation to its cylinder renormalization of order n is built as follows. As the first step, we construct a fundamental crescent C_f^n as described above, and find the normalized solution g of the Beltrami equation

$$g_{\bar z}(z) = \mu(z) g_z(z),$$

with μ as in (3–4), as described in Sections 3 and 7.

Next, we choose a contour γ in the domain of g, and map this contour into the fundamental crescent by $\tau \circ g_1$:

$$\tilde\gamma \equiv \tau \circ g_1(\gamma).$$

Applying the first return map to the points of this contour, we obtain $\tilde\Gamma = R_f(\tilde\gamma)$, and obtain the images $g(\gamma)$ and $g(\Gamma)$ (where $\Gamma = g_1^{-1}(\tau^{-1}(\tilde\Gamma))$). The coefficients in a finite-order polynomial approximation to

$$\mathcal{R}_{\rm cyl} f = g \circ g_1^{-1} \circ \tau^{-1} \circ R_f \circ \tau \circ g_1 \circ g^{-1}$$

are then found via the Cauchy integral formula using the two contours $g(\gamma)$ and $g(\Gamma)$ (see Figure 2).

As seen in Theorem 2.4, the sequence of the cylinder renormalizations of the quadratic polynomial P_θ converges to a fixed point $\hat f = \mathcal{R}_{\rm cyl}\hat f$. We have used this fact to compute an approximate renormalization fixed point $\hat f_a$ as the cylinder renormalization $\mathcal{R}^k_{\rm cyl} P_\theta$ of order $k = 11$.

Further, we improved this approximation by iterating

$$\hat f_a \mapsto \mathbb{P}_s \circ \mathcal{R}_{\rm cyl} \hat f_a, \tag{4–1}$$

where $\mathbb{P}_s$ is the projection on the candidate stable manifold of $\hat f$,

$$\mathrm{W}^s = \{f \in \mathbf{L}^1_\rho : f'(0) = e^{2\pi\theta i}\},$$

defined by setting

$$\mathbb{P}_s[f](x) \equiv f(x) + (e^{2\pi\theta i} - f'(0))x.$$

In this way we have obtained a polynomial $\hat f_a$ of degree 17, and have obtained the estimate, not taking into account the errors in the solution of the Beltrami equation and those due to round-off,

$$\|\mathcal{R}_{\rm cyl}\hat f_a - \hat f_a\|_\rho \le 1.88\times 10^{-3} \approx 0.89\times 10^{-4}\|\hat f_a\|_\rho. \tag{4–2}$$

Moreover, the iteration (4–1) does not lead to a significant variation in the computed values for the coefficients of $\hat f_a$, which indicates that the original approximation is indeed quite accurate. The largest change is in the highest coefficient, which differs by 0.4% for $\hat f_a$ and its renormalization. Of course, this represents a negligible correction to the absolute value of the coefficient itself.

[1] We have implemented the procedure for the cylinder renormalization described in Section 3 and a particular method of solving the Beltrami equation (see Section 7) as a set of routines in the programming language Ada 95 (see [Taft and Duff 95] for the language standard). We have parallelized our programs and compiled them with the public version 3.15p of the GNAT compiler. The programs [Yampolsky 07] have been run on the computational cluster of 92 2.2-GHz AMD Opteron processors located at the University of Texas at Austin.

The approximate expression for $\hat{f}_a$ is as follows (all numbers truncated to show six significant digits):

$$\begin{aligned}\hat{f}_a(x) = {} & xe^{2\pi i\theta} + x^2\left(8.00882\times10^{-1} + i4.07682\times10^{-1}\right)\\ & + x^3\left(-4.12708\times10^{-1} + i2.97670\times10^{-2}\right)\\ & + x^4\left(1.02033\times10^{-1} - i9.83702\times10^{-2}\right)\\ & + x^5\left(2.61573\times10^{-5} + i4.13871\times10^{-2}\right)\\ & + x^6\left(-8.42868\times10^{-3} - i6.96474\times10^{-3}\right)\\ & + x^7\left(2.60095\times10^{-3} - i6.58544\times10^{-4}\right)\\ & + x^8\left(-2.01382\times10^{-4} + i5.95113\times10^{-4}\right)\\ & + x^9\left(-9.40057\times10^{-5} - i1.11237\times10^{-4}\right)\\ & + x^{10}\left(3.21762\times10^{-5} - i4.40144\times10^{-6}\right)\\ & + \cdots .\end{aligned}$$

5. DOMAIN OF ANALYTICITY OF THE RENORMALIZATION FIXED POINT

5.1 Compactness of $\mathcal{R}_{\text{cyl}}$

We have verified experimentally the compactness property of $\mathcal{R}_{\text{cyl}}$ stated in Theorem 2.4. More precisely, we observe the following empirical fact:

> Set $\rho = 2.266$ and $\rho' = 3$. Then we can take $U \equiv \mathbb{D}_\rho$ in Theorem 2.4. More specifically, the fixed point $\hat{f}$ is a well-defined analytic mapping in $\mathbf{C}_\rho$, and moreover, if we set $g \equiv \hat{f}|_{\mathbb{D}_\rho}$, then $\mathcal{R}_{\text{cyl}}\, g \in \mathbf{C}_{\rho'}$.

To verify the claim numerically, we have used the approximation $\hat{f}_a$ obtained in the previous section. To estimate ρ', we have chosen a curve $\tilde{\gamma}$ in the fundamental crescent such that $\Phi_{\hat{f}_a}(\tilde{\gamma})$ is a simple closed loop that encircles $\mathbb{D}_3$ (Figure 3). We then verify that the orbit under

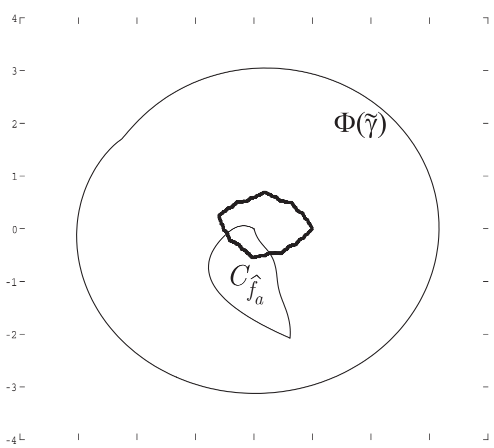

FIGURE 3. The fundamental domain $C_{\hat{f}_a}$ together with $\Phi(\tilde{\gamma})$.

the return map of the component $C^0_{\hat{f}_a}$ of the set $C_{\hat{f}_a} \setminus \tilde{\gamma}$ such that $0 \in \partial C^0_{\hat{f}_a}$ lies within $\mathbb{D}_{2.266}$ (see Figure 4).

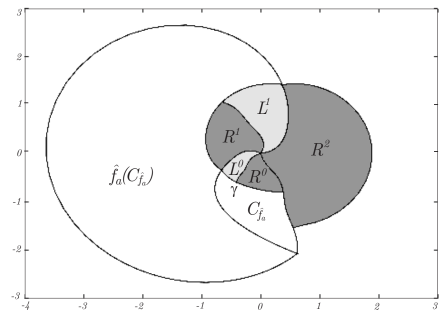

FIGURE 4. The orbit of $C^0_{\hat{f}_a}$: the orbit of $L^0_{\hat{f}_a}$, $L^k \equiv \hat{f}^k_a(L^0_{\hat{f}_a})$, is rendered in light gray, that of $R^0_{\hat{f}_a}$, $R^k \equiv \hat{f}^k_a(R^0_{\hat{f}_a})$ in dark gray.

For our choice of the curve $\tilde{\gamma}$, the return map of the set

$$C^0_{\hat{f}_a} = L^0_{\hat{f}_a} \cup R^0_{\hat{f}_a}$$

is given by the second and third iterates of $\hat{f}_a$ on $L^0_{\hat{f}_a}$ and $R^0_{\hat{f}_a}$, respectively.

6. HYPERBOLIC PROPERTIES OF CYLINDER RENORMALIZATION

6.1 The Expanding Direction of $\mathcal{R}_{\text{cyl}}$

It is not difficult to see that the operator $\mathcal{R}_{\text{cyl}}$ possesses an expanding direction at $\hat{f}$ (cf. [Yampolsky 06]):

Proof of Theorem 2.4(iii)*:* Let $v(z)$ be a vector field in $\mathbf{C}_U$,

$$v(z) = v'(0)z + o(z).$$

Denote by γ_v the quantity

$$\gamma_v = \frac{v'(0)}{\hat{f}'(0)} = e^{-2\pi i\theta}v'(0).$$

For a smooth family

$$\hat{f}_t(z) = \hat{f}(z) + tv(z) + o(t),$$

we have

$$\hat{f}_t(z) = \alpha^v_t(z)(\hat{f}'(0)z + o(z)),$$

where $\alpha_t(0) = 1 + t\gamma_v + o(t)$. The q_{m+1}st iterate is given by

$$\hat{f}_t^{q_{m+1}}(z) = (\alpha^v_t(z))^{q_{m+1}}((\hat{f}'(0))^{q_{m+1}}z + o(z)).$$

In a neighborhood of 0, the renormalized vector field $\mathcal{L}v$ is obtained by applying a uniformizing coordinate

$$\Phi(z) = (z + o(z))^{\beta}, \quad \text{where } \beta = \frac{1}{\theta q_m \bmod 1}.$$

Hence

$$\alpha_t^{\mathcal{L}v}(0) = [(\alpha_t^v(0))^{q_{m+1}}]^{\beta},$$

so

$$\gamma_{\mathcal{L}v} = \Lambda \gamma_v, \quad \text{where } \Lambda = \beta q_{m+1} > 1.$$

Hence the spectral radius $R_{\mathrm{Sp}}(\mathcal{L}v)$ is greater than 1, and since every nonzero element of the spectrum of a compact operator is an eigenvalue, the claim follows. $\square$

6.2 Numerical Verification of Hyperbolicity of $\mathcal{R}_{\mathrm{cyl}}$

It is natural to make the following conjecture:

Conjecture 6.1. *There exists an open neighborhood $\mathcal{U} \subset \mathbf{C}_U$ containing $\hat{f}$ such that $\mathcal{R}_{\mathrm{cyl}}$ is a strong contraction in*

$$W = \{f \in \mathcal{U} \mid f'(0) = e^{2\pi i \theta}\}.$$

Thus, $W = W^s_{\mathrm{loc}}(\hat{f})$.

To verify this conjecture numerically, we have to justify using a finite-dimensional approximation to $\mathcal{L}$ to test for contraction. For this we rely on a numerical observation discussed in the previous section:

$$\mathcal{L} : \mathbf{L}^1_{\rho} \to \mathbf{L}^1_{\rho'}, \text{ with } \rho = 2.266 \text{ and } \rho' = 3.$$

This implies that the finite-dimensional approximations of $\mathcal{L}$ obtained by truncating all powers higher than z^N will converge geometrically fast in N.

Set h_j to be the coordinate vectors $h_j(z) = z^j/\rho^j$, so that

$$\|\mathcal{L}\|_{\rho} = \sup \|\mathcal{L}h_j\|_{\rho}.$$

Since a perturbation $\hat{f}+\epsilon h_j$ does not lie in $\mathbf{L}^1_{\rho}$, we perturb along a different set of vectors:

$$e_j = \frac{g_j}{\|g_j\|_{\rho}}, \quad g_j(z) = z^j - \frac{j}{j+1} z^{j+1}, \quad j \geq 1,$$

which form a basis in $\mathbf{L}^1_{\rho}$.

Numerically, to estimate the spectral radius

$$R_{\mathrm{Sp}}\left(\mathcal{L}|_{T_{\hat{f}}W}\right),$$

we can fix a large enough N and a small ϵ (we have used the value $\epsilon = 0.01$); compute for each e_j, $2 \leq j \leq N$, the finite difference

$$\frac{1}{\epsilon}(\mathcal{R}_{\mathrm{cyl}}(\hat{f}_a + \epsilon e_j) - \mathcal{R}_{\mathrm{cyl}}\hat{f}_a);$$

0.45879 $-0.97624i$	0.68789 $-0.46254i$	0.11338 $-0.09738i$	0.13041 $+0.07490i$	0.15824 $+0.11616i$
-0.13666 $+1.72834i$	-0.64474 $+0.54306i$	0.33937 $-0.50837i$	0.14710 $-0.13849i$	-0.21006 $+0.02552i$
-0.90634 $-1.37322i$	0.27155 $-0.38270i$	-0.05765 $+1.09081i$	-0.22948 $+0.14700i$	0.18338 $-0.39078i$
1.23970 $+0.20634i$	-0.08549 $+0.23219i$	-0.68227 $-0.81153i$	0.12817 $-0.14685i$	0.10981 $+0.47861i$
-0.63443 $+0.58168i$	-0.02489 $-0.18893i$	0.80205 $+0.00357i$	-0.04014 $+0.12864i$	-0.34685 $-0.19936i$

TABLE 1. The matrix A_6.

truncate past the Nth power; and expand over the basis vectors e_j to obtain an $(N-1) \times (N-1)$ matrix A_N. In Table 1, we present the approximate expression for A_6 (the numbers have been truncated to the fifth decimal place).

This matrix has spectral radius

$$R_{\mathrm{Sp}}(A_6) \approx 0.53.$$

6.3 Estimating the Spectral Radius

We now proceed to produce a justification for the above numerical experiment. We will equip $\mathbf{L}^1_{\rho}$, viewed as a vector space, with a new l_1-norm

$$|f|_{\rho} = \sum_{k=1}^{\infty} |f_k|,$$

where f_k are the coefficients in the expansion of f in the basis $\{e_j\}$, $f = \sum_{k=1}^{\infty} f_k e_k$, and denote the new Banach space by $\tilde{\mathbf{L}}^1_{\rho}$. The projection $\mathbb{P}_{\leq N}$ on $\mathrm{span}_{1 \leq j \leq N}\{e_j\}$ will be defined by setting

$$\mathbb{P}_{\leq N} f = \sum_{j=1}^{N} f_j e_j. \tag{6–1}$$

We will also abbreviate $\mathbb{I} - \mathbb{P}_{\leq N}$ as $\mathbb{P}_{>N}$.

We would like to emphasize that $\mathbb{P}_{\leq N} f \in \tilde{\mathbf{L}}^1_{\rho}$ whenever $f \in \tilde{\mathbf{L}}^1_{\rho}$, and therefore the operator

$$\mathcal{A} = \mathbb{P}_{\leq N} \mathcal{L} \mathbb{P}_{\leq N} \tag{6–2}$$

serves as a finite-dimensional approximation to the action of $\mathcal{L}$ on $\tilde{\mathbf{L}}^1_{\rho}$. We will now make the latter statement more precise.

To this end, observe that

$$\mathcal{L} = \mathcal{A} + \mathcal{L}\mathbb{P}_{>N} + \mathbb{P}_{>N}\mathcal{L}\mathbb{P}_{\leq N} = \mathcal{A} + \mathcal{H}. \tag{6–3}$$

The following lemma demonstrates how one can obtain an upper bound on the spectral radius of the differential $\mathcal{L}$ at the fixed point in terms of the norm of a power of the finite-rank operator $\mathcal{A}$ and the magnitude of the norm of $\mathcal{H}$.

Lemma 6.2. *Let $\mathcal{L} = \mathcal{A} + \mathcal{H}$ be a bounded operator on some Banach space such that $\|\mathcal{A}^k\| < \gamma < 1$ for some $k \geq 1$ and $\|\mathcal{H}\| < \delta < 1$. Then the spectral radius $R_{\rm Sp}(\mathcal{L})$ satisfies*

$$R_{\rm Sp}(\mathcal{L}) \leq \gamma^{1/k}(1 + C\delta/\gamma)^{1/k} \tag{6–4}$$

for some (explicit) constant C.

Proof: The claim follows from the spectral radius formula. First,

$$R_{\rm Sp}(\mathcal{L}) = \overline{\lim}_{n\to\infty}\|\mathcal{L}^n\|^{1/n} = \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]+k\left\{\frac{n}{k}\right\}}\right\|^{1/n}$$
$$\leq \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]}\right\|^{1/n}\overline{\lim}_{n\to\infty}\|\mathcal{L}^k\|^{1-k\left[\frac{n}{k}\right]/n}.$$

The norm $\|\mathcal{L}^k\|$ is finite, and therefore

$$\overline{\lim}_{n\to\infty}\|\mathcal{L}^k\|^{\frac{n-k\left[\frac{n}{k}\right]}{n}} = 1.$$

Then

$$R_{\rm Sp}(\mathcal{L}) \leq \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]}\right\|^{\frac{1}{n}}$$
$$\leq \overline{\lim}_{n\to\infty}\left\|\mathcal{L}^{k\left[\frac{n}{k}\right]}\right\|^{\frac{1}{\left[\frac{n}{k}\right]k}}\overline{\lim}_{n\to\infty}\|\mathcal{L}^k\|^{\frac{\left[\frac{n}{k}\right]k-n}{nk}}$$
$$\leq \overline{\lim}_{m\to\infty}\left\|\mathcal{L}^{km}\right\|^{\frac{1}{mk}}.$$

Let ${}_nC_k$ denote the binomial coefficients, and let $C = \sum_{i=1}^k {}_kC_i\|\mathcal{A}^{k-i}\|\|\mathcal{H}\|^{i-1}$. Then

$$R_{\rm Sp}(\mathcal{L}) \leq \overline{\lim}_{n\to\infty}\left[\|\mathcal{A}^k\|^n + \sum_{i=1}^n {}_nC_i\|\mathcal{A}^k\|^{n-i}(C\delta)^i\right]^{\frac{1}{kn}}$$
$$\leq \overline{\lim}_{n\to\infty}\left[\gamma^n(1 + ((C\delta/\gamma+1)^n - 1)\right]^{\frac{1}{kn}}$$
$$= \gamma^{1/k}(1 + C\delta/\gamma)^{1/k},$$

and the proof is complete. □

It is left now to bound the difference of $\mathcal{L}$ from $\mathcal{A}$. First, we state the following Cauchy-type estimate, whose straightforward proof will be left to the reader:

Proposition 6.3. *Assume that an operator $\mathcal{R}_{\rm cyl}$ is analytic in an open ball $B_r(\hat f) \subset \tilde{\mathbf{L}}^1_\rho$. Let $\epsilon < 1$ and let $h \in \tilde{\mathbf{L}}^1_\rho$ be such that $|h|_\rho < r$. Then*

$$|\mathcal{R}_{\rm cyl}(\hat f + \epsilon h) - \mathcal{R}_{\rm cyl}\hat f - \epsilon\mathcal{L}h|_\rho \tag{6–5}$$
$$\leq \frac{\epsilon^2}{1-\epsilon}\sup_{|s|\leq 1}|\mathcal{R}_{\rm cyl}(\hat f + sh) - \mathcal{R}_{\rm cyl}\hat f|_\rho.$$

Note that

$$|\mathcal{L}|_{T_{\hat f}W}|_\rho \leq \sup_{j\geq 2}|\mathcal{L}e_j|_\rho.$$

This, together with the preceding proposition and the compactness property of renormalization, immediately implies that $|\mathcal{L}|_{T_{\hat f}W}|_\rho$ can be bounded by a finite difference. Specifically, for all $j > N$,

$$\begin{aligned}
|\mathcal{L}e_j|_\rho &\leq \epsilon^{-1}\sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\leq 1}}|\mathcal{R}_{\rm cyl}(\hat f + \epsilon h) - \hat f|_\rho \\
&\quad + \frac{\epsilon}{1-\epsilon}\sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\leq 1}}|\mathcal{R}_{\rm cyl}(\hat f + h) - \hat f|_\rho \\
&\leq \epsilon^{-1}\sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\leq 1}}|\mathbb{P}_{\leq N}[\mathcal{R}_{\rm cyl}(\hat f + \epsilon h) - \hat f]|_\rho \\
&\quad + \frac{\epsilon}{1-\epsilon}\sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\leq 1}}|\mathbb{P}_{\leq N}[\mathcal{R}_{\rm cyl}(\hat f + h) - \hat f]|_\rho \\
&\quad + \left(\frac{\rho}{\rho'}\right)^{N+1}\epsilon^{-1} \\
&\qquad \times \sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\leq 1}}|\mathbb{P}_{>N}[\mathcal{R}_{\rm cyl}(\hat f + \epsilon h) - \hat f]|_{\rho'} \\
&\quad + \frac{\epsilon}{1-\epsilon}\sup_{\substack{h\in\mathbb{P}_{>N}\tilde{\mathbf{L}}^1_\rho\\ |h|_\rho\leq 1}}|\mathbb{P}_{>N}[\mathcal{R}_{\rm cyl}(\hat f + h) - \hat f]|_{\rho'} \\
&\equiv C_1.
\end{aligned} \tag{6–6}$$

Similarly, for all $2 \leq j \leq N$,

$$\begin{aligned}
|\mathbb{P}_{>N}\mathcal{L}e_j|_\rho &\leq \epsilon^{-1}\sup_{\substack{h\in T_{\hat f}W\\ |h|_\rho\leq 1}}|\mathbb{P}_{>N}[\mathcal{R}_{\rm cyl}(\hat f + \epsilon h) - \hat f]|_\rho \\
&\quad + \frac{\epsilon}{1-\epsilon}\sup_{\substack{h\in T_{\hat f}W\\ |h|_\rho\leq 1}}|\mathbb{P}_{>N}[\mathcal{R}_{\rm cyl}(\hat f + h) - \hat f]|_\rho \\
&\leq \left(\frac{\rho}{\rho'}\right)^{N+1}\epsilon^{-1} \\
&\qquad \times \sup_{\substack{h\in T_{\hat f}W\\ |h|_\rho\leq 1}}|\mathbb{P}_{>N}[\mathcal{R}_{\rm cyl}(\hat f + \epsilon h) - \hat f]|_{\rho'} \\
&\quad + \frac{\epsilon}{1-\epsilon}\sup_{\substack{h\in T_{\hat f}W\\ |h|_\rho\leq 1}}|\mathbb{P}_{>N}[\mathcal{R}_{\rm cyl}(\hat f + h) - \hat f]|_{\rho'} \\
&\equiv C_2.
\end{aligned} \tag{6–7}$$

We would like to emphasize that these bounds use the fact that $\mathcal{L}$ is a compact operator in an essential way. The bounds (6–6) and (6–7) can be used to estimate $|(\mathcal{L} - \mathcal{A})|_{T_{\hat f}W}|_\rho$. In particular, according to (6–3),

$$|(\mathcal{L} - \mathcal{A})|_{T_{\hat f}W}|_\rho \leq \max\left\{|\mathcal{L}\mathbb{P}_{>N}|_\rho, |\mathbb{P}_{>N}\mathcal{L}|_{T_{\hat f}W}\mathbb{P}_{\leq N}|_\rho\right\}$$
$$\leq \max\{C_1, C_2\}.$$

This expression provides a bound on δ in (6–4).

We have chosen $N = 14$, and experimentally bounded C_1 and C_2 by testing on vectors $h = e_{15}$ and $h = e_2$, which empirically maximize the respective suprema. Choosing $k = 80$ in (6–4), we have the following values for the constants that enter the estimate (6–4):

$$\gamma < 2.07 \times 10^{-18}, \quad \delta < 0.24, \quad C < 8.4 \times 10^{-6}.$$

Therefore, according to (6–4),

$$R_{\mathrm{Sp}}(\mathcal{L}|_{T_{\hat{f}}W}) < 0.85.$$

A better bound can be obtained if one computes all relevant constants for a larger value of N, which requires more computer time. It is plausible that the spectral radius is close to 0.58, since we have observed that as N increases, the largest eigenvalue of the operator

$$\mathbb{P}_{\leq N}\mathcal{L}|_{T_{\hat{f}}W}\mathbb{P}_{\leq N}$$

converges to

$$\lambda = 0.15 + i0.56.$$

This eigenvalue has been truncated to two decimal places.

As a final comment, note that the following simple observation implies that perturbations in the directions of the vectors h_j can also be used for estimating the spectral radius of $\mathcal{L}|_{T_{\hat{f}}W}$.

Proposition 6.4. *We have*

$$\mathrm{Spec}(\mathcal{L}|_{\mathbf{B}^1_\rho}) = \mathrm{Spec}(\mathcal{L}|_{\mathbf{L}^1_\rho}).$$

To see this, note that the only difference between the spectra is that 0 (contained in both spectra) is an eigenvalue of the operator $\mathcal{L}|_{\mathbf{B}^1_\rho}$ corresponding to linear rescalings. We leave the straightforward details to the reader.

7. APPENDIX

The objective of this appendix is to describe how the Cauchy (3–6) and Hilbert (3–5) transforms can be computed numerically.

The constructive measurable Riemann mapping theorem, Theorem 3.1, deals with L_p functions that generally do not need to be differentiable. Therefore, one has to choose an appropriate representation of the L_p functions that enter Theorem 3.1 possibly as a collection of values on a grid, or as a Fourier series with radially dependent coefficients. The latter choice has been made, for instance, in [Daripa 92, Gaidashev 07, Gaidashev and Khmelev 06], and will also be adopted in the present paper.

Represent h and $P[h]$ in (3–6) as

$$h(re^{i\theta}) = \sum_{k=-\infty}^{\infty} h_k(r)e^{ik\theta}, \tag{7–1}$$

$$P[h](re^{i\theta}) = \sum_{k=-\infty}^{\infty} p_k(r)e^{ik\theta}, \tag{7–2}$$

where the coefficients of the P-transform are given by

$$p_k(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}P[h](re^{i\theta})\ d\theta. \tag{7–3}$$

A classical theorem of analysis (cf. [Ahlfors 66, Fletcher and Markovic 06]) states that the Cauchy transform of an L_p function, $p > 2$, is well defined and is Hölder continuous with exponent $1 - 2/p$. In [Daripa 92] and [Gaidashev and Khmelev 06] this fact has been used to show that the Fourier coefficients of a Cauchy transform are given by the following equations:

$$p_k(r) = \begin{cases} 2\displaystyle\int_0^r \left(\frac{r}{\rho}\right)^k h_{k+1}(\rho)\, d\rho, & k < 0, \\ -2\displaystyle\int_r^\infty \left(\frac{r}{\rho}\right)^k h_{k+1}(\rho)\, d\rho, & k \geq 0. \end{cases} \tag{7–4}$$

To obtain similar formulas for the Hilbert transform, assume that h is a Hölder continuous function compactly supported in an open disk around zero of radius R, $B(0,R) \subset \mathbb{C}$. The Hilbert transform of such a function is known to exist as a Cauchy principal value (cf. [Ahlfors 66, Carleson and Gamelin 91]). As with the Cauchy transform, represent this transform as a Fourier series:

$$T[h](re^{i\theta}) = \sum_{k=-\infty}^{\infty} c_k(r)e^{ik\theta},$$

$$c_k(r) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}T[h](re^{i\theta})\ d\theta.$$

In [Daripa 93, Daripa and Mashat 98, Gaidashev and Khmelev 06], the authors arrive at the following expressions for these coefficients:

$$c_0(0) = -2\lim_{\epsilon\to 0}\int_\epsilon^R \frac{h_2(\rho)}{\rho}d\rho \quad \text{and} \quad c_k(0) = 0, \tag{7–5}$$

whenever $k \neq 0$, and

$$c_k(r) = A_k\int_0^r \frac{r^k}{\rho^{k+1}}h_{k+2}(\rho)d\rho + B_k\int_r^R \frac{r^k}{\rho^{k+1}}h_{k+2}(\rho)d\rho + h_{k+2}(r), \tag{7–6}$$

where

$$A_k = \begin{cases} 0, & k \geq 0, \\ 2(k+1), & k < 0, \end{cases}$$

and

$$B_k = \begin{cases} -2(k+1), & k \geq 0, \\ 0, & k < 0. \end{cases}$$

We would like to mention that the fact that the Hilbert transform is a singular integral operator makes a rigorous justification of formulas (7–5) and (7–6) significantly more involved than that of (7–4).

Formulas (7–4)–(7–6) can be used to construct an efficient algorithm for solving a Beltrami equation. Given values of h, for instance on a circular $N \times M$ grid that contains the compact support of h, one can use a fast Fourier transform (FFT)—see, for example, [Press et al. 92])—to find the values of the coefficients h_k at the radii r_i, $1 \leq i \leq M$.

Next, one can use these values to construct a piecewise constant, piecewise linear, or spline approximation of the functions h_k (the choice of approximation, of course, depends on the known or expected smoothness of h_k). This allows one to compute integrals in (7–4)–(7–6).

Armed with these implementations of the Hilbert and Cauchy transforms, one can try to solve the Beltrami equation (3–1), first by running iterations (3–8) for some time, and finally, applying (3–9). It is convenient to use the pointwise multiplication of grid values of h and $\mu+1$ inside the Hilbert transform in (3–8), rather than the multiplication of their Fourier series: The order of the computational complexity of the pointwise multiplication is $O(NM)$, as opposed to $O(NM^2)$ for the series. The transition from the representation of h as a Fourier series to point values at each iteration step can be performed with the help of the FFT. This way, the computational complexity of one iteration step becomes $O(NM \log_2 M)$.

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