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Cylindrical laser resonators*

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A general class of cylindrical laser resonators is described in which the radiation propagates partially in the radial direction. There is a strong focusing of energy at the axis of such resonators. The low-loss cavity modes are related to ordinary gaussian beam modes. Two limiting forms are the disk laser and the tube laser, and some applications are considered.
Index Headings: Resonant modes; Laser.

Conventional laser resonators generally consist of a pair of mirrors between which electromagnetic radiation may be made to propagate. Within this optical cavity is placed an amplifying medium and perhaps a variety of other optical elements, as well. Elementary matrix techniques may be employed to derive the important Laguerre–gaussian and Hermite–gaussian modes of such resonators as long as each optical element can be characterized by a 2×2 beam matrix.[1] The subject of this work is a class of laser resonators in which the radiation propagates partially in the radial direction toward and away from the axis of symmetry. The simplest cylindrical resonator consists of one wrap-around mirror, as indicated in Fig. 1. The mirror shown has a slight curvature in the z direction, which would be useful for reducing diffraction losses in resonators with no other focusing elements. In high-gain lasers, however, diffraction would be unimportant.

More-general resonator configurations are sketched in Fig. 2. For simplicity only a vertical cross section through the center of these resonators is drawn. The dashed lines indicate possible mode profiles, the shaded area indicates the amplifying medium, and the arrows show the direction of the output beams (assuming that the resonator mirrors are partially transmitting). The disk resonator of Fig. 2(a) is the same as that shown in Fig. 1 except that an additional concentric mirror has been added so that the laser output may be taken from the center of the resonator as well as through the outside mirror. Figure 2(b) shows a cone resonator with an angle θ between the cone surface and the $z=0$ plane. Figure 2(c) shows a tube resonator. The disk and tube resonators are limiting forms of the cone resonator. The modes of all of these resonators are determined by the same basic parameters as the modes in conventional lasers.

THEORY

In this section, the low-loss electromagnetic modes of the cylindrical resonators are derived from the wave equation. For harmonically varying electric and magnetic fields in an isotropic homogeneous medium, the wave equation is

$$(\nabla\times\nabla\times-k^2)\begin{Bmatrix}\bar{E}\\ \bar{H}\end{Bmatrix}=0, \tag{1}$$

where $k=\omega(\mu\epsilon)^{\frac{1}{2}}$ is the complex propagation constant. The complex permittivity ϵ is related to the conductivity, the background polarizability, and the resonant susceptibility $\chi=\chi'-i\chi''$. The real and imaginary parts of χ determine, respectively, the dispersion and gain due to the laser transition. In some laser media, the spatial variations of the permittivity are sufficient to affect significantly the resonator modes. Then, more-general methods must be employed, such as those developed by Kurtz and Streifer for wave guiding in radially inhomogeneous media.[2]

In cylindrical coordinates, the scalar wave equation for a cartesian component of the electric or magnetic

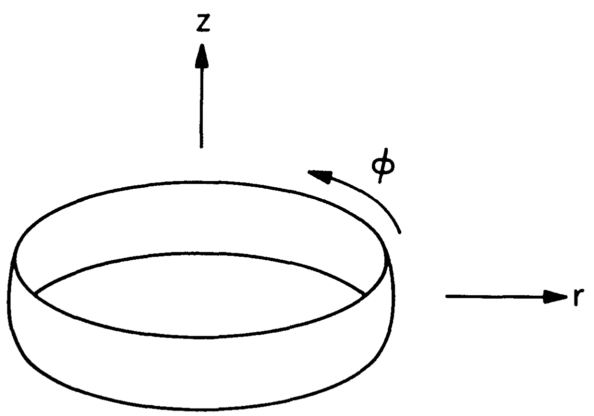

FIG. 1. Cylindrical laser resonator.

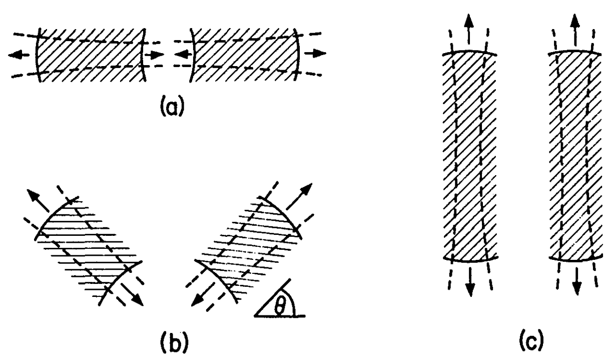

FIG. 2. Vertical cross section through the disk (a), cone (b), and tube (c) configurations of the general cylindrical resonator.

field is

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2}+\frac{\partial^2\psi}{\partial z^2}+k^2\psi=0. \tag{2}$$

The ϕ dependence can be separated out by means of the substitution

$$\psi=F(r,z)\Phi(\phi), \tag{3}$$

with the results

$$\frac{d^2\Phi}{d\phi^2}+m^2\Phi=0, \tag{4}$$

$$\frac{\partial^2 F}{\partial r^2}+\frac{1}{r}\frac{\partial F}{\partial r}-\frac{m^2F}{r^2}+\frac{\partial^2 F}{\partial z^2}+k^2F=0, \tag{5}$$

where m is a separation constant. The solutions of Eq. (4) are given, except for multiplicative constants, by

$$\Phi=\left\{\begin{matrix}\sin\\ \cos\end{matrix}\right\}m\phi. \tag{6}$$

If the fields are periodic about the z axis, then m must be an integer.

In solving Eq. (5) it is helpful to assume first that the outward-propagating wave modes will be given approximately by the cylindrical wave functions

$$F(r,z)=G(r,z)H_m^{(2)}[(k^2-h^2)^{\frac{1}{2}}r]e^{-ihz}, \tag{7}$$

where $G(r,z)$ is a slowly varying function and $H_m^{(2)}$ is a Hankel function of the second kind. The Hankel functions are governed by Bessel's equation

$$\frac{\partial^2 H}{\partial r^2}+\frac{1}{r}\frac{\partial H}{\partial r}+\left(k^2-h^2-\frac{m^2}{r^2}\right)H=0. \tag{8}$$

Substitution of Eq. (7) into Eq. (5) shows that $G(r,z)$ is governed by

$$\frac{\partial^2 G}{\partial r^2}+\left(\frac{2}{H_m^{(2)}}\frac{\partial H_m^{(2)}}{\partial r}+\frac{1}{r}\right)\frac{\partial G}{\partial r}+\frac{\partial^2 G}{\partial z^2}-2ih\frac{\partial G}{\partial z}=0. \tag{9}$$

The solutions of Eq. (9) are most easily obtained at distances $r\gg m/k$, where the Hankel functions can be replaced by their asymptotic value[3]

$$\lim_{r\to\infty} H_m^{(2)}[(k^2-h^2)^{\frac{1}{2}}r]=\sqrt{2}/[\pi r(k^2-h^2)^{\frac{1}{2}}]^{\frac{1}{2}}$$
$$\times\exp-i\left[(k^2-h^2)^{\frac{1}{2}}r-\frac{m\pi}{2}-\frac{\pi}{4}\right]. \tag{10}$$

With this approximation, Eq. (9) simplifies to

$$\frac{\partial^2 G}{\partial r^2}-2i(k^2-h^2)^{\frac{1}{2}}\frac{\partial G}{\partial r}+\frac{\partial^2 G}{\partial z^2}-2ih\frac{\partial G}{\partial z}=0. \tag{11}$$

We are interested here in the situation in which the radiation is confined to a region near the conical surface that makes an angle $\theta=\tan^{-1}[h/(k^2-h^2)^{\frac{1}{2}}]$ with the $z=0$ plane. Accordingly, a useful change of variables is

$$\begin{aligned} r'&=r\cos\theta+z\sin\theta,\\ z'&=-r\sin\theta+z\cos\theta. \end{aligned} \tag{12}$$

This transformation, in effect, rotates the coordinate system so that the radiation propagates primarily along the surface $z'=0$. Substitution of Eqs. (12) into Eq. (11) leads to

$$\frac{\partial^2 G}{\partial r'^2}-2ik\frac{\partial G}{\partial r'}+\frac{\partial^2 G}{\partial z'^2}=0. \tag{13}$$

We assume that G varies so slowly in the direction of propagation that the first term in Eq. (13) can be neglected, leaving

$$\frac{\partial^2 G}{\partial z'^2}-2ik\frac{\partial G}{\partial r'}=0. \tag{14}$$

Equation (14) can be solved by assuming the gaussian form

$$G(r',z')=\exp-i\left[P(r')+\frac{Q(r')z'^2}{2}\right] \tag{15}$$

and equating equal powers of z'. The results are

$$Q^2+k\frac{dQ}{dr'}=0, \tag{16}$$

$$\frac{dP}{dr'}=\frac{-iQ}{2k}. \tag{17}$$

Q is the familiar complex beam parameter, which is related to the spot size w and the radius of curvature of the phase fronts R (in the z' direction) by

$$\frac{Q}{k}=\frac{1}{q}=\frac{1}{R}-\frac{i2}{kw^2}, \tag{18}$$

where q is the complex beam radius. The solutions of Eq. (16) may be written

$$\frac{\pi w^2}{\lambda}=r_0\left[1+\left(\frac{r'}{r_0}\right)^2\right], \tag{19}$$

$$R=r'\left[1+\left(\frac{r_0}{r'}\right)^2\right], \tag{20}$$

where $r_0=\pi w_0^2/\lambda$ is the Rayleigh length. These expressions are essentially the same as the results for conventional beams. From Eq. (17), the amplitude of the cylindrical waves is proportional to $(w_0/w)^{\frac{1}{2}}$, and the real part of the phase is given by $P_r=-\frac{1}{2}\tan^{-1}(r'/r_0)$.

Higher-order modes can also be found,[1] and then the right-hand side of Eq. (15) is multiplied by the Hermite polynomial $H_n(\sqrt{2}z'/w)$ and $P_r=-(n+\frac{1}{2})\tan^{-1}(r'/r_0)$.

If Eqs. (3), (6), (7), (15), and (18) are combined, the outward-propagating cylindrical-beam modes are

$$\psi=\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi H_m{}^{(2)}[(k^2-h^2)^{\frac{1}{2}}r]e^{-ihz} \times H_n\left(\sqrt{2}\frac{z'}{w}\right)e^{-ikz'^2/2R}e^{-z'^2/w^2}e^{-iP}, \quad (21)$$

with a similar expression involving $H_m{}^{(1)}$ for inward-propagating waves. The variables r' and z' may be expressed in terms of r and z by means of the transformation given in Eqs. (12). From the approximation made in obtaining Eq. (11), it is clear that these solutions are valid as long as the Hankel functions may be replaced by their large-radius ($r\gg m/k$) asymptotic form. They are valid for small radii, as well, if the spot size is much greater than m/k.

The cylindrical modes given in Eq. (21) are the solutions of the scalar wave equation. If these represent the z components of the electric or magnetic fields, then the other components may be derived by use of the relations[4]

$$E_r=-\frac{i}{k^2-h^2}\left[h\frac{\partial E_z}{\partial r}+\frac{\omega\mu}{r}\frac{\partial H_z}{\partial\phi}\right], \quad (22)$$

$$E_\phi=-\frac{i}{k^2-h^2}\left[\frac{h}{r}\frac{\partial E_z}{\partial\phi}-\omega\mu\frac{\partial H_z}{\partial r}\right], \quad (23)$$

$$H_r=-\frac{i}{k^2-h^2}\left[-\frac{\omega\epsilon}{r}\frac{\partial E_z}{\partial\phi}+h\frac{\partial H_z}{\partial r}\right], \quad (24)$$

$$H_\phi=-\frac{i}{k^2-h^2}\left[\omega\epsilon\frac{\partial E_z}{\partial r}+\frac{h}{r}\frac{\partial H_z}{\partial\phi}\right]. \quad (25)$$

Two special cases of these solutions are considered in detail in the following sections.

THE DISK LASER

In the preceding section, a class of cylindrical laser modes was derived. These modes have, in general, both a z component and a radial component of propagation. Here we consider an important special case of those results in which the radiation propagates only in the radial direction in a disk-shaped region of space.[5] These disk lasers are not to be confused with other conventional lasers that have radiation propagating across the plane of a disk-shaped amplifying medium. An important feature of the disk lasers is the high energy density that is obtained at the axis. Some applications are considered.

In the limit of no propagation in the z direction ($h=0$) the results of the previous section simplify. In particularr z' is replaced by z and r' is replaced by r so that the scalar wave function given in Eq. (21) reduces to

$$\psi=\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi H_m{}^{(2)}(kr) \times\left\{H_n\left(\sqrt{2}\frac{z}{w}\right)e^{-ikz^2/2R}e^{-z^2/w^2}e^{-iP}\right\} \quad (26)$$

A useful relation for evaluating the field components is[3]

$$\frac{dC_m(lr)}{dr}=-lC_{m+1}(lr)+\frac{m}{r}C_m(lr), \quad (27)$$

where C_m is any linear combination of the Bessel functions J_m, Y_m, $H_m{}^{(1)}$, and $H_m{}^{(2)}$, and l is the appropriate propagation constant. Then from Eqs. (22)–(26) the nonzero components of the electric and magnetic fields for z-polarized modes are

$$E_z=\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi H_m{}^{(2)}(kr)\left\{\quad\right\}, \quad (28)$$

$$H_\phi=-\frac{i}{\mu\omega}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi\left[-kH_{m+1}{}^{(2)}(kr)+\frac{m}{r}H_m{}^{(2)}(kr)\right]\left\{\quad\right\}, \quad (29)$$

$$H_r=\frac{i}{\mu\omega}\frac{m}{r}\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix} m\phi H_m{}^{(2)}(kr)\left\{\quad\right\}, \quad (30)$$

where the empty brackets refer to the slowly varying bracketed quantity in Eq. (26). Similar equations hold for the incoming $H_m{}^{(1)}$ Hankel-function modes. For distances $r\gg m/k$, the magnetic field has only a simple $H_{m+1}{}^{(2)}$ ϕ component. Also, from Eqs. (22)–(26) we find that the nonzero field components of the modes with $E_z=0$ are

$$H_z=\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi H_m{}^{(2)}(kr)\left\{\quad\right\}, \quad (31)$$

$$E_\phi=\frac{i}{\omega\epsilon}\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi\left[-kH_{m+1}{}^{(2)}(kr)+\frac{m}{r}H_m{}^{(2)}(kr)\right]\left\{\quad\right\}, \quad (32)$$

$$E_r=\frac{-i}{\omega\epsilon}\frac{m}{r}\begin{Bmatrix}\cos\\ -\sin\end{Bmatrix} m\phi H_m{}^{(2)}(kr)\left\{\quad\right\}. \quad (33)$$

As indicated previously, one aspect of these results is the large energy density near the axis of the resonator. Because of the complicated form of the fields, only the lowest-order $m=0$ mode is considered here. Then, from Eqs. (28)–(30), the fundamental standing-wave z-polar-

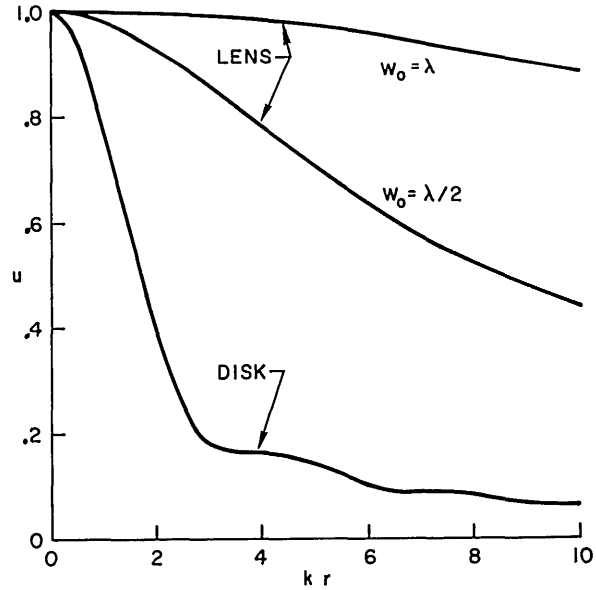

FIG. 3. Normalized energy density near the focus of a disk resonator and a cylindrical lens with $w_0=\lambda/2$ and $w_0=\lambda$.

ized mode has the form

$$E_z=\frac{2}{\sqrt{\epsilon}}J_0(kr), \tag{34}$$

$$H_\phi=\frac{i2}{\sqrt{\mu}}J_1(kr), \tag{35}$$

where the transverse structure of the beam has been assumed to be unimportant and the normalization has been changed. The average energy density is

$$\begin{aligned} u(r)&=\tfrac{1}{4}(\epsilon\bar{E}\cdot\bar{E}^*+\mu\bar{H}\cdot\bar{H}^*)\\ &=J_0^2(kr)+J_1^2(kr). \end{aligned} \tag{36}$$

Since $J_0(0)=1$ and $J_1(0)=0$, the energy density at the axis is unity with this choice of normalization. At large distances, the asymptotic form of the Bessel function is[3]

$$\lim_{r\to\infty} J_m(kr)=(2/\pi kr)^{\frac{1}{2}}\cos\left(kr-\frac{m\pi}{2}-\frac{\pi}{4}\right). \tag{37}$$

Therefore, the energy density at large distances is

$$\lim_{r\to\infty} u(r)=\frac{2}{\pi kr}\left[\cos^2\left(kr-\frac{\pi}{4}\right)+\sin^2\left(kr-\frac{\pi}{4}\right)\right]=\frac{2}{\pi kr}. \tag{38}$$

The corresponding expression for the energy density in a beam focused by a cylindrical lens is

$$u(r)=\left[1+\left(\frac{kr\lambda^2}{2\pi^2 w_0^2}\right)^2\right]^{-\frac{1}{2}}. \tag{39}$$

This result follows from Eq. (19). Here r measures the distance from the focal plane of a cylindrical lens. Equations (36) and (39) are plotted in Fig. 3. Values of the minimum spot size w_0 of less than about $\lambda/2$ are not possible with a conventional cylindrical lens. Focusing in the z direction could also be produced by appropriate choice of the z curvature of the disk resonator mirror.

The cavity-mode frequencies can be found from the requirement that the total round-trip phase delay be an integral multiple p of 2π. With Eq. (37), this requirement is

$$\left(kr-\frac{m\pi}{2}-\frac{\pi}{4}\right)-(n+\tfrac{1}{2})\tan^{-1}\frac{r}{r_0}=p\pi. \tag{40}$$

Therefore, the resonant frequencies are

$$\nu=\frac{c}{2r}\left[\frac{m}{2}+\tfrac{1}{4}+p+\frac{1}{\pi}(n+\tfrac{1}{2})\tan^{-1}\frac{r}{r_0}\right]. \tag{41}$$

Thus, the radial mode spacing associated with the integer p is $c/2r$. The azimuthal mode spacing associated with the integer m is $c/4r$. Two modes having the same value of $p+m/2$ are degenerate in frequency. If the resonator is confocal ($r=r_0$), then the transverse mode spacing associated with the index n of the Hermite polynomial is $c/8r$. A disk laser can be smaller for a given power output than a conventional laser, so the mode spacings are correspondingly greater. As in ordinary lasers, it should be possible to phase lock the radial and transverse modes. Besides the standing-wave sinusoidal azimuthal modes, traveling-wave azimuthal modes are also possible with a phase velocity $r\omega/m$ about the z axis.

Another property of the disk resonator is the uniformity of illumination. An object placed at the focus of an ordinary laser beam is illuminated from only one side, whereas an object at the axis of a disk resonator would be illuminated about its entire circumference. Thus, the region near the axis might be useful in applications involving the excitation, vaporization, ionization, or fusion of samples. For example, the active medium in the cylindrical resonator could be a nitrogen discharge with an ultraviolet output, so that an appropriate dye cell at the axis of the resonator would have gain for visible radiation propagating along the z axis. The result would be an efficiently coupled dye laser.

There are also various possible ways of combining cylindrical resonators to achieve higher energy densities. For example, a disk laser could be cut in half, and a number of such half-resonators could be assembled together (like assembling an orange from its segments). Alternatively, conical resonators like that shown in Fig. 2(b) having differing cone angles could be nested together.

THE TUBE LASER

In this section, another limit of the general cylindrical laser modes is considered. Here the conical surface of propagation is reduced to a tubular region in which the radiation propagates only in the z direction. The modes are determined by multiplying Eq. (21) by the constant factor $m![(k^2-h^2)^{\frac{1}{2}}r_t/2]^{-m}$ and then taking the limit $h\rightarrow k$ for propagation in the z direction. Since the Bessel functions for small arguments are given by[3]

$$J_m[(k^2-h^2)^{\frac{1}{2}}r]\sim[(k^2-h^2)^{\frac{1}{2}}r/2]^m/m!,$$

the Bessel-function part of the modes is now replaced by $(r/r_t)^m$. This factor is essentially unity for thin tubular modes. It is also useful to take the limits $z\rightarrow\infty$, $h\rightarrow k$ in such a way that the term $z\cos\theta\equiv r_t$ in the second of Eqs. (12) remains constant. Then we find the transformation $r'\rightarrow z, z'\rightarrow r_t-r$, and the tubular modes are

$$\psi=\begin{Bmatrix}\sin\\ \cos\end{Bmatrix} m\phi H_n\left[\sqrt{2}\frac{(r-r_t)}{w}\right] \times e^{-ikz}e^{-ik(r-r_t)^2/2R}e^{-(r-r_t)^2/w^2}e^{-iP}. \quad (42)$$

These modes can also be derived directly from the wave equation. This form for the modes is valid as long as the mode thickness is much less than the radius r_t. A complete set of fields is obtained by regarding the wave function ψ as either the x or y component of linearly polarized electric or magnetic fields. Alternatively, ψ may be interpreted as the amplitude of a field having only an r or ϕ component.

The modes described by Eq. (42) are in agreement with the ring modes, which are sometimes observed in ion lasers.[6–8] The ring modes may also be obtained from disk lasers by either of the coupling schemes shown in Fig. 4. The arrangement shown in Fig. 4(b) could be reversed so that the output of any tubular laser (or conventional laser) would be focused to the axis of a wrap-around mirror yielding a large energy density at the axis.

The tubular geometry might have an advantage in some lasers involving pumping by tungsten lamps or flashlamps. The lamp could be placed at the axis of a tubular laser medium the outside surface of which was reflecting. This simple arrangement would not provide as high a pumping level as is possible with a high-quality elliptical pump cavity, but for some applications it should be adequate.

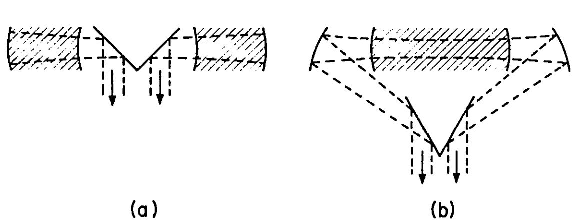

FIG. 4. Schemes for coupling disk modes to ring modes and vice versa.

CONCLUSION

A new class of cylindrical laser resonators has been considered, in which the radiation propagates partially in the radial direction. The electromagnetic modes of these cavities have been derived from the wave equation. Two limiting resonator configurations are the disk laser and the tube laser. An important feature of the disk laser is the high field strength at the resonator axis. The propagation characteristics of all of these beam modes are governed by essentially the same complex parameters that arise in the study of conventional laser resonators. Similar methods could be employed for investigating the modes of elliptic cylindrical resonators, but the resulting wave functions are less familiar and lack obvious practical applications. Experiments are in progress to verify the results described here.

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