

D. Blackwell's conjecture on power series with random coefficients

by

C. RYLL-NARDZEWSKI (Wrocław).

E. BOREL¹⁾ was the first to assert that Taylor's series with mutually independent random coefficients have, with probability 1, a *coupure* on their circle of convergence, which is to say that almost certainly the circle is a singular line for the function defined by such a series. This assertion has been proved since for certain classes of random power series²⁾, nevertheless its general validity is easily refuted by the following example:

Let $\{\vartheta_n\}$ be a sequence of random variables, each ϑ_n being chosen independently from the interval $(0, 2\pi)$ with uniform probability. The power series

$$\sum_{n=1}^{\infty} a_n z^n \quad \text{with} \quad a_n = e^{i\vartheta_n}$$

has 1 as its radius of convergence; it has independent random coefficients. The series

$$\sum_{n=1}^{\infty} \beta_n z^n \quad \text{with} \quad \beta_n = a_n + 2^n$$

has also independent random coefficients. Its radius of convergence is $1/2$ and as the first series is regular for $|z|=1/2$ the only singular point of the second series on the circle $|z|=1/2$ is the pole $z=1/2$ of the series $\sum 2^n z^n$, against Borel's assertion.

It is obvious that it is sufficient to add to the second series the series $\sum (-2^n) z^n$ to get the first one; now, it is known that the first series has, with probability 1, the circle $|z|=1$ as its singular line.

The conjecture we have to prove in this Note is, roughly speaking, that the example given above exhibits already the features of the general case: given any power series with random coefficients we have only to add to it an appropriate series with fixed coefficients (in our example the series $\sum (-2^n) z^n$) to get a series obeying Borel's assertion³⁾.

Preliminary remarks and explanations.

We will follow here the vocabulary of the theory of independent functions⁴⁾. Thus we will have to deal with series of the form

$$(1) \quad F(z) = \sum_{n=1}^{\infty} a_n(t) z^n;$$

the complex-valued and (L) -measurable functions $a_n(t)$ of the stochastic variable t ($0 \leq t \leq 1$) are assumed to be *stochastically independent* as a whole (*en bloc*). It can be shown that, under such circumstances, the series (1) has a *definite radius* of convergence $r(F)$ ($0 \leq r \leq \infty$), which means that for all values of t , a set of measure 0 if any excepted, $|z|=r$ is the circle of convergence of (1), r being a number and not a function of t . The proof is based on the fact that $1/r$ equals the

$$\limsup_{n \rightarrow \infty} \sqrt[n]{|a_n(t)|}$$

and the *lim sup* of a sequence of independent functions is constant almost everywhere.

We will have also to consider series of the form

$$(2) \quad G(z) = \sum_{n=1}^{\infty} b_n(s, t) z^n;$$

¹⁾ The conjecture has been formulated in 1947 by Prof. D. Blackwell of the Howard University, in a conversation with Prof. H. Steinhaus and communicated by the latter to the author.

²⁾ *Sur les séries de Taylor*, C. R. 123 (1896), p. 1051-1052.

³⁾ Cf. H. Steinhaus, *Über die Wahrscheinlichkeit dafür, dass der Konvergenzkreis einer Potenzreihe ihre natürliche Grenze ist*, Math. Zeitschrift 31 (1930), p. 408-416.

⁴⁾ *Sur les fonctions indépendantes I-IX*: Studia Mathematica 6 (1936), p. 46-58, p. 59-66, p. 89-97; 7 (1938), p. 1-15, p.96-100; 9 (1940), p.121-132; 10 (1948), p. 1-20; 11 (1949), p. 133-144; 12 (1951), p. 102-107.

here the coefficients b_n are (L) -measurable functions of two stochastic variables s, t ($0 \leq s \leq 1, 0 \leq t \leq 1$); assuming their independence it can be shown as in the case (1) the existence of a definite radius of convergence $r(G)$, the exceptional set of points (s, t) having a plane measure 0.

In both cases, (1) and (2), the series is called *singular* if its circle of convergence (defined as above) is a singular line for the function defined by the series, this being true for all points t , respectively for all points (s, t) , a set of linear measure 0, respectively of plane measure 0, of such points, if any, being excepted.

Series of the form (1) or (2) will be called *random series*. A special case of a random series is an ordinary power series

$$(3) \quad f(z) = \sum_{n=1}^{\infty} c_n z^n$$

with fixed coefficients c_n . For clarity's sake we will denote the sums of random series by capital letters F, G, \dots , the sums of ordinary series by small letters f, g, \dots

The radius of convergence of any series S will be denoted by $r(S)$ putting $S = F, G, \dots, f, g, \dots$. For almost all t of the interval $(0, 1)$ or for almost all (s, t) of the square $(0, 1)^2$ means that the property in question fails to be true only in a set of points t of linear measure 0, respectively in a set of points (s, t) of plane measure 0.

Theorem. *To every random series $F(z)$ an ordinary series $f_0(z)$ can be associated in such a manner that the random series*

$$(4) \quad H(z) = F(z) + f_0(z)$$

has the following properties:

- (i) $r(F+f) \leq r(H)$ for every ordinary series $f(z)$;
- (ii) $H(z)$ is singular;
- (iii) if an ordinary series $f(z)$ satisfies the equation

$$(5) \quad r(F+f) = r(H),$$

the series $F(z) + f(z)$ is singular.

To prove the theorem we have to establish two lemmas of which it is an immediate consequence.

Lemma 1. *Let $F(z)$ be the random series (1) and $G(z)$ the random series defined by*

$$(6) \quad G(z) = \sum_{n=1}^{\infty} (a_n(t) - a_n(s)) z^n \quad (0 \leq s \leq 1, 0 \leq t \leq 1).$$

Then we will have for every ordinary series $f(z)$ the inequalities

$$1^\circ \quad r(F+f) \leq r(G),$$

$$2^\circ \quad r(G+f) \leq r(G),$$

and there will exist an ordinary series $f_0(z)$ satisfying the equation

$$3^\circ \quad r(F+f_0) = r(G).$$

Proof of 1° . Let us write out $f(z)$ as in (3). It is then obvious that the series

$$\sum_{n=1}^{\infty} (a_n(t) + c_n) z^n$$

is convergent for almost all t of the interval $(0, 1)$ for $|z| < r(F+f)$, and the series

$$\sum_{n=1}^{\infty} (a_n(s) + c_n) z^n$$

is convergent for almost all s of the interval $(0, 1)$ for $|z| < r(F+f)$. Subtraction gives the convergence of the series (6) for almost all points (s, t) of the square $(0, 1)^2$ for $|z| < r(F+f)$, q. e. d.

Proof of 2° . The series

$$\sum_{n=1}^{\infty} (a_n(t) - a_n(s) + c_n) z^n$$

being convergent for almost all points (s, t) of the square $(0, 1)^2$ for $|z| < r(G+f)$, there exists an s_0 ($0 < s_0 < 1$) such that the series

$$(7) \quad \sum_{n=1}^{\infty} (a_n(t) - a_n(s_0) + c_n) z^n$$

is convergent for almost all t of $(0, 1)$ for $|z| < r(G+f)$. Calling f_0 the ordinary series

$$\sum_{n=1}^{\infty} (c_n - a_n(s_0)) z^n,$$

the series (7) becomes $F+f_0$ and thus we get

$$(8) \quad r(G+f) \leq r(F+f_0) \leq r(G),$$

the second inequality resulting from 1° .



Proof of 3°. Put $f \equiv 0$ in (8).

Lemma 2. If a random series $F(z)$ has on its circle of convergence a point which is regular for almost all values of the stochastic variable, there exists an ordinary series $f_1(z)$ such that

$$(9) \quad r(F) < r(F + f_1).$$

Proof. Let us assume the regular point z_0 to be $r(F)$; this assumption does not diminish the generality, because $z_0 = e^{i\theta_0} r(F)$ (with any real θ_0) is the general case and the series

$$F^* = \sum_{n=1}^{\infty} a_n(t) e^{in\theta_0} z \quad (i = \sqrt{-1})$$

has independent coefficients, the same radius of convergence as $F(z)$, and $z = r(F) = r(F^*)$ is a regular point for $F^*(z)$. The assumption implies the existence of real numbers u, v such that

$$(10) \quad 0 < u < r(F) < u + v,$$

and such that the series

$$(11) \quad \sum_{m=0}^{\infty} \frac{F^{(m)}(t, u)}{m!} v^m$$

is convergent for almost all t of $(0, 1)$. $F^{(m)}$ means here $d^m F(z) / dz^m$; we write $F(t, z)$ instead of $F(z)$ to recall the fact that the coefficients of $F(z)$ are functions of t .

The series (11) can be written as a double series:

$$\sum_{m=0}^{\infty} \sum_{n=m}^{\infty} [a_n(t) (u+v)^n] \binom{n}{m} \left(\frac{v}{u+v}\right)^m \left(\frac{u}{u+v}\right)^{n-m};$$

this transformation together with the definition

$$(12) \quad A_{N,n} = \sum_{m=0}^{\min(N,n)} \binom{n}{m} \left(\frac{v}{u+v}\right)^m \left(\frac{u}{u+v}\right)^{n-m}$$

enables us to replace the convergence of (11) by the equivalent condition of the existence of the limit

$$(13) \quad \lim_{N \rightarrow \infty} \sum_{n=0}^{\infty} A_{N,n} [a_n(t) (u+v)^n]$$

for almost all t of $(0, 1)$.

Now we have to apply to (13) a theorem of J. MARGINKIEWICZ and A. ZYGMUND⁵⁾ about series of independent functions: $\sum x_k(t)$ being such a series summable (T) almost everywhere, there exists a numerical sequence $\{\lambda_k\}$ which makes the series $\sum (x_k(t) + \lambda_k)$ convergent almost everywhere. Summability (T) means here the existence of the limit

$$(14) \quad \lim_{i \rightarrow \infty} \sum_{k=1}^{\infty} a_{ik} x_k,$$

the matrix $T = \|a_{ik}\|$ being any matrix with the property

$$(15) \quad \lim_{i \rightarrow \infty} a_{ik} = 1 \quad (k = 1, 2, 3, \dots).$$

To apply the theorem we have to verify the independence of the functions $a_n(t) (u+v)^n$, these functions playing the role of $x_n(t)$, and the property (15) for $a_{ik} = A_{i,k}$ as defined by (12). Both properties being evidently valid (for the second one notice that $A_{N,n} = 1$ for $N \geq n$), the existence of the limit (13) yields a numerical sequence $\{\lambda_n\}$ which makes the series

$$(16) \quad \sum_{n=0}^{\infty} [a_n(t) (u+v)^n + \lambda_n]$$

convergent for almost all t in $(0, 1)$. If we define $f_1(z)$ by

$$f_1(z) = \sum_{n=0}^{\infty} \frac{\lambda_n}{(u+v)^n} z^n,$$

the convergence of (16) implies the convergence of the series $F + f_1$ for $|z| < u + v$. Thus we get

$$r(F + f_1) \geq (u + v) > r(F),$$

the second inequality resulting from (10). This implies (9), q. e. d.

Proof of the theorem. We choose $f_0(z)$ to satisfy 3° of lemma 1. We have to show that this choice gives to the random series $H(z)$ defined by (4) the properties (i), (ii) and (iii) spoken of in the text of the theorem. (i) follows from 1° and 3° of lemma 1. Suppose $H(z)$ not to be singular; lemma 2 gives then an ordinary series $f_1(z)$ with $r(H + f_1) > r(H) = r(F + f_0) = r(G)$, which leads to $r(F + f_0 + f_1) > r(G)$

⁵⁾ Quelques théorèmes sur les fonctions indépendantes, Studia Mathematica 7 (1938), p. 104-120; p. 116, théorème 7.

against 1° of lemma 1; this contradiction yields (ii). To prove (iii) let us suppose (5) and $F(z)+f(z)$ not singular. Lemma 2 assures the existence of $f_1(z)$ with $r(F+f) < r(F+f+f_1)$; comparing with (5) we get $r(F+g) > r(H)$, where g means the ordinary series $f+f_1$; the last inequality contradicts (i) and thus (iii) is established.

(Reçu par la Rédaction le 6. 10. 1951)

Sur la convergence des séries de puissances de l'opérateur différentiel

par

C. RYLL-NARDZEWSKI (Wrocław).

1. Considérons la série

$$(1) \quad \gamma_0 + \gamma_1 s \lambda + \gamma_2 s^2 \lambda^2 + \dots,$$

où γ_n sont des nombres complexes, s l'opérateur différentiel¹⁾ et λ une variable complexe.

Pour la convergence de cette série, on a le critère suivant:

S'il existe un nombre $\delta > 1$, tel que

$$(2) \quad \limsup_{n \rightarrow \infty} (n^n)^\delta |\gamma_n| < \infty,$$

la suite (1) converge pour tout λ complexe. Si

$$(3) \quad \limsup_{n \rightarrow \infty} n^n |\gamma_n| > 0,$$

la suite (2) diverge pour tout λ complexe non nul.

Démonstration. Soit $1 < 1/\alpha < \beta < \delta$. Posons pour t réel

$$f(t) = \int_J \exp(zt - z^\alpha) dz,$$

où l'intégrale est prise le long de l'axe imaginaire J . On a

$$(4) \quad f^n(t) = \int_J z^n \exp(zt - z^\alpha) dz \quad (n=0, 1, 2, \dots),$$

car chacune des intégrales (4) converge uniformément pour tout t , ce qui résulte de l'inégalité

¹⁾ Voir [1], p. 47.