# D-Branes on Calabi-Yau Manifolds 

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#### Abstract

In this review we study BPS D-branes on Calabi-Yau threefolds. Such D-branes naturally divide into two sets called A-branes and B-branes which are most easily understood from topological field theory. The main aim of this paper is to provide a self-contained guide to the derived category approach to B-branes and the idea of $\Pi$-stability. We argue that this mathematical machinery is hard to avoid for a proper understanding of B-branes. A-branes and B-branes are related in a very complicated and interesting way which ties in with the "homological mirror symmetry" conjecture of Kontsevich. We motivate and exploit this form of mirror symmetry. The examples of the quintic 3 -fold, flops and orbifolds are discussed at some length. In the latter case we describe the rôle of McKay quivers in the context of D-branes.


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## 1 Introduction

There can be no doubt that the most important development in string theory in recent years is the discovery of D-branes. In flat spacetime a D-brane is regarded as a subspace on which open strings may end. ${ }^{1}$ Since string theory modifies classical notions of geometry at short distances, it is natural to assume that such a simple picture of a D-brane as a subspace is too naïve for more general backgrounds. A more abstract notion of a D-brane is required, one which coincides with the notion of a subspace when viewed in the context of only large distances. The aim of these lectures is to study how this can happen.

It is probably of profound importance in string theory to know a robust definition of D-branes in the most general space-time background, but this problem is far too difficult with our present understanding of string theory. Instead we look for the simplest context in which one might observe nontrivial D-brane behaviour. We render our model as simple as possible by the following steps:

1. Get rid of the enormous complications introduced by time by using a compactification model. We will assume our string theory has a target space $\mathbb{R}^{1,3} \times X$ for some compact space $X$. We focus our attention on $X$.
2. Send the string coupling $g_{s}$ to zero and consider only quantum corrections arising from nonzero $\alpha^{\prime}$ effects.
3. Use just as much supersymmetry as we can while keeping the problem nontrivial. This amounts to an $N=(2,2)$ supersymmetric theory on the worldsheet with $X$ a Calabi-Yau threefold.
4. Consider only the "topological sector" of the worldsheet theory. This results in a finite-dimensional Hilbert space of open strings and we remove all oscillator modes.

As we will see, after such dramatic simplifications, a very rich model remains which requires sophisticated mathematical tools to analyze. One can only wonder at how abstruse a more realistic D-brane, with the above assumptions removed, must be!

Much has already been written about D-branes. We refer to [2], for example, for a review of many aspects of D-branes. In this paper we chart a slightly different course to usual to achieve our aims. Firstly we try wherever possible to avoid the D-brane world-volume approach since this assumes that the D-brane really is a subspace of the target spacetime. Our ideas are then planted on the worldsheet which forces us to take the string coupling to zero. Having put ourselves in the worldsheet, we will avoid much of the boundary-state formalism that one often employs here. Whether this is a judicious choice is up to the reader to decide, but it does not seem to be of great importance in the context of our discussions.

By restricting attention to the topological sector of $N=(2,2)$ worldsheet theories we are in the land of mirror symmetry. In order to keep these lecture notes a manageable

[^0]length we will have to assume at least some familiarity with mirror symmetry for closed strings. We refer to the TASI 1996 lectures, and in particular [3], for a review. The more mathematical reader is referred to [4]. We will assume a rudimentary knowledge of the geometry of Calabi-Yau manifolds. The reviews [3,5] should suffice.

These lectures are primarily intended to review the ideas of the derived category and $\Pi$ stability for B-branes. These subjects have been reviewed by Douglas in [6] from a somewhat different direction than we employ here. Douglas also has a shorter, more mathematicallyoriented review in the ICM proceedings [7]. In order to motivate and better understand our constructions, a good deal of our discussion will also involve mirror symmetry for open strings. This latter topic has been extensively studied and reviewed in many places. In particular, the reader can consult [8] and references therein for a very detailed review of most of the aspects of the subject.

In section 2 we review the basic ideas one needs from topological field theory in the context of closed strings. This leads into mirror symmetry which will be a central tool in our analysis. Section 3 is then a guide to adding boundaries to the string worldsheet. By the end of this section we will realize that there are some difficulties in maintaining mirror symmetry without broadening our concept of D-branes.

Further analysis requires a degree of mathematical sophistication. We review the algebraic geometry that we require for further progress in section 4 . We would like to claim that only the necessary mathematics has been included here, with no complications introduced for their own sake. The fact remains however that pretty esoteric notions in cohomology due to Grothendieck do seem to be directly applicable to D-brane physics, and so we need to delve fairly deeply into this abstract world.

In section 5 it is then a straight-forward process to apply the machinery of section 4 to the case of B-branes. We derive the fact that B-branes are described by the derived category of coherent sheaves.

The notion of $\Pi$-stability, which is essential in relating the derived category to "physical" D-branes, is reviewed in section 6. Much of the motivation for this comes from A-branes and mirror symmetry which we also discuss at length. Finally in section 7 we give a few examples of the derived category and $\Pi$-stability.

## 2 Worldsheet Models of Closed Strings

### 2.1 The $N=(2,2)$ non-linear $\sigma$-model

Let $\Sigma$ be the string worldsheet. We consider a field theory based on all possible maps $\phi: \Sigma \rightarrow X$, where $X$ is the target manifold. This non-linear $\sigma$-model has an action

$$
\begin{equation*}
\frac{i}{8 \pi i} \int_{\Sigma} d^{2} z g_{I J}(\phi) \frac{\partial \phi^{I}}{\partial z} \frac{\partial \phi^{J}}{\partial \bar{z}}, \tag{1}
\end{equation*}
$$

where $z$ is a complex coordinate on $\Sigma$ and $\phi^{I}$ are local coordinates for the map $\phi$. The letters $I$ and $J$ are associated with real coordinates here. The object $g_{I J}$ may be viewed as a metric
on $X$ but it does not need to be symmetric for the non-linear $\sigma$-model to be well-defined. The antisymmetric part of $g_{I J}$ is usually called the "B-field".

This 2-dimensional field theory only defines string theory to an extent. We know that nonperturbative effects in the string coupling are invisible from this point of view. Since the entire content of these lectures is based on this worldsheet definition of string theory, one must realize that our results are only completely valid in the zero string coupling limit.

Assuming $X$ is a Kähler manifold, we may construct the $N=(2,2)$ supersymmetric version of the non-linear $\sigma$-model by adding worldsheet fermions. We now switch to complex coordinates denoted by $\phi^{i}$ and its complex conjugate $\phi^{\bar{\imath}}$. The action is ${ }^{2}$

$$
\begin{align*}
\frac{i}{4 \pi i} \int_{\Sigma} d^{2} z\left\{g_{i \bar{\jmath}}\left(\frac{\partial \phi^{i}}{\partial z} \frac{\partial \phi^{\bar{\jmath}}}{\partial \bar{z}}+\frac{\partial \phi^{i}}{\partial \bar{z}} \frac{\partial \phi^{\bar{\jmath}}}{\partial z}\right)\right. & +i B_{i \bar{\jmath}}\left(\frac{\partial \phi^{i}}{\partial z} \frac{\partial \phi^{\bar{\jmath}}}{\partial \bar{z}}-\frac{\partial \phi^{i}}{\partial \bar{z}} \frac{\partial \phi^{\bar{\jmath}}}{\partial z}\right) \\
& \left.+i g_{i \bar{\jmath}} \psi_{-}^{\bar{\jmath}} D \psi_{-}^{i}+i g_{i \bar{\jmath}} \psi_{+}^{\bar{\jmath}} \bar{D} \psi_{+}^{i}+R_{i \bar{\jmath} \bar{\jmath}} \psi_{+}^{i} \psi_{+}^{\bar{\imath}} \psi_{-}^{j} \psi_{-}^{\bar{\jmath}}\right\} \tag{2}
\end{align*}
$$

where $g_{i \bar{\jmath}}$ is the Kähler metric and $B_{i \bar{\jmath}}$ is a real $(1,1)$-form encoding the B-field degree of freedom. The fermions are defined as sections of bundles on $\Sigma$ as follows:

$$
\begin{align*}
\psi_{+}^{i} & \in \Gamma\left(K^{\frac{1}{2}} \otimes \phi^{*} T_{X}\right) \\
\psi_{+}^{\bar{J}} & \in \Gamma\left(K^{\frac{1}{2}} \otimes \phi^{*} \bar{T}_{X}\right) \\
\psi_{-}^{i} & \in \Gamma\left(\bar{K}^{\frac{1}{2}} \otimes \phi^{*} T_{X}\right)  \tag{3}\\
\psi_{-}^{\bar{\jmath}} & \in \Gamma\left(\bar{K}^{\frac{1}{2}} \otimes \phi^{*} \bar{T}_{X}\right),
\end{align*}
$$

where $K$ is the canonical bundle on $\Sigma$, i.e., the holomorphic cotangent bundle ${ }^{3}, T_{X}$ is the holomorphic tangent bundle on $X$ and bar denotes the corresponding antiholomorphic bundle. $D$ represents the covariant derivative $D \psi_{-}^{i}=\partial \psi_{-}^{i}+\partial \phi^{j} \Gamma_{j k}^{i} \psi_{-}^{j}$, where $\partial$ is the holomorphic part of the de Rham differential as usual.

Let $B=\frac{i}{2} B_{i \bar{j}} d \phi^{i} d \phi^{\bar{J}}$ and assume $d B=0 .{ }^{4}$ In section 2.2 it will become clear that the (continuous) $B$-field degree of freedom lies in $H^{2}(X, \mathbb{R}) / H^{2}(X, \mathbb{Z})$.

The supersymmetries are given by the following transformations:

$$
\begin{align*}
\delta \phi^{i} & =i \alpha_{-} \psi_{+}^{i}+i \alpha_{+} \psi_{-}^{i} \\
\delta \phi^{\bar{\imath}} & =i \tilde{\alpha}_{-} \psi_{+}^{\bar{\imath}}+i \tilde{\alpha}_{+} \psi_{-}^{\bar{\imath}} \\
\delta \psi_{+}^{i} & =-\tilde{\alpha}_{-} \partial \phi^{i}-i \alpha_{+} \psi_{-}^{j} \Gamma_{j k}^{i} \psi_{+}^{k} \\
\delta \psi_{+}^{\bar{\imath}} & =-\alpha_{-} \partial \phi^{\bar{i}}-i \tilde{\alpha}_{+} \psi_{-}^{\bar{\jmath}} \Gamma_{\overline{j k}}^{\bar{i}} \psi_{+}^{\bar{k}}  \tag{4}\\
\delta \psi_{-}^{i} & =-\tilde{\alpha}_{+} \bar{\partial} \phi^{i}-i \alpha_{-} \psi_{+}^{j} \Gamma_{j k}^{i} \psi_{-}^{k} \\
\delta \psi_{-}^{\bar{z}} & =-\alpha_{+} \bar{\partial} \phi^{\bar{\imath}}-i \tilde{\alpha}_{-} \psi_{+}^{\bar{\jmath}} \Gamma_{\bar{j} \bar{k}}^{\bar{i}} \psi_{-}^{\bar{k}}
\end{align*}
$$

[^1]with fermionic parameters $\alpha_{-}$and $\tilde{\alpha}_{-}$as sections of $K^{-\frac{1}{2}}$ and $\alpha_{+}$and $\tilde{\alpha}_{+}$as sections of $\bar{K}^{-\frac{1}{2}}$.
If $X$ is a Calabi-Yau manifold then it is well-known (see, for example, chapters 3 and 17 of [10]) that there will be a metric (close to the Ricci-flat metric if $X$ is large) such that this supersymmetry is extended to an $N=(2,2)$ superconformal symmetry. We restrict to this case from now on.

Let us quickly review some basic facts about $N=(2,2)$ superconformal field theories for Calabi-Yau threefolds in order to fix notation. We urge the reader to consult other sources (such as $[3,11,12]$ and chapter 19 of [10]) for a fuller account of this important subject if they are not familiar with it.

A closed string state forms a representation of the superconformal algebra. This is often encoded in the from of an operator product relationship between the generators of the algebra and the vertex operators associated to the closed strings. The generators of the left-moving algebra are then given by

$$
\begin{align*}
& T(z)=-g_{i \bar{\jmath}} \frac{\partial \phi^{i}}{\partial z} \frac{\partial \phi^{\bar{\jmath}}}{\partial z}+\frac{1}{2} g_{i \bar{\jmath}} \psi_{+}^{i} \frac{\partial \psi_{+}^{\bar{J}}}{\partial z}+\frac{1}{2} g_{i \bar{\jmath}} \psi_{+}^{\bar{\jmath}} \frac{\partial \psi_{+}^{i}}{\partial z} \\
& G(z)=\frac{1}{2} g_{i \bar{\jmath}} \psi_{+}^{i} \frac{\partial \phi^{\bar{\jmath}}}{\partial z}  \tag{5}\\
& \tilde{G}(z)=\frac{1}{2} g_{i \bar{\jmath}} \psi_{+}^{\bar{j}} \frac{\partial \phi^{i}}{\partial z} \\
& J(z)=\frac{1}{4} g_{i \bar{\jmath}} \psi_{+}^{i} \psi_{+}^{\bar{\jmath}}
\end{align*}
$$

with similar expressions for the right-moving $\bar{T}(\bar{z}), \bar{G}(\bar{z}), \tilde{G}(\bar{z})$ and $\bar{J}(\bar{z})$.
Two elements of the superconformal algebra are of interest to us. The first concerns the generator of dilatations of the worldsheet associated to $T(z)$. The eigenvalue of this operation is the conformal weight $h$ of a given state. The second is the charge $q$ associated with the $\mathrm{O}(2)=\mathrm{U}(1)$ R-symmetry of the superconformal algebra associated to $J(z)$. Since we have both a left-moving and a right-moving $N=2$ algebra, we have left-moving weight and charge which we denote $h$ and $q$, and a right-moving weight and charge which we denote $\bar{h}$ and $\bar{q}$.

The R-symmetry part of the superconformal algebra can essentially be "factored out" in the following sense. The $U(1)$ currents can be bosonized using bosons $\varphi$ and $\bar{\varphi}$ :

$$
\begin{equation*}
J(z)=i \sqrt{3} \frac{\partial \varphi}{\partial z}, \quad \bar{J}(\bar{z})=i \sqrt{3} \frac{\partial \bar{\varphi}}{\partial \bar{z}} \tag{6}
\end{equation*}
$$

If we have a vertex operator in the left-moving sector with charge $q$, then we can essentially write it as ${ }^{5}$

$$
\begin{equation*}
f=f_{0} \exp \left(\frac{i}{\sqrt{3}} q \varphi\right) \tag{7}
\end{equation*}
$$

where the operator $f_{0}$ will have charge 0 .

[^2]Periodic or anti-periodic boundary conditions on the fermions lead to the Ramond and Neveu-Schwarz sectors respectively as usual. The NS sector has $q \in \mathbb{Z}$ whilst the R sector has $q \in \mathbb{Z}+\frac{1}{2}$.

Naturally there are an infinite number of string states in this theory but there is a very interesting finite subset which is of central importance. Unitarity forces certain constraints on the allowed weights and charges. In the NS sector we have a set of states lying on the boundary of this set of unitary representations which satisfy

$$
\begin{equation*}
h=|q / 2|, \quad q=-3,-2, \ldots, 3 \tag{8}
\end{equation*}
$$

The operators producing these states from the vacuum are called "chiral primary" operators for $q>0$ and "antichiral primary" operators for $q<0$. We refer to [11, 13, 14] for more details. For simplicity of notation we will usually refer to both the chiral and antichiral operators as chiral.

The key feature of the chiral operators is that they close nicely under the operator product to form the "chiral algebra" (or, less precisely, the "chiral ring"). This finite-dimensional subalgebra of the full infinite-dimensional algebra of closed string vertex operators seems to encompass a good deal of information about the full superconformal field theory. It is best analyzed using methods of topological field theory as we will see in the following sections. One may also use methods of "gauged linear sigma models" as pioneered in [15]. Indeed, linear sigma models may be used to analyze open strings and D-branes as in [16-18]. We will not pursue the linear sigma model in these lectures.

An operator in the Ramond sector of particular interest is the "spectral flow operator" with $q=3 / 2$ :

$$
\begin{equation*}
\Sigma(z)=\exp \left(i \frac{\sqrt{3}}{2} \varphi\right) \tag{9}
\end{equation*}
$$

This has an operator product expansion with itself as

$$
\begin{equation*}
\Sigma(z) \Sigma(w)=(z-w)^{\frac{3}{4}} \Upsilon(z)+\ldots \tag{10}
\end{equation*}
$$

where

$$
\begin{equation*}
\Upsilon(z)=\exp (i \sqrt{3} \varphi)=\Omega_{i j k} \psi_{+}^{i} \psi_{+}^{j} \psi_{+}^{k} \tag{11}
\end{equation*}
$$

is the chiral primary operator with $q=3$ and $\Omega=\Omega_{i j k} d \phi^{i} d \phi^{j} d \phi^{k}$ is the (3,0)-form on $X$ which is unique up to normalization. The spectral flow operator is responsible for spacetime supersymmetry. Again we refer the reader to $[11,14]$ for details. Note that we have two spectral flow operators, $\Sigma(z)$ and $\bar{\Sigma}(\bar{z})$, which give us $N=2$ supersymmetry in the uncompactified spacetime directions.

### 2.2 The A model

The chiral algebra is best studied by passing to a topological field theory associated to the $N=(2,2)$ superconformal field theory described in section 2.1. There are two topological field theories that naturally occur this way - the "A model" and the "B model" discovered
by Witten $[9,19]$ which we now review. We will generally denote the target space for the A-model by $Y$, and use $X$ as the target space of the B-model. ${ }^{6}$

We "twist" the superconformal field theory by modifying the bundles in which the fermions take values. We set

$$
\begin{align*}
& \chi^{i}=\psi_{+}^{i} \in \Gamma\left(\phi^{*} T_{Y}\right) \\
& \chi^{\bar{\imath}}=\psi_{-\bar{i}}^{\bar{\imath}} \in \Gamma\left(\phi^{*} \bar{T}_{Y}\right)  \tag{12}\\
& \psi_{z}^{\bar{\imath}}=\psi_{+}^{\bar{\imath}} \in \Gamma\left(K \otimes \phi^{*} \bar{T}_{Y}\right) \\
& \psi_{\bar{z}}^{i}=\psi_{-}^{i} \in \Gamma\left(\bar{K} \otimes \phi^{*} T_{Y}\right) .
\end{align*}
$$

Note that the action (1) still makes sense (i.e., it is invariant under rotations of the worldsheet) with this assignment.

The "supersymmetry" (4) still holds but notice that the four $\alpha$ parameters are no longer worldsheet spinors. We consider a restricted version of this symmetry by setting $\alpha=\alpha_{-}=$ $\tilde{\alpha}_{+}$and $\alpha_{+}=\tilde{\alpha}_{-}=0$. That is, we have a symmetry depending on a single scalar parameter $\alpha$. Let us also denote the operator which generates this symmetry $Q$. To be precise,

$$
\begin{equation*}
\delta W=-i \alpha\{Q, W\} \tag{13}
\end{equation*}
$$

for any operator $W$. It follows that (up to equations of motion)

$$
\begin{equation*}
Q^{2}=0 . \tag{14}
\end{equation*}
$$

In other words, $Q$ generates a "BRST symmetry". Furthermore, we may write the action in a simplified form (where $\alpha^{\prime}$ has been chosen suitably):

$$
\begin{equation*}
S=\int_{\Sigma} i\{Q, V\}-2 \pi i \int_{\Sigma} \phi^{*}(B+i J) \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
V=2 \pi g_{i \bar{\jmath}}\left(\psi_{z}^{\bar{\jmath}} \bar{\partial} \phi^{i}+\partial \phi^{\bar{\jmath}} \psi_{\bar{z}}^{i}\right), \tag{16}
\end{equation*}
$$

and $B+i J \in H^{2}(Y, \mathbb{C})$ is the complexified Kähler form.
The next step is to restrict attention only to operators $W$, which are $Q$-closed, i.e., $\{Q, W\}=0$. The effect of the twisting (12) is to mix the notion of weight $h$ and $\mathrm{U}(1)$-charge $q$ from the original untwisted superconformal field theory. It follows that by restricting to $Q$-closed states we are effectively restricting attention to the case $h=q / 2, \bar{h}=-\bar{q} / 2$. That is, to a particular chiral algebra.

Now, suppose we have an operator $W$ which is $Q$-exact in the sense that $W=\left\{Q, W^{\prime}\right\}$ for some $W^{\prime}$. By standard methods one can show that any correlation function involving this operator and other $Q$-closed operators will vanish. In other words, a $Q$-exact operator is equivalent to zero in the chiral algebra.

[^3]```
This means we are restricting attention to \(Q\)-cohomology.
```

The reason we have put this statement in a pretentious little box is that it is the most important mathematical statement in these lectures. The fact that cohomology is essential will lead to a proliferation of homological algebra in the later lectures.

Note that the triviality of $Q$-exactness extends to the action too. That is, under the shift $S \mapsto S+\left\{Q, S^{\prime}\right\}$, correlation functions are invariant. One can show that a change in the worldsheet metric leads to a $Q$-exact shift in the action (15). This means that the location of the vertex operators on the worldsheet are not important (assuming the locations are distinct of course) when computing the correlation functions.

The action (15) manifestly depends on the complexified Kähler form but any change in complex structure merely changes $V$ and is thus trivial. So the correlation functions in this topological A-model depend only on the complexified Kähler form $B+i J$. Furthermore it is manifest from the action that it is only the cohomology class of $B+i J$ that is of importance, and that a shift in $B$ by an element of integral cohomology will not affect the correlation functions.

Operators will be general functions of the fields $\phi$ and $\psi$. We first consider "local operators" in $\Sigma$, i.e., scalars. This means we cannot use $\psi_{z}^{\bar{z}}$ or $\psi_{\bar{z}}^{i}$ as they are 1 -forms on $\Sigma$. A basis for the vector space of local operators is therefore given by operators of the form

$$
\begin{equation*}
W_{a}=a_{I_{1} I_{2} \ldots I_{p}} \chi^{I_{1}} \chi^{I_{2}} \ldots \chi^{I_{p}} \tag{17}
\end{equation*}
$$

where $a=a_{I_{1} I_{2} \ldots I_{p}} d \phi^{I_{1}} d \phi^{I_{2}} \ldots d \phi^{I_{p}}$ is a $p$-form on $Y$. The $I_{n}$ 's represent real indices - in other words they may be holomorphic or antiholomorphic. One can then compute

$$
\begin{equation*}
\left\{Q, W_{a}\right\}=-W_{d a} \tag{18}
\end{equation*}
$$

That is, for the A-model, $Q$-cohomology is de Rham cohomology and the space of operators is given by $H^{*}(Y, \mathbb{C})$.

Let us now address the question of how we might compute a correlation function between such operators:

$$
\begin{equation*}
\left\langle W_{a} W_{b} W_{c} \ldots\right\rangle=\int \mathscr{D} \phi \mathscr{D} \chi \mathscr{D} \psi e^{-S} W_{a} W_{b} W_{c} \ldots \tag{19}
\end{equation*}
$$

The fact that the action (15) splits naturally into two pieces makes life particularly easy when analyzing the space of all maps $\phi: \Sigma \rightarrow Y$. Let us assume $\Sigma$ is a sphere. The space of all maps then breaks up into connected components corresponding to elements of $\pi_{2}(Y)$. On a given component the second term in the action (15) is constant and can be pulled out of the path integral.

The term in the action that remains is $Q$-exact and is therefore trivial. Although one's first temptation might be to replace something that is trivial by zero, we do the opposite and rescale it by a factor that tends to infinity! Then the fact that this integrand is positive semi-definite means that we effectively restrict the path integral to maps $\phi$ where this part
of the action is zero. These are "worldsheet instantons". In other words, the saddle-point approximation of instantons is exact for topological field theories. The worldsheet instantons are given by $V=0$ in (16). These are holomorphic maps $\bar{\partial} \phi^{i}=0$.

The infinite-dimensional space of all maps $\phi: \Sigma \rightarrow Y$ is therefore replaced by the finitedimensional space of holomorphic maps when we perform the path integral. Supersymmetry then cancels the Pfaffians associated with the fermionic path integral and the remaining determinants from the $\phi$ integrals. We refer to [20] for more details on this cancellation process.

We focus on the $p$-forms for $p$ even since the odd forms do not directly correspond to operators in the untwisted superconformal field theory. The 0 -form clearly represents the identity operator. The simplest case is therefore to consider correlation functions between operators associated to 2 -forms. One can show that

$$
\begin{equation*}
\left\langle W_{a} W_{b} W_{c}\right\rangle=\int_{Y} a \wedge b \wedge c+\sum_{\alpha \in I} N_{a b c}^{\alpha} e^{2 \pi i \int_{\Sigma} \phi^{*}(B+i J)} \tag{20}
\end{equation*}
$$

where $I$ is the set of instantons and $N_{a b c}^{\alpha}$ are integers given by the intersection theory on the moduli space of rational curves (i.e., holomorphic embeddings of $\Sigma$ ) in $Y$, including the possibility of multiple covers [21,22].

The knowledge of these correlation functions between 2-forms is sufficient to define an algebra (i.e., multiplicative) structure on $H^{\text {even }}(Y, \mathbb{C})$. This is equivalent to the operator algebra. In the large radius limit, where $J \rightarrow \infty$, this coincides with the cohomology ring given by the wedge product. At finite volume the deformed ring is called the "quantum cohomology ring" of $Y$. We have impinged on a vast subject here which we do not have space to explore more fully. We refer to [4] and references therein for a more detailed account of this important subject together with more recent developments.

The operator algebra is graded by the degree of the forms. Viewing $Q$ as a generator of a BRST symmetry we can also refer to the grading as a "ghost number". That is, if $a$ is a $p$-form then the operator $W_{a}$ has ghost number $p$. The ghost number maps naturally back to the $\mathrm{U}(1)$ charges in the untwisted theory. In this case $W_{a}$ would map to an operator with $(q, \bar{q})=(p / 2,-p / 2)$. Note also that the correlation function of a product of operators is only nonzero if the total ghost number is 6 . This means that the grading of the operator algebra is preserved under multiplication mod 6 .

An important aspect of the A-model for our purposes concerns deformations of the theory. An operator within the theory may be used to deform the Lagrangian density if it makes sense to integrate such an operator over $\Sigma$ to deform the action. To find such operators we need to look beyond the local operators considered so far. Suppose $W_{a}$ is a local operator with ghost number $p$. The operator $d W_{a}$ (where " $d$ " is the worldsheet de Rham operator) will have trivial correlation functions with other operators since the location of the vertex operator insertions is unimportant. It follows that it must be $Q$-exact, i.e.,

$$
\begin{equation*}
d W_{a}=\left\{Q, W_{A}^{(1)}\right\} \tag{21}
\end{equation*}
$$

for some operator $W_{A}^{(1)}$ with ghost number $p-1$. We may repeat this process again by setting

$$
\begin{equation*}
d W_{a}^{(1)}=\left\{Q, W_{A}^{(2)}\right\}, \tag{22}
\end{equation*}
$$

for some operator $W_{A}^{(2)}$ with ghost number $p-2$. But $W_{A}^{(2)}$ is a 2 -form and so we can naturally integrate it over $\Sigma$. We may therefore consider a deformation of the theory given by

$$
\begin{equation*}
S \mapsto S+t \int_{\Sigma} W_{A}^{(2)} d^{2} z \tag{23}
\end{equation*}
$$

for some infinitesimal $t$. In order to preserve the grading of the operator algebra given by the ghost number, the deformation of the action should have ghost number zero, i.e., $p=2$.

It is not hard to see that this deformation of the field theory corresponds to deforming $B+i J$ by a 2 -form proportional to $t A$. Since the only dependence of the A-model was on $B+i J$, we see that we have described all the deformations of the A-model. (Other deformations that violate ghost number conservation were considered in [9].)

It is important to realize that the topological A-model is a different quantum field theory to the original $N=(2,2)$ superconformal field theory. Even though the vector space of primary chiral operators is naturally a subspace of the infinite-dimensional space of operators in the untwisted theory, the operator products may be quite different. There is an exception however. If the worldsheet is flat then the twisting has no affect. If the $N=(2,2)$ were used to compactify a heterotic string (rather than the type II strings we consider in these lectures) then the products which determine the effective superpotential of the resulting $N=1$ theory in four dimensions are unchanged in the topological field theory. We refer to [9] for more details.

We emphasize again that the structure of the operator algebra depends only upon $B+i J$ and not the complex structure of $Y$. In fact, as explored in [19], we don't need any complex structure on $Y$, nor do we require the Calabi-Yau condition. $Y$ can be any symplectic manifold with a compatible almost complex structure. Instantons then correspond to pseudoholomorphic curves. Since the topological A-model knows about only a small subset of the data of the untwisted theory, it should not come as a surprise that it can be applied to a wider class of target spaces.

In this section we considered a fixed worldsheet of genus zero mapping into $Y$. If higher genera are considered, the A-model becomes fairly trivial because of ghost number conservation constraints. A variant of the A-model that is commonly considered consists of coupling the worldsheet theory to gravity. In other words one includes all metrics on $\Sigma$ in the path integral. Such a theory now contains nontrivial information about higher genus worldsheets as discovered in [23]. This "topological gravity" is also important in the "large N" duality of $[24,25]$.

If we were going to do a full treatment of mirror symmetry for open strings we would certainly have to wade into many of the technicalities of the A-model coupled to gravity. However, in these lectures, which focus on the issues of stability, we can get away with largely ignoring this topic.

### 2.3 The B model

We may relabel the fermions in the superconformal field theory in a different way to obtain the "B-model" which was also introduced by Witten [9].

Let $\psi_{ \pm}^{\bar{j}}$ be sections of $\phi^{*}\left(\bar{T}_{X}\right)$, while $\psi_{+}^{j}$ is a section of $K \otimes \phi^{*}\left(T_{X}\right)$ and $\psi_{-}^{j}$ is a section of $\bar{K} \otimes \phi^{*}\left(T_{X}\right)$. Define scalars

$$
\begin{align*}
& \eta^{\bar{\jmath}}=\psi_{+}^{\bar{\jmath}}+\psi_{-}^{\bar{\jmath}} \\
& \theta_{j}=g_{j \bar{k}}\left(\psi_{+}^{\bar{k}}-\psi_{-}^{\bar{k}}\right), \tag{24}
\end{align*}
$$

and define a 1 -form $\rho^{j}$ with (1,0)-form part given by $\psi_{+}^{j}$ and ( 0,1 )-form part given by $\psi_{-}^{j}$.
Now consider a variation corresponding to the original supersymmetric variation with $\alpha_{ \pm}=0$ and $\tilde{\alpha}_{ \pm}=\alpha$. As in the A-model, this produces a BRST-like variation $Q$ satisfying $Q^{2}=0$ (up to equations of motion).

Now, for a suitable choice of $\alpha^{\prime}$, we may rewrite the action in the form

$$
\begin{equation*}
S=i \int\{Q, V\}+U \tag{25}
\end{equation*}
$$

where

$$
\begin{align*}
V & =g_{j \bar{k}}\left(\rho_{z}^{j} \bar{\partial} \phi^{\bar{k}}+\rho_{\bar{z}}^{j} \partial \phi^{\bar{k}}\right) \\
U & =\int_{\Sigma}\left(-\theta_{j} D \rho^{j}-\frac{i}{2} R_{j j k \bar{k}} \rho^{j} \wedge \rho^{k} \eta^{\bar{j}} \theta_{l} g^{l \bar{k}}\right) \tag{26}
\end{align*}
$$

An additional complication arises in the B-model because the fermions are twisted in a more asymmetric fashion than in the A-model. For a general target space $X$ one has a chiral anomaly associated with a problem properly defining the phase of the Pfaffian associated to the fermionic path integrals. This anomaly is zero if we require $c_{1}\left(T_{X}\right)=0$, i.e., if $X$ is a Calabi-Yau manifold.

It is not immediately obvious from (26) but $U$ depends only upon the complex structure of $X$. It is independent of both the metric on $\Sigma$ and the complexified Kähler form on $X$, $B+i J$. Thus the correlation functions have a similar independence.

Local observables are now written

$$
\begin{equation*}
W_{A}=\eta^{\bar{k}_{1}} \ldots \eta^{\bar{k}_{q}} A_{\bar{k}_{1} \ldots \bar{k}_{q}}^{j_{1} \ldots j_{p}} \theta_{j_{1}} \ldots \theta_{j_{p}} \tag{27}
\end{equation*}
$$

where

$$
\begin{equation*}
A=d \bar{z}^{\bar{k}_{1}} \ldots d \bar{z}^{\bar{k}_{q}} A_{\bar{k}_{1} \ldots \bar{k}_{q}}^{j_{1} \ldots j_{p}} \frac{\partial}{\partial z_{j_{1}}} \ldots \frac{\partial}{\partial z_{j_{p}}}, \tag{28}
\end{equation*}
$$

is a $(0, q)$-form on $X$ valued in $\bigwedge^{p} T_{X}$. One might call $A$ a " $(-p, q)$-form". Note that we can use contraction with the holomorphic 3 -form $\Omega$ to give an isomorphism between the spaces of $(-p, q)$-forms and $(3-p, q)$-forms. This isomorphism is often used implicitly and explicitly in discussions of mirror symmetry as we will see in section 2.4.

Now,

$$
\begin{equation*}
\left\{Q, W_{A}\right\}=-W_{\bar{\partial} A}, \tag{29}
\end{equation*}
$$

and so, for the B-model, $Q$-cohomology is Dolbeault cohomology on forms valued in exterior powers of the holomorphic tangent bundle.

The instantons in the B-model are trivial. Setting $V=0$ in (26) requires $\bar{\partial} \phi^{\bar{k}}=\partial \phi^{\bar{k}}=0$, i.e., $\phi$ is a constant map mapping $\Sigma$ to a point in $X$. Thus the correlation functions do not consist of some infinite sum.

The generators of the operator algebra of interest in the B -model are given by $(-1,1)$ forms. The three-point functions can be shown to be

$$
\begin{equation*}
\left\langle W_{A} W_{B} W_{C}\right\rangle=\int_{X} \Omega^{j k l} A_{j} \wedge B_{k} \wedge C_{k} \wedge \Omega \tag{30}
\end{equation*}
$$

where $A=A^{j} \frac{\partial}{\partial \phi^{j}} \in H_{\bar{\partial}}^{1}\left(X, T_{X}\right)$ etc. The object $\Omega^{j k l}$ can be obtained from the antiholomorphic 3-form $\bar{\Omega}$ using the Kähler metric to raise indices.

A $(-p, p)$-form in the B-model has ghost number $2 p$ and maps to an operator with $(q, \bar{q})=(p, p)$ in the untwisted model. Just as in the A-model we may consider deforming the theory by adding operators to the Lagrangian density. This time such operators correspond to $(-1,1)$-forms, i.e., elements of $H_{\bar{\partial}}^{1}\left(X, T_{X}\right)$. That this cohomology group corresponds to deformations of complex structure of $X$ is well-known (see chapter 15 of [26] for a nice account of this).

Note that the B-model does require that $X$ has a complex structure and that it be Calabi-Yau. However, it does not require any mention of $B+i J$. This means that the B-model requires only "algebraic" knowledge of $X$ in the following sense. Suppose that $X$ is an "algebraic variety" i.e., a subspace of $\mathbb{P}^{N}$ defined by the intersection of various (homogeneous) equations $f_{1}=f_{2}=\ldots=0$ in the homogeneous coordinates. Then the B -model is defined completely by the equations $f_{1}, f_{2}, \ldots$

The fact that the B-model has no instanton corrections together with the above algebraic nature means that one should think of the B-model as being the "easy" topological field theory and the A-model as the "difficult" theory. When we discuss open strings in section 5 the reader may decide that the B-model is not so "easy" after all but no one can deny that it is a good deal easier than the A-model!

### 2.4 Mirror Symmetry

There are several definitions of mirror symmetry varying in strength. We require only a fairly weak definition which asserts that two Calabi-Yau threefolds $X$ and $Y$ are mirror if the operator algebra of the A-model with target space $Y$ is isomorphic to the operator algebra of the B-model with target space $X$.

The original definition is stronger and is a statement concerning conformal field theories. The strongest definition would be that the type IIA string compactified on $Y$ yields "isomorphic" physics in four dimensions to the type IIB string compactified on $X$.

A simple analysis of the dimensions of the vector spaces of the operator algebra yields the simple statement that $h^{p, q}(Y)=h^{3-p, q}(X)$ and thus $\chi(Y)=-\chi(X)$.

The operator algebra for the A-model on $Y$ depends on a choice of $B+i J$ on $Y$ and the operator algebra for the $\mathrm{B}-$ model on $X$ depends on a choice of complex structure for $X$. Thus a precise statement of mirror symmetry must map the moduli space of $B+i J$ of $Y$ to the moduli space of complex structures of $X$. This mapping is called the "mirror map" and we now discuss it in some detail for a simple key example.

Let us introduce the most-studied example of a mirror pair of Calabi-Yau threefolds following $[21,27] . Y$ is the "quintic threefold, i.e., defined as a hypersurface in $\mathbb{P}^{4}$ given by the vanishing of an equation of degree 5 in the homogeneous coordinates. Since $h^{1,1}(Y)=1$, the moduli space of complexified Kähler classes is only one-dimensional. Let $e$ denote the positive ${ }^{7}$ generator of $H^{2}(Y, \mathbb{Z})$. Then, by an abuse of the notation, we will refer to the cohomology class of the complexified Kähler form as $(B+i J) e$, i.e., $B$ and $J$ are real numbers in the context of the quintic. Basically we can think of the size of $Y$ (i.e., $J$ ) being determined purely by the size of the ambient $\mathbb{P}^{4}$.
$Y$ has $h^{2,1}=101$ and thus 101 deformations of complex structure but this is of no interest to us here.

The mirror $X$ of the quintic is constructed by dividing $Y$ by a $\left(\mathbb{Z}_{5}\right)^{3}$ orbifold action. We refer to [3] for a review of why this orbifold yields the mirror. $Y$ has orbifold singularities which should be resolved yielding many degrees of freedom for $B+i J$. However, all we care about is the complex structure of $X$ which may be defined by specifying the exact quintic polynomial used. The most general quintic compatible with the $\left(\mathbb{Z}_{5}\right)^{3}$ orbifold action is given by

$$
\begin{equation*}
x_{0}^{5}+x_{1}^{5}+x_{2}^{5}+x_{3}^{5}+x_{4}^{5}-5 \psi x_{0} x_{1} x_{2} x_{3} x_{4} . \tag{31}
\end{equation*}
$$

Thus the complex structure is determined by the single complex parameter $\psi$. The mirror map we desire will be a mapping between $B+i J$ on the A-model side and $\psi$ on the B-model side. This map turns out to be quite complicated and is actually a many-to-many mapping.

Because the mirror map is not globally well-defined one generally starts with a basepoint, which is usually the large radius limit on the A-model side, and finds the mirror map in some neighbourhood of this basepoint. One can then try to analytically continue the mirror map to a larger region.

One may analyze the moduli space intrinsically without any reference to a specific compactification by studying the general features of scalar fields in $N=2$ theories of supergravity in four dimensions. The result is that the moduli space is a so-called "special Kähler manifold" [28-30]. For a nice mathematical treatment of this subject see [31].

The special Kähler structure of the moduli space leads to the existence of favoured (but no uniquely defined) coordinates, the "special coordinates" which obey certain flatness constraints. On the A-model side, the components of $B+i J$ form such special coordinates. On the B-model side, the natural complex parameters such as $\psi$ in (31) do not form special coordinates. Instead, the special coordinates are formed from periods as follows. Let $\alpha_{m}, \beta^{m}$

[^4]for $m=0 \ldots h^{2,1}(Y)$ form a symplectic basis of $H_{3}(X, \mathbb{Z})$ in the sense that we have the following intersection numbers
\[

$$
\begin{equation*}
\alpha_{m} \cap \alpha_{n}=0, \quad \alpha_{m} \cap \beta^{n}=\delta_{m}^{n}, \quad \beta^{m} \cap \beta^{n}=0 . \tag{32}
\end{equation*}
$$

\]

A "period" of the holomorphic 3-form

$$
\begin{equation*}
\varpi_{m}=\int_{\alpha_{m}} \Omega \tag{33}
\end{equation*}
$$

is not intrinsically defined as we may rescale $\Omega$ by a constant. However, ratios of periods, $\varpi_{m} / \varpi_{0}$ for $m=1 \ldots h^{2,1}$ are well-defined. These ratios do form special coordinates and these are naturally mapped to components of $B+i J$ by the mirror map.

To find exactly which periods are mapped to which components of $B+i J$ one looks at the monodromy of these coordinates around the large radius limit induced by the symmetry $B \mapsto B+1$. A systematic method for doing this was analyzed in [32]. The criteria we have described so far almost determines the mirror map uniquely. To nail down the last constants one really needs to explicitly count some rational curves on $Y$ and map the correlation functions of the A-model to that of the B-model directly. Having said that, there is a conjectured form of the mirror map (which was implicitly used in [21]) which appears to work in all known cases. We refer to [4] for more details.

Let's see how all this works for the case of the quintic. We first need to find the relationship between the periods of $X$ and the parameter $\psi$. This relationship is encoded in a differential equation called the "Picard-Fuchs" equation. This is a differential equation whose solutions are the periods (33). There are various ways of deriving this equation. A fairly tortuous method was originally pursued in [21] with a more direct way discussed in [33]. The nicest method was given in [34] (see also [35]) in terms of toric geometry.

First introduce a coordinate $z=(5 \psi)^{-5}$ on the B-model moduli space. The method of [34] yields a differential equation

$$
\begin{equation*}
\left(z \frac{d}{d z}\right)^{4} \Phi-5^{5} z\left(z \frac{d}{d z}+\frac{1}{5}\right)\left(z \frac{d}{d z}+\frac{2}{5}\right)\left(z \frac{d}{d z}+\frac{3}{5}\right)\left(z \frac{d}{d z}+\frac{4}{5}\right) \Phi=0 . \tag{34}
\end{equation*}
$$

Expanding around $z=0$ we obtain a basis of solutions in the following form:

$$
\begin{align*}
& \Phi_{0}=\sum_{n=0}^{\infty} \frac{(5 n)!}{n!^{5}} z^{n}  \tag{35}\\
& \Phi_{k}=\frac{1}{(2 \pi i)^{k}} \log (z)^{k} \Phi_{0}+\ldots, \quad k=1,2,3
\end{align*}
$$

The monodromy of this set of solutions around $z=0$ is precisely the right form to be associated with the large radius limit $J=\infty$ on the A-model side [32]. The mirror map is then given by ${ }^{8}$

$$
\begin{equation*}
B+i J=\frac{\Phi_{1}}{\Phi_{0}}=\frac{1}{2 \pi i}\left(\log (z)+770 z+717825 z^{2}+\ldots\right) . \tag{36}
\end{equation*}
$$

[^5]

Figure 1: Five fundamental regions for the moduli space of the quintic.

There are three points of particular interest in the moduli space where the Picard-Fuchs equation becomes singular. As we have just stated, the point $z=0$ corresponds to the large radius limit. The point $z=\infty($ or $\psi=0)$ corresponds to the "Gepner model" [36]. It may also be interpreted as a $\mathbb{Z}_{5}$-orbifold of the Landau-Ginzburg theory $[15,37,38]$. The solutions to the Picard-Fuchs equation have a branch point of order 5 at this point. Finally there is a singularity at $z=5^{-5}$ (or $\psi=\exp (2 \pi i n / 5)$ ) usually referred to as the "conifold" point.

A nice way to visualize the mirror map is to plot fundamental regions of the moduli space in the $(B+i J)$-plane. To do this we put branch cuts along $\psi=R \exp (2 \pi i n / 5)$ for real $R>0$ and $n=0,1$. In figure 1 we show the "scorpion" diagram from [21] which shows five fundamental regions. These consist of the one containing the large radius limit in the region $-1<B<0$ together with 4 other fundamental regions obtained by analytically continuing around the Gepner point.

It is very important to note that the fundamental regions do not tesselate in general. Monodromy around the large radius limit induces a shift $B \mapsto B+1$ and it is clear from figure 1 that such shifts cause overlaps between fundamental regions.

If we stick to the region containing the large radius limit we see that the Gepner point represents the "smallest" possible quintic threefold. For further discussion of minimal sizes in this context see $[35,39]$.

A more typical example of a mirror pair will require analysis of moduli spaces of more than one complex dimension. This makes the problem a good deal more complicated than the quintic but we do not require any more basic concepts to solve this problem. The PicardFuchs equations are now a set of simultaneous linear partial differential equations. We refer to [39], for example, for an efficient way of dealing with this situation.

## 3 Boundaries

### 3.1 The A-model

In this section we consider a worldsheet $\Sigma$ with boundaries. A careful analysis of this gets quite technical quite quickly, taking us beyond where we need to be for these talks. We refer the reader to [40] (and also [41,42] for the most thorough treatment. One should also consult [8] which is based on the analysis of [43]. In the following we will make rough and ready assumptions which are quite adequate for our purposes.

### 3.1.1 A-branes

As stated earlier one of the main purposes of these lectures is to demonstrate the existence of D-branes which do not correspond simply to subspaces. Despite this, we will initially assume that D-branes are subspaces. Thus we assume that we have a collection of subspaces $L_{a} \subset Y$ and that our maps $\phi: \Sigma \rightarrow Y$ obey the condition

$$
\begin{equation*}
\phi(\partial \Sigma) \subset \bigcup_{a} L_{a} \tag{37}
\end{equation*}
$$

i.e., the open strings end on the D-branes $L_{a}$. We have not yet constrained the dimensions of the D-branes and one might be free to consider the case that one of the $L_{a}$ 's fills $Y$ in which case we have imposed no condition at all. A D-brane that can appear in the A-model will have to satisfy certain constraints which we now discuss. Such a D-brane is called an "A-brane".

The first step is to apply the variational principle to the problem. Applying a variation of the fields and then integrating by parts divides the variation of the action into two parts - the bulk and the boundary. Setting the variation of the bulk to zero yields the EulerLagrange equations in the usual way. Demanding that the variation of the boundary is zero imposes further conditions.

In flat space the vanishing of the variation of the boundary imposes either Dirichlet or Neumann conditions for the fields $\phi^{I}$ (see [44] for example). More generally [40, 45] we set

$$
\begin{equation*}
\frac{\partial \phi^{I}}{\partial z}=R_{J}^{I}(\phi) \frac{\partial \phi^{J}}{\partial \bar{z}}+\text { fermions } \tag{38}
\end{equation*}
$$

where $R$ is a matrix orthogonal with respect to the metric $g_{I J}$. Eigenvectors of $R$ with eigenvalue -1 give Dirichlet conditions and are thus associated with directions normal to $L$.

To be completely general, one need not assume that directions tangent to the D-branes are associated to eigenvectors with eigenvalue $+1[45,46]$, but for our purposes we may make this assumption.

It is impossible to preserve all the $N=(2,2)$ supersymmetry of section 2.1 once $\Sigma$ has a boundary. This is because we must have a reflection condition at the boundary which mixes the left-moving and right-moving fermions. The best we can do is use the same reflection matrix as above:

$$
\begin{equation*}
\psi_{+}^{I}=R_{J}^{I}(\phi) \psi_{-}^{J} \tag{39}
\end{equation*}
$$

Now, referring to the A-model twist of (12), such a reflection only really makes sense in the A-model if $R_{j}^{i}=R_{\bar{j}}^{\bar{i}}=0$ when we use holomorphic coordinates. That is, only the off-diagonal terms $R_{j}^{\bar{i}}$ and $R_{\bar{j}}^{i}$ are nonzero.

Now choose a vector $v$ which has eigenvalue +1 with respect to $R$, i.e., a tangent vector in the D-brane. Let us introduce the almost complex structure $J$, which in holomorphic coordinates is of the form

$$
\begin{equation*}
J_{n}^{m}=i \delta_{n}^{m}, \quad J_{\bar{n}}^{\bar{m}}=-i \delta_{\bar{n}}^{\bar{m}} \tag{40}
\end{equation*}
$$

with off-diagonal entries equal to zero. It is then easy to see that the vector $J v$ has eigenvector -1 . Furthermore, $J^{2} v=-v$, so a further application of $J$ restores us to the tangent direction. Thus $J$ exchanges the directions tangent and normal to the D-brane $L$. Clearly then $L$ must be of middle dimension, i.e., real dimension 3 .

Note that if $v$ and $w$ are two tangent vectors in $L$ with eigenvalue +1 under $R$, then $w$ is orthogonal to $J v$ with respect to the metric $g_{I J}$. Since, by definition, the Kähler form on $Y$ is $\frac{1}{2} g_{L M} J_{N}^{M} d \phi^{L} d \phi^{N}$, we see that the Kähler form restricted to $L$ is zero.

A middle-dimensional manifold on which the Kähler form restricts to zero is called a $L a$ grangian submanifold. Thus we have argued that the simplest D-branes compatible with the A-model twist appear to consist of Lagrangian submanifolds. There are further constraints which we discuss shortly.

A more careful analysis $[45,46]$ shows that a Calabi-Yau $n$-fold may have "coisotropic" submanifolds of real dimension $n+2 p$ for non-negative integer $p$. Such submanifolds will be of no interest to us in the case of Calabi-Yau threefolds since $b_{5}=0$ so long as the holonomy is not a proper subgroup of $\mathrm{SU}(3)$.

Thus far we have taken care of the analysis of the theory that pertains to the metric. We should also consider the effect of the boundary on the $B$-field.

When $\Sigma$ had no boundary, it was apparent from the A-model action (15) that only the cohomology class of $B$ affected any correlation functions. This is no longer true when $\Sigma$ has a boundary and so there are more degrees of freedom associated to the $B$-field than would arise from $H^{2}(Y, \mathbb{R})$. Naturally these degrees of freedom must be associated to the boundary and so should show up in the D-brane.

This extra freedom may be written in the guise of 1-form $A$ on $Y$ and an addition of a term

$$
\begin{equation*}
S_{\partial \Sigma}=-2 \pi i \oint_{\partial \Sigma} \phi^{*}(A) \tag{41}
\end{equation*}
$$

to the action. In order to maintain supersymmetry and/or BRST invariance it is also necessary to add some terms involving fermions to this boundary contribution.

A shift of the $B$-field by an exact 2 -form $d \Lambda$ is then an invariance of the theory if it is accompanied by a shift $A \mapsto A-\Lambda$. Thus, with this symmetry understood, the $B$-field is restored to living in $H^{2}(Y, \mathbb{R})$ and we have a new parameter $A$. Setting $F=d A$, we note that $B+F$ is invariant under the $\Lambda$-symmetry and thus, unlike $B$ or $F$ alone, can be a physically meaningful parameter.

It is also important to realize that the theory is no longer invariant under a lone shift of the $B$-field by an integral 2 -form. An invariance is obtained by accompanying such a shift by a similar shift in $F$. We will see this effect clearly in section 7.1.4.

Typically one thinks of $A$ as the connection on a $\mathrm{U}(1)$-bundle associated to the boundary of $\Sigma$ in the form of "Chan-Paton" factors.

Like the $B$-field, the only contribution from $A$ to the correlation functions will arise from worldsheet instantons. As in section 2.2, worldsheet instantons correspond to holomorphic maps of $\Sigma$ into $Y$. The action of such an instanton will receive a contribution from (41) in the form of a line integral of $A$ around the boundary of $\Sigma$ in the D-brane $L$. As this integral contains only directions tangent to $L$, it is only the projection of $A$ into the cotangent bundle of $L$ that matters. Thus the $\mathrm{U}(1)$-bundle may be considered as living purely on the D-brane even though we defined $A$ as living in the cotangent bundle of $Y$.

To derive this notion that $A$ is a connection one should really follow the "gerbe" description of the $B$-field [47]. We will not attempt to do this here. As we will do later, one is free to associate larger gauge groups than $\mathrm{U}(1)$ to the D -branes. One should always be aware, however, that there is a natural diagonal $\mathrm{U}(1)$ in this gauge group which is associated to the $B$-field by the $\Lambda$ symmetry.

The condition that the BRST symmetry (or supersymmetry of the untwisted theory) is preserved puts a condition on the connection $A$. In the case that $B=0$ one can show $[40,43,45,48]$ that $F=0$, i.e., the connection must be flat. We will generally restrict to this case.

The statement that an A-brane consists of a Lagrangian submanifold with a flat bundle is a purely classical statement. Quantum considerations impose two further constraints. The first arises due to an anomaly. We would like to preserve the ghost number grading of the operator product algebra once we include A-branes. It turns out that an arbitrary Lagrangian submanifold can break this symmetry which, in physics language, is due to an anomaly.

This anomaly is carefully analyzed in chapter 40 of [8] and is tied to the problem of grading Floer cohomology [49]. Since this subject is rather technical we will simply state the result here. Let us fix a particular choice of a holomorphic 3-form $\Omega$ on $Y$. At any point $p$ on a Lagrangian submanifold $L$ the volume form of $L$ may be written as a restriction

$$
\begin{equation*}
d V_{L}=\left.R e^{-i \pi \xi(p)} \Omega\right|_{L}, \tag{42}
\end{equation*}
$$

where $R$ is a positive real number. $\xi$ gives a map from $L$ to a circle $\xi: L \rightarrow S^{1}$. This in


Figure 2: Loops which do and do not give an anomaly.
turn induces a map on the fundamental group

$$
\begin{equation*}
\xi_{*}: \pi_{1}(L) \rightarrow \pi_{1}\left(S^{1}\right) \cong \mathbb{Z} \tag{43}
\end{equation*}
$$

known as the "Maslov class" of $L$.
The anomaly is absent precisely when the Maslov class of $L$ is zero. Clearly this is always the case when $\pi_{1}(L)=0$. For a nontrivial example we can picture, consider the case where $Y$ is a one-complex-dimensional torus. Any line of $Y$ is trivially Lagrangian. As shown in figure 2, a contractible loop has a nontrivial Maslov class and so is ruled out as an A-brane.

The second condition on A-branes which arises from quantum effects concerns destabilizing from open string tadpoles. This requires knowledge about the open string states to which we now turn.

### 3.1.2 Open strings for one A-brane

The Hilbert space of closed string states in the A-model is unaffected by the presence of Abranes and is still given by the De Rham cohomology of $Y$. In addition we need to consider open strings stretching between two D-branes. Clearly there are two possibilities:

1. The ends of the open string may lie on the same D-brane.
2. The string stretches between two distinct D-branes.

These two cases are best analyzed differently. The first case was analyzed by Witten in [48].
Suppose we have a Lagrangian cycle $L$ with a $\operatorname{GL}(N, \mathbb{C})$ vector bundle $E \rightarrow L$. By the usual methods of Chan-Paton factors [10], the open strings will lie in the set $\operatorname{End}(E)$, i.e., endomorphisms of $E$. In the case of a line bundle we simply have $\operatorname{End}(E)=\mathbb{C} . Q$-invariance allows us to take a scaling limit [48] effectively taking the string tension to infinity, which implies that the open string states are only associated with constant maps $\phi$ and the fermions take values in the tangent bundle of $L$.

Thus, local operators corresponding to the insertion of open string states on the boundary of $\Sigma$ which maps to $L$ are given by objects of the form

$$
\begin{equation*}
a_{I_{1} I_{2} \ldots}(\phi) \chi^{I_{1}} \chi^{I_{2}} \ldots \tag{44}
\end{equation*}
$$

where $a_{I_{1} I_{2} \ldots}(\phi) \in \operatorname{End}(E), \phi \in L$, and $\chi^{I_{k}}$ lies in the tangent bundle of $L$. The BRST operator $Q$ acts similarly to section 2.2 and so the Hilbert space of open string states is given by the total de Rham cohomology group

$$
\begin{equation*}
\bigoplus_{n=0}^{3} H^{n}(L, \operatorname{End}(E)) \tag{45}
\end{equation*}
$$

where the ghost number is given by $n$.
The discussion of deformations of the theory induced by operators in section 2.2 applies similarly to boundary states. One difference is that the deforming operator will naturally be integrated along the one-dimensional $\partial \Sigma$ rather than the two-dimensional $\Sigma$. Thus we look for ghost number one boundary operators to give the deformation. These correspond to elements of $H^{1}(L, \operatorname{End}(E))$.

It is interesting to explicitly match these deformations coming from open string vertex operators to the parameters that define the A-model. First we note that the number of deformations of the connection on a flat vector bundle are given by $H^{1}\left(L, \operatorname{End}\left(E_{\mathbb{R}}\right)\right)$ as shown in chapter 15 of [26] for example. Note that to count degrees of freedom correctly we restrict to a real form of the gauge group. Since our open string operators are complexvalued, these deformations of the connection $A$ account for exactly half of the degrees of freedom present in $H^{1}(L, \operatorname{End}(E))$.

So what do the other half of the deformations correspond to? Clearly deformations of $L$ itself should correspond to deformations of the A-model. We will now show that such deformations indeed account for the remaining degrees of freedom.

Let us assume initially the minimal case, i.e., that $E$ is a line bundle. A deformation of $L$ corresponds to a section of the normal bundle of $L$. We saw in section 3.1.2 that the Kähler form provides a perfect pairing between vectors normal to $L$ and vectors tangent to $L$. We may thus use the Kähler form to provide a one-to-one mapping between sections of the normal bundle of $L$ and sections of the cotangent bundle of $L$. Thus a deformation is given by a 1 -form on $L$. Of course, we would like the deformed submanifold to still be Lagrangian. A simple calculation reveals that this condition dictates that the 1-form on $L$ be closed.

The result is that Lagrangian deformations of $L$ are in one-to-one correspondence with closed 1 -forms on $L$. In contrast, the degrees of freedom coming from the A-model consist of cohomology classes of 1 -forms on $L$. Thus, if a deformation of $L$ corresponds to a 1form which is exact, then it does not affect the A-model. Such a deformation is called a "Hamiltonian" deformation of $L$. Thus we see that A-branes are really only defined up to Hamiltonian deformation.

If the rank of $E$ is greater than one then some of the deformations of $L$ corresponding to $H^{1}(L, \operatorname{End}(E))$ are associated to breaking $L$ up into a collection of branes with bundles of lower rank. This all ties together with the picture of enhanced gauge symmetry for coincident D-branes as discussed in [50]. We should therefore think of the generic A-brane as comprising of a line bundle $E \rightarrow L$ with higher rank bundles obtained by allowing such basic A-branes to coalesce.

There is one more piece of information about the properties of A-branes that the open strings in $H^{1}(L, \operatorname{End}(E))$ can tell us. If an A-brane background defines a truly stable vacuum for the topological A-model then the one-point function $\left\langle W_{a}\right\rangle$ will be zero for any vertex operator associated with $H^{1}(L, \operatorname{End}(E))$. A nonzero value, called a "tadpole", would force the operator to acquire an expectation value which would move the D-brane to another location.

We therefore need to know how to compute $\left\langle W_{a}\right\rangle$ exactly. Fortunately Witten [48] discovered a beautiful way of computing all the correlation functions between open string operators associated to $H^{1}(L, \operatorname{End}(E)) .{ }^{9}$

Without instanton corrections, Witten showed that the correlation functions could be determined by a Chern-Simons field theory on the Lagrangian $L$. The effect of instantons is to add an additional term into the action. At tree level an instanton will consist of a holomorphic map of a disk into $Y$ with the boundary of the disk mapped to $L$.

It is this instanton contribution to the effective action that has the potential to generate tadpoles $\left\langle W_{a}\right\rangle$. Restricting to the case of a line bundle, the condition that such tadpoles vanish is that

$$
\begin{equation*}
\sum_{\alpha \in I} \exp \left(2 \pi i \int_{D_{\alpha}}(B+i J)+2 \pi i \oint_{\partial D_{\alpha}} A\right)\left[\partial D_{\alpha}\right]=0 \quad \text { in } H_{1}(L), \tag{46}
\end{equation*}
$$

where the sum is over all holomorphic disks $D_{\alpha}$ with $\partial D_{\alpha} \subset L$ (including multiple covers). The notation $\left[\partial D_{\alpha}\right]$ refers to the cohomology class of $\partial D_{\alpha}$. It is an interesting exercise to show that (46) is invariant under Hamiltonian deformations of $L$.

Any Lagrangian violating (46) should not be considered to be an A-brane. This condition on A-branes has been explored in some cases (in $[51,52]$ for example) but a general geometric understanding appears to be missing. For example, it is not known if the 3-torus fibrations of SYZ [53] satisfy this condition. Note that the condition (46) depends on $B+i J$ and the value of the connection $A$. Thus there can be A-branes which are good for a specific value of these parameters but will, in general, be killed by tadpoles. Note also that an $S^{3}$, which is simply-connected, always trivially satisfies the vanishing tadpole condition.

It is worth summarizing the definition of an A-brane that we have finally settled on:
An A-brane is an element of the equivalence class of Lagrangian 3-manifolds in $Y$ modulo Hamiltonian deformations, which satisfies the tadpole cancellation property (46) and has trivial Maslov class.
${ }^{9}$ Note that Witten's method applies to the topological field theory coupled to gravity.

Although we will be doing our best to evade the issue, we really should mention the $A_{\infty}$-algebra structure associated to the A-branes. The correlation functions for the open string vertex operators which arise from Witten's Chern-Simons theory are not consistent with an associative algebra in the usual way. Instead one defines a series of products

$$
\begin{equation*}
m_{k}\left(a_{1}, a_{2}, \ldots, a_{k}\right) \tag{47}
\end{equation*}
$$

for which $m_{2}$ would be the usual product. These higher products are related in a specific way. We refer to [54-56], for example, for more details. The recent paper [57] explains carefully how the $A_{\infty}$ structure appears directly in the topological field theory.

### 3.1.3 Open strings for many A-branes

Suppose we have a set of A-branes $L_{a}$. For simplicity of exposition, let us initially assume that we just have line bundles over each brane. Given a pair of A-branes $L_{a}$ and $L_{b}$ we will have a Hilbert space of open strings beginning on $L_{a}$ and ending on $L_{b}$. This Hilbert space has a grading, which, up to an additive shift is the ghost number. This additive shift will turn out to be very important and we will discuss it extensively soon. We use the following notation for this graded Hilbert space:

$$
\begin{equation*}
\operatorname{Hom}^{*}\left(L_{a}, L_{b}\right)=\bigoplus_{m \in \mathbb{Z}} \operatorname{Hom}^{m}\left(L_{a}, L_{b}\right) \tag{48}
\end{equation*}
$$

We will also denote $\operatorname{Hom}^{0}\left(L_{a}, L_{b}\right)$ simply by $\operatorname{Hom}\left(L_{a}, L_{b}\right)$.
The reason for this notation is that the concept of open strings between branes fits naturally into the mathematical structure of a category. A category is defined as follows (as copied from [58])

Definition $1 A$ category $\mathcal{C}$ consists of the following: a class ${ }^{10}$ obj $(\mathcal{C})$ of objects, a set $\operatorname{Hom}_{\mathcal{C}}(A, B)$ of morphisms for every ordered pair $(A, B)$ of objects, an identity morphism $\operatorname{id}_{A} \in \operatorname{Hom}_{\mathcal{C}}(A, A)$ for every object $A$, and a composition function

$$
\begin{equation*}
\operatorname{Hom}_{\mathcal{C}}(A, B) \times \operatorname{Hom}_{\mathcal{C}}(B, C) \rightarrow \operatorname{Hom}_{\mathcal{C}}(A, C), \tag{49}
\end{equation*}
$$

for every ordered triple $(A, B, C)$ of objects. If $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$ and $g \in \operatorname{Hom}_{\mathcal{C}}(B, C)$, the composition is denoted $g f$. The above data is subject to two axioms:

1. Associativity axiom: $(h g) f=h(g f)$ for $f \in \operatorname{Hom}_{\mathcal{C}}(A, B), g \in \operatorname{Hom}_{\mathcal{C}}(B, C)$ and $h \in$ $\operatorname{Hom}_{\mathcal{C}}(C, D)$.
2. Unit axiom: $\operatorname{id}_{B} f=f=f \operatorname{id}_{A}$ for $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$.

There are many examples of a categories. Some of the obvious ones are as follows:

[^6]1. Objects are sets, morphisms are maps.
2. Objects are groups, morphisms are group homomorphisms.
3. Objects are rings (or modules, etc.), morphisms are ring homomorphisms (modules homomorphisms, etc.)
4. Objects are topological spaces, morphisms are continuous maps.

Note that in each case above an object is a set, or some glorified notion of a set, and so consists of elements. One of the key ideas in category theory is to phrase things so that you never make any mention of these elements. There are also categories whose objects are not composed of elements. The D-brane categories which will be of particular interest to us are examples of such "elementless" categories!

A morphism $f \in \operatorname{Hom}(A, B)$ is often written $f: A \rightarrow B$ for obvious reasons. As one might guess, we say that two objects, $A$ and $B$, in a category are isomorphic if there are morphisms $f: A \rightarrow B$ and $g: B \rightarrow A$ such that $g f=\operatorname{id}_{A}$ and $f g=\operatorname{id}_{B}$. That is, there exists an invertible morphism between $A$ and $B$.


Figure 3: Composition of morphisms.
Clearly we would like to form a category where the A-branes are the objects and the open strings of the A-model form the morphisms. While one is free to define the set of morphisms of the total Hilbert space as in (48), we will ultimately see that there is little difference between this and restricting just to the case of $\operatorname{Hom}^{0}\left(L_{1}, L_{2}\right)$.

The composition function corresponds precisely to the notion of two open strings joining together as shown in open string diagram of figure 3. The edges of this figure have labels showing to which D-brane the ends of the open string are attached. This data is encoded in the correlation functions of the topological field theory.

We see from the previous section that the identity operator $\operatorname{id}_{L}$ for a given D -brane $L$ is given by the identity operator in $H^{0}(L)$.

Just as in the case of a single D-brane, the correlation functions of the topological A-model coupled to gravity encode a more complicated product rule than the simple composition
(49). Thus the notion of an $A_{\infty}$-algebra is generalized to the notion of an $A_{\infty}$-category. This structure is very interesting and important but we do not really need to concern ourselves with it in these lectures. In particular, if one ignores the higher products, the category of A-branes that we wish to construct really does satisfy the axioms of a plain old category specified above.

The information content of the topological A-model with open strings is precisely the data associated to the category of A-branes. We already know exactly what the objects are. We now want to compute the dimensions of the Hilbert spaces of open strings and the correlation functions between such states.

To compute the Hilbert space of open strings stretched between two different D-branes $L_{1}$ and $L_{2}$, it is easiest to assume that $L_{1}$ and $L_{2}$ intersect transversely. As in section 3.1.2, the $Q$-invariance of the topological field theory can be used to argue that open strings can only arise from constant maps $\phi: \Sigma \rightarrow Y$. This means that an open string state is associated to a point of intersection $L_{1}$ and $L_{2}$.

The previous section suggests that locally the Hilbert space should be given by the De Rham cohomology of this intersection, i.e., the cohomology of a point. We therefore first guess that there is a one-dimensional Hilbert space associated with each point of intersection. Thus the dimension of $\operatorname{Hom}^{*}\left(L_{1}, L_{2}\right)$ would be given by the number of points of intersection between $L_{1}$ and $L_{2}$.

This cannot be right. We know the A-model is invariant under Hamiltonian deformation of $L_{1}$ or $L_{2}$ but the number of points of intersection is not such an invariant. Of course, the oriented intersection number $\#\left(L_{1} \cap L_{2}\right)$ is such an invariant as it depends only on homology classes but this turns out to be too crude for our purposes.

Let us introduce some notation. Let there be $M$ points of intersection between $L_{1}$ and $L_{2}$ and let the points be labeled $p_{a}, a=1 \ldots M$. Thus we have open string vertex operators $W_{p_{a}}$ that create an open string at the point $p_{a}$. Our putative Hilbert space will be denoted $V=\mathbb{C}^{M}$. Each vertex operator $W_{p_{a}}$ has a ghost number that we denote $\mu\left(p_{a}\right)$. This leads to a grading of $V$ by ghost number

$$
\begin{align*}
V_{i} & =\bigoplus_{\mu\left(p_{a}\right)=i} \mathbb{C}  \tag{50}\\
V & =\bigoplus_{i} V_{i} .
\end{align*}
$$

The way to determine the true Hilbert space lies in Witten's work on Morse theory [59] as generalized in the work of Floer [60] and, in particular, by Fukaya [61,62]. We also refer to chapters 10.5 and 40.4 of [8] for a nice review of this. Because these references are quite thorough, we will only outline the general picture in the following discussion.

The basic idea is that an instanton can "tunnel" from an open string state at one point of intersection to an open string at another point of intersection. The worldsheet of an instanton of such a tunneling process is shown in figure 4. As we saw earlier, at treelevel these worldsheet instantons are holomorphic disks in $Y$. These instantons produce a


Figure 4: Instanton Tunneling.
correction to the BRST operator resulting in:

$$
\begin{equation*}
\left\{Q, W_{p_{a}}\right\}=\sum_{b} n_{a b} W_{p_{b}} \tag{51}
\end{equation*}
$$

for some coefficients $n_{a b}$ to be determined. Thus the true Hilbert space will be determined as the $Q$-cohomology of some complex based on the vector space $V$. Since $Q$ has ghost number one, the complex looks like

$$
\begin{equation*}
\ldots \xrightarrow{Q} V_{-1} \xrightarrow{Q} V_{0} \xrightarrow{Q} V_{1} \xrightarrow{Q} \ldots \tag{52}
\end{equation*}
$$

We define $\operatorname{Hom}^{i}\left(L_{a}, L_{b}\right)$ as the cohomology of this complex at position $i$.
To compute $n_{a b}$ we must perform an integral of the moduli space of instantons. This integral must be performed over the fermionic parameters as well as the obvious bosonic maps $\phi$. By the usual rules of fermionic integration such an integral vanishes unless the fermionic parameters cancel in some way, i.e., we have no net fermionic zero modes. To be more precise, we require that the index of the Dirac operator for the instanton is equal to the ghost number of $Q$, i.e., one [59].

The index of the Dirac operator also measures the generic (or, to be precise, virtual) dimension of the moduli space of holomorphic maps. We refer to [63] for a nice account of what happens in the non-generic situation. In the generic case, we thus compute $n_{a b}$ simply by counting the number of points in the zero-dimensional instanton moduli space.

For an instanton connecting $p_{a}$ to $p_{b}$, the index of the Dirac operator is given by the difference in ghost numbers $\mu\left(p_{b}\right)-\mu\left(p_{a}\right)$. Thus we expect that the generic dimension of the moduli space of instantons is given by

$$
\begin{equation*}
\operatorname{dim} \mathscr{M}=\mu\left(p_{b}\right)-\mu\left(p_{a}\right)-1 . \tag{53}
\end{equation*}
$$

We refer the reader to [64] for further information on this point.
The astute reader should have noticed that we have nowhere specified a way that one can actually compute $\mu\left(p_{a}\right)$. Given the dimensions of moduli spaces of instantons, the relation


Figure 5: Disk instanton associated to three-point functions.
(53) only gives enough information to compute the relative ghost number of two points of intersection of $L_{a}$ and $L_{b}$. Indeed, we have the following very important fact:

The topological A-model does not contain enough information to determine the absolute ghost number of an open string associated to a point of intersection of two A-branes.

Just how much ambiguity in the ghost number do we actually have? Given a pair of Dbranes $L_{1}$ and $L_{2}$ we are free to shift the ghost numbers of the open strings from $L_{1}$ to $L_{2}$ by some fixed integer. We also saw in section 3.1.2 that if $L_{1}=L_{2}$ then the ghost number was given by the degree of de Rham cohomology which is perfectly well-defined. Furthermore, we would like to preserve ghost number in the operator product

$$
\begin{equation*}
\operatorname{Hom}^{i}\left(L_{1}, L_{2}\right) \otimes \operatorname{Hom}^{j}\left(L_{2}, L_{3}\right) \rightarrow \operatorname{Hom}^{i+j}\left(L_{1}, L_{3}\right) . \tag{54}
\end{equation*}
$$

The ambiguity in the ghost number can then be accounted for by assigning a ghost number $\mu(L)$ to each D-brane itself. One then defines the ghost number of an element of $\operatorname{Hom}^{i}\left(L_{a}, L_{b}\right)$ as

$$
\begin{equation*}
i+\mu\left(L_{b}\right)-\mu\left(L_{a}\right) \tag{55}
\end{equation*}
$$

It is easy to see that this definition has all the properties we desire.
We may restate the above as follows. The topological A-model has a symmetry which allows us to shift the ghost numbers of the open string states by assigning arbitrary ghost numbers to the A-branes and defining the ghost number as in (55). Note that this idea of assigning integers to Lagrangian submanifolds to fix this ambiguity was studied carefully in [65].

We will not give details on how to compute the correlation functions. It should be clear however that there will be instanton corrections involved. For example, if we compute the three-point function associated to figure 3, at tree-level we will consider holomorphic disks in $Y$ with boundary conditions shown in figure 5 . The cancellation of fermion zero modes will enforce ghost number conservation as usual.

In this section we have outlined the definition of the category of A-branes in the case that the objects $L_{a}$ and $L_{b}$ intersect transversely. Actually one may always use Hamiltonian deformations to deform any pair of Lagrangian into this case. Thus we actually have a complete definition of the category of A-branes.

This category is named after Fukaya who introduced it. The reader should note that our discussion of the Fukaya category in this section has omitted a vast number of technical details that have made this subject the object of a good deal of attention for the past ten years. We refer to [56,62, 66-68], for example, for more of the gory details. We also refer to $[69,70]$ where the Fukaya category (complete with its $A_{\infty}$ structure) is determined explicitly for the 2-torus. Recently, in a remarkable paper [71], Seidel has described the Fukaya category for the quartic K3 surface. No other examples are known.

The generalization of the Fukaya category to the case of higher rank bundles over each A-brane should be fairly obvious. Rather than associating $\mathbb{C}$ with each point of intersection, we have a matrix representing a linear map from the fibre of one bundle to the fibre of the other over the point of intersection.

We emphasize that nothing in A-model depends on the complex structure of $Y$. Indeed, the Fukaya category is usually defined purely in terms of the symplectic geometry of $Y$ thus explicitly removing any possible dependence on the complex structure. The Fukaya category depends on $B+i J$ for both its objects and its composition of morphisms. The tadpole condition (46) has a $B+i J$ dependence and so certain objects might only exist for particular values of this parameter. The correlation functions depend on $B+i J$ through instanton corrections and so the composition of morphisms are similarly dependent.

Finally we should point out that worldsheet instantons are generally expected to adversely affect notions based on the concept of a spacetime metric. Thus it would be reasonable to expect that the concept of a Lagrangian submanifold is only really valid at large radius limit. The composition rules in the Fukaya category are based on power series associated to instanton effects. Beyond the radius of convergence of these power series it is reasonable to think that the Lagrangian submanifold description of A-branes has broken down.

### 3.2 The B-model

It should be with relief that we turn attention to the B-model on $X$. Unfortunately we will see that there is a subtlety concerning the set of all possible B-branes that will occupy us for most of the remaining lectures.

### 3.2.1 B-branes

We may repeat the analysis of the beginning of section 3.1.1. The difference for the case of B-branes is that the B-model twist implies that we should impose $R_{j}^{\bar{\imath}}=R_{\bar{j}}^{i}=0$ for the reflection matrix in (38) and (39). That is, only the diagonal terms $R_{j}^{i}$ and $R_{\bar{j}}^{i}$ are nonzero.

This means that the almost complex structure now preserves the tangent and normal directions to the D-brane, rather than exchanging them. It follows that the D-brane is
a holomorphically embedded submanifold of $X$. Clearly this forces the dimension of the D-brane to be even, i.e., $0,2,4$ or 6 .

Although 0, 2 and 4-dimensional B-branes exist, we will at first restrict attention only to the 6 -dimensional case, where the D-brane fills $X$. That is we put purely Neumann conditions on the open string. The complexities of B-branes will allow us to deduce the properties of all the B-branes purely from a knowledge of 6 -branes. ${ }^{11}$

As in the A-brane, consideration of the $B$-field forces us to consider the possibility of a bundle over the B -brane, i.e., a bundle $E \rightarrow X$. Setting the $B$-field equal to zero we may consider the constraint on this bundle from the requirement that the variation of the action from the boundary term is zero. In this case, we find that the curvature, $F$, of the bundle is a 2-form purely of type $(1,1)[8,43,48]$. In other words, $E \rightarrow X$, is a holomorphic bundle.

We refer again to chapter 15 of [26] for a very readable account of holomorphic vector bundles. The basic idea is that the transition functions for the bundle may be written as holomorphic functions of the coordinates of $X$. Thus the bundles may be described very naturally in the language of algebraic geometry. In section 4 this will allow us to move from the language of bundles to the language of sheaves which, although alien to most physicists, is definitely the right language for B-branes.

### 3.2.2 Open strings for B-branes

Since we have chosen purely Neumann boundary conditions on the open string, we have effectively set the matrix $R$ equal to the identity in (39). Thus, from (24) we have, on the boundary

$$
\begin{equation*}
\theta_{j}=g_{j \bar{k}}\left(\psi_{+}^{\bar{k}}-\psi_{-}^{\bar{k}}\right)=0 \tag{56}
\end{equation*}
$$

and so a local operator will depend only on $\phi$ and $\eta^{\bar{j}}$. It follows that local operators look like $(0, q)$-forms.

Suppose we have two B-branes in the form of two bundles $E_{1} \rightarrow X$ and $E_{2} \rightarrow X$. The Chan-Paton degrees of freedom are associated with maps from $E_{1}$ to $E_{2}$. We denote the space of such maps as $\operatorname{Hom}\left(E_{1}, E_{2}\right)$.

We saw in section 2.3 that the BRST operator $Q$ looks like the Dolbeault operator in the B-model. Adding all these ingredients together, we see that an open string vertex operator for a string stretching from $E_{1}$ to $E_{2}$ is given by the cohomology groups

$$
\begin{equation*}
H_{\bar{\partial}}^{0, q}\left(X, \operatorname{Hom}\left(E_{1}, E_{2}\right)\right) . \tag{57}
\end{equation*}
$$

In contrast to the A-brane case, we can choose to declare the ghost number of an operator in (57) to be $q$ without ambiguity.

[^7]As always, the B-model has no instanton corrections. If

$$
\begin{align*}
& a \in H_{\bar{\partial}}^{0,1}\left(X, \operatorname{Hom}\left(E_{1}, E_{2}\right)\right) \\
& b \in H_{\bar{\partial}}^{0,1}\left(X, \operatorname{Hom}\left(E_{2}, E_{3}\right)\right)  \tag{58}\\
& c \in H_{\bar{\partial}}^{0,1}\left(X, \operatorname{Hom}\left(E_{3}, E_{1}\right)\right),
\end{align*}
$$

then we may compute the 3 -point function exactly from

$$
\begin{equation*}
\left\langle W_{a} W_{b} W_{c}\right\rangle=\int_{X} \operatorname{Tr}(a \wedge b \wedge c) \wedge \Omega \tag{59}
\end{equation*}
$$

and deduce an operator product algebra. The Hom matrices are composed in the obvious way.

If the B-model is coupled to gravity one may analyze higher $n$-point functions. In [48] Witten showed that these correlation functions could be deduced from a "holomorphic" Chern-Simons theory. This $A_{\infty}$ structure was analyzed more abstractly by Merkulov [72]. See also [57] for more discussion of the $A_{\infty}$ structure in topological field theories.

### 3.2.3 A failure of mirror symmetry

Given the dreadful complexities one is forced to endure to define the Fukaya category (most of which we omitted) the reader is probably shocked at how easy the B-branes were to analyze.

It would be remarkable if one could now invoke mirror symmetry and say that the category of A-branes on $Y$ is equivalent to the category of B -branes on $X$ at this point. Unfortunately this equivalence would be wrong with our current definition of B-branes. The problem is that we simply do not have enough B-branes.

Clearly our assumption that B-branes are 6-branes is too strong. The lower-dimensional branes certainly exist and one might hope that such branes account for the missing B-branes. Sadly we still fall far short of the number of objects in the Fukaya category.

There is a lack of symmetry between the A-branes and B-branes which is key in the failure of mirror symmetry. In section 3.1.3 we had a real problem when we tried to assign an intrinsic ghost number to an open string which we solved by labeling the A-branes with a ghost number. The B-branes did not have this problem. We will essentially restore mirror symmetry by inflicting the ghost number ambiguity on the B-branes!

As Kontsevich proposed as far back as 1994 [73], the answer involves going to the "derived category" as we will explain in section 5 .

## 4 Some Mathematical Tools

Before continuing with the story of B-branes we need some more mathematical weapons. As these ideas are not familiar to a typical physicist we will try to be fairly thorough. Most of the ideas in this section are taken from [58, 74, 75].

As defined in section 3.2, a B-brane is associated to a vector bundle over $X$. In section 2.3 we noted that the B-model for a closed string can be described in purely algebraic terms. In order to do the same for closed strings we need to replace vector bundles by something purely algebraic, namely sheaves. This mathematical construction appears to be unavoidable if one wants to fully understand B-branes. Anyone who ignores the language of sheaves would be forced to reinvent it!

In order to discuss sheaves properly we use quite a bit of categorical language. This will also prove useful later on when we discuss the derived category.

### 4.1 Categories of sheaves

### 4.1.1 Holomorphic functions

Let us begin with $\mathbb{P}^{N}$ with homogeneous coordinates $\left[z_{0}, z_{1}, \ldots, z_{N}\right]$. Let $X \subset \mathbb{P}^{N}$ be an "algebraic variety", i.e., $X$ is defined as the intersection of the zeroes of a set of polynomials $F_{1}, F_{2}, \ldots$ in the homogeneous coordinates. These polynomials define the space $X$ purely in terms of algebraic data. Other than $\mathbb{P}^{N}$ itself, the simplest case consists of a hypersurface $X \subset \mathbb{P}^{N}$ defined by a single equation. We never consider more than one defining equation in these lectures.

How might we put some more "stuff" on $X$ that is described purely in terms of algebraic structures built on the homogeneous coordinates? The obvious thing to do is to define functions $f: X \rightarrow \mathbb{C}$. In terms of the homogeneous coordinates, the natural way to define such a function is

$$
\begin{equation*}
f=\frac{g}{h} \tag{60}
\end{equation*}
$$

where $g$ and $h$ are polynomials in the homogeneous coordinates. Clearly we require $g$ and $h$ to have the same homogeneous degree so that the function is well-defined on the projective space $\mathbb{P}^{N}$ and thus $X$. Any function $f: X \rightarrow \mathbb{C}$ that can be written in the form (60) in a neighbourhood of a point $p \in X$, such that $h$ never vanishes in this neighbourhood, is called regular at $p$. A function is regular on $X$ if it is regular at all the points in $X$. We will also refer to such functions on $X$ as holomorphic. It is these regular functions on $X$ which form the prototype of "extra data" on $X$ which will be used to replace the unalgebraic notion of a vector bundle.

### 4.1.2 Sheaves

Let $X$ be a topological space. We make the following ${ }^{12}$
Definition $2 A$ presheaf $\mathscr{F}$ on $X$ consists of the following data
a) For every open set $U \subset X$ we associate an abelian group $\mathscr{F}(U)$.

[^8]b) If $V \subset U$ are open sets we have a "restriction" homomorphism $\rho_{U V}: \mathscr{F}(U) \rightarrow \mathscr{F}(V)$.

This data is subject to the conditions
0. $\mathscr{F}(\emptyset)=0$.

1. $\rho_{U U}$ is the identity map.
2. If $W \subset V \subset U$ then $\rho_{U W}=\rho_{V W} \rho_{U V}$.

If $\sigma \in \mathscr{F}(U)$ then we use the notation $\left.\sigma\right|_{V}$ for the restriction $\rho_{U V}(\sigma)$. An element of $\mathscr{F}(U)$ is called a section of $\mathscr{F}$ over $U . X$ is an open subset of itself and an element of $\mathscr{F}(X)$ is called a global section.

We then make a more restrictive
Definition $3 A$ sheaf $\mathscr{F}$ on $X$ is a presheaf satisfying the conditions
3. If $U, V \subset X$ and $\sigma \in \mathscr{F}(U), \tau \in \mathscr{F}(V)$ such that $\sigma_{U \cap V}=\tau_{U \cap V}$, then there exists $\nu \in \mathscr{F}(U \cup V)$ such that $\left.\nu\right|_{U}=\sigma$ and $\left.\nu\right|_{V}=\tau$.
4. If $\sigma \in \mathscr{F}(U \cup V)$ and $\sigma_{U}=\sigma_{V}=0$, then $\sigma=0$.

This definition makes the data defining a sheaf essentially contained in very small open sets. That is, the sheaf is defined by "local" information.

A simple example of a presheaf is given by associating some fixed abelian group, such as $\mathbb{Z}$, to every non-empty open set in $X$. The restriction maps are set equal to the identity. This example is not a sheaf as it violates condition 3 above when we consider disconnected open sets. The closest sheaf we can find to this presheaf would be to associate $\mathbb{Z}^{n}$ to each open set $U$ where $n$ is the number of connected components of $U$. This latter sheaf will be useful and we denote it simply by $\mathbb{Z}$.

Another important sheaf is constructed by making $\mathscr{F}(U)$ the group (under addition) of holomorphic functions over $U$. The restriction map is the obvious restriction map in the usual sense. This "sheaf of holomorphic functions", also known as the structure sheaf, is denoted $\mathscr{O}_{X}$ (or just $\mathscr{O}$ ).

Yet another example is given by $\mathscr{O}^{*}$ - the sheaf of nonzero ${ }^{13}$ holomorphic functions. This time the abelian group structure is given by multiplication.

It will be useful to make a category of sheaves. Obviously the sheaves on $X$ form the objects. We define a morphism $\phi: \mathscr{F} \rightarrow \mathscr{G}$ of sheaves as something which associates a homomorphism $\phi(U): \mathscr{F}(U) \rightarrow \mathscr{G}(U)$ to each open set $U \subset X$ such that, for any $V \subset U$, the following diagram commutes:


[^9]where $\rho$ and $\rho^{\prime}$ are the restriction maps on $\mathscr{F}$ and $\mathscr{G}$ respectively. The fact that this forms a category follows immediately from the properties of homomorphisms of groups. Note that the objects, $\mathscr{F}$, in the category of sheaves are not directly composed of elements. Having said that, for any open set, $\mathscr{F}(U)$ is a group and so is composed of elements.

We note that two sheaves are said to be isomorphic if there is an invertible morphism from one to the other.

### 4.1.3 Locally free sheaves

Having defined the structure sheaf $\mathscr{O}_{X}$, we would like to define more complicated sheaves that are equally algebraic in nature. First recall the definition of a module:

Definition 4 Let $R$ be a ring with a multiplicative identity 1. An $R$-module is an abelian group $M$ with an $R$-action given by a mapping $R \times M \rightarrow M$ such that

1. $r(x+y)=r x+r y$
2. $(r+s) x=r x+s x$
3. $(r s) x=r(s x)$
4. $1 x=x$
for any $r, s \in R$ and $x, y \in M$.
Now, the set of regular functions over $U, \mathscr{O}_{X}(U)$, is an abelian group under addition as noted earlier, but multiplication of functions also gives it a ring structure. This allows us to introduce the concept of a sheaf of $\mathscr{O}_{X}$-modules. That is, let $\mathscr{E}$ be a sheaf such that $\mathscr{E}(U)$ is an $\mathscr{O}_{X}(U)$-module for any open $U \in X$. By a common abuse of notation we will refer to a sheaf of $\mathscr{O}_{X}$-modules as an $\mathscr{O}_{X}$-module.

Clearly $\mathscr{O}_{X}$ is itself an $\mathscr{O}_{X}$-module. We may also take a sum of copies

$$
\begin{equation*}
\mathscr{O}_{X}^{\oplus n}=\underbrace{\mathscr{O}_{X} \oplus \mathscr{O}_{X} \oplus \ldots \oplus \mathscr{O}_{X}}_{n} \tag{62}
\end{equation*}
$$

to give another $\mathscr{O}_{X}$-module. This is called the free $\mathscr{O}_{X}$-module of rank $n$. We call a sheaf $\mathscr{E}$ locally free of rank $n$ if there is an open covering $\left\{U_{\alpha}\right\}$ of $X$ such that $\mathscr{E}\left(U_{\alpha}\right) \cong \mathscr{O}_{X}\left(U_{\alpha}\right)^{\oplus n}$ for all $\alpha$.

There is a one-to-one correspondence between holomorphic vector bundles of rank $n$ on $X$ and locally free sheaves of rank $n$ on $X$. To see this first consider the trivial complex line bundle over $X$. We may regard $\mathscr{O}_{X}(U)$ as the group of holomorphic sections of this bundle over $U$. Thus, if the covering $\left\{U_{\alpha}\right\}$ trivializes a vector bundle $E \rightarrow X$, then the group of holomorphic sections of $E$ over $U_{\alpha}$ is given by $\mathscr{O}_{X}\left(U_{\alpha}\right)^{\oplus n}$.

Conversely, let $\mathscr{E}$ be a locally free sheaf and let

$$
\begin{equation*}
\phi_{\alpha}: \mathscr{E}\left(U_{\alpha}\right) \rightarrow \mathscr{O}_{X}\left(U_{\alpha}\right)^{\oplus n} \tag{63}
\end{equation*}
$$

be the explicit isomorphism. On $U_{\alpha} \cap U_{\beta}$ we may define the $n \times n$ matrix of holomorphic functions

$$
\begin{equation*}
\phi_{\beta} \phi_{\alpha}^{-1}: \mathscr{O}_{X}\left(U_{\alpha} \cap U_{\beta}\right)^{\oplus n} \rightarrow \mathscr{O}_{X}\left(U_{\alpha} \cap U_{\beta}\right)^{\oplus n} \tag{64}
\end{equation*}
$$

which defines a holomorphic bundle $E \rightarrow X$.
So locally free sheaves are the algebraic way of describing holomorphic vector bundles. Clearly a trivial bundle corresponds to a free $\mathscr{O}_{X}$-module and, as we stated above, the trivial line bundle corresponds to the structure sheaf $\mathscr{O}_{X}$.

To get a better feel for locally free sheaves, let us consider some very simple cases where $X$ is $\mathbb{P}^{1}$ with homogeneous coordinates $\left[z_{0}, z_{1}\right]$. We define an open set $U_{0}$ by $z_{0} \neq 0$ with an affine coordinate $y_{0}=z_{1} / z_{0}$. Similarly $U_{1}$ has $z_{1} \neq 0$ with an affine coordinate $y_{1}=z_{0} / z_{1}$. Thus $\mathbb{P}^{1}$ has an open cover $\left\{U_{0}, U_{1}\right\}$.

Consider a holomorphic line bundle on $\mathbb{P}^{1}$ with a fibre coordinate $w_{i}$ over the open set $U_{i}$. Then the transition function in $U_{0} \cap U_{1}$ may be written in the form

$$
\begin{equation*}
w_{1}=y_{1}^{n} w_{0} \tag{65}
\end{equation*}
$$

for some integer $n .{ }^{14}$ We define $\mathscr{O}(n)$ as the locally free sheaf associated to this line bundle. Clearly $\mathscr{O}(0)$ is $\mathscr{O}$.

Now consider an example of a morphism $\mathscr{O} \rightarrow \mathscr{O}(n)$. On $U_{0}$ we use the identity map 1 . To keep the transition function valid, this would force us to make the morphism look like $g\left(y_{1}\right) \mapsto y_{1}^{n} g\left(y_{1}\right)$ on $U_{1}$ for any $g \in \mathscr{O}\left(U_{1}\right)$. We will write a morphism as

$$
\begin{equation*}
\mathscr{O} \xrightarrow{f} \mathscr{O}(n), \tag{66}
\end{equation*}
$$

where $f$ is a homogeneous function in $\left[z_{0}, z_{1}\right]$ and we understand the morphism in $U_{\alpha}$ to be given by a multiplication by $f / z_{\alpha}^{n}$. Thus, in the case we just described, $f=z_{0}^{n}$. Clearly, so long as $f$ is homogeneous of degree $n$, it will be compatible with the transition functions of the vector bundle. Indeed, all morphisms from $\mathscr{O}$ to $\mathscr{O}(n)$ are of this form.

Grothendieck [76] proved that any locally free sheaf of finite rank on $\mathbb{P}^{1}$ is isomorphic to a sum $\mathscr{O}\left(n_{1}\right) \oplus \mathscr{O}\left(n_{2}\right) \oplus \ldots$ For $\mathbb{P}^{N}$ we may define analogous sheaves $\mathscr{O}(n)$ which will serve as our basic building blocks later on. Note that Grothendieck's theorem is not valid for $N>1$.

### 4.1.4 Kernels and cokernels

Given a morphism between two objects in a category we would like to define the notion of kernel and cokernel of this map.

If we have a map $f: B \rightarrow C$ between two groups we would usually define the kernel of $f, \operatorname{Ker}(f)$, to be the subgroup of $B$ which $f$ maps to the identity of $C$. In the world of categories it is taboo to talk about elements and so this definition is, in general, no good.

The first thing we need to define is the notion of the identity in some categorical way. This is done as follows

[^10]Definition $5 A$ zero object in a category is an object 0 such that for any object $B$ there is precisely one morphism in $\operatorname{Hom}(0, B)$ and precisely one morphism in $\operatorname{Hom}(B, 0)$. If the zero object exists, then for any pair of objects, $B$ and $C$, we define the zero morphism (also denoted as 0 ) in $\operatorname{Hom}(B, C)$ as the composition $B \rightarrow 0 \rightarrow C$.

It is easy to show that all zero objects are isomorphic. For the category of sets the zero object does not exist. For groups, rings, etc., it is the trivial group, ring, etc. In the category of sheaves it is the sheaf that associates the trivial group to every set $U$. In the category of D-branes, it represents the absence of a D-brane!

Secondly, we are going to restrict attention to additive categories. This is a category with a zero object and an abelian group structure (written as addition) on the set of morphisms $\operatorname{Hom}(B, C)$ such that the distributive law $\left(f+f^{\prime}\right) g=f g+f^{\prime} g$ and $f\left(g+g^{\prime}\right)=f g+f g^{\prime}$ is true for compositions. ${ }^{15}$ It is easy to see that the zero morphism is the identity in the group $\operatorname{Hom}(B, C)$. Clearly the category of abelian groups and the category of sheaves has an additive structure. In addition, we saw that for D-branes $\operatorname{Hom}(B, C)$ represents a Hilbert space, thus endowing the category of D-branes with an additive structure too.

Now we may make the following
Definition 6 The kernel of a morphism $f: B \rightarrow C$ is a morphism $i: A \rightarrow B$ such that fi $=0$ and which satisfies the following "universal" property: For any morphism e $: A^{\prime} \rightarrow B$ such that $f e=0$, there is a unique morphism $e^{\prime}: A^{\prime} \rightarrow A$ such that $e=i e^{\prime}$. That is, the map $e^{\prime}$ can be constructed such that the following commutes: ${ }^{16}$


Note that the kernel may not always exist for a general additive category. We emphasize that, in this categorical language, a kernel is a morphism - not an object as one might first think. Of course, since a morphism must specify the object it is mapping from, we do intrinsically define a kernel object, $A$, too. If the reader is unfamiliar with universal properties of maps such as the above, they should convince themselves that this definition of kernel coincides with the usual one for, say, the category of groups. To be precise, the kernel, as defined above, is the inclusion map of the kernel subgroup into the group itself.

We say $f$ is a monomorphism if the kernel of $f$ is zero. It is easy to prove that the uniqueness of $e^{\prime}$ implies that the kernel itself is a monomorphism. For groups, etc., a monomorphism is the same thing as an injective map. If the kernel object exists, it is unique up to isomorphism.

While we're at it, we can reverse all the arrows in the above to define the cokernel:

[^11]Definition 7 The cokernel of a morphism $f: B \rightarrow C$ is a morphism $p: C \rightarrow D$ such that $p f=0$ and which satisfies the following "universal" property: For any morphism $g: C \rightarrow D^{\prime}$ such that $g f=0$, there is a unique morphism $g^{\prime}: D \rightarrow D^{\prime}$ such that $g=g^{\prime} p$. That is, the map $g^{\prime}$ can be constructed such that the following commutes:


The object $D$ is called the cokernel object. Recall that the "old" definition of the cokernel of a map $f: B \rightarrow C$ is the quotient $C / \operatorname{Im}(f)$. The reader should again check that this agrees with the categorical definition.

We say $f$ is an epimorphism if the cokernel of $f$ is zero. For groups, etc., an epimorphism is the same thing as a surjective map. Again, the cokernel itself is an epimorphism and the cokernel object is unique, if it exists, up to isomorphism.

So why have we dragged ourselves through all this mathematical nonsense? The answer is that the category of sheaves does not quite behave as one might expect when this categorical machinery is applied to it. One might be forgiven for thinking that given a map between sheaves, $\phi: \mathscr{F} \rightarrow \mathscr{G}$, one could apply the old ideas of kernel and cokernel to the group maps $\phi(U): \mathscr{F}(U) \rightarrow \mathscr{G}(U)$ to get a definition of kernel and cokernel of $\phi$. While this works for the kernel, it can fail for the cokernel. In particular, the cokernel defined this way, while always a presheaf, need not be a sheaf.

To illustrate what can go wrong, consider the morphism of sheaves

$$
\begin{equation*}
\phi: \mathscr{O} \rightarrow \mathscr{O}^{*} \tag{69}
\end{equation*}
$$

on $\mathbb{C}^{*}=\mathbb{C}-\{0\}$ by the $\operatorname{map} \phi(U)(f)=\exp (2 \pi i f)$ for any function $f \in \mathscr{O}(U)$. The coordinate $z$ on the complex plane gives us a global section $z \in \mathscr{O}^{*}\left(\mathbb{C}^{*}\right)$. This section is clearly not in the image of $\phi\left(\mathbb{C}^{*}\right)$. However, if we were to consider a simply-connected subspace $U \in \mathbb{C}^{*}$, then $\left.z\right|_{U}$ would lie in the image of $\phi(U)$. We can define a nontrivial presheaf $\mathscr{F}$ such that $\mathscr{F}(U)$ is given by $\mathscr{O}^{*}(U) / \operatorname{Im}(\phi(U))$, but it violates property number 4 of the definition of a sheaf in section 4.1.3.

Conversely, the categorical definition of cokernel tells us that cokernel of $\phi$ in this example is zero. Thus $\phi$ is an epimorphism.

### 4.1.5 Abelian categories

A particular kind of category will be of particular importance to us - namely an abelian category. While the D-brane category itself will turn out not to be abelian, these special categories will be an essential building block in describing D-branes.

We defined the kernel and cokernel morphisms in section 4.1.4. Now let us further define the image of a map as the kernel of its cokernel (if it exists) and the coimage of a map as
the cokernel of its kernel. Again, this defines the image and coimage as morphisms but each has a naturally associated object too.

Given any map $f: B \rightarrow C$ such that the image and coimage exist, chasing through the definitions of these various morphisms shows that we may construct a map $h$ to make the following diagram commute:


Now we may define the category of interest:
Definition 8 An abelian category is an additive category satisfying the following axioms:

1. Every morphism has a kernel and a cokernel (and thus an image and coimage).
2. The map $h$ in (70) is an isomorphism for any $f$.

Any category for which the objects are made up of elements, such as the category of $R$-modules, is abelian. Since the coimage, etc., are defined only up to an isomorphism, in the case of an abelian category we may assume that $E$ and $F$ are the same objects in (70). That is, every map $f$ factors into its coimage (which is epic) followed by its image (which is monic).

As usual, an exact sequence may be defined as a sequence of maps

$$
\begin{equation*}
\ldots \xrightarrow{f_{n-2}} A^{n-1} \xrightarrow{f_{n-1}} A^{n} \xrightarrow{f_{n}} A^{n+1} \xrightarrow{f_{n+1}} \ldots, \tag{71}
\end{equation*}
$$

such that the image of $f_{n-1}$ is the same morphism as the kernel of $f_{n}$.
A short exact sequence is then an exact sequence of the form

$$
\begin{equation*}
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow \text {. } \tag{72}
\end{equation*}
$$

If the reader is finding all this category stuff a bit confusing, they should check through the above definitions to prove that this short exact sequence implies that $f$ is the kernel of $g$, and $g$ is the cokernel of $f$.

Finally in this section we can use the category machinery to define cohomology abstractly for a complex as follows:

$$
\begin{gather*}
\cdots \xrightarrow{f_{n-2}} A^{n-1} \xrightarrow{f_{n-1}} A^{n} \xrightarrow{f_{n}} A^{n+1} \xrightarrow{f_{n+1}} \cdots  \tag{73}\\
\operatorname{im(f_{n-1})} \bigwedge_{\operatorname{ker}\left(f_{n}\right)}^{L^{n}-{ }_{-}^{k}>K^{n}}
\end{gather*}
$$

If $f_{n} f_{n-1}=0$, the definition of the kernel of $f_{n}$ guarantees the existence of the map $k$ above. The cohomology of this complex at position $n$ is then defined as the cokernel object of the map $k$. We generally denote the cohomology $H^{n}\left(A^{\bullet}\right)$ in the case that it is an abelian group, or $\mathscr{H}^{n}\left(A^{\bullet}\right)$ if it is a sheaf.

There is a remarkable theorem due to Freyd [77] that says that, if the objects in an abelian category can be described as a set, the category may be embedded in the category of $R$-modules for some ring $R$. This effectively means that the objects in such a category can be thought of as consisting of elements.

We will see eventually that the categories we build for A-branes and B-branes are not abelian - hence the necessity for the abstractions of category theory.

### 4.1.6 Coherent sheaves

We saw in section 4.1.3 that the concept of vector bundles can be replaced by locally free sheaves in algebraic geometry. We want to be able to do things like compute cohomology for complexes of these things so it would be nice if the category of locally free sheaves were an abelian category. Sadly it is not.

The problem is that, while this category contains all of its kernels, it does not contain its cokernels. The solution is to start with the category of locally free sheaves, which is a subcategory of the category of $\mathscr{O}_{X}$-modules. Then add in all the cokernel objects together with all the possible morphisms between these new objects and the objects we already had. The resulting category is abelian. Thus we end up with a minimal abelian full ${ }^{17}$ subcategory of the category of $\mathscr{O}_{X}$-modules containing locally free sheaves. This is the category of coherent sheaves. ${ }^{18}$

Let's give a simple example of a coherent sheaf that is not locally free. We work locally on part of $X$ and pretend it looks like $\mathbb{C}^{3}$ with affine coordinates $(x, y, z)$. Consider the following morphism

$$
\begin{equation*}
\mathscr{O}^{\oplus 3} \xrightarrow{(x y z)} \mathscr{O} \tag{74}
\end{equation*}
$$

that is, three functions $f_{1}, f_{2}$, and $f_{3}$ are mapped to $x f_{1}+y f_{2}+z f_{3}$ by this morphism. Naïvely speaking, the cokernel object should consist of functions on $\mathbb{C}$ modded out by functions in the image of the map. This quotient should kill all functions away from the origin. We define $\mathscr{O}_{p}$ as the sheaf such that $\mathscr{O}_{p}(U)$ is the trivial group if $U$ does not contain the origin. If $U$ does contain the origin we set $\mathscr{O}_{p}(U)=\mathbb{C}$. This is an $\mathscr{O}_{X}$-module and is called the skyscraper sheaf of the origin. There is also a natural map

$$
\begin{equation*}
\mathscr{O} \longrightarrow \mathscr{O}_{p}, \tag{75}
\end{equation*}
$$

${ }^{17} \mathrm{~A}$ subcategory is full if, for any pair of objects $A$ and $B$, the set of morphisms $\operatorname{Hom}(A, B)$ is the same in the subcategory as it was in the original category.
${ }^{18}$ In these notes we are going to be somewhat careless about specifying whether ranks of free modules are infinite or finite. In particular we will make no effort to distinguish between coherent and quasi-coherent sheaves.
which takes a function to its value at the origin. The reader can carefully check through the definitions to see that (75) indeed represents the cokernel of (74). That is,

$$
\begin{equation*}
\mathscr{O}^{\oplus 3} \xrightarrow{(x y z)} \mathscr{O} \longrightarrow \mathscr{O}_{p} \longrightarrow 0 \tag{76}
\end{equation*}
$$

is an exact sequence.
Thus, the skyscraper sheaf $\mathscr{O}_{p}$ is a coherent sheaf. It clearly isn't locally free since its rank appears to jump at the origin. In a way, it looks like it could be associated to a vector bundle which has a fibre $\mathbb{C}$ over the origin and has trivial fibre elsewhere. We will see in section 5.4 that this is a 0 -brane on $X$.

We can also compute the image of the map (74), i.e., the kernel of the map (75). To phrase it a third way, we are looking for the $\mathscr{O}_{X}$-module $\mathscr{I}_{p}$ that completes the short exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathscr{I}_{p} \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}_{p} \longrightarrow 0 . \tag{77}
\end{equation*}
$$

One can easily show that $\mathscr{I}_{p}$ is a subsheaf of $\mathscr{O}$ and consists of functions which vanish at the origin. This sheaf is called the ideal sheaf of the origin.

Again, $\mathscr{I}_{p}$ is not locally free. The best we could say is that somehow it represents a trivial bundle with fibre $\mathbb{C}^{3}$ everywhere except at the origin, where it has no fibre. In Dbrane language you might say it is a 6 -brane on $X$ with an anti- 0 -brane glued in at the origin. We will see later that this isn't such a bad description of $\mathscr{I}_{p}$.

### 4.2 Cohomology

We have replaced the vector bundles of the B-model by coherent sheaves. We now need to replace the notion of Dolbeault cohomology with something more algebraic. To do this we use sheaf cohomology. Let us emphasize immediately that by sheaf cohomology we do not mean Čech cohomology, which is more topological than algebraic. ${ }^{19}$ Our sheaf cohomology is the version due to Grothendieck as reviewed in chapter III of [74] which is much more suited to the B-model. Having said that, we will start our discussion with Čech cohomology.

### 4.2.1 Čech cohomology

Here we give a lightning review. For more information and examples, the reader is referred to [79].

Suppose $\mathfrak{U}=\left\{U_{\alpha}\right\}$ is an open covering of a manifold $X$. Let us denote $U_{\alpha_{0}} \cap U_{\alpha_{1}} \cap \ldots \cap U_{\alpha_{p}}$ by $U_{\alpha_{0} \alpha_{1} \ldots \alpha_{p}}$. Given a sheaf $\mathscr{F}$ we define

$$
\begin{equation*}
\check{C}^{p}(\mathfrak{U}, \mathscr{F})=\prod_{\alpha_{0}<\alpha_{1}<\ldots<\alpha_{p}} \mathscr{F}\left(U_{\alpha_{0} \alpha_{1} \ldots \alpha_{p}}\right) . \tag{78}
\end{equation*}
$$

That is, an element $a \in \check{C}^{p}(\mathfrak{U}, \mathscr{F})$ is specified by giving an element $a_{\alpha_{0} \alpha_{1} \ldots \alpha_{p}} \in \mathscr{F}\left(U_{\alpha_{0} \alpha_{1} \ldots \alpha_{p}}\right)$ for each unordered ( $p+1$ )-tuple of open sets in $\mathfrak{U}$.

[^12]Now define a coboundary $\delta: \check{C}^{p} \rightarrow \check{C}^{p+1}$ by

$$
\begin{equation*}
(\delta a)_{\alpha_{0} \alpha_{1} \ldots \alpha_{p} \alpha_{p+1}}=\sum_{k=0}^{p+1}(-1)^{k} a_{\alpha_{0} \alpha_{1} \ldots \hat{\alpha}_{k} \ldots \alpha_{p+1}}, \tag{79}
\end{equation*}
$$

where the notation $\hat{\alpha}_{k}$ means omit $\alpha_{k}$.
It is easy to prove that $\delta^{2}=0$ and so we can define Čech cohomology, $\check{H}^{p}(\mathfrak{U}, \mathscr{F})$, as the cohomology of the complex

$$
\begin{equation*}
0 \longrightarrow \check{C}^{0}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} \check{C}^{1}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} \check{C}^{2}(\mathfrak{U}, \mathscr{F}) \xrightarrow{\delta} \ldots \tag{80}
\end{equation*}
$$

This definition of Čech cohomology depends on the open covering $\mathfrak{U}$ but one can show that, as the covering becomes finer and finer, one approaches a well-defined limit which we call $\check{H}^{p}(X, \mathscr{F})$. An open cover which yields $\check{H}^{p}(\mathfrak{U}, \mathscr{F}) \cong \check{H}^{p}(X, \mathscr{F})$ is a so-called good cover where each finite intersection $U_{\alpha_{0}} \cap U_{\alpha_{1}} \cap \ldots \cap U_{\alpha_{p}}$ is diffeomorphic to $\mathbb{R}^{n}$. We refer to [79] for more details.

As is well-known, given a short exact sequence of sheaves

$$
\begin{equation*}
0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{F} \longrightarrow \mathscr{G} \longrightarrow 0, \tag{81}
\end{equation*}
$$

we have an associated long exact sequence of cohomology:

$$
\begin{equation*}
\ldots \longrightarrow \check{H}^{k}(X, \mathscr{E}) \longrightarrow \check{H}^{k}(X, \mathscr{F}) \longrightarrow \check{H}^{k}(X, \mathscr{G}) \longrightarrow \check{H}^{k+1}(X, \mathscr{E}) \longrightarrow \ldots \tag{82}
\end{equation*}
$$

Also, if $X$ is an $n$-dimensional manifold, then $\check{H}^{k}(X, \mathscr{F})=0$ for any $\mathscr{F}$ if $k<0$ or $k>n$.
It will prove useful to know all the cohomology groups $\breve{H}^{k}\left(\mathbb{P}^{n}, \mathscr{O}(m)\right)$ for the locallyfree sheaves of rank one introduced in section 4.1.3. The zeroth Cech cohomology consists of sections defined over each $U_{\alpha}$ such that the differences of these sections over the pairwise intersections vanishes. In other words, the zeroth Čech cohomology consists of global sections.

The global sections of a sheaf $\mathscr{F}$ which is an $\mathscr{O}_{X}$-module are in one-to-one correspondence with morphisms $\mathscr{O}_{X} \rightarrow \mathscr{F}$. This important fact will be used many times below. The morphism may be produced from the section simply by multiplication by the section. Given the morphism, the definition of morphisms between $\mathscr{O}_{X}$-modules forces it to be given by multiplication by a global section.

The analysis around (66) therefore shows that global sections of $\mathscr{O}(m)$ are given by homogeneous functions of degree $m$ in the homogeneous coordinates $\left[z_{0}, z_{1}, \ldots, z_{n}\right]$. Thus,

$$
\begin{equation*}
\operatorname{dim} \check{H}^{0}\left(\mathbb{P}^{n}, \mathscr{O}(m)\right)=\binom{n+m}{m} \tag{83}
\end{equation*}
$$

where the binomial coefficient is defined to be zero if either of its entries are negative.

The method of Čech cohomology may be applied to compute the higher cohomologies. This is tedious and there are other ways to do the computation (see [74, 78] for example). The result is that

$$
\begin{align*}
& \operatorname{dim} \check{H}^{k}\left(\mathbb{P}^{n}, \mathscr{O}(m)\right)=0 \quad \text { for } k \neq 0, n \\
& \operatorname{dim} \check{H}^{n}\left(\mathbb{P}^{n}, \mathscr{O}(m)\right)=\binom{-m-1}{-n-m-1} \tag{84}
\end{align*}
$$

Now suppose $X \subset \mathbb{P}^{n}$ is an algebraic variety of dimension $n-1$ corresponding to the zeroes of a polynomial $f$ of homogeneous degree $d$. We denote the embedding $i: X \hookrightarrow \mathbb{P}^{n}$.

Let us introduce a little notation. Suppose we have a map $f: X \rightarrow Y$ between two algebraic varieties and a sheaf $\mathscr{F}$ on $X$. We may define the sheaf $f_{*} \mathscr{F}$ on $Y$ by $f_{*} \mathscr{F}(U)=$ $\mathscr{F}\left(f^{-1} U\right)$. More specifically, suppose we have an embedding $i: X \hookrightarrow \mathbb{P}^{n}$, and we are given a sheaf $\mathscr{E}$ on $X$. The sheaf $i_{*} \mathscr{E}$ is therefore given by $i_{*} \mathscr{E}(U)=\mathscr{E}\left(U \cap \mathbb{P}^{n}\right)$ for all open subsets $U \subset \mathbb{P}^{n}$. This naturally embeds the set of sheaves on $X$ into the sheaves on $\mathbb{P}^{n}$.

The structure sheaf $\mathscr{O}_{X}$ may thus be pushed forward into a sheaf $i_{*} \mathscr{O}_{X}$ on $\mathbb{P}^{4}$. By a modest abuse of notation we refer to this latter sheaf as $\mathscr{O}_{X}$. Clearly $\mathscr{O}_{X}$ is then a quotient sheaf of $\mathscr{O}$. In fact, it is not hard to see that we have a short exact sequence of sheaves

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}(-d) \stackrel{f}{\longrightarrow} \mathscr{O} \longrightarrow \mathscr{O}_{X} \longrightarrow 0 . \tag{85}
\end{equation*}
$$

Using (82), and the fact that $H^{k}(X, \mathscr{F})=H^{k}\left(\mathbb{P}^{n}, i_{*} \mathscr{F}\right)$, this allows the computation of the cohomology of $\mathscr{O}_{X}$. Similarly we may tensor (85) by $\mathscr{O}(m)$ to compute the cohomology of $\mathscr{O}_{X}(m)=\mathscr{O}_{X} \otimes \mathscr{O}(m)$.

### 4.2.2 Spectral sequences

We would like to compare Čech cohomology with something we already know about such as Dolbeault cohomology. A very quick way of doing this is to use "spectral sequences" which we now review. The idea of spectral sequences will also be a recurring topic in the rest of lectures. Again, a comprehensive review of this important subject is beyond the scope of these lectures and we recommend the interested reader consult [79,80], for more information. Here we focus on how a spectral sequence is used rather than why it works.

Suppose we are given a double complex, $E_{0}^{p, q}$,

where each row and each column forms a complex. We assume that the two derivatives anticommute $d \delta+\delta d=0$.

Clearly we may make a new complex

$$
\begin{equation*}
0 \longrightarrow E^{0} \xrightarrow{D} E^{1} \xrightarrow{D} E^{2} \xrightarrow{D} \ldots, \tag{87}
\end{equation*}
$$

where

$$
\begin{equation*}
E^{n}=\bigoplus_{p+q=n} E_{0}^{p, q} \tag{88}
\end{equation*}
$$

and $D=d+\delta$. The big question is, how to we compute the cohomology of (87)?
The spectral sequence method is to inductively form a sequence of stages $E_{r}^{p, q} . E_{r+1}^{p, q}$ is defined as the cohomology of $E_{r}^{p, q}$ with respect to a differential

$$
\begin{equation*}
d_{r}: E_{r}^{p, q} \rightarrow E_{r}^{p+r, q-r+1} \tag{89}
\end{equation*}
$$

where $d_{0}$ is given by $d$ in figure (86) and $d_{1}$ is given by the image of $\delta$ in $E_{1}^{p, q}$. We won't need to know how to compute $d_{r}$ for $r \geq 2$ since any problem that requires such knowledge is essentially too difficult for these lectures!

For large enough $r$ ( $r \geq 2$ if we're lucky) the differentials $d_{r}$ are all zero. This means that, for large $r$, the $E_{r}^{p, q}$ 's become independent of $r$ and so are written $E_{\infty}^{p, q}$. The cohomology of (87) is then given as ${ }^{20}$

$$
\begin{equation*}
H_{D}^{n}=\bigoplus_{p+q=n} E_{\infty}^{p, q} \tag{90}
\end{equation*}
$$

### 4.2.3 Dolbeault cohomology

Let's try the spectral sequence construction to prove that Dolbeault cohomology is equal to a particular example of Čech cohomology. We consider the sheaf $\mathscr{A}^{m, n}$ of $(m, n)$-forms on $X$.

[^13]That is, $\mathscr{A}^{m, n}(U)$ is the group of differentiable (but not necessarily holomorphic) sections of the bundle $\bigwedge^{m} T_{X} \otimes \bigwedge^{n} \bar{T}_{X}$ over the open set $U \subset X$.

Now, build the double complex

$$
\begin{equation*}
E_{0}^{p, q}=\check{C}^{p}\left(\mathfrak{U}, \mathscr{A}^{m, q}\right) \tag{91}
\end{equation*}
$$

where $\mathfrak{U}$ is a good cover of $X$. The "horizontal" $\delta$ operator in (86) is given by the Čech coboundary (79) and the "vertical" $d$ operator is given by the Dolbeault operator $\bar{\partial}: \mathscr{A}^{m, n} \rightarrow$ $\mathscr{A}^{m, n+1}$.

The first step is to take the cohomology under the vertical map $d_{0}=\bar{\partial}$ to obtain the $E_{1}^{p, q}$ s. On $\mathbb{R}^{N}$ any $(m, n)$-form which is $\bar{\partial}$-closed must be $\bar{\partial}$-exact. Thus all the cohomology groups vanish except the bottom row given by $q=0$. The groups $E_{1}^{p, 0}$ are given by $(p, 0)$ forms which are killed by $\bar{\partial}$. In other words, the $E_{1}^{p, q}$ stage looks like

$$
q\left[\begin{array}{c}
\uparrow  \tag{92}\\
\vdots \\
0 \xrightarrow{\delta} 0 \xrightarrow{\delta} 0 \xrightarrow{\delta} 0 \xrightarrow{\delta} 0 \xrightarrow{\delta} \cdots \\
0 \xrightarrow{\delta} 0 \\
\check{C}^{0}\left(\mathfrak{U}, \Omega^{m}\right) \xrightarrow{\delta} \check{C}^{1}\left(\mathfrak{U}, \Omega^{m}\right) \xrightarrow{\delta} \check{C}^{2}\left(\mathfrak{U}, \Omega^{m}\right) \xrightarrow{\delta} \cdots
\end{array}\right.
$$

where $\Omega^{m}$ is the sheaf of holomorphic $m$-forms. That is, $\Omega^{m}(U)$ is the group of holomorphic sections of $\bigwedge^{m} T_{X}$ over $U$.

Now we take the $\delta$-cohomology to form the $E_{2}^{p, q}$ stage. Clearly we end up with $E_{2}^{p, q}=$ $\check{H}^{p}\left(X, \Omega^{m}\right)$ for $q=0$ and $E_{2}^{p, q}=0$ for $q>0$.

For the next stage we note that $d_{2}$ cannot map between nonzero entries and therefore must be zero. Similarly $d_{r}=0$ for $r>2$. Thus, in this case, $E_{2}^{p, q}=E_{\infty}^{p, q}$. Applying (90) we see

$$
\begin{equation*}
H_{D}^{n}=\check{H}^{n}\left(X, \Omega^{m}\right) \tag{93}
\end{equation*}
$$

The alert reader will be wondering why, when the definition of the double complex in (86) is symmetric under interchange of $d$ and $\delta$, the spectral sequence procedure clearly treated them differently. We can exchange the rôles of rows and columns to define another spectral sequence $\tilde{E}_{r}^{p, q}$ with $\tilde{E}_{0}^{p, q}=E_{0}^{p, q}$ and differential

$$
\begin{equation*}
\tilde{d}_{r}: \tilde{E}_{r}^{p, q} \rightarrow \tilde{E}_{r}^{p-r+1, q+r} \tag{94}
\end{equation*}
$$

with $\tilde{d}_{0}=\delta$. This spectral sequence may also be used to compute the total cohomology $H_{D}^{n}$.

Under such a reversal we compute the Čech cohomology first. This may be done by using the fact that the following "Mayer-Vietoris sequence" is exact:

$$
\begin{equation*}
0 \longrightarrow \mathscr{A}^{m, n}(X) \longrightarrow \check{C}^{0}\left(\mathfrak{U}, \mathscr{A}^{m, n}\right) \xrightarrow{\delta} \check{C}^{1}\left(\mathfrak{U}, \mathscr{A}^{m, n}\right) \xrightarrow{\delta} \check{C}^{2}\left(\mathfrak{U}, \mathscr{A}^{m, n}\right) \xrightarrow{\delta} \ldots . \tag{95}
\end{equation*}
$$

Elements of the group $\mathscr{A}^{m, n}(X)$ are globally defined $(m, n)$-forms. The exactness of the start of this sequence should be pretty clear. For the other terms one uses a trick using a partition of unity. We refer to page 94 of [79] for the proof in an essentially identical situation.

The exactness of (95) implies that the $\tilde{E}_{1}^{p, q}$ stage of the spectral sequence looks like


The first column is nothing but the usual Dolbeault complex. Thus $\tilde{E}_{2}^{0, q}=H_{\bar{\partial}}^{m, q}(X)$ and $H_{D}^{n}=H_{\bar{\partial}}^{m, n}(X)$. Comparing to (93) and relabeling a little, we obtain Dolbeault's theorem:

$$
\begin{equation*}
H_{\bar{\partial}}^{p, q}(X)=\check{H}^{q}\left(X, \Omega^{p}\right) \tag{97}
\end{equation*}
$$

Thus Dolbeault cohomology can be rewritten as Čech cohomology.
Suppose $E$ is a holomorphic vector bundle over $X$. The above argument can be generalized to

$$
\begin{equation*}
H_{\bar{\partial}}^{p, q}(X, E)=\check{H}^{q}\left(X, \Omega^{p} \otimes \mathscr{E}\right) \tag{98}
\end{equation*}
$$

where $\mathscr{E}$ is the locally-free sheaf associated to $E$ and the tensor product " $\otimes$ " is defined as a sheaf of tensor products of $\mathscr{O}_{X}$-modules.

### 4.2.4 Sheaf cohomology

Now we give a definition of cohomology which is couched purely in terms of category language. An object $I$ in a category is called injective if, given a monomorphism $f: A \rightarrow B$ and any map $g: A \rightarrow I$, we may construct a map $g^{\prime}: B \rightarrow I$ such that $g^{\prime} f=g$. This may be pictured as the following diagram:


The reader might like to check that $\mathrm{U}(1)$ is injective in the category of abelian groups for example.

In the category of $\mathscr{O}_{X}$-modules, injective objects have an interesting property as follows. Given an open set $U \subset X$ we define $\mathscr{O}_{U}$ as the sheaf $\mathscr{O}_{X}$ restricted to $U$ and then "extended by zero" outside $U$. Roughly speaking, this is the sheaf of holomorphic functions on $U$. We refer to page 68 of [74] for the precise definition of extending by zero. Using a similar argument to that in section 4.2.1, for any $\mathscr{O}_{X}$-module $\mathscr{F}$ one may argue that

$$
\begin{equation*}
\operatorname{Hom}\left(\mathscr{O}_{U}, \mathscr{F}\right)=\mathscr{F}(U) . \tag{100}
\end{equation*}
$$

Now, if $V \subset U$ then $\mathscr{O}_{V}$ is a subsheaf of $\mathscr{O}_{U}$, i.e., there is a monomorphism $\mathscr{O}_{V} \rightarrow \mathscr{O}_{U}$. If $\mathscr{I}$ is an injective $\mathscr{O}_{X}$-module, then, by the definition above, we have a surjective map $\operatorname{Hom}\left(\mathscr{O}_{U}, \mathscr{I}\right) \rightarrow \operatorname{Hom}\left(\mathscr{O}_{V}, \mathscr{I}\right)$. That is, from (100), the restriction map $\rho_{U V}: \mathscr{I}(U) \rightarrow$ $\mathscr{I}(V)$ is surjective.

A sheaf whose restriction maps are all surjective is called flabby. ${ }^{21}$ Thus we have shown that injective $\mathscr{O}_{X}$-modules are flabby sheaves.

Given any object $A$ in an abelian category, an injective resolution of $A$ is a long exact sequence of the form

$$
\begin{equation*}
0 \longrightarrow A \longrightarrow I^{0} \longrightarrow I^{1} \longrightarrow I^{2} \longrightarrow \ldots, \tag{101}
\end{equation*}
$$

where the $I^{k}$ are all injective objects. Injective resolutions need not exist. In the category of $\mathscr{O}_{X}$-modules, injective resolutions do always exist (see page 207 of [74] for example).

At this point we need to introduce the notion of a functor in a category. In our categories of D-branes we will have no direct physical manifestation of a functor but the concept still proves valuable.

Definition $9 A$ functor $F: \mathcal{C} \rightarrow \mathcal{D}$ is a rule that associates an object $F(C)$ of $\mathcal{D}$ to every object $C$ of $\mathcal{C}$ and a morphism $F(f): F\left(C_{1}\right) \rightarrow F\left(C_{2}\right)$ in $\mathcal{D}$ to every morphism $f: C_{1} \rightarrow C_{2}$ in $\mathcal{C}$. It satisfies

1. $F\left(\mathrm{id}_{\mathcal{C}}\right)=\mathrm{id}_{\mathcal{D}}$
2. $F(f g)=F(f) F(g)$.

Suppose we have a short exact sequence

$$
\begin{equation*}
0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0, \tag{102}
\end{equation*}
$$

in an abelian category. A functor $F$ is said to be left-exact if

$$
\begin{equation*}
0 \longrightarrow F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C), \tag{103}
\end{equation*}
$$

[^14]is also exact, and right-exact if the following is exact:
\[

$$
\begin{equation*}
F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \longrightarrow 0 . \tag{104}
\end{equation*}
$$

\]

An example of a functor from the category of abelian groups to itself is $\operatorname{Hom}(G,-)$ for some fixed group $G$. This maps a group $A$ to $\operatorname{Hom}(G, A)$. If $f \in \operatorname{Hom}(G, A)$, then any homomorphism $h: A \rightarrow B$ yields a map $h f: G \rightarrow B$ thus inducing the required map $\operatorname{Hom}(G, A) \rightarrow \operatorname{Hom}(G, B)$. The functor $\operatorname{Hom}(G,-)$ is easily shown to be left-exact but not right-exact.

Given an injective resolution (101) and a left-exact functor $F$, we may construct a complex ${ }^{22}$

$$
\begin{equation*}
0 \longrightarrow F\left(I^{0}\right) \longrightarrow F\left(I^{1}\right) \longrightarrow F\left(I^{2}\right) \longrightarrow \ldots \tag{105}
\end{equation*}
$$

The cohomology of this complex at position $n$ is defined as the $n$th right-derived functor of $A$ and is denoted $\mathbf{R}^{n} F(A)$. The reader is invited to check that the left-exactness of $F$ means that $\mathbf{R}^{0} F(A)=F(A)$. While it is not obvious at first sight, these derived functors do not depend on the choice of an injective resolution.

In the category of $\mathscr{O}_{X}$-modules, the functor of interest is $\operatorname{Hom}\left(\mathscr{O}_{X},-\right)$. As we saw in (100), $\operatorname{Hom}\left(\mathscr{O}_{X}, \mathscr{F}\right)=\mathscr{F}(X)$, i.e., the group of global sections of $\mathscr{F}$. Thus we may also view $\operatorname{Hom}\left(\mathscr{O}_{X},-\right)$ as the global section functor. We may then define sheaf cohomology for $\mathscr{O}_{X}$-modules as the right-derived functors of the global section functor. That is,

$$
\begin{equation*}
H^{n}(X, \mathscr{F})=\mathbf{R}^{n} \operatorname{Hom}\left(\mathscr{O}_{X},-\right)(\mathscr{F}) . \tag{106}
\end{equation*}
$$

Since $\mathbf{R}^{0} F(A)=F(A), H^{0}(X, \mathscr{F})$ corresponds to the group of global sections of $\mathscr{F}$ - just like Cech cohomology.

OK, so this is all pretty abstract! At this point the reader would probably like some examples to work through to get a feel for sheaf cohomology. The truth is that this definition of cohomology is awful for practical calculations. The best one can generally do is show that it is equivalent to some other form of cohomology that can be computed realistically. The reason we have bothered to introduce sheaf cohomology is that its definition is very powerful in the context of the B-model as we see later.

Sheaf cohomology is equivalent to Čech cohomology as can be seen as follows. Given an $\mathscr{O}_{X}$-module $\mathscr{F}$, construct an injective resolution

$$
\begin{equation*}
0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^{0} \longrightarrow \mathscr{I}^{1} \longrightarrow \mathscr{I}^{2} \longrightarrow \ldots \tag{107}
\end{equation*}
$$

Consider the double complex given by

$$
\begin{equation*}
E_{0}^{p, q}=\check{C}^{p}\left(\mathfrak{U}, \mathscr{I}^{q}\right) \tag{108}
\end{equation*}
$$

It follows from (107) that at the first stage of the spectral sequence we have that $E_{1}^{p, q}=$ $\check{C}^{p}(\mathfrak{U}, \mathscr{F})$ for $q=0$ and is zero otherwise. Thus $E_{2}^{p, q}=E_{\infty}^{p, q}=\check{H}^{p}(X, \mathscr{F})$ for $q=0$ and is zero otherwise. This yields $H_{D}^{n}=\check{H}^{n}(X, \mathscr{F})$.

[^15]Applying the spectral sequence the other way requires us to compute the Čech cohomology of injective $\mathscr{O}_{X}$-modules. We saw above that an injective $\mathscr{O}_{X}$-module is a flabby sheaf. It can be shown (e.g., page 221 of [74]) that if $\mathscr{F}$ is flabby, then the Čech cohomology groups $\check{H}^{n}(X, \mathscr{F})$ are zero for $n>0$. As always, $\check{H}^{0}(X, \mathscr{F})$ is given by the global sections $\mathscr{F}(X)$.

This means that the $\tilde{E}_{1}^{p, q}$ stage of the spectral sequence looks like

and $\tilde{E}_{2}^{p, q}$ computes the sheaf cohomology of $\mathscr{F}$. Thus $H_{D}^{n}=H^{n}(X, \mathscr{F})$ and we obtain the equivalence

$$
\begin{equation*}
H^{n}(X, \mathscr{F})=\check{H}^{n}(X, \mathscr{F}) . \tag{110}
\end{equation*}
$$

Note that we have used the language of $\mathscr{O}_{X}$-modules in the section. Everything works equally well if we restrict attention to coherent sheaves since we may always form injective resolutions using coherent sheaves. ${ }^{23}$

We may take the definition of sheaf cohomology a little further. The functor $\operatorname{Hom}(\mathscr{E},-)$ is left-exact for any $\mathscr{O}_{X}$-module $\mathscr{E}$. We denote its right derived functors by "Ext":

$$
\begin{equation*}
\mathbf{R}^{n} \operatorname{Hom}(\mathscr{E},-)(\mathscr{F})=\operatorname{Ext}^{n}(\mathscr{E}, \mathscr{F}) \tag{111}
\end{equation*}
$$

Let us note the following obvious statements:

$$
\begin{align*}
\operatorname{Ext}^{0}(\mathscr{E}, \mathscr{F}) & =\operatorname{Hom}(\mathscr{E}, \mathscr{F})  \tag{112}\\
\operatorname{Ext}^{n}\left(\mathscr{O}_{X}, \mathscr{F}\right) & =H^{n}(X, \mathscr{F}) .
\end{align*}
$$

A very useful fact about these Ext groups is that they satisfy "Serre duality". We refer to [74] for more details. In the case of a smooth Calabi-Yau $m$-fold, Serre duality states that

$$
\begin{equation*}
\operatorname{Ext}^{n}(\mathscr{E}, \mathscr{F}) \cong \operatorname{Ext}^{m-n}(\mathscr{F}, \mathscr{E}) \tag{113}
\end{equation*}
$$

Now suppose we have two vector bundles $E$ and $F$ over $X$. The bundle $\operatorname{Hom}(E, F)$ is also a vector bundle as we saw in section 3.2.2. We may associate locally free sheaves

[^16]$\mathscr{E}$ and $\mathscr{F}$ to the bundles $E$ and $F$ respectively. The locally-free sheaf we associate to $\operatorname{Hom}(E, F)$ is denoted $\mathscr{H} o m(\mathscr{E}, \mathscr{F})$. We emphasize that $\mathscr{H} o m(\mathscr{E}, \mathscr{F})$, which is a sheaf, should not be confused with $\operatorname{Hom}(\mathscr{E}, \mathscr{F})$ which is the abelian group of morphisms from $\mathscr{E}$ to $\mathscr{F} . \operatorname{Hom}(\mathscr{E}, \mathscr{F})$ is actually the group of global sections of $\mathscr{H}$ om $(\mathscr{E}, \mathscr{F})$.

From (98) it follows that

$$
\begin{align*}
H^{0, q}(X, \operatorname{Hom}(E, F)) & =\check{H}^{q}(X, \mathscr{H} \operatorname{Com}(\mathscr{E}, \mathscr{F})) \\
& =H^{q}(X, \mathscr{H} \operatorname{com}(\mathscr{E}, \mathscr{F})) . \tag{114}
\end{align*}
$$

Since sheaf cohomology is the right-derived functor of global section, and a global section of $\mathscr{H} \operatorname{lom}(\mathscr{E}, \mathscr{F})$ is given by $\operatorname{Hom}(\mathscr{E}, \mathscr{F})$, we may further deduce that

$$
\begin{equation*}
H^{q}(X, \mathscr{H} o m(\mathscr{E}, \mathscr{F}))=\operatorname{Ext}^{q}(\mathscr{E}, \mathscr{F}) \tag{115}
\end{equation*}
$$

Thus we have achieved our goal. We have converted the Dolbeault cohomology language of differential geometry into purely algebraic ideas. The statement in section 3.2.2 that an open string from a B -brane $E \rightarrow X$ to a B -brane $F \rightarrow X$ is given by an element of the Dolbeault cohomology group $H_{\bar{\rho}}^{0, q}(X, \operatorname{Hom}(E, F))$ is now restated in the form

An open string from the B-brane associated to the locally-free sheaf $\mathscr{E}$ to another B-brane given by the locally-free sheaf $\mathscr{F}$ is given by an element of the group $\operatorname{Ext}^{q}(\mathscr{E}, \mathscr{F})$.

The reader is probably thoroughly unimpressed at this point given the lengths of abstraction we went to. Hopefully the later lectures will convince the reader that it is all worthwhile!

## 5 The Category of B-branes

### 5.1 Deformations and complexes

The problem with the B-model we have thus far is that it doesn't contain enough B-branes. The first thing to try to do is to see if we can deform the B-branes we already know about into something new.

Looking at the topological field theory, we already saw that we could use vertex operators as deformations. The closed string operators are required to have ghost number two and correspond to $H^{1}\left(X, T_{X}\right)$. These give the expected deformations of complex structure.

As in section 3.1.2 the open strings vertex operators which deform the theory must be ghost number one. For a theory with a single D-brane given by the locally free sheaf $\mathscr{E}$ these correspond to $\operatorname{Ext}^{1}(\mathscr{E}, \mathscr{E})$. Ignoring potential obstructions in the moduli space this agrees with the expected deformations of the sheaf $\mathscr{E} .{ }^{24}$

[^17]What about the ghost number one open strings stretched between two distinct B-branes $\mathscr{E}$ and $\mathscr{F}$ ? The first guess would be to look at vertex operators in $\operatorname{Ext}^{1}(\mathscr{E}, \mathscr{F})$. One can do this but it turns out that one still doesn't generate enough B-branes. To get the right answer we need to be more general.

Our experience with the A-model in section 3.1.3 tells us that assigning a ghost number to an open string stretched between two distinct D-branes is a little troublesome. The Bmodel has no right to be so unambiguous in its knowledge of the ghost number and so we should inflict the same ignorance on it. That is, let us label a B-brane $\mathscr{F}$ with some ghost number $\mu(\mathscr{F})$. An open string stretching from $\mathscr{E}$ to $\mathscr{F}$ in the group $\operatorname{Ext}^{q}(\mathscr{E}, \mathscr{F})$ is then given a ghost number

$$
\begin{equation*}
q+\mu(\mathscr{F})-\mu(\mathscr{E}) \tag{116}
\end{equation*}
$$

in agreement with (55). We are certainly free to attach such ghost number labels to the B-branes without any effect on the B-model. We will also see in section 6 that associating a ghost number to the B-branes themselves is essential if we want to understand the untwisted superconformal field theory.

We may construct a general collection of D-branes in terms of a locally-free sheaf $\mathscr{E}$ in which we have a decomposition:

$$
\begin{equation*}
\mathscr{E}=\bigoplus_{n \in \mathbb{Z}} \mathscr{E}^{n} \tag{117}
\end{equation*}
$$

where $\mathscr{E}^{n}$ is a B-brane with ghost number $n$. The ghost number one operators in this B-model are therefore elements of $\operatorname{Ext}^{k}\left(\mathscr{E}^{n}, \mathscr{E}^{n-k+1}\right)$ for any $n$ and $k$.

We have already noted that the case $k=1$ corresponds to deformations of the sheaves that we already know about. The case $k=0$ concerns open strings

$$
\begin{align*}
d & =\sum_{n} d_{n}  \tag{118}\\
d_{n} & \in \operatorname{Ext}^{0}\left(\mathscr{E}^{n}, \mathscr{E}^{n+1}\right)=\operatorname{Hom}\left(\mathscr{E}^{n}, \mathscr{E}^{n+1}\right)
\end{align*}
$$

i.e., morphisms $d_{n}: \mathscr{E}^{n} \rightarrow \mathscr{E}^{n+1}$ of locally-free sheaves. It will turn out that, by just studying this case, we will obtain the deformations for the other values of $k$ for free.

Let $W_{d}^{(1)}$ be the operator obeying

$$
\begin{equation*}
\left\{Q, W_{d}^{(1)}\right\}=d_{\Sigma} d \tag{119}
\end{equation*}
$$

where $d_{\Sigma}$ is the worldsheet de Rham operator. We deform the action by

$$
\begin{equation*}
S=S_{0}+\oint_{\partial \Sigma} W_{d}^{(1)} \tag{120}
\end{equation*}
$$

Following the usual Noether method, we may show that this results in a change in the BRST charge

$$
\begin{equation*}
Q=Q_{0}+d \tag{121}
\end{equation*}
$$

So, to maintain the relation $Q^{2}=0$, we are required to impose

$$
\begin{equation*}
\left\{Q_{0}, d\right\}+d^{2}=0 \tag{122}
\end{equation*}
$$

We are assuming that $d$ was an open string vertex operator in our original theory before deformation and so we assume $\left\{Q_{0}, d\right\}=0$. Naturally this can only be justified if the deformation of the theory is infinitesimal in some way. Indeed the analysis we perform below only really describes the tangent space for the deformations. There can be obstructions to these deformations.

Anyway, with the assumption $\left\{Q_{0}, d\right\}=0$, we require $d^{2}=0$. What does this mean exactly? $d$ is an open string vertex operator given by a sum in (118). Multiplication of $d$ by itself means we use the operator product algebra for open strings. The first thing to note is that the boundary conditions must make sense in order to obtain a nonzero result: an open string $A \rightarrow B$ can only combine with an open string $C \rightarrow D$ to produce a string $A \rightarrow D$ if $B$ and $C$ represent the same D-brane. Secondly, we stated in section 2.3 that the operator product was given by simple wedge product between forms. Here we are simply multiplying zero forms valued in a group of homomorphisms. Thus the operator product is simply a composition of these homomorphisms. The result is that $d^{2}=0$ implies that

$$
\begin{equation*}
d_{n+1} d_{n}=0 \quad \text { for all } n \tag{123}
\end{equation*}
$$

In other words we have a complex

$$
\begin{equation*}
\ldots \xrightarrow{d_{n-1}} \mathscr{E}^{n} \xrightarrow{d_{n}} \mathscr{E}^{n+1} \xrightarrow{d_{n+1}} \mathscr{E}^{n+2} \xrightarrow{d_{n+2}} \ldots, \tag{124}
\end{equation*}
$$

which we denote $\mathscr{E}^{\bullet}$ for short.
A B-brane is therefore more generally represented by a complex of locally-free sheaves. The maps in the complex represent a deformation from the initial simple collection of sheaves. Note that a sheaf $\mathscr{E}$ itself is a complex in a rather trivial way:

$$
\begin{equation*}
\ldots \xrightarrow{0} 0 \xrightarrow{0} \mathscr{C} \xrightarrow{0} 0 \xrightarrow{0} \ldots \tag{125}
\end{equation*}
$$

Our convention, in this context, will be to assume that $\mathscr{E}$ is in position 0 of the complex.
So, a contender for the objects in our category of B-branes appears to be complexes of locally-free sheaves. This will turn out to be the correct answer but we will need to quotient out by a large set of equivalences. That is, two different complexes may represent the same B-brane. In the language of categories this means that two complexes are related by an invertible morphism, i.e., they are isomorphic. Thus, it is by analyzing the morphisms, i.e., open strings, that we will know if two complexes represent the "same" B-brane.

### 5.2 Open strings

The deformation above will also affect the spectrum of open strings between the B-branes. In this section we compute the corresponding Hilbert spaces of open string states.

Initially let us assume that the open strings come from a B-brane that is a complex and goes to a B-brane that is just a locally-free sheaf. To be more precise, suppose, for simplicity, we have a collection of locally free sheaves $\mathscr{E}^{0}, \mathscr{E}^{1}, \ldots$ and another locally-free sheaf $\mathscr{F}$. Let $\mathscr{E}^{n}$ have ghost number $n$ and $\mathscr{F}$ have ghost number 0 . Now deform the theory by turning the collection of $\mathscr{E}$ 's into a complex (124) with boundary maps $d_{n}^{\mathscr{E}}$. We want to consider the open strings $\mathscr{E} \bullet \rightarrow \mathscr{F}$.

Suppose $\mathscr{F}$ has an injective resolution

$$
\begin{equation*}
0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{I}^{0} \longrightarrow \mathscr{I}^{1} \longrightarrow \mathscr{I}^{2} \longrightarrow \ldots \tag{126}
\end{equation*}
$$

We now construct the double complex $E_{0}^{p, q}=\operatorname{Hom}\left(\mathscr{E}^{-p}, \mathscr{I}^{q}\right)$ :


The maps in this sequence are the obvious ones induced by the complexes (124) and (126). We label the vertical maps $Q_{0}$ since we know, by the definition of Ext in section 4.2.4, that cohomology in this direction produces the open string Hilbert spaces before the deformation $d$ is turned on.

Clearly we have something that looks just like the spectral sequence construction of section 4.2 .2 . The only difference is that we have made $p$ negative to make the horizontal maps point to the right. Note that nothing in the spectral sequence construction in section 4.2.2 depended on the positivity of $p$ and $q$. The required anti-commutivity of the differentials is given by $Q^{2}=0$ as we saw above.

Thus, a spectral sequence construction applied to (127) yields the total cohomology, i.e., the cohomology of $Q=Q_{0}+d$, which is exactly what we are after! In keeping with the notation before the deformation, we denote the Hilbert space of open strings of ghost number $q$ by $\operatorname{Ext}^{q}\left(\mathscr{E}^{\bullet}, \mathscr{F}\right)$.

Note that there is no reason why we couldn't include $\mathscr{E}^{n}$ for $n<0$ in the complex. It just made the diagram a little easier to draw.

We need to work a little harder in the case that the string starts on a sheaf $\mathscr{E}$ and ends on a complex $\mathscr{F}{ }^{\bullet}$ given by

$$
\begin{equation*}
\ldots \longrightarrow \mathscr{F}^{0} \xrightarrow{d_{0}^{\mathscr{E}}} \mathscr{F}^{1} \xrightarrow{d_{1}^{\mathscr{F}}} \mathscr{F}^{2} \xrightarrow{d_{2}^{\mathscr{F}}} \ldots, \tag{128}
\end{equation*}
$$

where each $\mathscr{F}^{p}$ has an injective resolution

$$
\begin{equation*}
0 \longrightarrow \mathscr{F}^{p} \longrightarrow \mathscr{I}^{p, 0} \longrightarrow \mathscr{I}^{p, 1} \longrightarrow \mathscr{I}^{p, 2} \longrightarrow \ldots \tag{129}
\end{equation*}
$$

From the definition of an injective object in section 4.1.6 we may prove the following generalization

Theorem 1 Given any maps $f: A \rightarrow B, g: A \rightarrow I$ in an abelian category with $I$ an injective object, a map $g^{\prime}$ can be constructed to make the following commutative:

$$
\begin{equation*}
\underset{\substack{v^{\prime} g^{\prime} \\ I}}{A \xrightarrow{\prime}} \tag{130}
\end{equation*}
$$

so long as $g \operatorname{ker}(f)=0$.
From this, we may prove that any map $d_{p}^{\mathscr{F}}: \mathscr{F}^{p} \rightarrow \mathscr{F}^{p+1}$ may be extended to a map between resolutions

where the vertical set of maps form a chain map, i.e., every square commutes.
We may build a double complex from $\mathscr{I}^{p, q}$ if we switch the sign of every vertical map in (131) to make the squares anticommute. So we build a single complex of injective objects

$$
\begin{equation*}
\ldots \longrightarrow \mathscr{I}^{n-1} \longrightarrow \mathscr{I}^{n} \longrightarrow \mathscr{I}^{n+1} \longrightarrow \ldots, \tag{132}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathscr{I}^{n}=\bigoplus_{p+q=n} \mathscr{I}^{p, q} . \tag{133}
\end{equation*}
$$

Applying the $\operatorname{Hom}(\mathscr{E},-)$ functor, we obtain $\operatorname{Ext}^{n}\left(\mathscr{E}, \mathscr{F}^{\bullet}\right)$ as the cohomology of the induced complex

$$
\begin{equation*}
\ldots \longrightarrow \operatorname{Hom}\left(\mathscr{E}, \mathscr{I}^{n-1}\right) \longrightarrow \operatorname{Hom}\left(\mathscr{E}, \mathscr{I}^{n}\right) \longrightarrow \operatorname{Hom}\left(\mathscr{E}, \mathscr{I}^{n+1}\right) \longrightarrow \ldots \tag{134}
\end{equation*}
$$

Clearly the general case, $\operatorname{Ext}^{n}\left(\mathscr{E}^{\bullet}, \mathscr{F} \bullet\right)$, must be computed by a triple complex $E_{0}^{p, q, s}=$ $\operatorname{Hom}\left(\mathscr{E}^{p}, \mathscr{I}^{q, s}\right)$. The Hilbert space of open strings is then given by the cohomology of this with respect to $Q=Q_{0}+d^{\mathscr{E}}+d^{\mathscr{F}}$.

To compute this we collapse the double complex $\mathscr{I}^{q, s}$ into a single complex $\mathscr{I}^{q}$ as in (132). Now we have a double complex given by $E_{0}^{p, q}=\operatorname{Hom}\left(\mathscr{E}^{p}, \mathscr{I}^{q}\right)$ from which $\operatorname{Ext}^{n}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ may be found.

Note that this Hilbert space $\operatorname{Ext}^{n}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ of open strings of ghost number $n$ from $\mathscr{E}^{\bullet}$ to $\mathscr{F}^{\bullet}$ occurs commonly in homological algebra and is known as the hyperext group (see chapter 10 of [58] for example).

Let us introduce the useful notion of shifting for complexes. Let $\mathscr{F} \bullet[n]$ denote the complex $\mathscr{F}^{\bullet}$ shifted $n$ places to the left. ${ }^{25}$ Thus, if the $q$ th position of $\mathscr{F}^{\bullet}$ contains $\mathscr{F}^{q}$, the $q$ th position of $\mathscr{F} \cdot[n]$ contains $\mathscr{F}^{q+n}$. It is then easy to convince oneself that

$$
\begin{equation*}
\operatorname{Ext}^{q}(\mathscr{E} \bullet[m], \mathscr{F} \bullet[n])=\operatorname{Ext}^{q-m+n}\left(\mathscr{E}^{\bullet}, \mathscr{F} \bullet\right), \tag{135}
\end{equation*}
$$

i.e., these shift operators just change the ghost number of the B-branes.

It would seem sensible to define $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)=\operatorname{Ext}^{0}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$. Doing this actually defines the category of B-branes. We know what the objects are, namely complexes of locally-free sheaves, and now we've defined the morphisms. We should check, of course, that the morphisms satisfy the axioms of a category. This is not hard to do and we leave it as an exercise for the reader.

### 5.3 The derived category

Logically speaking, we have achieved our goal. Section 5.2 completely defined a category of B-branes. Practically speaking, however, we need to analyze the mathematical structure of this category in order to extract useful information about it. In particular, we would like a more intrinsic description of it.

We constructed the category of B-branes by using the right-derived functor Ext. A category in which the morphisms are obtained from the derived functors of some other category, as above, is called a derived category for obvious reasons.

The definition of the derived category proceeds as follows. We begin with an abelian category $\mathcal{C}$. The derived category of $\mathcal{C}$, denoted $\mathbf{D}(\mathcal{C})$ has objects consisting of complexes of objects of $\mathcal{C}$. We will denote these chain complexes $\mathscr{E} \bullet$, etc., as in the last section. If we were being careful, we would distinguish the case where these complexes had finite length and call it the "bounded derived category". As it is, we will be sloppy and implicitly assume this finiteness condition most of the time.

We will build up the set of morphisms in two stages. As we saw in section 5.2, a chain map is defined as a map between complexes such that all squares commute. Given two chain maps $f, g: \mathscr{E}^{\bullet} \rightarrow \mathscr{F}^{\bullet}$ we define a chain homotopy from $f$ to $g$ as a set of maps $\left\{h_{n}\right\}$ such that we have a diagram


[^18]with $f_{n}-g_{n}=d_{n-1}^{\mathscr{F}} h_{n}+h_{n+1} d_{n}^{\mathscr{E}}$ for all $n$.
The first set of morphisms that we include in the derived category consists of the set of chain maps modulo chain homotopies.

Comparing to section 5.2, the reader might like to check that, given a general map between two chains $\mathscr{E}^{\bullet}$ and $\mathscr{F}^{\bullet}$, a map that is a chain map will be $Q$-closed, and that two chain maps differing by a chain homotopy differ by something $Q$-exact. Thus these morphisms do give part of the set of morphisms in the category of B-branes.

Given a chain map $f: \mathscr{E}^{\bullet} \rightarrow \mathscr{F}^{\bullet}$, we induce a map $f_{*}^{n}: \mathscr{H}^{n}\left(\mathscr{E}^{\bullet}\right) \rightarrow \mathscr{H}^{n}(\mathscr{F} \bullet)$ between the cohomologies of the complexes. We should emphasize that we do not mean sheaf cohomology, but rather the cohomology in the sense of abelian categories in section 4.1.5. Thus, for the category of $\mathscr{O}_{X}$-modules, the objects $\mathscr{H}^{n}\left(\mathscr{E}^{\bullet}\right)$ are sheaves - hence the notation.

A chain map is called a quasi-isomorphism if the induced morphisms $f_{*}^{n}$ are isomorphisms in the category $\mathcal{C}$ for all $n$. If the morphism $f$ is a quasi-isomorphism, we add another morphism $f^{-1}$ to the derived category which composes with $f$ to give the identity. Adding in all these inverse morphisms finally constructs the derived category $\mathbf{D}(\mathcal{C})$. Thus the derived category looks somewhat like

where $A, B, \ldots$ are chain complexes; $f, g, \ldots$ are equivalence classes of chain maps modulo homotopy; and $\approx$ denotes a quasi-isomorphism.

Adding in these inverses makes a lot of objects in $\mathbf{D}(\mathcal{C})$ isomorphic. To be precise, any two objects are isomorphic if the complexes are related by a sequence of quasi-isomorphisms $\mathscr{E}_{1}^{\bullet} \rightarrow \mathscr{E}_{2}^{\bullet} \leftarrow \mathscr{E}_{2}^{\bullet} \leftarrow \mathscr{E}_{3}^{\bullet} \rightarrow \ldots \leftarrow \mathscr{E}_{m}^{\bullet}$, where the arrows may point in either direction.

Note that any complex $\mathscr{E}^{\bullet}$ obviously has the same cohomology as the sequence given by the cohomology itself, i.e., the following complex with zero morphisms:

$$
\begin{equation*}
\ldots \xrightarrow{0} \mathscr{H}^{0}\left(\mathscr{E}^{\bullet}\right) \xrightarrow{0} \mathscr{H}^{1}\left(\mathscr{E}^{\bullet}\right) \xrightarrow{0} \mathscr{H}^{2}\left(\mathscr{E}^{\bullet}\right) \xrightarrow{0} \ldots, \tag{138}
\end{equation*}
$$

However, it is not necessarily true that there is a chain map in either direction between $\mathscr{E}^{\bullet}$ and (138). Thus, in general, a complex need not be quasi-isomorphic to its cohomology. This very important fact leads to the complicated structure of the derived category.

Suppose $\mathcal{C}$ is the abelian category with objects corresponding to complex linear vector spaces and morphisms corresponding to linear maps. In this case, there is always a quasiisomorphism between a complex and its cohomology and thus the derived category takes on a simple form. Every isomorphism class of objects is determined by its cohomology. We emphasize again though that this simplification does not happen in a more general case, such as sheaves.

Let's see what these quasi-isomorphisms do in the B-brane category. The first thing we should note is that the category of locally-free sheaves is not abelian. In order to compute
the cohomology of a complex we need a larger abelian category in which the category of locally-free sheaves is embedded. We may take this to be $\mathscr{O}_{X}$-modules for example.

Consider the double complex $E_{0}^{p, q}=\mathscr{I}^{p, q}$ that we constructed in (131) for the complex $\mathscr{F}^{\bullet}$. Applying the spectral sequence construction we obtain $E_{1}^{p, 0}=\mathscr{F}^{p}$ and $E_{1}^{p, q}=0$ for $q>0$. It follows that $E_{2}^{p, 0}=E_{\infty}^{p, 0}=\mathscr{H}^{p}\left(\mathscr{F}^{\bullet}\right)$ and thus the total cohomology of this complex is given by $\mathscr{H}^{n}\left(\mathscr{F}^{\bullet}\right)$. Of course, the total cohomology of this double complex is, by construction, the cohomology of the single combined complex given in (132). The means that the injective resolution in (131) is equivalent to the statement that

is a quasi-isomorphism. We also say that this quasi-isomorphism represents an injective resolution of the complex $\mathscr{F}^{\bullet}$.

Now, if we have a quasi-isomorphism $\mathscr{E}^{\bullet} \rightarrow \mathscr{F}^{\bullet}$, we may compose this chain map with the quasi-isomorphism (injective resolution) $\mathscr{F}^{\bullet} \rightarrow \mathscr{I}^{\bullet}$ to obtain another quasi-isomorphism $\mathscr{E}^{\bullet} \rightarrow \mathscr{J}^{\bullet}$ - but this is clearly an injective resolution again. Thus $\mathscr{I} \bullet$ represents an injective resolution of both $\mathscr{E} \bullet$ and $\mathscr{F} \bullet!$

Referring back to all the computations in section 5.2, where we used these injective resolutions, it should now be fairly clear that any two complexes related by a quasi-isomorphism are isomorphic objects in the category of B-branes. If $\mathscr{E} \bullet$ and $\mathscr{F}^{\bullet}$ are quasi-isomorphic, the construction of $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)=\operatorname{Ext}^{0}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ is identical to the construction of $\operatorname{Hom}\left(\mathscr{F}^{\bullet}, \mathscr{F}^{\bullet}\right)$ and thus contains a natural "identity" element, as does $\operatorname{Hom}\left(\mathscr{F}^{\bullet}, \mathscr{E}^{\bullet}\right)$ and these elements are naturally inverses to each other. Thus the quasi-isomorphisms are invertible - just like the derived category.

Given two objects $\mathscr{E}^{\bullet}$ and $\mathscr{F}^{\bullet}$ in the derived category, how might we go about computing the set of morphisms $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ ? We can chase the inverted quasi-isomorphisms as follows. Suppose we have a third object $\mathscr{E}_{\boldsymbol{\bullet}}$ with the following chain maps

where $\approx$ denotes a quasi-isomorphism. These maps do not imply the existence of a chain map $\mathscr{E}^{\bullet} \rightarrow \mathscr{F}^{\bullet}$, but in the derived category the map $f$ will contribute to $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ since it may be composed with the inverse of the quasi-isomorphism.

Thus to compute $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ we need to look at chain maps between all objects quasiisomorphic to $\mathscr{E}^{\bullet}$ and $\mathscr{F}^{\bullet}$. To actually carry this process out is hopelessly impractical. Thankfully there is a often a better way.

Suppose the abelian category $\mathcal{C}$ is such that all objects have an injective resolution. Thus, for any complex $\mathscr{F}^{\bullet}$, we have a quasi-isomorphism $\mathscr{F}^{\bullet} \rightarrow \mathscr{I}^{\bullet}$, where $\mathscr{I}^{\bullet}$ is a complex of
injective objects. With a bit of effort, one may then show (see sections 3.10 and 3.11 of [81] or chapter 10 of [58]) that $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ is equal to the set of chain maps from $\mathscr{E}^{\bullet}$ to $\mathscr{I}^{\bullet}$ modulo chain homotopies.

But wait! This is exactly how we were computing the Hilbert space of open strings $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ in section 5.2. Thus, to be almost precise, the category of B-branes is the derived category of locally-free sheaves.

### 5.4 Coherent sheaves

So what's wrong with the last statement in the previous section? The objects in the category of B-branes are indeed complexes of locally-free sheaves and the morphisms are computed exactly in the manner of the derived category.

The only problem is that the way we defined the derived category, we had to begin with an abelian category. This was necessary so that we could take the cohomology of the complex and thus define the notion of a quasi-isomorphism. Locally-free sheaves do not form an abelian category since they do not contain their own cokernels. The way we defined quasi-isomorphisms was to embed the category of locally-free sheaves into the category of $\mathscr{O}_{X}$-modules where the cohomology was defined.

This is only really a cosmetic problem. To be pedantic we should replace the category of locally-free sheaves by the minimal abelian full subcategory of $\mathscr{O}_{X}$-modules containing locally-free sheaves. In section 4.1 .6 we saw that this is the category of coherent sheaves. We have thus proven that ${ }^{26}$

The category of B-branes is the derived category of coherent sheaves $\mathbf{D}(X)$.

This was first conjectured by Kontsevich [73]. This proof is an improved version of an argument in [82] which, in turn, was based on ideas by Douglas [83].

We should emphasize that we have added nothing by going from locally-free sheaves to coherent sheaves. On a smooth space, any coherent sheaf $\mathscr{A}$ has a locally-free resolution, i.e., an exact sequence ${ }^{27}$

$$
\begin{equation*}
0 \longrightarrow \mathscr{F}^{-3} \longrightarrow \mathscr{F}^{-2} \longrightarrow \mathscr{F}^{-1} \longrightarrow \mathscr{F}^{0} \longrightarrow \mathscr{A} \longrightarrow 0, \tag{141}
\end{equation*}
$$

where each $\mathscr{F}^{k}$ is locally free. This is nothing but a quasi-isomorphism, $\mathscr{F}^{\bullet} \rightarrow \mathscr{A}$, between a complex of locally-free sheaves and a coherent sheaf. Similarly, any complex of coherent sheaves is quasi-isomorphic to a complex of locally-free sheaves.

We saw in section 4.1.6 that an example of a coherent sheaf looked a lot like a 0-brane. Since we have now shown that all coherent sheaves are B-branes, we will assert that it really is the 0 -brane.

[^19]Suppose we have an embedding $i: S \hookrightarrow X$, and we are given a sheaf $\mathscr{E}$ on $S$. In section 4.2.1 we defined a sheaf $i_{*} \mathscr{E}$ on $X$. This naturally embeds the set of sheaves on $S$ into the sheaves on $X$.

One might be forgiven for thinking that, if $\mathscr{E}$ is a locally-free sheaf associated to a vector bundle $E$, then $i_{*} \mathscr{E}$ would represent a B-brane wrapping the cycle $S$ with vector bundle $E \rightarrow S$. It turns out that this is not true. This may be traced to the FreedWitten anomaly [84]. To get the correct answer requires an explicit analysis of the vertex operators in the topological field theory for the 2 -cycles and 4 -cycles as was done by Katz and Sharpe $[85,86]$. The sheaf $i_{*} \mathscr{E}$ corresponds to a "bundle" $E \otimes K_{S}^{-\frac{1}{2}}$, where $K_{S}$ is the canonical line bundle of $S$. Note that if $S$ does not admit a spin structure, then $E \otimes K_{S}^{-\frac{1}{2}}$ is a "twisted bundle" in the sense that its first Chern class is not integral. This is in agreement with [84].

To recap, we only needed to consider 6-branes in order to find the correct category for all B-branes. That said, the precise identification of which sheaves correspond to 2-branes and 4 -branes requires the further analysis of $[85,86]$.

It should be noted that there are many many more coherent sheaves on $X$ than these wrapped branes $i_{*} \mathscr{E}$. Indeed, the derived category $\mathbf{D}(X)$ itself is a vast thing encompassing a good deal more than one would expect for B-branes. This is because we have yet to analyze the stability of the B-branes - something the B-model knows nothing about. A physical D-brane in the untwisted theory will only correspond to stable objects in some sense and this condition will rule out the vast majority of objects in $\mathbf{D}(X)$.

In this proof of B-branes being described by the derived category $\mathbf{D}(X)$ we should note we assumed that B-branes really are described by a category. In particular, we assumed that two B-branes which are isomorphic in the category are the "same" B-brane. It is probably a deep philosophical question as to when two abstractly-defined D-branes are the "same" in a strict sense. All we can say is that, within the language of the data of topological field theory, two B-branes which are isomorphic in $\mathbf{D}(X)$ are indistinguishable. If someone wishes to add extra data beyond the topological field theory, then it could be that two quasi-isomorphic complexes represent different D-branes.

### 5.5 More deformations

In section 5.1 we only considered deformations arising from $\operatorname{Hom}\left(\mathscr{E}^{n}, \mathscr{E}^{n+1}\right)$. The obvious question to ask is whether there are any more deformations which can take us outside the derived category.

For example, we could turn on some open strings corresponding to $g_{n} \in \operatorname{Ext}^{3}\left(\mathscr{E}^{n}, \mathscr{E}^{n-2}\right)$ to produce a more complicated "complex":


This actually produces nothing new. To see this first replace the complex $\mathscr{E}^{\bullet}$ by a quasiisomorphic complex $\mathscr{I}^{\bullet}$ of injective sheaves. Now use the definition of Ext in section 4.2.4 and we see that the strings $g_{n}$ are converted into maps $g_{n}: \mathscr{I}^{n} \rightarrow \mathscr{I}^{n+1}$ returning us to the case considered in section 5.1.

About the most general deformation we may consider is as follows. Suppose we have two D-branes given by complexes $\mathscr{E}^{\bullet}$ and $\mathscr{F}^{\bullet}$. Assuming the ghost numbers of the components were not affected by turning on the differentials, an open string corresponding to $f \in \operatorname{Hom}\left(\mathscr{E}^{\bullet} \bullet[-1], \mathscr{F} \bullet\right)$ will have ghost number one. Thus we may consider a deformation given by $f$. It is easy to see that this produces a new combined complex:


This construction is well-known in the context of the derived category and is known as the mapping cone of $f$. We refer to [87] for a nice account of why it has this name. We denote the new complex in (143) as $\operatorname{Cone}\left(f: \mathscr{E}^{\bullet}[-1] \rightarrow \mathscr{F}^{\bullet}\right)$ or just Cone $(f)$.

The cone construction encompasses almost all the deformations we can consider. For example, a complex itself can be considered an iterated cone:

$$
\begin{equation*}
\mathscr{E}^{\bullet}=\ldots \operatorname{Cone}\left(d_{2}: \operatorname{Cone}\left(d_{1}: \operatorname{Cone}\left(d_{0}: \mathscr{E}^{0} \rightarrow \mathscr{E}^{1}\right) \rightarrow \mathscr{E}^{2}\right) \rightarrow \mathscr{E}^{3}\right) \ldots, \tag{144}
\end{equation*}
$$

where we think of a sheaf as a complex with a single entry.
The only exception to this rule is the case of deformations given by $\operatorname{Ext}^{1}\left(\mathscr{E}^{n}, \mathscr{E}^{n}\right)$. Adding such a vertex operator to the action simply deforms $\mathscr{E}^{n}$ itself. This is not quite the same thing as forming $\operatorname{Cone}\left(f: \mathscr{E}^{\bullet} \rightarrow \mathscr{E} \bullet[1]\right)$ in the derived category although the concepts are very closely related. In the latter case we are turning on a string between $\mathscr{E}^{\bullet}$ and a second copy of this D-brane whereas in the former case there was an open string beginning and ending on the same D-Brane. What $\operatorname{Cone}\left(f: \mathscr{E}^{\bullet} \rightarrow \mathscr{E} \bullet[1]\right)$ actually represents is a family of infinitesimal deformations of $\mathscr{E}^{\bullet}$ rather than the deformed $\mathscr{E}^{\bullet}$. We refer to [88] for more details on the theory of deformations.

There is an interesting feature of the deformations we are considering which is worth discussing. Suppose we turn on a nonzero map $f: \mathscr{E}^{0} \rightarrow \mathscr{E}^{1}$. Clearly we can rescale this map by a nonzero number $c \in \mathbb{C}$. There is now a quasi-isomorphism

and so the deformations $f$ and $c f$ represent the same B-brane. ${ }^{28}$ We may use this feature to justify our assumption that the maps $d$ in section 5.1 were infinitesimal since their scale doesn't matter at all!

[^20]Thus, as the deformation $f$ is turned on from zero, we suddenly obtain the new D-brane and any increase in $f$ makes no difference. Such discontinuous behaviour is common in algebraic geometry and therefore it should come as no surprise that B-brane exhibit such behaviour. In the untwisted theory one might expect more continuous behaviour and in section 6.2.3 we will see that this is so.

### 5.6 Anti-branes and K-Theory

One of the key steps in arriving at the derived category picture was associating a ghost number with each B-brane in section 5.1. How physical is this? That is, is there much of a difference between a B-brane associated to the complex $\mathscr{E} \bullet$ and a B-brane associated to the shifted complex $\mathscr{E} \bullet[n]$ for some $n$ ?

Relative shifts certainly matter. We have

$$
\begin{equation*}
\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right) \neq \operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}[n]\right), \tag{146}
\end{equation*}
$$

for generic $\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}$ and nonzero $n$.
If we shift all the complexes by the same $n$ then there is no change in the physics. That is,

$$
\begin{equation*}
\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)=\operatorname{Hom}\left(\mathscr{E}^{\bullet}[n], \mathscr{F}^{\bullet}[n]\right), \tag{147}
\end{equation*}
$$

and there is no change in any of the operator products. Thus, it looks like there is a gauge symmetry of the theory associated to a global shift of the complexes by any integer.

While this is essentially correct, there is a subtlety used in the language of D-branes that makes it preferable to state the gauge symmetry in a different way. Consider a complex as follows:

$$
\begin{equation*}
\ldots \longrightarrow 0 \longrightarrow \mathscr{E} \xrightarrow{c} \mathscr{E} \longrightarrow 0 \longrightarrow \ldots, \tag{148}
\end{equation*}
$$

where the nontrivial map is given by multiplication by $c \in \mathbb{C}$. If $c \neq 0$, the complex (148) is quasi-isomorphic to zero. That is, the two $\mathscr{E}$ 's in (148) cancel out. In other words, the $\mathscr{E}$ on the left is the "anti-brane" of the $\mathscr{E}$ on the right. Turning on $c$ must represent a "tachyon condensate" in the sense of Sen [89] which performs the cancellation. We will have much more to say about such tachyons in section 6.1.3 and 6.2.3.

The generalization of this cancellation is that the mapping cone of the identity map Cone(id : $\mathscr{E}^{\bullet} \rightarrow \mathscr{E}^{\bullet}$ ) is quasi-isomorphic to zero for any complex $\mathscr{E}^{\bullet}$. From section 5.5 it follows that $\mathscr{E}^{\bullet}[1]$ represents the anti-brane to $\mathscr{E}^{\bullet}$. The gauge symmetry is therefore stated as follows [83]

The B-model is subject to a gauge symmetry generated by simultaneously shifting all the B-brane complexes one place to the right (or left) and exchanging the notion of D-brane and anti-D-brane.

We should warn that this anti-brane language is a little crude and can lead to misleading statements. For example $\mathscr{E}^{\bullet}[1]$ is the anti-brane to $\mathscr{E}^{\bullet}$, and $\mathscr{E} \bullet[2]$ is the anti-brane to $\mathscr{E}^{\bullet}[1]$ in the above sense. It does not follow that $\mathscr{E}^{\bullet}$ and $\mathscr{E}^{\bullet}[2]$ are the same D-brane, however, since $\operatorname{Hom}\left(\mathscr{E}^{\bullet}, \mathscr{F}^{\bullet}\right)$ is not generically equal to $\operatorname{Hom}\left(\mathscr{E}^{\bullet}[2], \mathscr{F} \bullet\right)$.

This gauge symmetry means that any intrinsic physical property associated to a D-brane $\mathscr{E} \bullet$ is also given to $\mathscr{E} \bullet[n]$ for any $n$. This would include mass, stability (to be discussed later), etc. Thus, if a particular D-brane $\mathscr{E} \bullet$ becomes massless at a given point in moduli space, all the D-branes $\mathscr{E} \bullet[n]$ become massless. However, it does not mean that an infinite number of D-branes has become massless in a physically meaningful way, since all these D-branes are gauge equivalent. For counting purposes the collection $\left\{\mathscr{E}^{\bullet}[n]: n \in \mathbb{Z}\right\}$ is one D-brane!

The fact that D-brane/anti-D-brane annihilation is built into the derived category descriptions means that we can map the derived category to Witten's K-theory language for D-branes [90]. To do this we basically disregard all the information contained in the morphisms. We saw above that we could deform two D-branes $\mathscr{E}^{\bullet}[-1]$ and $\mathscr{F}^{\bullet}$ into a single D-brane represented by the cone of a morphism $f: \mathscr{E}^{\bullet}[-1] \rightarrow \mathscr{F}^{\bullet}$. Thus the B-brane Cone $(f)$ is composed of $\mathscr{E}^{\bullet}[-1]$ and $\mathscr{F}^{\bullet}$, where $\mathscr{E}^{\bullet}[-1]$ is an anti- $\mathscr{E}^{\bullet}$. We may therefore assert that

$$
\begin{align*}
{[\text { Cone }(f)] } & =\left[\mathscr{F}^{\bullet}\right]+\left[\mathscr{E}^{\bullet}[-1]\right] \\
& =\left[\mathscr{F}^{\bullet}\right]-\left[\mathscr{E}^{\bullet}\right], \tag{149}
\end{align*}
$$

where [ ] represents some kind of "class" of a D-brane. We may define an abelian group $\mathscr{K}(X)$ which is generated by all the objects in $\mathbf{D}(X)$ and we divide out by all relationships of the form (149) for all possible mapping cones. This group $\mathscr{K}(X)$ is called the "Grothendieck group" of $X$ (see page 77 of [75] for more details). The Grothendieck group was also discussed in [91] in the context of D-branes.

We may naturally map the derived category to K-theory as follows. Using locally-free resolutions we may replace any complex by a quasi-isomorphic complex $\mathscr{E}^{\bullet}$ of locally-free sheaves. We may then construct the K-theory object

$$
\begin{equation*}
\ldots \ominus E^{-1} \oplus E^{0} \ominus E^{1} \oplus E^{2} \ominus \ldots \tag{150}
\end{equation*}
$$

where $E^{i}$ is the holomorphic vector bundle associated to $\mathscr{E}^{i}$. One can show that this leads to a well-defined map $\mathscr{K}(X) \rightarrow K(X)$. Note that this map need not be surjective. $K(X)$ is generated by all vector bundles whereas we have restricted attention to holomorphic vector bundles. This is because we have focused only on B-branes, which are essentially BPS. The full K-theory might require some non-BPS branes in order to generate all possible classes.

Anyway, we should emphasize that K-theory contains much less information than the derived category. For example, all 0-branes on $X$ would be represented by the same K-theory element. In contrast, two 0 -branes corresponding to distinct points in $X$ are associated to non-isomorphic objects in $\mathbf{D}(X)$. We like to think of K-theory as a "poor man's derived category" that knows only about D-brane charge.

A more precise notion of D-brane charge may be defined from the world-volume of the Dbrane. This may be computed by anomaly inflow arguments following [92-94]. This subject
is covered by Harvey's lectures at this TASI meeting and so we may refer to [95] and be brief here. A D-brane corresponding to a vector bundle $E$ is given a charge

$$
\begin{equation*}
Q(E)=\operatorname{ch}(E) \sqrt{t d(X)} \tag{151}
\end{equation*}
$$

where $\operatorname{ch}(E)$ is the Chern character of $E$ and $t d(X)$ is the Todd class of the tangent bundle of $X$. Note that $Q(E)$ is an element of $H^{\text {even }}(X, \mathbb{Q})$. It follows from above that this extends to the derived category by

$$
\begin{equation*}
Q\left(\mathscr{E}^{\bullet}\right)=\operatorname{ch}\left(\mathscr{E}^{\bullet}\right) \sqrt{\operatorname{td}(X)} \tag{152}
\end{equation*}
$$

where

$$
\begin{equation*}
\operatorname{ch}\left(\mathscr{E}^{\bullet}\right)=\ldots-\operatorname{ch}\left(E^{-1}\right)+\operatorname{ch}\left(E^{0}\right)-\operatorname{ch}\left(E^{1}\right)+\operatorname{ch}\left(E^{2}\right)-\ldots \tag{153}
\end{equation*}
$$

In section 5.4 we argued that a D-brane wrapped on $S$ was given by a coherent sheaf $i_{*} \mathscr{E}$. The charge of such a D-brane can be computed using the Grothendieck-Riemann-Roch theorem. In the special case that we have an embedding $i: S \rightarrow X$, this asserts that

$$
\begin{equation*}
\operatorname{ch}\left(i_{*} \mathscr{E}\right) \operatorname{td}(X)=i_{!}(\operatorname{ch}(\mathscr{E}) \operatorname{td}(S)) \tag{154}
\end{equation*}
$$

where $i_{!}$is defined on cohomology as $P \cdot i_{*} \cdot P^{-1}$, where $P$ is Poincaré duality and $i_{*}$ in this latter context is the natural map induced by $i$ on homology. It follows that, for any $C \in H^{\text {even }}(X)$, we have the following formula

$$
\begin{equation*}
\int_{X} C \cdot Q\left(i_{*} \mathscr{E}\right)=\int_{S} \operatorname{ch}(\mathscr{E}) \sqrt{\frac{t d(S)}{t d(N)}} \cdot i^{*} C \tag{155}
\end{equation*}
$$

where $N$ is the normal bundle of $S$ in $X$.
That said, in section 5.4 we also saw that $i_{*} \mathscr{E}$ corresponds to a B-brane given by a (twisted) bundle $E^{\prime}=E \otimes K_{S}^{-\frac{1}{2}}$ over $S$, where $E$ is the bundle associated to $\mathscr{E}$. Using the relation $t d=\exp \left(\frac{1}{2} c_{1}\right) \hat{A}$ and the fact that $c_{1}(X)=0$, we may therefore rewrite (155) as

$$
\begin{equation*}
\int_{X} C \cdot Q\left(i_{*} \mathscr{E}\right)=\int_{S} \operatorname{ch}\left(E^{\prime}\right) \sqrt{\frac{\hat{A}(S)}{\hat{A}(N)}} \cdot i^{*} C \tag{156}
\end{equation*}
$$

which is the formula one would arrive at via anomaly consideration [95].

### 5.7 Mirror symmetry restored?

If the A-model on $Y$ is "the same" as the B-model on $X$, then it would appear that we have motivated the proposal that the Fukaya category on $Y$ is equivalent to $\mathbf{D}(X)$, the derived category on $X$. That was Kontsevich's original proposal. It turns out that there is still a small fly in the ointment, as we discuss in section 6.2.2, but this proposal seems to be very close to the truth. In this section we note a few miscellaneous features of how well this mirror symmetry works.

The D-brane charge of an A-brane is simply given by its homology class in $\left[L_{i}\right] \in H_{3}$ multiplied by the rank of the bundle over the Lagrangian. Let us denote the Poincaré dual of $\left[L_{i}\right]$ by $l_{i} \in H^{3}(X)$. Note that we have a natural symplectic inner product on these charges given by the oriented intersection number $\#\left(\left[L_{1}\right] \cap\left[L_{2}\right]\right)$. It can be shown [64] that the orientations of the points of intersection in figure 4 are opposite if the difference in their ghost numbers is odd. Thus the intersection number is given by the Euler characteristic of the complex (52). That is,

$$
\begin{align*}
\#\left(\left[L_{1}\right] \cap\left[L_{2}\right]\right) & =\int_{Y} l_{1} \cdot l_{2} \\
& =\sum_{i}(-1)^{i} \operatorname{dim} \operatorname{Hom}^{i}\left(L_{1}, L_{2}\right) . \tag{157}
\end{align*}
$$

If the Lagrangian $L_{i}$ is mirror to a complex $\mathscr{E}_{i}^{\bullet}$ then the right-hand-side of (157) is clearly mirror to the alternating sum of $\operatorname{dim} \operatorname{Ext}^{i}\left(\mathscr{E}_{1}^{\bullet}, \mathscr{E}_{2}^{\bullet}\right)$. The Hirzebruch-Riemann-Roch theorem says that this is given by

$$
\begin{align*}
\sum_{i}(-1)^{i} \operatorname{dim} \operatorname{Ext}^{i}\left(\mathscr{E}_{1}^{\bullet}, \mathscr{E}_{2}^{\bullet}\right) & =\int_{X} \operatorname{ch}\left(\mathscr{E}_{1}^{\bullet}\right)^{\vee} \cdot \operatorname{ch}\left(\mathscr{E}_{2}^{\bullet}\right) \cdot t d(X)  \tag{158}\\
& =\int_{X} Q\left(\mathscr{E}_{1}^{\bullet}\right)^{\vee} \cdot Q\left(\mathscr{E}_{2}^{\bullet}\right)
\end{align*}
$$

where, if $\omega$ is a $2 p$-form, then $\omega^{\vee}=(-1)^{p} \omega$. Thus we see a very nice agreement for the pairing between A-brane charges and B-branes charges. Note that the " V " is necessary in (158) to get a symplectic inner product.

The tadpole cancellation condition in section 3.1.3 produces two interesting aspects of the moduli space of A-branes:

1. First-order deformations of the Lagrangian, which correspond to $H^{1}(L)$, may be obstructed and do not lead to genuine A-brane deformations.
2. Some A-branes may depend on very special values for $B+i J$ and disappear completely for generic $B+i J$.

The mirror statements in the B-model are both true:

1. The first-order deformations of coherent sheaves, which correspond to $\operatorname{Ext}^{1}(\mathscr{E}, \mathscr{E})$ can be obstructed. We refer to [96] for examples.
2. There are some sheaves which only exist for special values of complex structure. An example of this is given by 2-branes wrapped around an algebraic curve of high genus in $X$ [97].

This subject was also analyzed in [51,52]. Note that this is a typical example of mirror symmetry in that instanton effects (i.e., tadpoles) in the A-model are mapped to effects in the B-model that can be understood from classical geometry.

|  | A-model | B-model |
| :--- | :--- | :--- |
| Geometry | Symplectic (no complex structure) | Algebraic (no metric) |
| Category | Fukaya category | Derived category |
| D-branes | Lagrangians | Complexes of coherent sheaves |
| Open strings | Floer cohomology | Ext's |
| Dependence | $B+i J$ | complex structure |
| Charges | $l_{i} \in H^{3}$ | $\operatorname{ch}(\mathscr{E}) \sqrt{t d(X)} \in H^{\text {even }}(X)$ or $K(X)$ |

Table 1: Mirror symmetry for A-branes and B-branes.

In table 1 we review the picture of mirror symmetry that we have obtained so far. The reader might be a little disappointed to note that we haven't actually used the more exotic elements of the derived category in this discussion of mirror symmetry - everything was done for coherent sheaves. We will give a very explicit example that requires a nontrivial complex in section 7.1.3.

## 6 Stability

So far we have dealt with D-branes in the context of topological field theory. This was sufficient to understand the origins of the Fukaya category in the case of A-branes and the derived category in the case of B-branes. In the untwisted theory it is the D-branes that correspond to BPS states that descend to D-branes in the topological field theory. Having said that, the BPS condition is stronger than that imposed on branes in the topological field theories. In order for an A-brane or a B-brane to correspond to a BPS state in the untwisted theory we need to impose a further condition - namely "stability".

The purpose of studying stability is two-fold. As we just said, in order to make contact with the "real world", i.e., the untwisted theory, a D-brane must be stable. In addition, stability makes us study a mathematical structure on the categories of D-branes, i.e., "distinguished triangles", that provides further insight into the intrinsic structure of the D-brane categories.

Stability of D-Branes was also studied in $[98,99]$ using a quite different method than we employ here.

### 6.1 A-Branes

### 6.1.1 Special Lagrangians

The spacetime supersymmetry arises from the spectral flow operators discussed at the end of section 2.1. These are associated with the holomorphic 3 -form $\Omega$ on $X$ as in (11). The $N=2$ spacetime supersymmetry arises because we have a spectral flow operator in the left-
moving and right-moving sector. The boundary conditions on the open string destroys the independence of these sectors and the best we can do is to preserve an $N=1$ supersymmetry. To do this we can set

$$
\begin{equation*}
\Sigma=\exp (-i \pi \xi) \bar{\Sigma} \tag{159}
\end{equation*}
$$

on the ends of the string, where $0 \leq \xi<2$. The parameter $\xi$ measures "which" $N=1$ spacetime supersymmetry is preserved from the original $N=2 .{ }^{29}$

This boundary conditions given by $R_{\bar{\jmath}}^{i}$ in section 3.1.1 imply that this is given by

$$
\begin{equation*}
\left.\Omega\right|_{L}=\left.\exp (-2 i \pi \xi) \bar{\Omega}\right|_{L}, \tag{160}
\end{equation*}
$$

on the A-brane $L$. In section 3.1.1 we noted that $\left.\Omega\right|_{L}$ was equivalent to the real volume form on $L$ up to some complex constant. This means that our choice of the real parameter $\xi$ coincides with that of (42). That is,

$$
\begin{equation*}
\xi=\frac{1}{\pi} \arg \frac{\left.\Omega\right|_{L}}{d V_{L}} . \tag{161}
\end{equation*}
$$

The key issue is that the parameter $\xi$ must be the same at all points on the Lagrangian $L$ in order for the same spacetime supersymmetry to be preserved everywhere. A Lagrangian for which $\xi$ is a constant is called a special Lagrangian. Thus it is the special Lagrangians which correspond to BPS states as first observed in [100].

Note that if $\xi$ is a constant, then we may rewrite (161) as

$$
\begin{equation*}
\xi=\frac{1}{\pi} \arg \int_{L} \Omega \tag{162}
\end{equation*}
$$

Note also that $\Omega$ is only defined up to a complex constant so the value of $\xi$ might appear somewhat meaningless. Indeed, the standard definition of a special Lagrangian is to put $\xi=0$ and so assert that the real part of $\left.\Omega\right|_{L}$ is zero. We will need the idea of comparing values of $\xi$ between different special Lagrangians and so we retain the notion here, although one should always bear in mind that only relative values of $\xi$ have any meaning.

In section 3.1.2 we gave a very specific definition of an A-brane. In addition to being a Lagrangian it had to satisfy two extra condition. It should be obvious that the map $\xi_{*}$ in (43) is trivial and thus the Maslov class condition is automatically satisfied for a special Lagrangian. The second condition concerned the tadpole cancellation. This, in general, is not automatically satisfied in the special Lagrangian case and so remains an extra condition to be imposed.

It is very easy to motivate the idea that special Lagrangians are BPS states. A special Lagrangian is a calibrated submanifold in the sense of Harvey and Lawson [101]. The details of this definition need not concern us but the useful fact is that any calibrated submanifold

[^21]automatically minimizes the volume of any manifold in its homology class. Thus, if we think of a D-brane as some kind of membrane with a tension, a D-brane that wraps a special Lagrangian submanifold is clearly stable. Note that there is no reason to suppose that all minimal 3-manifolds in a Calabi-Yau are special Lagrangians, reflecting the fact that not all stable D-branes are necessarily BPS.

### 6.1.2 A geometrical decay

Tadpoles aside, in section 3.1.2 we saw that an A-brane (with a line bundle) has a moduli space given by $H^{1}(L)$. It can be shown [102] that the moduli space of special Lagrangians is also given by $H^{1}(L)$. Thus, locally, the moduli space of special Lagrangians agrees with the moduli space of Lagrangians modulo Hamiltonian deformation. At first sight, this might suggest that in each equivalence class of Lagrangians modulo Hamiltonian deformation there is a unique special Lagrangian.

This is not actually true. It turns out the vast majority of Lagrangians have no special Lagrangian equivalent to them by Hamiltonian deformation. From our perspective, the best way to see this is to consider how special Lagrangians can "disappear", or "decay", as the complex structure of the target space $Y$ is deformed. Note that a Lagrangian submanifold is defined purely in terms of the symplectic structure of $Y$ induced by the Kähler form and so has no dependence on the complex structure. Adding the "special" in special Lagrangian does introduce a dependence on the complex structure.

A quite explicit picture for the decay of special Lagrangians was given by Joyce [103] which we follow here.

Let us first consider special Lagrangian planes $\mathbb{R}^{m} \subset \mathbb{C}^{m}$. We may specify such a plane by

$$
\begin{equation*}
\Pi^{\phi}=\left\{\left(e^{i \phi_{1}} x_{1}, e^{i \phi_{2}} x_{2}, \ldots, e^{i \phi_{m}} x_{m}\right): x_{j} \in \mathbb{R}\right\} \tag{163}
\end{equation*}
$$

This plane is determined by the real numbers $\phi_{j}$. Using the standard holomorphic $m$-form $\Omega=d z_{1} \wedge d z_{2} \wedge \ldots \wedge d z_{m}$ we obtain

$$
\begin{align*}
\xi\left(\Pi^{\phi}\right) & =\frac{1}{\pi} \arg \int_{\Pi^{\phi}} \Omega \\
& =\frac{1}{\pi} \sum_{j=1}^{m} \phi_{j} \quad(\bmod 2) . \tag{164}
\end{align*}
$$

If we reverse the orientation of $\Pi^{\phi}$ we shift $\xi\left(\Pi^{\phi}\right)$ by one. Such a reversal of orientation may be viewed as replacing an A-brane by an anti-A-brane. If we forget about the orientation we are free to restrict the $\phi_{j}$ 's to the range $0 \leq \phi_{j}<\pi$. In this case we have

$$
\begin{equation*}
\sum_{j=1}^{m} \phi_{j}=k \pi, \tag{165}
\end{equation*}
$$

for $0 \leq k<m$. We say such an intersection of planes is of type $k$.

Let us denote by $\Pi^{0}$ the plane for which $\phi_{1}=\phi_{2}=\ldots=\phi_{m}=0$. Now consider two Dbranes intersecting transversely at the origin in the form $\Pi^{0} \cup \Pi^{\phi}$. The transverse condition amounts to $\phi_{j}>0$ for all $j$. One may therefore interpret these two D -branes as one singular special Lagrangian so long as the type of the intersection is integral. The fact that such a configuration is a BPS state was first noted in [104]

If the type of the intersection is 1 , this singular A-brane can be written as a limit in a family of smooth special Lagrangians following Lawlor [105]. A smooth member of this family is called a "Lawlor neck" and can be described as follows $[103,106]$. Let

$$
\begin{equation*}
P(x)=\frac{\prod_{j=1}^{m}\left(1+a_{j} x^{2}\right)-1}{x^{2}} \tag{166}
\end{equation*}
$$

Now fix a positive real number $A$. The positive real numbers $a_{1}, \ldots, a_{m}$ are then implicitly and uniquely determined by $A$ and $\phi_{1}, \phi_{2}, \ldots, \phi_{m}$ by the following equations:

$$
\begin{align*}
\phi_{j} & =a_{j} \int_{-\infty}^{\infty} \frac{d x}{\left(1+a_{j} x^{2}\right) \sqrt{P(x)}}  \tag{167}\\
A & =\frac{\omega_{m}}{\sqrt{a_{1} \cdots a_{m}}},
\end{align*}
$$

where $\omega_{m}$ is the volume of a unit sphere in $\mathbb{R}^{m}$. These equations impose the condition $\sum \phi_{j}=\pi$, i.e., type 1 . Note that if we include orientations, since the type is odd, we would say that we have unbroken spacetime supersymmetry if one of the planes is viewed as a D-brane and the other plane is viewed as an anti-D-brane.

Now define functions $\eta_{j}: \mathbb{R} \rightarrow \mathbb{C}$ by

$$
\begin{equation*}
\eta_{j}(y)=\exp \left(i a_{j} \int_{-\infty}^{y} \frac{d x}{\left(1+a_{j} x^{2}\right) \sqrt{P(x)}}\right) \sqrt{\frac{1}{a_{j}}+y^{2}} \tag{168}
\end{equation*}
$$

This allows the Lawlor neck to be defined as

$$
\begin{equation*}
L^{\phi, A}=\left\{\left(\eta_{1}(y) x_{1}, \eta_{2}(y) x_{2}, \ldots, \eta_{m}(y) x_{m}\right): y \in \mathbb{R}, x_{j} \in \mathbb{R}, x_{1}^{2}+\ldots+x_{m}^{2}=1\right\} . \tag{169}
\end{equation*}
$$

One can then show that this is a smooth special Lagrangian submanifold of $\mathbb{C}^{m}$ which approaches $\Pi^{0} \cup \Pi^{\phi}$ as $A \rightarrow 0$. This submanifold also asymptotically approaches $\Pi^{0} \cup \Pi^{\phi}$ as one moves far from the origin. For the precise analytical details of this we refer again to [103]. Topologically this space looks like a cylinder $S^{m-1} \times \mathbb{R}$. It is impossible to sketch this space completely realistically since, for $m=1$, the condition $\sum \phi_{j}=0$ makes the case trivial, and for $m>1$ we are in at least four dimensions. The case $m=2$ is shown roughly in figure 6 .

Now suppose we have a Calabi-Yau $m$-fold $Y$ containing two A-branes $L_{1}$ and $L_{2}$ which intersect transversely at a point $p$. We may use a $\mathrm{U}(m)$ transformation to rotate the tangent plane of $L_{1}$ into the standard plane $\Pi^{0}$ as above. This same $\mathrm{U}(m)$ matrix rotates the tangent


Figure 6: The Lawlor Neck.
plane of $L_{2}$ into $\Pi^{\phi}$ thus defining a type for the intersection as above. Again assume that the type of intersection is equal to 1 .

Let us consider a small deformation of complex structure of $Y$. Joyce [103] proved that, close to the point of intersection of $L_{1}$ and $L_{2}$, the local geometry could produce a Lawlor neck to smooth out the neighbourhood of the point of intersection. Let us now spell out in details exactly what happens.

Suppose $L_{1}$ and $L_{2}$ are smooth submanifolds of $Y$. As we deform the complex structure of $Y$ we would like to find special Lagrangian submanifolds of the deformed $Y$ which are the deformed versions of $L_{1}$ and $L_{2}$. That is, we would like to be able to follow $L_{1}$ and $L_{2}$ as we deform $Y$ without these D-branes disappearing for some reason. As long as we consider sufficiently small deformations, we are guaranteed to be able to do this [103, 107]. This means that we may define $\xi\left(L_{1}\right)$ and $\xi\left(L_{2}\right)$ as real numbers which vary continuously over the moduli space for small deformations of $Y$. To be precise, fix the mod 2 ambiguity in $\xi\left(L_{1}\right)$ arbitrarily, then we set $\xi\left(L_{2}\right)=\xi\left(L_{1}\right)+1$ to reflect the fact that the intersection is type 1. Now $\xi\left(L_{1}\right)$ and $\xi\left(L_{2}\right)$ are defined over the moduli space of $Y$ at least in some neighbourhood of the starting point. We will refer to the value of $\xi$ defined in $\mathbb{R}$ as the "grading" of a special Lagrangian.

A wall divides the moduli space into $\mathscr{M}^{+}$and $\mathscr{M}^{-}$corresponding to the sign of $\xi\left(L_{1}\right)$ $\xi\left(L_{2}\right)+1$. We begin at a point in the wall. As we deform the complex structure to the $\mathscr{M}^{+}$ side of this wall, the point of intersection is smoothed out by a Lawlor neck for $A>0$. On the other side, $\mathscr{M}^{-}$, no smoothing occurs.

On the $\mathscr{M}^{+}$side of the wall we use the notation $L_{1} \hookrightarrow L_{2}$ to denote the smoothed new special Lagrangian. Very close to the wall, $L_{1} \leftrightarrow L_{2}$ looks asymptotically like $L_{1} \cup L_{2}$ away from the point of intersection and the geometry near the point of intersection is replaced by a Lawlor neck. The notation is intentionally asymmetric since $L_{1} \rightarrow L_{2}$ is quite different from $L_{2} \rightarrow L_{1}$. Note that away from the wall, $L_{1} \cup L_{2}$ is no longer a special Lagrangian since the two components have a different value of grading $\xi$. The smooth space $L_{1} \rightarrow L_{2}$ is the smooth space homological to $L_{1} \cup L_{2}$ which minimizes the volume, i.e., energy of a D-brane

| $\mathscr{M}_{+}$ |  | $\xi\left(L_{2}\right)<\xi\left(L_{1} \rightarrow L_{2}\right)<\xi\left(L_{1}\right)+1$ | $L_{1} \rightarrow L_{2}$ is stable. |
| :---: | :---: | :---: | :---: |
| Wall |  | $\xi\left(L_{2}\right)=\xi\left(L_{1} \rightarrow L_{2}\right)=\xi\left(L_{1}\right)+1$ | $L_{1} \rightarrow L_{2}$ is marginally stable. |
| $\mathscr{M}_{-}$ | $L_{1} \circlearrowleft L_{2}$ | $\xi\left(L_{2}\right)>\xi\left(L_{1}\right)+1$ | $L_{1} \rightarrow L_{2}$ is unstable. |

Figure 7: A-Brane Decay.
wrapped around these cycles. It follows that

$$
\begin{equation*}
\left|\int_{L_{1} \nrightarrow L_{2}} \Omega\right|<\left|\int_{L_{1}} \Omega\right|+\left|\int_{L_{2}} \Omega\right|, \tag{170}
\end{equation*}
$$

and we may choose

$$
\begin{equation*}
\xi\left(L_{2}\right)<\xi\left(L_{1} \rightarrow L_{2}\right)<\xi\left(L_{1}\right)+1 . \tag{171}
\end{equation*}
$$

In $\mathscr{M}^{-}$there is no smooth special Lagrangian minimizing the volume of $L_{1} \cup L_{2}$ and $L_{1} \cup L_{2}$ itself is not a BPS state. Thus spacetime supersymmetry is broken in $\mathscr{M}^{-}$.

To recap, in $\mathscr{M}^{+}$we have a BPS state $L_{1} \rightarrow L_{2}$. We also have BPS states $L_{1}$ and $L_{2}$ but the mass of $L_{1} \rightarrow L_{2}$ is less than the sum of the masses of $L_{1}$ and $L_{2}$. As we hit the wall, $L_{1} \rightarrow L_{2}$ becomes $L_{1} \cup L_{2}$. Beyond the wall in $\mathscr{M}^{-}$we only have BPS states $L_{1}$ and $L_{2}$ which together break supersymmetry. What we have just described is a decay of a BPS state $L_{1} \leftrightarrow L_{2}$ into its products $L_{1}$ and $L_{2}$ as we pass from $\mathscr{M}^{+}$into $\mathscr{M}^{-}$. In $\mathscr{M}^{+}$we view $L_{1} \leftrightarrow L_{2}$ as a bound state of $L_{1}$ and $L_{2}$. We depict this story in figure 7 .

This is familiar from the standard properties of BPS states in $N=2$ theories in four dimensions as studied, for example, by Seiberg and Witten [108]. We refer to [109] for background in this subject.

In the above discussion, the period of the holomorphic 3-form is playing the rôle of the central charge, Z , of the BPS state. That is

$$
\begin{equation*}
Z(L)=\int_{L} \Omega \tag{172}
\end{equation*}
$$

This relationship may also be derived from the fact that both $Z$ and the periods are subject to the same rules of special geometry $[29,31,108] .{ }^{30}$

### 6.1.3 Tachyon condensates

In the previous section we saw a fairly concrete geometrical picture for A-brane decay. The same decay process can be motivated from a different perspective by using the idea of tachyon condensation from [89] (see also [110] and references therein).

Consider an intersection of two special Lagrangian planes $\Pi^{0}$ and $\Pi^{\phi}$ in $\mathbb{C}^{m}$ as in section 6.1.2. One may analyze the masses of the open strings which begin on $\Pi^{0}$ and end on $\Pi^{\phi}$ following [104] or section 13.4 of [10] (see also [8]). The result is that there are R sector strings which are always massless and NS sector scalar fields which have a mass

$$
\begin{equation*}
M^{2}=\frac{1}{2 \pi}\left(\sum_{j=1}^{m} \phi_{j}-\pi\right) . \tag{173}
\end{equation*}
$$

These scalars fields are not projected out by the GSO process if one of the D-branes is viewed as an anti-D-brane.

We now propose an equation which generalizes (164) to remove the mod 2 ambiguity. In section 6.1 .2 we gave a way of defining $\xi$ to be valued in $\mathbb{R}$ by demanding continuity over the moduli space of complex structures at least as long as the associated A-brane did not decay. If $L_{1}$ and $L_{2}$ intersect at a point $p$ with Floer index (i.e., ghost number) $\mu(p)$ then

$$
\begin{equation*}
\xi\left(L_{2}\right)-\xi\left(L_{1}\right)+\mu(p)=\frac{1}{\pi} \sum_{j=1}^{m} \phi_{j} . \tag{174}
\end{equation*}
$$

This follows from continuity and the following fact. Suppose $L_{1}$ and $L_{2}$ intersect at two points $p_{1}$ and $p_{2}$ and that each Lagrangian has a trivial Maslov class as in section 3.1.1. Using the arguments of [64] one can show that the difference in $\mu\left(p_{1}\right)$ and $\mu\left(p_{2}\right)$ is equal to the difference in $\sum \phi_{j}$ for each point.

The equation (174) ties the ambiguity in defining the ghost number $\mu(p)$, which we discussed in section 3.1.3, to the ambiguity in the definition of the grading $\xi$. The ambiguity in $\mu(p)$ was fixed by labeling each A-brane $L$ with some integral ghost number $\mu(L)$. Borrowing some notation from the derived category, let $L[n]$ be exactly the same A-brane as $L$ except that we have increased its ghost number by $n$. It follows from (55) and (174) that

$$
\begin{equation*}
\xi(L[n])=\xi(L)+n . \tag{175}
\end{equation*}
$$

Restricting attention to open strings with ghost number 0, i.e., $\operatorname{Hom}\left(L_{1}, L_{2}\right)$ in the Fukaya category, we see that

$$
\begin{equation*}
2 M^{2}=\xi\left(L_{2}\right)-\xi\left(L_{1}\right)-1 \tag{176}
\end{equation*}
$$

[^22]Thus, comparing to the last section, in $\mathscr{M}^{+}$we have $M^{2}<0$ and so the open string in $\operatorname{Hom}\left(L_{1}, L_{2}\right)$ is tachyonic. This is entirely consistent with the fact that there is a ground state $L_{1} \leftrightarrow L_{2}$ lower in energy than $L_{1} \cup L_{2}$. This tachyon condenses to form $L_{1} \rightarrow L_{2}$. In $\mathscr{M}^{-}$ the open string is not tachyonic and no condensation occurs.

The tachyonic condensation picture therefore gives a very simple description of the hard analysis performed by Joyce reviewed in section 6.1.2. What we would like to conjecture is that this tachyon picture gives a complete criterion for how A-branes decay as one moves in the moduli space of complex structures. This is certainly well-motivated from a physics point of view but the differential geometry required to make such a statement rigorous is difficult. Progress has been made in this direction in $[111,112]$ for example.

Note that since $0 \leq \phi_{j}<\pi$, it follows from (174) that, if $Y$ is a Calabi-Yau $m$-fold,

$$
\begin{equation*}
0 \leq \xi\left(L_{2}\right)-\xi\left(L_{1}\right)+\mu(p)<m \tag{177}
\end{equation*}
$$

This relation is nicely consistent with the unitarity of representations of the superconformal algebra as studied in [13]. Any open string vertex operator in the topological field theory corresponds to a chiral primary field (in the NS sector) in the untwisted theory in the sense of [14]. This field must have conformal weight $h$ between 0 and $c / 6$, where $c$ is the central charge. For a non-linear $\sigma$-model, $c / 3=m$. Finally, a vertex operator for a primary chiral field of conformal weight $h$ is associated to a mass of $M^{2}=h-\frac{1}{2}$. Thus, comparing to (173) and (174), we see agreement.

We emphasize that the A-brane decay process occurs due to deformations of complex structure of $Y$. This makes it essentially invisible to the topological A-brane since the latter depends only upon $B+i J$. This is consistent with the fact that, as far as the A-model is concerned, A-branes are Lagrangian with no special condition applied. Without the special condition there is no decay process.

Finally in this section we recall from standard string theory analysis that there are open string states in the NS sector corresponding to vector particles in the uncompactified dimensions. These have mass

$$
\begin{equation*}
2 M^{2}=\xi\left(L_{2}\right)-\xi\left(L_{1}\right) \tag{178}
\end{equation*}
$$

These are therefore always massless when $L_{1}=L_{2}$. In other words we have vectors associated to $L_{1}$ given by $\operatorname{Hom}\left(L_{1}, L_{1}\right)$. These are the vectors associated to gauge group present in the D-brane - to be precise, $\operatorname{Hom}\left(L_{1}, L_{1}\right)$ is the complexification of the gauge algebra. In the case of a single irreducible D-brane we expect a $\mathrm{U}(1)$ gauge group and thus $\operatorname{Hom}\left(L_{1}, L_{1}\right)=\mathbb{C}$. If the gauge group is enhanced, either because we have two distinct $D$-branes, or because we have coincident D -branes, the gauge group, and thus $\operatorname{Hom}\left(L_{1}, L_{1}\right)$ will be bigger. The fact that the irreducibility of D-brane is equivalent to $\operatorname{Hom}\left(L_{1}, L_{1}\right)=\mathbb{C}$ may be viewed as a version of Schur's Lemma in representation theory.

### 6.2 B-Branes

The message we keep repeating in these lectures is that the B-model should be easier to analyze than the A-model. While this is true, it doesn't necessarily mean that the B-model

|  | A-model | B-model |
| :--- | :--- | :--- |
| A/B-branes | Lagrangians | Complexes of coherent sheaves |
| BPS A/B-branes | Special Lagrangians | $\Pi$-stable complexes |
| Dependence of corr. funcs. | $B+i J$ | complex structure |
| Dependence of stability | complex structure | $B+i J$ |
| Bound state | $A \rightarrow B$ | $\operatorname{Cone}(A \rightarrow B)$ |

Table 2: Mirror symmetry for BPS A-branes and B-branes.
is easier to picture in classical terms. This is particularly true for D-brane decay. In the case of the A-branes of section 6.1 we have a direct picture of how special Lagrangians decay as the complex structure is varied. It should be pointed out however that this picture is very difficult to make explicit in concrete cases. Conversely we will see in this section that B-brane decay is not classical at all, thanks essentially due to nonperturbative $\alpha^{\prime}$ corrections. This makes it hard to picture and we are forced to introduce more mathematics not familiar to the typical physicist. Having said that, we can give fairly explicit examples of B-brane decay.

In table 2 we review how mirror symmetry relates the ideas of stability between the A-model and B-model. Notice in particular that the rôles of complex structure and $B+i J$ are exchanged between the topological field theory dependence and the stability criteria.

### 6.2.1 Triangles

Just like the A-model, the B-model itself should not know about any stability issue. What we do demand from the B-model though is some criterion of whether a given object in the derived category can potentially decay into two other objects.

The discussion of A-brane decay via tachyon condensates in section 6.1.3 showed that, when we were on the wall of marginal stability, the open string was massless. Thus it acts like a marginal (but probably not truly marginal) operator in the conformal field theory. In this sense a decay (or binding) process looks like a deformation. Thus it is the mapping cone of (143) which defines a potential bound state of two D-branes.

The mapping cone construction in the derived category gives rise to a triangulated structure on the category. This mathematical structure turns out to be central to the notion of D-brane stability. The fact we ignored it in section 6.1 turns out to be a problem as we will see later. So let us now turn to the definition of this triangulated structure.

A triangulated category $\mathcal{C}$ is an additive category with two further ingredients:

1. A translation functor $T: \mathcal{C} \rightarrow \mathcal{C}$ which is an isomorphism. If $A$ is an object (or morphism) in $\mathcal{C}$ we will denote $T^{n}(A)$ by $A[n]$.
2. A collection of distinguished triangles. A triangle is a set of three objects and three
morphisms in the form

where the " $[1]$ " on the arrow denotes that $c$ is a map from $C$ to $A[1]$.
A triangle may also be written as

$$
\begin{equation*}
A \xrightarrow{a} B \xrightarrow{b} C \xrightarrow{c} A[1] . \tag{180}
\end{equation*}
$$

A morphism between two triangles is simply a commutative diagram of the form


This data is subject to the following axioms:
TR1: a) For any object $A$, the following triangle is distinguished:

b) If a triangle is isomorphic to a distinguished triangle then, it too, is distinguished.
c) Any morphism $a: A \rightarrow B$ can be completed to a distinguished triangle of the form (179).

TR2: The triangle (179) is distinguished if and only if

is also distinguished. That is, we may shuffle the edge containing "[1]" around the triangle translating the objects and morphisms accordingly.

TR3: Given two triangles and the vertical maps $f$ and $g$ in (181), we may construct a morphism $h$ to complete (181).

TR4: The Octahedral Axiom:


Four faces of the octahedron are distinguished triangles and the other four faces commute. The relative orientations of the arrows obviously specify which is which.
The octahedral axiom specifies that, given $A, B, C, D, E$ and the solid arrows in the octahedron, there is an object $F$ such that the octahedron may be completed with the dashed arrows. The pairs of maps that combine to form maps between $B$ and $F$ also commute.

For any abelian category $\mathcal{C}$, the derived category $\mathbf{D}(\mathcal{C})$ is a triangulated category. The translation functor is the same as the shift functor that we introduced earlier of course. The distinguished triangles are provided by the mapping cone - any vertex of a distinguished triangle is isomorphic to the mapping cone of the opposite edge when the " $[1]$ " is shuffled around to the appropriate edge. We refer to $[58,81]$ for the proof that $\mathbf{D}(\mathcal{C})$ satisfies the above axioms.

Given a short exact sequence

$$
\begin{equation*}
0 \longrightarrow A \xrightarrow{a} B \xrightarrow{b} C \longrightarrow 0, \tag{185}
\end{equation*}
$$

in $\mathcal{C}$, we induce a distinguished triangle (179) in $\mathbf{D}(\mathcal{C})$ for the corresponding single-entry complexes with $a$ and $b$ induced in the obvious way. This short exact sequence can be also phrased " $B$ is an extension of $C$ by $A$ ". The group of extensions of $C$ by $A$ is given by $\operatorname{Ext}(C, A)=\operatorname{Ext}^{1}(C, A)$ (see [113] for example). Thus, the short exact sequence determines an element $c$ which is a morphism in $\operatorname{Ext}^{1}(C, A)$ which equals $\operatorname{Hom}(C, A[1])$ in $\mathbf{D}(\mathcal{C})$. This is the map $c$ in (179). The derived category is not an abelian category since kernels and cokernels do not always exist. Thus one cannot define short exact sequences in $\mathbf{D}(\mathcal{C})$. In a sense, distinguished triangles are a weaker notion of short exact sequences that are "the best one can do" for the derived category.

This definition of a triangulated category was invented by Verdier [114] for completely abstract reasons of course, but it turns out to be precisely what is needed for the rules of D-brane decay. The basic triangle (179) should be read as the $D$-branes $A$ and $C$ may bind via the potentially tachyonic open string $c$ to form $B$. We may then go through each axiom in turn and say what it means:

TR1: a) $A$ binds with 0 (the empty brane) to produce $A$.
b) We consider two objects in $\mathbf{D}(X)$ which are isomorphic to be the same D-brane. Thus this rule is required for consistency.
c) The existence of an open string from $A$ to $B$ means that $B$ can potentially decay into $A$ and some other decay product $C$. This is not obvious but this axiom may be rephrased after the following.

TR2: If $B$ can potentially decay into $A$ and $C$, then $C$ can potentially decay into $A[1]$ and $B$. This is consistent with the observation in section 5.6 that $A[1]$ could be interpreted as an anti- $A$.

Note that using this axiom we may now rephrase TR1: c) as follows. Given an open string from $A$ to $B$ we may potentially form a bound state of these two D-branes.

TR3: Given open strings between D-branes $A$ and $A^{\prime}$ and between $B$ and $B^{\prime}$, we may construct open strings between the corresponding bound states.

TR4: This formidable looking axiom is little more than a statement of associativity in the rules for combining D-branes. If we crudely write addition to represent rules for combining, the distinguished triangles in (184) can be read (using TR2) as

$$
\begin{align*}
C & =A[1]+B \\
& =A[1]+(E+D[-1]) \\
& =(A[1]+E)+D[-1]  \tag{186}\\
& =F+D[-1] .
\end{align*}
$$

One may choose to regard these rules for D-brane decay as mainly self-evident, or as proven since we have proven that the category of B-branes is the derived category and therefore triangulated.

The triangulated structure encodes the long exact sequences associated to cohomology as follows. A functor between two triangulated categories is exact if it preserves triangles. An example of such an exact functor is $\operatorname{Hom}(M,-)$ for some fixed object $M$. This is a functor from an arbitrary category to the category of vector spaces. As mentioned in section 5.3 , the derived category of vector spaces is rather trivial in the sense that every complex is quasi-isomorphic to a complex where all the differential maps are zero. Using this fact, let us write the object $\operatorname{Hom}(M, A)$ in the derived category of vector spaces as

$$
\begin{equation*}
\ldots \xrightarrow{0} \operatorname{Ext}^{-1}(M, A) \xrightarrow{0} \operatorname{Ext}^{0}(M, A) \xrightarrow{0} \operatorname{Ext}^{1}(M, A) \xrightarrow{0} \ldots, \tag{187}
\end{equation*}
$$

where the Ext's are vector spaces. The triangle

then becomes the usual long exact sequence of vector spaces

$$
\begin{equation*}
\ldots \longrightarrow \operatorname{Ext}^{0}(M, A) \longrightarrow \operatorname{Ext}^{0}(M, B) \longrightarrow \operatorname{Ext}^{0}(M, C) \longrightarrow \operatorname{Ext}^{1}(M, A) \longrightarrow \ldots \tag{189}
\end{equation*}
$$

### 6.2.2 Categorical mirror symmetry at last

Of course, there is a some vagueness in the word "potentially" whenever we refer to binding or decay in section 6.2.1. We have stated explicitly above that if there is an open string from $A$ to $B$ then we regard $A+B$ as a potential bound state. In order for this to actually happen there must be some region of moduli space where $A$ and $B$ are both themselves stable and the open string from $A$ to $B$ is tachyonic. This is not guaranteed. Thus, the triangulated structure appears when one has an optimistic view (which is as much as the topological field theory can know) about what can bind to what.

Our discussion of A-brane stability in section 6.1 was approached directly rather than using the topological field theory language. Because of this the Fukaya category need not have a triangulated structure - it certainly knows about the A-branes which really are stable but it need not include the potentially stable branes in the topological field theory which never actually make it to stability. In particular there is no reason to suppose that the Fukaya category is actually triangulated. That is, it may well violate axiom TR1: c). ${ }^{31}$

If the Fukaya category is not triangulated then the mirror symmetry proposal in section 5.7 cannot possibly be correct. The derived category $\mathbf{D}(X)$ is triangulated and thus cannot be equivalent to a category which is not triangulated. The solution, of course, is to add the extra "potentially stable" A-branes to the Fukaya category so that the result is triangulated. This can be done in a precise mathematical way by following the procedure of Bondal and Kapranov [115].

If $\mathcal{F}(Y)$ is the Fukaya category of $Y$, then let $\operatorname{Tr} \mathcal{F}(Y)$ be the triangulated category produced by the method of Bondal and Kapranov. ${ }^{32}$ The current state-of-the-art conjecture for mirror symmetry which follows from our topological field theory constructions is then:

If $X$ and $Y$ are mirror Calabi-Yau threefolds then the category $\mathbf{D}(X)$ is equivalent to the category $\operatorname{Tr} \mathcal{F}(Y)$.

We should warn that even this statement is subject to corrections when we go outside the class of Calabi-Yau threefolds with zero $b_{1}$ because of the appearance of extra coisotropic A-branes [46].

This "homological mirror symmetry" statement has been demonstrated for 2-tori [69] and quartic K3 surfaces [71].

[^23]
### 6.2.3 П-Stability

Assuming mirror symmetry to be true we may now copy the description of the stability of A-branes in section 6.1 over to the case of B-branes.

The first ingredient we need is how to compute the central charge $Z$ of a given B-brane. This author is not aware of a complete argument in the literature for how to compute this, but we may proceed a little less than rigorously as follows.

The first step is to note that $Z$ is given in the mirror by a period $\int_{L} \Omega$ for some 3 -cycle $L$. This varies with the complex structure of $Y$. Thus, $Z$ must depend upon $B+i J$ for $X$. We saw in section 2.4 that we can derive the periods exactly from the Picard-Fuchs equations. These differential equations are written in terms of parameters that specify the complex structure algebraically, such as the " $\psi$ " in section 2.4. However, we also saw how to relate such parameters to the complexified Kähler form $B+i J$ of $X$. Thus, a knowledge of the Picard-Fuchs equations is sufficient to obtain exact (but transcendental) expressions for the set of $Z$ 's for objects in $\mathbf{D}(X)$.

Now, given a particular object $\mathscr{E}^{\bullet}$ in $\mathbf{D}(X)$, the next step is to find which particular period computed above should be associated to the central charge $Z\left(\mathscr{E}^{\bullet}\right)$. In the case of the A-model we may choose some basis $\gamma_{k}$ of $H_{3}(Y)$ and compute a basis of periods $\varpi_{k}=\int_{\gamma_{k}} \Omega$. The charge, $Q$, of an A-brane is given by its homology class in $H_{3}$. Thus, if

$$
\begin{equation*}
Q=\sum b_{k} \gamma_{k} \tag{190}
\end{equation*}
$$

then the central charge is given by $\sum b_{k} \varpi_{k}$. We saw in section 5.6 that the D-brane charge of a B-brane is given by $\operatorname{ch}\left(\mathscr{E}^{\bullet}\right) \sqrt{t d(X)}$. Thus, the formula for the central charge of a B-brane must contain this expression linearly.

We saw in section 3.1.1 that the curvature, $F$, of a bundle associated to a D-brane is not really a physical quantity by itself since it is not invariant under the gauge symmetry we introduced. Instead we must always have the combination $B-F$. Since $\operatorname{ch}(E)=\operatorname{Tr} e^{F}$, the Chern character by itself it not gauge invariant and must always appear in the combination $e^{-B} \operatorname{ch}(E)$ in any physical quantity. Furthermore, holomorphy of supersymmetric theories demands that $B$ always appears in the combination $B+i J$. All this suggests that the simplest expression for the central charge should be

$$
\begin{equation*}
Z\left(\mathscr{E}^{\bullet}\right)=\int_{X} e^{-(B+i J)} \operatorname{ch}\left(\mathscr{E}^{\bullet}\right) \sqrt{t d(X)} \tag{191}
\end{equation*}
$$

Unfortunately this is not, in general, a solution to the Picard-Fuchs equations! However, it is a familiar situation in mirror symmetry that any expression depending on $B+i J$ is subject to quantum $\alpha^{\prime}$ corrections. Thus we should regard (191) as the asymptotic form of $Z$ near the large radius limit. If we know exactly just how asymptotic this formula is, we have enough information to determine exactly which combination of periods give $Z$ exactly.

The periods are associated to the prepotential in special geometry and so one expects them to receive corrections in the same way. Such corrections were discussed in [21] and appear in two ways:

1. The perturbative corrections will be due to the 4 -loop correction in the non-linear $\sigma$ model as analyzed in [116]. These corrections will be three powers of $B+i J$ less than the leading term in (191). Thus, in the case of Calabi-Yau threefolds this produces at most a constant term.
2. The nonperturbative corrections will produce a power series, with no constant term, in $q_{i}=\exp 2 \pi i\left(\int_{C_{i}} B+i J\right)$ for some basis $\left\{C_{i}\right\}$ of $H_{2}(X)$.

This finally provides enough information to determine the precise expression for $Z\left(\mathscr{E}^{\bullet}\right)$.
Having found the central charge we may now proceed with the next stage in copying the stability condition from the A-brane model. Given $\mathscr{E}^{\bullet}$ we may choose a grading $\xi\left(\mathscr{E}^{\bullet}\right)$ such that

$$
\begin{equation*}
\xi\left(\mathscr{E}^{\bullet}\right)=\frac{1}{\pi} \arg Z\left(\mathscr{E}^{\bullet}\right) \quad(\bmod 2) \tag{192}
\end{equation*}
$$

and demand that $\xi\left(\mathscr{E}^{\bullet}\right)$ vary continuously with $B+i J$ so long as $\mathscr{E}^{\bullet}$ is stable. Following (175) we have

$$
\begin{equation*}
\xi\left(\mathscr{E}^{\bullet}[n]\right)=\xi\left(\mathscr{E}^{\bullet}\right)+n . \tag{193}
\end{equation*}
$$

Finally we copy the picture in figure 7 by asserting that if we have a distinguished triangle in $\mathbf{D}(X)$ of the form

with $A$ and $B$ stable, then $C$ is stable with respect to the decay represented by this triangle if and only if $\xi(B)<\xi(A)+1$. Also, if $\xi(B)=\xi(A)+1$ then $C$ is marginally stable and we may state that

$$
\begin{equation*}
\xi(C)=\xi(B)=\xi(A)+1 \tag{195}
\end{equation*}
$$

We may use axiom TR2 in section 6.2 .1 to rephrase this as follows. If $A$ and $C$ are stable then $B$ is stable with respect to decay into $A$ and $C$ so long as $\xi(A)<\xi(C)$.

These criteria for stability are known as $\Pi$-stability and were studied in [83, 117-120].
As stated these rules are not sufficient to determine the set of stable B-branes at a given point in the moduli space of $B+i J$. If we happen to know the set of stable Bbranes (including their gradings) at some basepoint in the moduli space then these rules are sufficient to determine how the stable spectrum changes as we move along some path in the moduli space. The following rules are applied

- We begin with a stable set of B-branes together with a value of the grading $\xi$ for each B-brane. This set must be consistent with the rules of $\Pi$-stability. That is, no distinguished triangle may allow a stable B-brane to decay into two other stable B-branes.
- As we move along a path in moduli space the gradings will change continuously.
- Two stable B-branes may bind to form a new stable state.
- A stable B-brane may decay into other (marginally) stable states.

Note in the last case that a brane may decay into another state which becomes unstable at exactly the same point in moduli space. This certainly can happen. We also emphasize that we never make any reference to a value of $\xi$ for an unstable object. This is probably not defined.

These rules certainly do not imply that the stable set of B-branes is uniquely determined by a point in the moduli space of $B+i J$. In order to determine the stable set we explicitly specified a path from the basepoint to the desired path. We will see in section 7.1.4 very explicitly that there is nontrivial monodromy in $\mathbf{D}(X)$ which changes the set of stable objects as we go around loops in the moduli space.

The monodromy occurs because of monodromy in the gradings. This corresponds to Bbranes acquiring zero $Z$, i.e., zero mass. If a stable B-brane becomes massless it necessarily induces a singularity in the conformal field theory following the arguments of $[121,122]$. Let us define the Teichmüller space $\mathscr{T}$ as the universal cover of the moduli space of $B+i J$ with these singular CFT points deleted. Thus there should be no monodromy in $\mathscr{T}$ and we expect the set of stable B-branes to be well-defined at any point in $\mathscr{T}$.

The tachyon condensation picture sets the rather discontinuous behaviour of algebraic geometry that we saw in section 5.5 in a more natural setting. There we found that the cone of any map $\operatorname{Cone}(f: A \rightarrow B)$ is invariant under a rescaling of $f$ by a nonzero complex number. In the tachyon condensation picture, the scale of $f$ is determined by minimizing the tachyon potential. Thus, when the cone is unstable, $f$ is fixed at zero and when the cone becomes stable $f$ acquires a definite value depending on the modulus $B+i J$. Thus, the only discontinuity appears as the cone decays - as one should expect.

There is a consistency condition that any set of stable B-branes must obey. In section 6.1.3 we found a condition (177) equivalent to the unitarity constraint on the conformal field theory. In B-model language this amounts to

$$
\begin{equation*}
\xi(A)>\xi(B) \Rightarrow \operatorname{Hom}(A, B)=0 \tag{196}
\end{equation*}
$$

This removes any states of conformal weight $h<0$. By Serre duality (113) it also removes states with $h>m / 2$ for a Calabi-Yau $m$-fold.

### 6.2.4 Multiple decays

Every object in $\mathbf{D}(X)$ is either stable or unstable for a given point in the Teichmüller space of $B+i J$. A particle which is unstable must be unstable because it decays into a set of stable objects. Thus the set of stable objects must be big enough to account for this property. This puts a stronger constraint on stability than the previous section. For example, having no stable objects at all would have been consistent with our earlier definition of $\Pi$-stability.

If an unstable object decays into 2 stable objects we know how to describe the decay by a distinguished triangle. We now want to describe a decay of an object into 3 stable objects.


Figure 8: Walls of marginal stability for a decay into 3 objects.

We use the following octahedron to describe the process:


Suppose that we begin at a point $p_{0}$ in the Teichmüller space where $\xi\left(A_{1}\right)<\xi\left(A_{2}\right)<\xi\left(A_{3}\right)$ and end at a point $p_{1}$ where $\xi\left(A_{1}\right)>\xi\left(A_{2}\right)>\xi\left(A_{3}\right)$. Thus the open strings corresponding to $f_{1}$ and $f_{2}$ in (197) go from tachyonic to massive as we pass from $p_{0}$ to $p_{1}$.

At $p_{0}$, with respect to the triangles in this octahedron, $E_{2}$ and $F$ are stable. We may also declare that $E_{3}$ is stable (but this isn't really necessary). Suppose there are two walls $W_{1}$ and $W_{2}$ between $p_{0}$ and $p_{1}$ such that $\xi\left(A_{i}\right)-\xi\left(A_{i+1}\right)$ is negative on the $p_{0}$ side of $W_{i}$ and positive on the $p_{1}$ side of $W_{i}$. We depict this in figure 8. Then there are two possibilities to consider as we move from $p_{0}$ to $p_{1}$ :

1. We cross $W_{1}$ and then $W_{2}$. As we cross $W_{2}$ the object $F$ will decay into $A_{2}$ and $A_{3}$. At this instant $\xi(F)=\xi\left(A_{2}\right)<\xi\left(A_{1}\right)$ so we know that $E_{3}$ must have already decayed into $F$ and $A_{1}$. Thus $E_{3}$ decays into $A_{1}, A_{2}$ and $A_{3}$ by the time we reach $p_{1}$.
2. We cross $W_{2}$ and then $W_{1}$. As we cross $W_{1}$ the object $E_{2}$ will decay into $A_{1}$ and $A_{2}$. At this instant $\xi\left(E_{2}\right)=\xi\left(A_{2}\right)>\xi\left(A_{3}\right)$ so we know that $E_{3}$ must have already decayed into $E_{2}$ and $A_{3}$. Thus $E_{3}$ decays into $A_{1}, A_{2}$ and $A_{3}$ by the time we reach $p_{1}$.

Either way, the condition for $E_{3}$ to decay into $A_{1}, A_{2}$ and $A_{3}$ is that

$$
\begin{equation*}
\xi\left(A_{1}\right)>\xi\left(A_{2}\right)>\xi\left(A_{3}\right) \tag{198}
\end{equation*}
$$

We may generalize this to the case of decays into any number of objects. For any object $E$ we define the following set of distinguished triangles


Then $E$ decays into $A_{1}, A_{2}, \ldots, A_{n}$ so long as

$$
\begin{equation*}
\xi\left(A_{1}\right)>\xi\left(A_{2}\right)>\ldots>\xi\left(A_{n}\right) \tag{200}
\end{equation*}
$$

Thus we motivate the following
Conjecture 1 At every point in the Teichmüller space of $B+i J$ there is a set of stable objects in $\mathbf{D}(X)$ such that every object $E$ can be written in the form (199) for some $n$ (meaning it decays into $n$ stable objects) and for stable objects $A_{k}$ satisfying (200).

Together with the unitarity constraint (196), this is the form of $\Pi$-stability proposed by Bridgeland [123].

Note that we can't claim to have proven this conjecture since there are many objects in $\mathbf{D}(X)$ which are never stable. The argument at the start of the subsection cannot then be used to follow the decay. We would also like to include a finiteness condition on $n$ in (199). This appears to be an additional assumption too.

All that said, Bridgeland's form of the stability conditions does seem to work very nicely $[123,124]$ although many aspects are still poorly-understood in the case of Calabi-Yau threefolds.

### 6.2.5 $\mu$-stability

In order to determine the set of $\Pi$-stable objects it is best if we have a basepoint in the moduli space of $B+i J$ from which we may follow paths as in section 6.2.3. The obvious choice for such a basepoint is near the large radius limit of the Calabi-Yau threefold $X$.

At the large radius limit we should expect that the classical analysis of D-branes is valid and therefore that D-branes correspond to vector bundles supported over subspaces $S \subset X$. Furthermore, we may assume that the world-volume approach to D-branes should be accurate. We refer to the general literature on D-branes such as [2] for more details.

For simplicity let us assume that we have a 6-brane wrapping $X$ associated to a holomorphic vector bundle $E \rightarrow X$ with curvature (1,1)-form $F$. At large radius, the BPS condition reduces to the Hermitian-Yang-Mills condition [125]. That is, the curvature tensor obeys the relation

$$
\begin{equation*}
g^{j \bar{k}} F_{\alpha j \bar{k}}^{\beta}=\mu(E) \cdot \delta_{\alpha}^{\beta}, \tag{201}
\end{equation*}
$$

where $\alpha, \beta$ are indices in the fibre of $E$ and $\mu(E)$ is a real number called the "slope" of $E$. Following the analysis of [126], for example, one can integrate (201) and obtain ${ }^{33}$

$$
\begin{equation*}
\mu(E)=\frac{\operatorname{deg}(E)}{k \cdot \operatorname{Vol}(X)} \tag{202}
\end{equation*}
$$

where $k$ is the rank of $E$ and

$$
\begin{equation*}
\operatorname{deg}(E)=\int_{X} J \wedge J \wedge c_{1}(E) \tag{203}
\end{equation*}
$$

is the degree of the bundle $E$.
As usual, this condition for a supersymmetric vacuum is a first-order differential equation (in the connection) and is a sufficient condition for a solution of the equations of motion, i.e., the Yang-Mills condition, which is a second order differential equation.

The Hermitian-Yang-Mills condition (201) depends explicitly on the metric and thus the Kähler form $J$. As such, whether we have a BPS solution can depend upon $J$. The existence of Hermitian-Yang-Mills connections has been studied by Donaldson [127], and Uhlenbeck and Yau [126], who proved the following theorem. Let $\mathscr{E}$ be the locally-free sheaf associated to $E$. The bundle $E$ is said to be $\mu$-stable if every subsheaf $\mathscr{F}$ of $\mathscr{E}$ satisfies $\mu(\mathscr{F})<\mu(\mathscr{E})$. We then have:

Theorem 2 A bundle is $\mu$-stable if and only if it admits an irreducible Hermitian-YangMills connection.

The stability of the BPS B-brane is thus equivalent to $\mu$-stability in the classical limit where $\alpha^{\prime}$ corrections are ignored.

If $\mathscr{F}$ is a subsheaf of $\mathscr{E}$ then we have a short exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathscr{F} \longrightarrow \mathscr{E} \longrightarrow \mathscr{G} \longrightarrow 0, \tag{204}
\end{equation*}
$$

for some sheaf $\mathscr{G}$. If $\mathscr{E}$ is unstable then it decays into $\mathscr{F}$ and $\mathscr{G}$.
The lower-dimensional branes can be analyzed similarly. We can focus on a subspace $S \subset X$ and look for stable vector bundles (or twisted bundles if $S$ is not spin) within the class of bundles on $S$. The subspaces do not interfere with each other in the following sense. A bundle on $X$ associated to $\mathscr{E}$ cannot decay into a subsheaf $\mathscr{F}$ supported only on $S$ since

[^24]there is no homomorphism $\mathscr{F} \rightarrow \mathscr{E}$. Equally, a bundle on $S$ cannot decay into a subsheaf on $X$ since the quotient sheaf $\mathscr{G}$ in (204) would have negative rank.

It follows that $\mu$-stability establishes a set of stable B-branes at the large radius limit. We should now check that this set of stable B-branes is consistent with $\Pi$-stability.

From (191) we see that, for large $J$, the leading contribution to the central charge $Z(A)$ will be given by the lowest degree differential form in $\operatorname{ch}(A)$. The Grothendieck-RiemannRoch formula (154) can be used to show that the lowest component of $\operatorname{ch}\left(i_{*} \mathscr{F}\right)$ is given by the $(6-2 \operatorname{dim}(S))$-form $s$ which is Poincaré dual to $S .{ }^{34}$ Thus, for large $J$,

$$
\begin{align*}
Z & \sim \int_{X}(-i J)^{\operatorname{dim}(S)} \wedge s \\
& \sim \int_{S}\left(-\left.i J\right|_{S}\right)^{\operatorname{dim}(S)}  \tag{205}\\
& \sim(-i)^{\operatorname{dim}(S)} \operatorname{Vol}(S)
\end{align*}
$$

yielding

$$
\begin{equation*}
\xi\left(i_{*} \mathscr{E}\right)=-\frac{1}{2} \operatorname{dim}(S) \quad(\bmod 2) \tag{206}
\end{equation*}
$$

If we choose the values of $\xi$ to fix the mod 2 ambiguity arbitrarily we will violate the unitarity condition (177). For example, let $\mathscr{O}_{X}$ be the 6 -brane wrapping $X$ and let $\mathscr{O}_{p}$ be the 0 -brane (skyscraper sheaf) at a point $p \in X$. Thus $\xi\left(\mathscr{O}_{X}\right)=-\frac{3}{2}(\bmod 2)$ and $\xi\left(\mathscr{O}_{p}\right)=0(\bmod 2)$. By restricting the value of a function on $X$ to its value at $p$ we see that $\operatorname{Hom}\left(\mathscr{O}_{X}, \mathscr{O}_{p}\right)=\mathbb{C}$ and so we must insist $\xi\left(\mathscr{O}_{X}\right)<\xi\left(\mathscr{O}_{p}\right)$ if these B-branes are stable. Furthermore, by Serre duality, $\operatorname{Hom}\left(\mathscr{O}_{p}, \mathscr{O}_{X}[3]\right)=\mathbb{C}$ and so $\xi\left(\mathscr{O}_{p}\right)<\xi\left(\mathscr{O}_{X}\right)+3 .{ }^{35}$ So the only possibility at large radius is that $\xi\left(\mathscr{O}_{X}\right)=\xi\left(\mathscr{O}_{p}\right)-\frac{3}{2}$.

A consistent choice is to set

$$
\begin{equation*}
\xi\left(i_{*} \mathscr{E}\right)=-\frac{1}{2} \operatorname{dim}(S) \tag{207}
\end{equation*}
$$

Let us see what happens at the subleading order in $J$. We restrict attention to the case of 6 -branes, i.e., locally-free sheaves. Now $\operatorname{ch}(\mathscr{E})=k+c_{1}+\ldots$, where $k$ is the rank of the associated vector bundle. Applying (191) we now obtain

$$
\begin{equation*}
\xi(\mathscr{E})=-\frac{3}{2}+\frac{1}{\pi} \tan ^{-1} \frac{\mu(\mathscr{E})}{2}+\ldots \tag{208}
\end{equation*}
$$

where $\mu(\mathscr{E})$ is, again, the slope of the sheaf $\mathscr{E}$ introduced above. Note from (202) that $|\mu(\mathscr{E})| \ll 1$ in the large radius limit.

The short exact sequence (204) induces the triangle


[^25]in $\mathbf{D}(X)$ as explained in section 6.2.1. If we are near the large radius limit, then $\xi(\mathscr{E})$ and $\xi(\mathscr{F})$ are both very close to $-\frac{3}{2}$ since they are locally-free sheaves. $\mathscr{G}$ is either locally free or supported on a complex codimension one subspace. Thus $\xi(\mathscr{G})$ is close to either $-\frac{3}{2}$ or -1 . Since the D-brane charges add according to section 5.6, we have $Z(\mathscr{F})=Z(\mathscr{E})+Z(\mathscr{G})$, which implies that $\xi(\mathscr{F})$ lies between $\xi(\mathscr{E})$ and $\xi(\mathscr{G})$. The $\Pi$-stability condition for $\mathscr{F}$ is $\xi(\mathscr{E})<\xi(\mathscr{G})$, which is therefore equivalent to $\xi(\mathscr{E})<\xi(\mathscr{F})$. By (208) this, in turn, is equivalent to $\mu(\mathscr{E})<\mu(\mathscr{F})$. Thus $\Pi$-stability reduces to $\mu$-stability as first observed in [117].

In [125] some $\alpha^{\prime}$ corrections to the Hermitian-Yang-Mills condition were computed. This brings the D-brane stability condition closer to $\Pi$-stability in a sense near the radius limit but there is a very important qualitative difference between $\Pi$-stability and any version of $\mu$-stability as follows.
$\mu$-stability is defined for the abelian category of coherent sheaves whereas $\Pi$-stability is defined for the triangulated category $\mathbf{D}(X)$. When checking for $\mu$-stability one looks for decay into subobjects $\mathscr{E} \subset \mathscr{F}$. The notion of subobjects leads to a definite hierarchy within the category of coherent sheaves. If $\mathscr{F}$ can decay into the subobject $\mathscr{E}$ (and some other decay product $\mathscr{G}$ ) then there is no way $\mathscr{E}$ can decay into something including $\mathscr{F}$. The notion of a subobject is given by the injective map $\mathscr{E} \rightarrow \mathscr{F}$, i.e., a map with zero kernel. This is well-defined since kernels always exist in an abelian category.

This hierarchy does not extend to the derived category since we no longer have an abelian category. Each vertex of a distinguished category may decay into the other two vertices. This essentially arises due to the possibility of anti-branes as discussed in section 5.6. Thus, generically within the moduli space of $B+i J$ we are forced to use the more difficult notion of $\Pi$-stability on a triangulated category. At the large radius limit, however, the stability structure simplifies and we may use an abelian category instead.

Does this simplification to a useful abelian structure exist elsewhere in the moduli space? In section 7.3 we will see that this indeed happens for orbifolds. Douglas [83] has suggested that this might be a key ingredient in a full understanding of $\Pi$-stability and some progress in this direction has been made by Bridgeland [123]. We will not pursue this idea in these lectures.

## 7 Applications

### 7.1 The Quintic Threefold

We are now in a position to give some examples of B-branes and, in particular, how $\alpha^{\prime}$ corrections modify the naïve picture of a B-brane as simply a holomorphic submanifold of $X$. The obvious place to start is the quintic threefold, as introduced in section 2.4 , since the moduli space of $B+i J$ is one-dimensional. As emphasized in table 2 , since we are focusing on stability, rather than the structure of the topological field theory, we need to exchange the rôles of $X$ and $Y$ relative to section 2.4. That is, in this section $X$ is the quintic threefold and $Y$ is its mirror.

### 7.1.1 Periods

The first thing we analyze is how to compute the exact form of the central charge $Z$ for any object $\mathscr{E}^{\bullet}$ in $\mathbf{D}(X)$. We argued in section 6.2 .3 that $Z$ was always a period of $\Omega$ on the mirror $Y$.

Since $\operatorname{dim} H_{3}(Y)=4$, we have 4 independent periods $\int_{\Gamma} \Omega$ for the holomorphic 3-form $\Omega$. These periods satisfy the Picard-Fuchs equation (34). In terms of the moduli space coordinate $z$ introduced in section 2.4 we may use the following solutions (as in [21]):

$$
\begin{equation*}
\varpi_{j}=-\frac{1}{5} \sum_{m=1}^{\infty} \frac{\alpha^{(2+j) m} \Gamma\left(\frac{m}{5}\right)}{\Gamma(m) \Gamma\left(1-\frac{m}{5}\right)^{4}} z^{-\frac{m}{5}} . \tag{210}
\end{equation*}
$$

Since $\varpi_{0}+\varpi_{1}+\ldots+\varpi_{4}=0$, we may use $\varpi_{0}, \ldots, \varpi_{3}$ as a basis.
These solutions may be analytically continued by using the Barnes' integral method (see $[35,39]$ for the precise method we have used here) to match this basis with the series around the large radius limit. We use the following basis for solutions near $z=0$ :

$$
\begin{align*}
\Phi_{0} & =\frac{1}{2 \pi i} \int \frac{\Gamma(5 s+1) \Gamma(-s)}{\Gamma(s+1)^{4}}\left(e^{\pi i} z\right)^{s} d s \\
& =\sum_{n=0}^{\infty} \frac{(5 n)!}{n!5} z^{n}  \tag{211}\\
& =1+O\left(e^{2 \pi i t}\right) \\
& =\varpi_{0}, \\
\Phi_{1} & =-\frac{1}{2 \pi i} \cdot \frac{1}{2 \pi i} \int \frac{\Gamma(5 s+1) \Gamma(-s)^{2}}{\Gamma(s+1)^{3}} z^{s} d s \\
& =\frac{1}{2 \pi i} \log z+O(z)  \tag{212}\\
& =t+O\left(e^{2 \pi i t}\right) \\
& =-\frac{1}{5}\left(\varpi_{0}-3 \varpi_{1}-2 \varpi_{2}-\varpi_{3}\right), \\
\Phi_{2}= & -\frac{1}{2 \pi^{2}} \cdot \frac{1}{2 \pi i} \int \frac{\Gamma(5 s+1) \Gamma(-s)^{3}}{\Gamma(s+1)^{2}}\left(e^{\pi i} z\right)^{s} d s \\
= & -\frac{1}{4 \pi^{2}}(\log z)^{2}+\frac{1}{2 \pi i} \log z-\frac{5}{6}+O(z)  \tag{213}\\
= & t^{2}+t-\frac{5}{6}+O\left(e^{2 \pi i t}\right) \\
= & \frac{2}{5}\left(-2 \varpi_{0}+\varpi_{2}+\varpi_{3}\right),
\end{align*}
$$

$$
\begin{align*}
\Phi_{3} & =\frac{1}{(2 \pi i)^{3}} \cdot(-6) \cdot \frac{1}{2 \pi i} \int \frac{\Gamma(5 s+1) \Gamma(-s)^{4}}{\Gamma(s+1)} z^{s} d s \\
& =\frac{i}{8 \pi^{3}}(\log z)^{3}+\frac{7 i}{4 \pi} \log z-\frac{30 i \zeta(3)}{\pi^{3}}+O(z)  \tag{214}\\
& =t^{3}-\frac{7}{2} t-\frac{30 i \zeta(3)}{\pi^{3}}+O\left(e^{2 \pi i t}\right) \\
& =-\frac{6}{5}\left(2 \varpi_{1}+2 \varpi_{2}+\varpi_{3}\right),
\end{align*}
$$

where $t=B+i J=\Phi_{1} / \Phi_{0}$ by the mirror map (and abuse of notation) of section 2.4. In each case the contour integral is along the line $s=\epsilon+i y$, where $\epsilon$ is a fixed real number such that $-\frac{1}{5}<\epsilon<0$ and $y$ goes from $-\infty$ to $+\infty$. This contour is completed to the left or to the right in order to get the analytic continuation. These contour integrals converge (and thus the analytic continuation is valid) for

$$
\begin{equation*}
-2 \pi<\arg z<0 \tag{215}
\end{equation*}
$$

Now consider the sheaves $\mathscr{O}(m)$ on $\mathbb{P}^{4}$ as discussed in section 4.1.3. These restrict to $X$ to form locally-free sheaves of rank one which we denote $\mathscr{O}_{X}(m)$. Denoting the generator of $H^{2}(X, \mathbb{Z})$ by $e$,

$$
\begin{align*}
\int_{X} \exp (-t e) \operatorname{ch}\left(\mathscr{O}_{X}(m)\right) \sqrt{t d(X)} & =\int_{X} \exp (-t e+m e) \sqrt{1+\frac{5}{6} e^{2}}  \tag{216}\\
& =\frac{5}{12}(m-t)\left(2 m^{2}+2 t^{2}-4 m t+5\right) .
\end{align*}
$$

Assuming that the term proportional to $\zeta(3)$ in (214) arises from the 4-loop correction in section 6.2.3, we may determine the period associated to $\mathscr{O}_{X}(m)$ exactly from this leading behaviour. The result is that

$$
\begin{align*}
Z\left(\mathscr{O}_{X}(m)\right) & =\frac{5}{6} m\left(m^{2}+5\right) \Phi_{0}-\frac{5}{2}\left(m^{2}+m+2\right) \Phi_{1}+\frac{5}{2} m \Phi_{2}-\frac{5}{6} \Phi_{3} \\
& =\frac{1}{6}\left(5 m^{3}+3 m^{2}+16 m+6\right) \varpi_{0}-\frac{1}{2}\left(3 m^{2}+3 m+2\right) \varpi_{1}-m^{2} \varpi_{2}-\frac{1}{2} m(m-1) \varpi_{3}, \tag{217}
\end{align*}
$$

and, in particular, $Z\left(\mathscr{O}_{X}\right)=\varpi_{0}-\varpi_{1}$. The conifold point, $z=5^{-5}$ lies at the edge of the radius of convergence of the various power series above. One can show that the power series (211) for $\Phi_{0}=\varpi_{0}$ in terms of $z$ is convergent at the conifold point and is clearly real. For real $z^{\frac{1}{5}},(210)$ tells us that $\varpi_{1}$ is the complex conjugate of $\varpi_{0}$. Thus, at the conifold point, $\varpi_{0}=\varpi_{1}$ and so $Z\left(\mathscr{O}_{X}\right)=0$, i.e., $\mathscr{O}_{X}$ becomes massless.

We know that the conifold point corresponds to a singular conformal field theory since a soliton, i.e., D-brane, becomes massless $[121,122]$. Thus it seems natural to assume that the B-brane $\mathscr{O}_{X}$ is the one in question. Note that we have not quite proven this since we haven't proven that $\mathscr{O}_{X}$ is stable at the conifold point. We will assume this stability in order to proceed.

We will have much to say about monodromy in section 7.1.4 but for now let us note the following simple observation. Around the large radius limit of the quintic, monodromy corresponds to $B \rightarrow B+1$. From the gauge invariance discussed in section 3.1.1, such a shift is accompanied by a shift $F \rightarrow F+1$ in the curvature of the bundle over any D-brane. Thus $\mathscr{O}_{X}(m)$ becomes $\mathscr{O}_{X}(m+1)$. This means that the D-brane corresponding to $\mathscr{O}_{X}(m)$ becomes massless at the conifold point if we circle the large radius limit $m$ times before proceeding towards the conifold point.

### 7.1.2 4-branes

We can now study the stability of 4 -branes on the quintic. Consider the following short exact sequence of sheaves:

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}_{X}(a) \xrightarrow{f} \mathscr{O}_{X}(b) \longrightarrow \mathscr{O}_{S}(b) \longrightarrow 0, \tag{218}
\end{equation*}
$$

where $b>a$ and $f$ is a function of homogeneous degree $b-a$ as discussed in section 4.1.1. $S$ is a divisor, i.e., a subspace of complex codimension one and $\mathscr{O}_{S}(b)$ is the sheaf corresponding to a line bundle of degree $b$ over $S$. All 4-branes corresponding to B-branes on the quintic correspond to $\mathscr{O}_{S}(b)$ for some $f$ and some $b$.

To leading order at large radius we have

$$
\begin{align*}
\xi\left(\mathscr{O}_{X}(m)\right) & =\frac{1}{\pi} \arg \int_{X} \exp ((m-B-i J) e) \\
& =\frac{1}{\pi} \arg \left(5(m-B-i J)^{3}\right)  \tag{219}\\
& =\frac{3}{\pi} \theta_{m}-3
\end{align*}
$$

where $\theta_{m}$ is the angle in the complex $(B+i J)$-plane made between the positive real axis and the line from $m$ to $B+i J$. Applying the $\Pi$-stability criterion to the distinguished triangle associated to (218) gives $\xi\left(\mathscr{O}_{X}(b)\right)-\xi\left(\mathscr{O}_{X}(a)\right)<1$ which yields

$$
\begin{equation*}
\theta_{b}-\theta_{a}<\frac{\pi}{3} \tag{220}
\end{equation*}
$$

When this is satisfied, the open string corresponding to $f$ in (218) is tachyonic. Simple geometry yields that this corresponds to the points above a circular arc in the upper $(B+i J)$ plane with centre $\frac{1}{2}(a+b)+\frac{1}{2 \sqrt{3}}(b-a) i$ and radius $\frac{1}{\sqrt{3}}(b-a)$.

As expected, these 4 -branes are stable in the large radius limit. Below this arc of marginal stability the 4 -brane decays into $\mathscr{O}_{X}(b)$ and $\mathscr{O}_{X}(a)[-1]$, that is a 6 -brane and anti- 6 -brane with some 4 -brane charges.

The classical $\mu$-stability criterion of section 6.2 .5 would imply that a 4 -brane is always stable since it has no subobjects. We emphasize that $\mu$-stability does not fail because we have not taken enough $\alpha^{\prime}$ corrections into account - after all (219) was only an approximation


Figure 9: Stability of various D4-branes in the $B+i J$-plane.
anyway. Rather $\mu$-stability fails because of the qualitative aspect that it is defined for abelian categories. This means it can never see decays caused by anti-branes.

We also draw attention to the fact that it is misleading to think that $\Pi$-stability corrections can always be ignored at large radius. For very large values of $b-a$ this line of marginal stability can extend to large values of $J$.

Of course, we do not have the precise form of the line of marginal stability since we used the large radius approximation in (219). We may use numerical computation techniques and the exact form of the periods from (217) to plot these curves more precisely as was done in [120]. The result is shown in figure 9 for $a=0$ and $b=N$. The shaded areas of this figure represent fundamental regions of the moduli space as in figure 1.

We see that the lines of marginal stability in figure 9 are not so far from being circular arcs. Note that the lines of marginal stability end on the conifold points precisely where one of the decay products becomes massless, and thus the grading becomes poorly-defined.

It should be emphasized that we have not strictly proven that either the 4 -branes are stable above the lines in figure 9 or that the 4 -branes are unstable below the lines. The 4 branes might decay by other channels before these lines are reached, and the decay products might also be unstable by the time we reach a line of marginal stability. Having said that, it seems hard to imagine that figure 9 is incorrect. We know the B-branes $\mathscr{O}_{X}(m)$ are stable
near the conifold points since they are supposed to be responsible for the singularities at the conifold. Other more contrived modes of decay of 4-branes always seem to decay at smaller radii.

### 7.1.3 Exotic B-branes

Having spent most of the lectures extolling the virtues of the derived category, we have yet to see an example of a B-brane that doesn't correspond to a single term complex - i.e., a coherent sheaf. It is time to rectify this situation.

The idea, as suggested in [128], is to apply Serre duality to the 4 -brane decay of section 7.1.2. The potential tachyons of that discussion lie in $\operatorname{Hom}\left(\mathscr{O}_{X}(a), \mathscr{O}_{X}(b)\right)$ which is nonzero for $b>a$. By the Serre duality of section 4.2.4, this Hilbert space of open strings is isomorphic to $\operatorname{Hom}\left(\mathscr{O}_{X}(b), \mathscr{O}_{X}(a)[3]\right)$. Thus, we may consider the distinguished triangle

where $\mathscr{X}_{a, b}$ is defined as the cone of the map $g$.
The stability of $\mathscr{X}_{a, b}$ can now be determined (at least relative to the triangle (221)) from the analysis above together with the relation (193). For stability we require $\xi\left(\mathscr{O}_{X}(a)[3]\right)-$ $\xi\left(\mathscr{O}_{X}(b)\right)=3+\xi\left(\mathscr{O}_{X}(a)\right)-\xi\left(\mathscr{O}_{X}(b)\right)<1$. Using the approximation (219) this yields that $\mathscr{X}_{a, b}$ is stable below a circular arc in the upper-half plane with centre $\frac{1}{2}(a+b)-\frac{1}{2 \sqrt{3}}(b-a) i$ and radius $\frac{1}{\sqrt{3}}(b-a)$. In particular, $\mathscr{X}_{a, b}$ is always unstable in the large radius limit, although we may make it stable at any given arbitrarily large radius by choosing a large enough value of $b-a$.

Again we may use numerical techniques to plot a more precise version of the lines of marginal stability. In figure 10 (again taken from [120]) we plot some examples. We should note that something very interesting happens for $\mathscr{X}_{0,2}$. It is an example of a case where a decay product can itself decay forcing the line of marginal stability to end on another line of marginal stability (rather that a conifold point). We refer to [120] for a full discussion of this case.

So, what, exactly is $\mathscr{X}_{a, b}$ ? It is a B-brane that exists purely because of $\alpha^{\prime}$ effects. It does not exist at large radius limit and does not have a world-volume interpretation. Let's get a feel of these exotic B-branes by examining $\mathscr{X}_{0,1}$, say, a little more closely. Suppose we have an injective resolution for $\mathscr{O}_{X}$ :

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}_{X} \longrightarrow \mathscr{I}^{0} \xrightarrow{i_{0}} \mathscr{I}^{1} \xrightarrow{i_{1}} \mathscr{I}^{2} \xrightarrow{i_{2}} \mathscr{I}^{3} \xrightarrow{i_{3}} \ldots \tag{222}
\end{equation*}
$$

Referring to section 4.2.4, an element of $\operatorname{Ext}^{3}\left(\mathscr{O}_{X}(1), \mathscr{O}_{X}\right)$ corresponds to a map $g: \mathscr{O}_{X}(1) \rightarrow$


Figure 10: Stability of the exotic objects $\mathscr{X}_{0, N}$ in the $B+i J$-plane.
$\mathscr{I}_{3}$ such that $i_{3} g=0$. Thus $\mathscr{X}_{0,1}=\operatorname{Cone}(g)$ corresponds to the complex

$$
\begin{equation*}
\cdots \longrightarrow \mathscr{I}^{0} \xrightarrow{i_{0}} \mathscr{I}^{1} \xrightarrow{\binom{0}{i_{1}}} \mathscr{O}_{X}(1) \mathscr{I}^{2} \xrightarrow{\left(g i_{2}\right)} \mathscr{I}^{3} \xrightarrow{i_{3}} \cdots, \tag{223}
\end{equation*}
$$

where we denote the zero position with a dotted underline.
The cohomology sheaves of this complex look like $\mathscr{H}^{-3}=\mathscr{O}_{X}$ and $\mathscr{H}^{-1}=\mathscr{O}_{X}(1)$, so naïvely this B-brane looks like an anti-6-brane added to another anti-6-brane with a 4 -brane charge. This is certainly true if $g$ is zero. More precisely, (223) is quasi-isomorphic to

$$
\begin{equation*}
\ldots \longrightarrow \mathscr{O}_{X} \longrightarrow 0 \longrightarrow \mathscr{O}_{X}(1) \longrightarrow 0 \longrightarrow \ldots, \tag{224}
\end{equation*}
$$

if and only if $g=0$. To prove this assertion we may use the discussion at the end of section 6.2.1 to compute $\operatorname{Hom}\left(\mathscr{X}_{0,1}, \mathscr{X}_{0,1}\right)$. After wading through several long sequences and using the cohomology groups of $\mathscr{O}_{X}(m)$ as defined in section 4.2.1 we obtain

$$
\operatorname{Hom}\left(\mathscr{X}_{0,1}, \mathscr{X}_{0,1}\right)= \begin{cases}\mathbb{C}^{2} & \text { if } g=0  \tag{225}\\ \mathbb{C} & \text { if } g \neq 0\end{cases}
$$

This ties in nicely with the comments at the end of section 6.1.3. If $g \neq 0$ we have a single irreducible D-brane. When $g=0$ we have a direct sum of two D-branes and thus a gauge group $\mathrm{U}(1) \times \mathrm{U}(1)$.
$\mathscr{X}_{0,1}$ thus becomes a distinct object when the tachyon is turned on. All this shows that $\mathscr{X}_{0,1}$ is a truly exotic object from the old point of view of D -branes. It cannot be written as a complex with a single coherent sheaf and so it cannot be viewed as a vector bundle supported on some subspace of $X$.

The reader should note that we were able to build these exotic D-branes because we were able to use Ext ${ }^{n}$ 's for $n>1$. This separated terms in the complexes far enough to avoid everything collapsing back to a single term complex. That such Ext's can ever be physically relevant is traced back to fact that the grading $\xi$ was defined to live in $\mathbb{R}$ rather than on a circle. If we didn't extend the definition of $\xi$ in this way, we would never see the exotic D-branes.

We should note that one can directly attack the question of identifying D-brane states at the Gepner point itself using the notion of boundary conformal field theories. This was done in [129-131]. We will not review this method here as it requires quite a bit of technology. However, this method is not guaranteed to obtain all of the D-branes at the Gepner point. It is possible to show that some of the charges of the exotic D-branes we found above can coincide with charges found using this direct method. We should add that there has also been recent progress in more advanced methods of analyzing the Gepner model [132-134] which appear to fill in the missing states of the boundary conformal field theory method.

### 7.1.4 Monodromy

One of the easiest ways of seeing that B-branes as subspaces is an inadequate picture is to think about monodromy in D-branes as we move around loops in the moduli space.

In the case of A-branes, such monodromy is purely classical. The periods of the holomorphic 3-form over integral 3-cycles undergo non-trivial monodromy as we go around noncontractible loops in the moduli space of complex structure. Such monodromy may be interpreted as an automorphism of $H_{3}(Y, \mathbb{Z})$ which preserves the intersection form between 3 -cycles. Thus the homology classes of A-branes undergo monodromy and thus the A-branes themselves undergo monodromy.

How is monodromy seen in the B-branes? In this case, the D-brane charge is an element of $H^{\text {even }}(X, \mathbb{Z})$. In the case of the quintic we know exactly how to map between $H_{3}(Y, \mathbb{Z})$ and $H^{\text {even }}(X, \mathbb{Z})$ thanks to the analysis of section 7.1.1. Thus we can copy the monodromy action from the A-model into the B -model. If one begins with a particular sheaf $\mathscr{F}$, one can then compute the monodromy action on $\operatorname{ch}(\mathscr{F})$ to get some idea of what this B-brane becomes under monodromy. In this way, it looks as if 2 -cycles may becomes mixtures of 2 -cycles and 0 -cycles etc., which is certainly not classical! Furthermore, it is not hard to find examples where the resulting element of $H^{\text {even }}(X, \mathbb{Z})$ cannot correspond to the Chern character of any sheaf (for example, the rank of the bundle might be negative). Fortunately going to the derived category solves these problems.

Upon going around a loop in the moduli space the physics must be unchanged of course. This means that monodromy may shuffle the objects in $\mathbf{D}(X)$ around but, at the end of the day, we should be able to relabel everything to restore the original spectra of open strings etc. In category language this means that monodromy induces an "autoequivalence" on $\mathbf{D}(X)$. That is, we have a functor from $\mathbf{D}(X)$ to $\mathbf{D}(X)$ that is a bijection on the (isomorphism classes of) objects and that preserves the corresponding morphisms.

Such an autoequivalence on $\mathbf{D}(X)$ is always induced by a so-called "Fourier-Mukai transform'. To define a Fourier-Mukai transform one begins with a fixed object $\mathcal{Z}$ (called the "kernel") in the derived category of $X \times X$. We define the projection maps from $X \times X$ to its first and second factor:


Given any object $A \in \mathbf{D}(X)$ we then define the transformed object (or morphism)

$$
\begin{equation*}
T_{\mathcal{Z}}(A)=\mathbf{R} p_{2 *}\left(\mathbf{L} p_{1}^{*} A \stackrel{\mathbf{L}}{\otimes} \mathcal{Z}\right) \tag{227}
\end{equation*}
$$

Some explanation of notation is required here. We sequentially apply three functors in (227) which have been either "left-derived" or "right-derived" hence the "L" or "R". This part is taken care of automatically by the derived category machinery as follows. In each case, we take an object in the derived category and choose a representative complex which satisfies some nice property required by the functor. The functor can then be applied to the complex. The three functors are:

1. $\mathbf{L} p_{1}^{*}$ : We take the complex to be a complex of locally-free sheaves or, equivalently, vector bundles. $p_{1}^{*}$ is then the usual pull-back map on vector bundles. ${ }^{36}$
2. $\stackrel{\mathrm{L}}{\otimes} \mathcal{Z}$ : Again the complex must be a complex of locally-free sheaves. $\otimes$ is then the usual tensor product of sheaves. Acting on complexes, $\otimes$ produces a double complex which can then be collapsed back to a single complex in the usual way.
3. $\boldsymbol{R} p_{2 *}$ : In this case the complex must be a complex of injective objects. $p_{2 *}$ is then the push-forward map for sheaves as defined in section 5.4.

Since the derived functor machinery is built into the derived category, we will usually drop the L's and R's from the notation from now on.

Note that if $p: X \rightarrow\{q\}$ is map to a point, then $p_{*}$ is the global section functor. Thus, its right derived functors $\mathbf{R} p_{*}$ amount to sheaf cohomology as explained in section 4.2.4. In the above, $p_{2}: X \times X \rightarrow X$ is a fibration and therefore, roughly speaking, its right-derived functors amount to taking cohomology in the fibres.

[^26]It was proven in [135] that any autoequivalence of $\mathbf{D}(X)$ could be written as (227) for a suitable choice of $\mathcal{Z}$. Conversely, Bridgeland [136] has established the conditions on $\mathcal{Z}$ for (227) to yield an autoequivalence of $\mathbf{D}(X)$.

As a warm-up exercise let us consider how we produce the identity Fourier-Mukai transform. Let the "diagonal" map $\Delta_{X}: X \rightarrow X \times X$ be given by $\Delta_{X}(x)=(x, x)$, and let us denote $\Delta_{X *} \mathscr{O}_{X}$ by $\mathscr{O}_{\Delta X}$. Suppose $\mathscr{F}$ is a sheaf on $X$. Then $p_{1}^{*} \mathscr{F} \otimes \mathscr{O}_{\Delta X}$ is a sheaf supported on the image of $\Delta_{X}$. In fact, it is precisely $\mathscr{F}$ on the image of $X$ under the map $\Delta_{X}$. This image is mapped identically back to $X$ by $p_{2}$ and so $T_{\mathscr{O}_{\Delta X}}(\mathscr{F})=\mathscr{F}$. Thus $\mathscr{O}_{\Delta X}$ generates the identity Fourier-Mukai transform of sheaves and therefore the derived category.

The (stringy) moduli space of $B+i J$ on the quintic is given by a 2 -sphere with 3 punctures corresponding to large radius limit, the Gepner point and the conifold point. Thus we just need to find the monodromy around two of these points and the third may then be deduced.

The monodromy around the large radius limit is easy. We know that this corresponds to $B \rightarrow B+1$ which is equivalent to a shift $F \rightarrow F+1$ in the curvature of the bundles. This shift in curvature can be achieved by a transformation $\mathscr{F} \rightarrow \mathscr{F} \otimes \mathscr{O}_{X}(1)$ on sheaves. In terms of a Fourier-Mukai transform, this may be achieved by a kernel $\mathcal{L}=\Delta_{*} \mathscr{O}_{X}(1)=\mathscr{O}_{\Delta X}(1)$ in a similar way to the identity transform discussed above. We will denote this transform $T_{\mathcal{L}}$ by $L$.

Let us next consider the Fourier-Mukai transform associated to monodromy around the conifold point. We will denote this transform $K$. Unlike the large radius limit, there is no direct argument that yields $K$ in a concrete way. Instead, let us first consider the action of $K$ on the D-brane charges. This may be deduced from the periods listed in section 7.1.1. We may compose a loop around the large radius limit (given by $z \mapsto e^{2 \pi i} z$ for small $z$ ) by a loop around the Gepner point (given by $z \mapsto e^{-2 \pi i / 5} z$ for large $z$ ) to obtain monodromy around the conifold point. For details of the explicit monodromy matrices see [21] and also [43,137-139] for further details about this monodromy. The result may be expressed in the following concise form:

$$
\begin{equation*}
\operatorname{ch}(K(\mathscr{F}))=\operatorname{ch}(\mathscr{F})-\left\langle\mathscr{O}_{X}, \mathscr{F}\right\rangle \operatorname{ch}\left(\mathscr{O}_{X}\right), \tag{228}
\end{equation*}
$$

where $\langle$,$\rangle is the natural inner product given by (158). We would like to do better than this$ however. We want to know the monodromy transform $K$ itself, rather than just the action on charges.

Clearly, since $Z\left(\mathscr{O}_{X}\right)$ has a simple zero at the conifold point, monodromy around this point will shift $\xi\left(\mathscr{O}_{X}\right)$ by 2 . This means that stability conditions on any decay involving $\mathscr{O}_{X}$ may well be affected by this monodromy. For example, consider the 0 -brane $i_{*} \mathscr{O}_{p}$ given by the inclusion of a point $i: p \hookrightarrow X$. For brevity, we will refer to this skyscraper sheaf as $\mathscr{O}_{p}$ as in section 4.1.6. From (77) we obtain the following distinguished triangle


At large radius limit we expect $\mathscr{O}_{X}$ and $\mathscr{O}_{p}$ to be stable (as they are the basic 6 -brane and 0 -brane respectively) and $\mathscr{I}_{p}$ (and thus $\mathscr{I}_{p}[1]$ ) to be unstable (since it doesn't correspond to a vector bundle over a subspace). The latter can be seen directly from the fact that $\xi\left(\mathscr{O}_{p}\right)-\xi\left(\mathscr{O}_{X}\right)=\frac{3}{2}>1$ at large radius, making the open string $a$ massive in (229).

Near the conifold point $Z\left(\mathscr{O}_{p}\right)$ and $Z\left(\mathscr{I}_{p}\right)$ are non-zero and so the $\xi$ 's are roughly fixed for these B-branes in a small neighbourhood of the conifold point when they are defined, i.e., when the B-branes are stable. Thus, as we orbit the conifold point in the direction of increasing $\xi\left(\mathscr{O}_{X}\right)$, we can make $\xi\left(\mathscr{O}_{p}\right)-\xi\left(\mathscr{O}_{X}\right)$ smaller and thus $a$ tachyonic. That is, $\mathscr{I}_{p}[1]$ becomes stable. At this instant $\xi\left(\mathscr{I}_{p}[1]\right)=\xi\left(\mathscr{O}_{p}\right)$ from our $\Pi$-stability rules. Thus when these B-branes are stable, they have approximately the same value of $\xi$ in a small neighbourhood of the conifold point. This implies that the open string $c$ in (229) is always tachyonic as expected.

As we continue around the conifold point increasing $\xi\left(\mathscr{O}_{X}\right)$ further, we can make $\xi\left(\mathscr{O}_{X}\right)-$ $\xi\left(\mathscr{I}_{p}[1]\right)>0$ and thus $\mathscr{O}_{p}$ unstable. It is easy to see that this will happen an angle $\pi$ later than the formation of $\mathscr{I}_{p}[1]$. The diagram (229) therefore encodes both the formation of $\mathscr{I}_{p}[1]$ and the decay of $\mathscr{O}_{p}$ as we encircle the conifold point. We refer to [120] for more analysis of precisely where these events occur.

The monodromy action of $\mathbf{D}(X)$ should be seen as relabeling of the B-branes as objects in the category so that the physics remains unchanged. Thus, since $\mathscr{O}_{p}$ was stable to begin with but has now become unstable, it must be replaced by something. The natural choice would seem to be $\mathscr{I}_{p}[1]$. Thus we conjecture $K\left(\mathscr{O}_{p}\right)=\mathscr{I}_{p}[1]$. This is consistent with the charges (228). Furthermore $\xi\left(\mathscr{O}_{X}\right)$ has increased by 2 upon looping once around the conifold. Thus, to restore physics, we should assert $K\left(\mathscr{O}_{X}\right)=\mathscr{O}_{X}[-2]$ to compensate for this.

Let us define the notation $A \boxtimes B$ to mean $p_{1}^{*} A \otimes p_{2}^{*} B$ for $A, B \in \mathbf{D}(X)$. Now, consider the Fourier-Mukai transform induced by

$$
\begin{equation*}
\mathcal{K}=\left(\mathscr{O}_{X} \boxtimes \mathscr{O}_{X} \xrightarrow{r} \mathscr{O}_{\Delta X}\right), \tag{230}
\end{equation*}
$$

where $r$ is the obvious restriction map. Let us apply this transform to a sheaf $\mathscr{F}$. We know the $\mathscr{O}_{\Delta X}$ in position zero of (230) acts as the identity. As $\mathscr{O}_{X} \boxtimes \mathscr{O}_{X}=\mathscr{O}_{X \times X}$, then $p_{1}^{*} \mathscr{F} \otimes \mathcal{K}$ is simply $p_{1}^{*} \mathscr{F}$. Pushing this down via $p_{2}$ will produce a sum of copies of $\mathscr{O}_{X}$ since $p_{1}^{*} \mathscr{F}$ is trivial (i.e., free) in this direction. As mentioned above, the act of pushing down takes sheaf cohomology in the fibre direction which in turn is equivalent to the functor $\operatorname{Hom}\left(\mathscr{O}_{X},-\right)$. This means that, in the derived category,

$$
\begin{equation*}
p_{2 *} p_{1}^{*} \mathscr{F}=\operatorname{Hom}\left(\mathscr{O}_{X}, \mathscr{F}\right) \otimes \mathscr{O}_{X} \tag{231}
\end{equation*}
$$

which implies

$$
\begin{equation*}
T_{\mathcal{K}}(\mathscr{F})=\operatorname{Cone}\left(\operatorname{Hom}\left(\mathscr{O}_{X}, \mathscr{F}\right) \otimes \mathscr{O}_{X} \xrightarrow{r} \mathscr{F}\right), \tag{232}
\end{equation*}
$$

where $r$ is now an obvious "evaluation" map. This Fourier-Mukai transform reproduces the desired monodromy above on $\mathscr{O}_{p}$ and $\mathscr{O}_{X}$. For example put $\mathscr{F}=\mathscr{O}_{X}$. Then, from section 4.2.1 we may compute

$$
H^{k}\left(X, \mathscr{O}_{X}\right)= \begin{cases}\mathbb{C} & \text { if } k=0 \text { or } 3  \tag{233}\\ 0 & \text { otherwise }\end{cases}
$$

which implies $\operatorname{Hom}\left(\mathscr{O}_{X}, \mathscr{F}\right)$ can be represented by a complex of vector spaces


Tensoring by $\mathscr{O}_{X}$ simply replaces the $\mathbb{C}$ 's by $\mathscr{O}_{X}$ 's. The cone construction then shifts this one place left and the first $\mathscr{O}_{X}$ cancels with the $\mathscr{O}_{X}$ on the right of the cone (since the evaluation map is the identity) leaving $\mathscr{O}_{X}[-2]$.

A result of Bridgeland and Maciocia [140] may now be used to show that (230) is actually the only Fourier-Mukai transform which produces the desired monodromy on $\mathscr{O}_{p}$ (for all $p \in X)$ and $\mathscr{O}_{X}$. We refer to [120] for more details. Thus we will assume that this gives the correct monodromy on $\mathbf{D}(X)$ for loops around the conifold point. Note that this particular transformation is well-known in the study of "helices and mutations" (see, for example, [141] and references therein). It was first conjectured to be applicable to monodromy in the context we are considering by Kontsevich [142] and was subsequently studied extensively by Seidel and Thomas [143]. It is also a special case of Horja's model of monodromy [144-146].

The monodromy around the Gepner point can then be constructed by composing the monodromies around the large radius limit and the conifold point. In general, two FourierMukai transforms based on kernels $\mathcal{A}$ and $\mathcal{B}$ may be composed to form a transform with a kernel

$$
\begin{equation*}
\mathcal{B} \star \mathcal{A}=p_{13 *}\left(p_{12}^{*} \mathcal{A} \otimes p_{23}^{*} \mathcal{B}\right), \tag{235}
\end{equation*}
$$

where $p_{i j}$ are the obvious projection maps from $X \times X \times X$ to $X \times X$. Therefore, if $G$ is the monodromy around the Gepner point, and $\mathcal{G}$ is the corresponding kernel, we may compute

$$
\begin{align*}
\mathcal{G} & =\mathcal{K} \star \mathcal{L} \\
& =\left(\mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X} \longrightarrow \mathscr{O}_{\Delta X}(1)\right) . \tag{236}
\end{align*}
$$

Consider the result of monodromy $G^{5}$, i.e., five times around the Gepner point. Given that the Gepner model is a $\mathbb{Z}_{5}$-orbifold one might be forgiven that thinking that $G^{5}$ induces a trivial monodromy, i.e., $\mathcal{G}^{\star 5}=\mathscr{O}_{\Delta X}$. Surprisingly this is not the case as we now show. ${ }^{37}$ We show the full details of this computation here so that the reader can get a good feel for manipulations in the derived category. Anyone not interested in these details should, of course, skip ahead to the answer!

Let $\mathcal{S}=\mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}$ so that $\mathcal{G}=\operatorname{Cone}\left(\mathcal{S} \rightarrow \mathscr{O}_{\Delta X}(1)\right)$. The functors in (235) preserve distinguished triangles in $\mathbf{D}(X \times X)$, so $\mathcal{S} \star \mathcal{G}=\operatorname{Cone}\left(\mathcal{S} \star \mathcal{S} \rightarrow \mathcal{S} \star \mathscr{O}_{\Delta X}(1)\right)$. Now

$$
\begin{align*}
\mathcal{S} \star \mathcal{S} & =p_{13 *}\left(\mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}\right) \\
& =\mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}^{\oplus 5}, \tag{237}
\end{align*}
$$

since,

$$
H^{k}\left(X, \mathscr{O}_{X}(1)\right)= \begin{cases}\mathbb{C}^{5} & \text { if } k=0  \tag{238}\\ 0 & \text { otherwise }\end{cases}
$$

[^27]Also, $\mathcal{S} \star \mathscr{O}_{\Delta X}(1)$ is easily seen to be $\mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}(1)$. Thus

$$
\begin{align*}
\mathcal{S} \star \mathcal{G} & =\left(\mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}^{\oplus 5} \longrightarrow \mathscr{O}_{X}(1) \boxtimes \mathscr{O}_{X}(1)\right)  \tag{239}\\
& =\left.\mathscr{O}_{X}(1) \boxtimes \Omega(1)\right|_{X}[1] .
\end{align*}
$$

This follows from the Euler exact sequence ${ }^{38}$

$$
\begin{equation*}
0 \longrightarrow \Omega(1) \longrightarrow \mathscr{O}^{\oplus 5} \longrightarrow \mathscr{O}(1) \longrightarrow 0 \tag{240}
\end{equation*}
$$

on $\mathbb{P}^{4}$ restricted to $X$. Here $\Omega$ is the cotangent sheaf of $\mathbb{P}^{4}$ and $\Omega(1)=\Omega \otimes \mathscr{O}(1)$. The latter sheaf is restricted to $X$ in (239). ${ }^{39}$ Also

$$
\begin{equation*}
\mathscr{O}_{\Delta X}(1) \star \mathcal{G}=\left(\mathscr{O}_{X}(2) \boxtimes \mathscr{O}_{X} \longrightarrow \mathscr{O}_{\Delta X}(2)\right) \tag{241}
\end{equation*}
$$

and so

$$
\begin{align*}
\mathcal{G} \star \mathcal{G} & =\operatorname{Cone}\left(\mathcal{S} \star \mathcal{G} \rightarrow \mathscr{O}_{\Delta X}(1) \star \mathcal{G}\right) \\
& =\left(\left.\mathscr{O}_{X}(1) \boxtimes \Omega(1)\right|_{X} \longrightarrow \mathscr{O}_{X}(2) \boxtimes \mathscr{O}_{X} \longrightarrow \mathscr{O}_{\Delta X}(2)\right) . \tag{242}
\end{align*}
$$

Similarly we may prove that

$$
\begin{equation*}
\mathcal{G} \star \mathcal{G} \star \mathcal{G}=\left(\left.\left.\mathscr{O}_{X}(1) \boxtimes \Omega^{2}(2)\right|_{X} \longrightarrow \mathscr{O}_{X}(2) \boxtimes \Omega(1)\right|_{X} \longrightarrow \mathscr{O}_{X}(3) \boxtimes \mathscr{O}_{X} \longrightarrow \mathscr{O}_{\Delta X}(3)\right), \tag{243}
\end{equation*}
$$

where $\Omega^{2}$ is the second exterior power of $\Omega$ which, from (240), fits into the following exact sequence:

$$
\begin{equation*}
0 \longrightarrow \Omega^{2}(2) \longrightarrow \mathscr{O}^{\oplus 10} \longrightarrow \mathscr{O}(1)^{\oplus 5} \longrightarrow \mathscr{O}(2) \longrightarrow 0 . \tag{244}
\end{equation*}
$$

Continuing this process yields the desired kernel

$$
\left.\begin{array}{rl}
\mathcal{G}^{\star 5}=\left(\left.\mathscr{O}_{X}(1) \boxtimes \Omega^{4}(4)\right|_{X}\right. & \left.\longrightarrow \mathscr{O}_{X}(2) \boxtimes \Omega^{3}(3)\right|_{X} \\
\left.\mathscr{O}_{X}(3) \boxtimes \Omega^{2}(2)\right|_{X} & \left.\longrightarrow \mathscr{O}_{X}(4) \boxtimes \Omega(1)\right|_{X} \tag{245}
\end{array} \mathscr{O}_{X}(5) \boxtimes \mathscr{O}_{X} \longrightarrow \mathscr{O}_{\Delta X}(5)\right) . .
$$

This looks very similar to an exact sequence due to Beilinson [148] for sheaves on $\mathbb{P}^{4} \times \mathbb{P}^{4}$ :

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}(-4) \boxtimes \Omega^{4}(4) \longrightarrow \ldots \longrightarrow \mathscr{O}(-1) \boxtimes \Omega(1) \longrightarrow \mathscr{O} \boxtimes \mathscr{O} \longrightarrow \mathscr{O}_{\Delta} \longrightarrow 0, \tag{246}
\end{equation*}
$$

where $\mathscr{O}_{\Delta}$ is the structure sheaf of the diagonally embedded $\mathbb{P}^{4}$. If we tensor (246) by $\mathscr{O}(5) \boxtimes \mathscr{O}$ and restrict to $X$ we obtain the complex $\mathcal{G}^{\star 5}$. This process does not preserve the exactness of (246) so we need to be a little careful. We claim

$$
\begin{equation*}
\mathcal{G}^{\star 5}=\operatorname{Cone}\left(\mathscr{O}_{\Delta}(5) \stackrel{\mathrm{L}}{\otimes} \mathscr{O}_{X \times X} \rightarrow \mathscr{O}_{\Delta X}(5)\right) \tag{247}
\end{equation*}
$$

[^28]This is seen as follows. As stated earlier in this section, the operation $\stackrel{\mathbf{L}}{\otimes}$ requires the object on the left or right of this symbol to be expressed in terms of locally free sheaves. Then the usual $\otimes$ operator may be applied. The exact sequence (246) tensored by $\mathscr{O}(5) \boxtimes \mathscr{O}$ gives the desired locally free resolution of $\mathscr{O}_{\Delta}(5)$. The functor $\stackrel{\mathrm{L}}{\otimes} \mathscr{O}_{X \times X}$ is then precisely restriction $X \times X$ thus yielding the complex (245).

Alternatively we can compute (247) using a locally-free resolution of $\mathscr{O}_{X \times X}$ in the form

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}(-5) \boxtimes \mathscr{O}(-5) \longrightarrow(\mathscr{O} \boxtimes \mathscr{O}(-5)) \oplus(\mathscr{O}(-5) \boxtimes \mathscr{O}) \longrightarrow \mathscr{O} \boxtimes \mathscr{O} \longrightarrow \mathscr{O}_{X \times X} \longrightarrow 0 . \tag{248}
\end{equation*}
$$

This yields

$$
\begin{align*}
\mathcal{G}^{\star 5} & =\left(\mathscr{O}_{\Delta}(-5) \longrightarrow \mathscr{O}_{\Delta} \oplus \mathscr{O}_{\Delta} \longrightarrow \mathscr{O}_{\Delta}(5) \longrightarrow \mathscr{O}_{\Delta X}(5)\right) \\
& =\left(\mathscr{O}_{\Delta}(-5) \longrightarrow \mathscr{O}_{\Delta} \longrightarrow 0 \longrightarrow 0\right)  \tag{249}\\
& =\mathscr{O}_{\Delta X}[2] .
\end{align*}
$$

This is, of course, the identity Fourier-Mukai transform shifted left 2 places. This means that we have proven, in general, that

The action of the monodromy on $\mathbf{D}(X)$ associated to looping five times around the Gepner point corresponds to shifting the complexes two places to the left.

So how worried should we be that this is not the identity? It doesn't contradict any statement about physics. Monodromy five times around the Gepner point does not affect physics - but then again monodromy once around the Gepner point is an invariance of physics too!

What this computation is telling us is that any form of the topological field theory associated to the Gepner model, or equivalently a Landau-Ginzburg orbifold theory, which explicitly exhibits the derived category language for D-branes must not have a $\mathbb{Z}_{5}$ quantum symmetry. This might make such a model rather awkward to construct.

While these notes were being completed, the paper [134] appeared which is based on the work of $[132,133]$ which, in turn, is based upon the ideas of [149]. These interesting papers apply the derived category description directly to Landau-Ginzburg theories and so should provide exactly the model we are looking for. Unfortunately the shift-of-two ambiguity above appears to be evaded in these papers by identifying such a shift with the identity. The complete understanding of how these models fit into the full picture has yet to be completely elucidated and so we will not attempt to review these ideas here, although clearly they have much to offer.

We will see in section 7.3.5 that the geometrical orbifolds do not appear to be plagued by this shift of two anomaly. It therefore seems to be intrinsic to the Landau-Ginzburg theories in some sense.

### 7.2 Flops

While the quintic Calabi-Yau threefold provides the simplest example of a compact CalabiYau threefold, we may consider simpler examples of stability by going to noncompact cases. Perhaps the easiest, and most interesting, is provided by the "flop".

The geometry of flops has been extensively reviewed elsewhere. We refer to [3] for example for more details. The general idea is that there is a singular algebraic variety $X_{0}$ that contains a conifold point. This singular space may be resolved by replacing the conifold point by a $\mathbb{P}^{1}$ in two different ways to form smooth Calabi-Yau manifolds $X$ or $X^{\prime}$. Generally $X$ and $X^{\prime}$ are topologically distinct.

Geometrically the process of blowing down $X$ or $X^{\prime}$ back to $X_{0}$ may be viewed as a deformation of the Kähler form $J$. Indeed, in the space of Kähler forms, $X$ and $X^{\prime}$ may be considered to live on the two sides of a wall corresponding to $X_{0}$. Let $C$ be the $\mathbb{P}^{1}$ inside $X$. As we approach the wall from the $X$ side, the area of $C$ shrinks down to zero. Continuing past this wall would give $C$ negative area but by reinterpreting the geometry in terms of $X^{\prime}$ we give positive area to the new $C^{\prime} \subset X^{\prime}$. The process of passing from $X$ to $X^{\prime}$ is called a "flop".

In the context of the non-linear $\sigma$-model, we have the $B$-field in addition to $J$. This has a profound effect as described in [150]. The conformal field theory associated to the singular target space $X_{0}$ is perfectly well-defined and finite so long as the component of $B$ associated to $C$ (or $C^{\prime}$ ) is nonzero. Thus, rather than having a real codimension one wall of singular spaces in the moduli space of $J$, we have a complex codimension one subspace of singular conformal field theories in the moduli space of $B+i J$. It follows that one can pass from the $X_{1}$ "phase" to the $X_{2}$ "phase" smoothly by going around this singular subset. The conformal field theory does not witness any "jump" as we move between these phases.

Once we bring D-branes into the picture we see that we must have a jump in some sense. At least to some degree of approximation, the Calabi-Yau target space is the moduli space of 0-branes. Thus the moduli space of 0 -branes must undergo some transition during the flop even if we avoid the singular conformal field theory at $B=0$. This discontinuity is provided by 0 -brane stability considerations as we now show. This calculation was first suggested in [83] and performed in [151] (modulo sign conventions). The flop was studied by Bridgeland [152] in the context of the derived category and many of the observations in that paper are relevant here.

We imagine that we are in a Calabi-Yau threefold $X$ where everything has a very large area except for the flopping $\mathbb{P}^{1}$. The periods on the mirror of this may be analyzed simply in this limit as explained in [35]. Essentially the only component of $B+i J$ of interest is given by

$$
\begin{equation*}
t=\int_{C} B+i J \tag{250}
\end{equation*}
$$

This has a moduli space given by $\mathbb{P}^{1}$ as shown in figure 11 . The flop takes place on the equator and the singular conformal field theory exists at the point labeled $O$. Let $z$ be the affine coordinate on this $\mathbb{P}^{1}$ so that $z=0$ in the large $C$ limit, $z=1$ corresponds to $O$ and


Figure 11: The moduli space of $B+i J$ for the flop.
$z=\infty$ gives the large $C^{\prime}$ limit after the flop transition. The analysis of [35] then yields the simple result that the periods on the mirror have a general form

$$
\begin{equation*}
\Phi=A_{1}+A_{2} \log (z) \tag{251}
\end{equation*}
$$

Thus we have the exact relation given by the mirror map

$$
\begin{equation*}
t=\frac{1}{2 \pi i} \log (z) \tag{252}
\end{equation*}
$$

Let $i: C \rightarrow X$ be the inclusion map and let $\mathscr{O}_{C}(m)$ denote $i_{*} \mathscr{O}(m)$. We will use $\mathscr{O}_{x}$ to denote the skyscraper sheaf, i.e., 0 -brane, associated with the point $x \in X$. Using the Grothendieck-Riemann-Roch theorem of section 5.6 we see

$$
\begin{equation*}
\int_{X} e^{-(B+i J)} \operatorname{ch}\left(\mathscr{O}_{C}(m)\right) \sqrt{t d\left(T_{X}\right)}=-t+m+1 \tag{253}
\end{equation*}
$$

Therefore $Z\left(\mathscr{O}_{C}(m)\right)=-t+m+1$ exactly since $t$ and 1 are periods. The most natural statement would seem to be therefore that the brane corresponding to $\mathscr{O}_{C}(-1)$ becomes massless at $O$. Actually we are free to choose the branch of the logarithm however we feel and we could, instead, say that $\mathscr{O}_{C}$ becomes massless for simplicity. This is equivalent to choosing a basepoint near the large radius limit but then going once around this large radius limit before heading towards $O$. With this choice, we focus on this neighbourhood of $O$ by putting $t=1+\epsilon e^{i \theta}$ for a small real and positive $\epsilon$. We sketch this neighbourhood in figure 12.

Suppose $x \in C$. Then we have a short exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}_{C}(-1) \longrightarrow \mathscr{O}_{C} \longrightarrow \mathscr{O}_{x} \longrightarrow 0 \tag{254}
\end{equation*}
$$

This leads to a distinguished triangle we write in the form



Figure 12: The neighbourhood of $O$.

Near $O, Z\left(\mathscr{O}_{C}\right)$ is very small and $Z\left(\mathscr{O}_{x}\right)=1$. It follows that $\xi\left(\mathscr{O}_{x}\right)=\xi\left(\mathscr{O}_{C}(-1)[1]\right)=0$ and $\xi\left(\mathscr{O}_{C}\right)=\theta / \pi-1$. The morphisms in (255) are labeled by the differences in the $\xi$ 's between the head and tail of the arrow. Thus a vertex is $\Pi$-stable (with respect to that triangle) when and only when the label on the opposite edge is $<1$. Therefore, the 0 -brane $\mathscr{O}_{x}$ decays into $\mathscr{O}_{C}$ and $\mathscr{O}_{C}(-1)[1]$ as $\theta$ increases beyond $\pi$.

We also have a distinguished triangle

which shows that $\mathscr{O}_{x}$ decays into $\mathscr{O}_{C}(1)$ and $\mathscr{O}_{C}[1]$ for $\theta<0$. Either way, we see that $\mathscr{O}_{x}$ decays as we move from the $X$ phase into the $X^{\prime}$ phase in figure 12 .

Suppose $y \notin C$. Then, even though $\mathscr{O}_{y}$ has exactly the same D-brane charge as $\mathscr{O}_{x}$ it does not decay by $(255)$ or $(256)$ since there are no morphisms from $\mathscr{O}_{C}$ or $\mathscr{O}_{C}(1)$ to $\mathscr{O}_{y}$. Indeed, we would not expect 0-branes away from $C$ to be affected by the flop transition. Note again the superiority of the derived category approach to D-brane physics. Any method based solely on the notion of D-brane charges would not be able to distinguish between $\mathscr{O}_{x}$ and $\mathscr{O}_{y}$ even though their properties are quite different with respect to a flop.

We should also be able to see the objects in the derived category of $X$ which play the rôle of points on $C^{\prime}$ after we do the flop. Before we do this we need to introduce a method of computing some of the relevant Ext's. Suppose we are given sheaves $\mathscr{E}$ and $\mathscr{F}$ on $C$. Given the embedding $i: C \rightarrow X$ with a normal bundle $N$ on $C$, there is a spectral sequence with

$$
\begin{equation*}
E_{2}^{p, q}=\operatorname{Ext}_{C}^{p}\left(\mathscr{E}, \mathscr{F} \otimes \wedge^{q} N\right), \tag{257}
\end{equation*}
$$

converging to $E_{\infty}^{p, q}$ with $\bigoplus_{p+q=n} E_{\infty}^{p, q}=\operatorname{Ext}_{X}^{n}\left(i_{*} \mathscr{E}, i_{*} \mathscr{F}\right) .{ }^{40}$
For example, consider $\operatorname{Ext}_{X}^{n}\left(\mathscr{O}_{C}, \mathscr{O}_{C}(-1)\right)$. Since $N=\mathscr{O}(-1) \oplus \mathscr{O}(-1)$, we have an $E_{2}$

[^29]stage of the spectral sequence given by $E_{2}^{p, q}=H^{p}\left(C, \mathscr{O}_{C}(-1) \otimes \wedge^{q} N\right)$ :

and thus
\[

\operatorname{Ext}_{X}^{n}\left(\mathscr{O}_{C}, \mathscr{O}_{C}(-1)\right)= $$
\begin{cases}\mathbb{C}^{2} & \text { if } n=2 \text { or } 3  \tag{259}\\ 0 & \text { otherwise }\end{cases}
$$
\]

Thus we may use morphisms $f \in \operatorname{Hom}\left(\mathscr{O}_{C}[-1], \mathscr{O}_{C}(-1)[1]\right) \cong \mathbb{C}^{2}$ to form new objects $D_{f}$ :

which become stable for $\theta>\pi$. As noted in section 5.5 , rescaling $f$ by a complex number has no effect on $D_{f}$, so we have a $\mathbb{P}^{1}$ 's worth of $D_{f}$ 's. These indeed represent the points on $C^{\prime}$ which become stable as we flop into $X^{\prime}$ by increasing $\theta$ through $\pi$ as argued in [152]. We leave it to the reader to find the objects corresponding to points on $C^{\prime}$ which become stable as we flop by decreasing $\theta$ through 0 .

The objects $D_{f}$ are exotic in the same sense as those in section 7.1.3. They were described as "perverse sheaves" in [152]. Note that it is a trivial observation that the derived category of $X$ is equal to that of $X^{\prime}$ in the context of the B-model. The B-branes yield $\mathbf{D}(X)$ and $X$ is converted into $X^{\prime}$ by a shift in $B+i J$ - but such a shift has no affect on the B -model. The equivalence of $\mathbf{D}(X)$ and $\mathbf{D}\left(X^{\prime}\right)$ was established rigorously in $[152,154]$.

### 7.3 Orbifolds

Let $G$ be a finite subgroup of $\operatorname{SU}(d)$. In this section we are interested in a string propagating on the orbifold $\mathbb{C}^{d} / G$. Of course, beginning with the seminal work of [155], orbifolds have played an enormously important rôle in the understanding of stringy geometry. It should therefore be no surprise that the subject of B-branes on orbifolds provides a rich laboratory for further insight into the properties of D-branes. Much of what follows is a review of the works [117-119, 156-159]. We also refer to $[139,160]$ for related work.

One of the key concepts in studying the geometry of orbifolds is that these singularities can be blown-up (at least in the case $d \leq 3$ ) to produce a smooth manifold. This blowup process introduces an "exceptional divisor" to replace the orbifold singularity. We will assume the reader is familiar with the basic ideas of this process. We refer to [161], for example, for a review of this process in the context of conformal field theory.

### 7.3.1 The McKay correspondence

We begin with the purely mathematical and beautiful observation of McKay [162] in the case $d=2$. Let $G \subset \mathrm{SU}(2)$ be a finite group. It is well-known that such groups have an A-D-E classification and that the resulting exceptional divisor in the blow-up is given by a collections of $\mathbb{P}^{1}$ 's intersecting according to the corresponding Dynkin diagram. We refer to section 2.6 of [163] for a review of these facts.

Let $V_{i}, i=1, \ldots, r$ be the irreducible representations of $G$. In addition let $Q$ be the fundamental 2-dimensional representation of $G$ induced by the embedding $G \subset \mathrm{SU}(2)$. Consider the following decompositions

$$
\begin{equation*}
Q \otimes V_{i}=\bigoplus_{j=1}^{r} a_{j i} V_{j} \tag{261}
\end{equation*}
$$

for non-negative integers $a_{i j}$. The "McKay quiver" is defined by drawing $r$ nodes, one for each irreducible representation, and then drawing $a_{i j}$ arrows from the node associated to $V_{i}$ to the node associated to $V_{j}$. In the case of $d=2, a_{i j}=a_{j i}$ but this is not necessarily true for $d>2$.

McKay's observation was that the resulting quiver is precisely the extended Dynkin diagram associated to $G$. For example, if $G$ is the binary icosahedral group, it is associated to $E_{8}$ and the McKay quiver is:

where the numbers refer to the dimensions of the irreducible representations. Thus, except for the extra node present in the extended diagram, the McKay quiver represents exactly the configuration of $\mathbb{P}^{1}$ 's in the exceptional divisor upon blowing-up the orbifold singularity. In this latter context, the numbers in (262) refer to the multiplicity of the $\mathbb{P}^{1}$ 's in the exceptional divisor.

Within algebraic geometry we may give a quite different interpretation of the McKay quiver which turns out to be very relevant for D-branes. We follow the description in [164]. Let $V$ be a finite-dimensional complex vector space. Let us denote $\mathbb{C}^{d}$ by $Q$ and, as usual, $Q^{*}=\operatorname{Hom}(Q, \mathbb{C})$. Now let $S \in \operatorname{Hom}\left(Q^{*}, \operatorname{End}(V)\right)=\operatorname{Hom}(V, Q \otimes V)$, i.e., $S$ defines a linear action of the coordinates of $Q$ on $V$. If $x$ and $y$ are any two coordinates in $Q$ we demand that $S(x)$ commutes with $S(y)$. We denote this condition $S \wedge S=0$.
$\mathbb{C}[Q]$ denotes the polynomial ring of functions on $Q$ and it is equal to $\oplus_{k} \operatorname{Sym}^{k} Q^{*}$. Thus $S$ defines an action of $\mathbb{C}[Q]$ on $V$. In other words it gives $V$ the structure of a $\mathbb{C}[Q]$-module.

Now let $G \subset \mathrm{SU}(d)$ act on $V$ and on $Q . V$ is an arbitrary representation of $G$ and $Q$ is the standard $d$-dimensional representation. We now demand that $S$ commutes with the $G$-action by putting ${ }^{41}$

$$
\begin{equation*}
S \in \operatorname{Hom}_{G}(V, Q \otimes V), \quad S \wedge S=0 \tag{263}
\end{equation*}
$$

[^30]Such an $S$ defines a "G-equivariant" $\mathbb{C}[Q]$-module structure on $V$. In fact, we have defined a $G$-equivariant sheaf on $Q \cong \mathbb{C}^{d}$. For a sheaf one would normally define a $\mathscr{O}_{X}(U)$-module structure for all open sets $U$. We have only done this for $U=\mathbb{C}^{d}$, i.e., we have only considered global sections. However, since $\mathbb{C}^{d}$ is contractible, there is no information content beyond these global sections.

Now decompose $V$ into irreducible representations, $V=\bigoplus_{k} m_{k} V_{k}$. Schur's lemma states $\operatorname{Hom}\left(V_{i}, V_{j}\right)=\mathbb{C} \delta_{i j}$. Thus, using (261), it follows that $S$ is represented by a collection of matrices of complex numbers. To be precise, there are $a_{i j}$ matrices representing a map from $\mathbb{C}^{m_{i}}$ to $\mathbb{C}^{m_{j}}$ for each $i, j$. In other words, we associate a number $m_{i}$ to each node in the McKay quiver and a matrix of dimension $m_{j} \times m_{i}$ to each arrow from the $i$ th node to the $j$ th node. This collection of linear maps associated to the arrows on a McKay quiver is called a representation of the quiver. We refer to [165] for more background in this subject.

The condition $S \wedge S=0$ puts constraints on these matrices. For example, suppose $G$ is abelian. Then every irreducible representation is one-dimensional and thus $Q^{*}=q_{1} \oplus q_{2} \oplus$ $\ldots \oplus q_{d}$ for suitable irreducible representations $q_{\alpha}$. It follows that there are $d$ arrows leaving each node in the McKay quiver. Let $M_{\alpha}^{i}$ represent the matrix associated to the $\alpha$ 'th arrow leaving node $i$. Let $\alpha(i)$ be the node at the head of this arrow. Then the $S \wedge S=0$ relations read

$$
\begin{equation*}
M_{\beta}^{\alpha(i)} M_{\alpha}^{i}=M_{\alpha}^{\beta(i)} M_{\beta}^{i} . \tag{264}
\end{equation*}
$$

These are said to be relations on the representation of the quiver.
Now we consider morphisms of $G$-equivariant sheaves. In the quiver language this amounts to a morphism of $\mathbb{C}[Q]$-modules respecting the $G$-action. It is not hard to find the explicit form of these morphisms. Let $W=\bigoplus_{k} n_{k} V_{k}$. A $G$-invariant morphism from a quiver representation associated to $V$ to one associated to $W$ will, by Schur's lemma, be a choice of a matrix in $\operatorname{Hom}\left(\mathbb{C}^{m_{k}}, \mathbb{C}^{n_{k}}\right)$ for each node of the quiver. If these maps commute with the arrows within each quiver then we preserve the $\mathbb{C}[Q]$-module structure. For example, suppose we have a quiver with two nodes and a single arrow connected them. A morphism

is equivalent to the following commutative diagram


We have constructed a $G$-invariant morphism of $\mathbb{C}[Q]$-modules. We can use this construction to define a morphism of quiver representations and thus the category of quiver representations.

Our definitions show that the category of $G$-equivariant sheaves on $\mathbb{C}^{d}$ is equivalent to the category of representations of the McKay quiver with the relations $S \wedge S=0$.

The category of representations of the McKay quiver (with relations) is an abelian category. Each quiver representation is associated to a vector space $V$ as defined above. Kernels and cokernels may easily be computed purely from this vector space structure. When we start to write down exact sequences we see a new interpretation of the arrows in a quiver as follows. Note that

is a short exact sequence of quiver representations where the map $f \in \operatorname{Hom}(\mathbb{C}, \mathbb{C})=\mathbb{C}$ can be multiplication by any complex number. Let $F_{i}$ be the quiver representation with $n_{j}=\delta_{i j}$, i.e., $V$ is simply the irreducible representation $V_{i}$. Assuming there are no arrows beginning and ending on the same node, all the maps in $F_{i}$ are obviously zero and so $F_{i}$ specifies a unique object. The short exact sequence (267) represents an extension of $F_{1}$ by $F_{2}$. That is $\operatorname{Ext}^{1}\left(F_{1}, F_{2}\right)=\mathbb{C}$. This easily generalizes to an arbitrary quiver to give the following result (even if relations are imposed): the number of arrows from node $i$ to node $j$ equals the dimension of $\operatorname{Ext}^{1}\left(F_{i}, F_{j}\right)$. Thus the quiver is often called an "Ext quiver".

The most impressive generalization of the McKay correspondence was given by Bridgeland, King and Reid (BKR) [166] who said the following.

Theorem 3 Suppose $X$ is a smooth resolution (i.e., blow-up) of the orbifold $\mathbb{C}^{d} / G$ with $G$ a finite subgroup of $\mathrm{SU}(d)$ and $d \leq 3 .{ }^{42}$ Then the derived category $\mathbf{D}(X)$ is equivalent to the derived category of $G$-equivariant sheaves on $\mathbb{C}^{d}$.

In other words, $\mathbf{D}(X)$ is equivalent to the derived category of McKay quiver representations associated to $G$ with relations $S \wedge S=0$. Thus, whereas the original McKay correspondence viewed the arrows in the Dynkin diagram (262) as something to do with the intersection theory of the components of the exceptional divisor, in the generalization to three dimensions it is best to view the arrows as a statement about Ext's in the derived category.

Of course, the appearance of the derived category in the latter version of the McKay correspondence is excellent news for B-brane physics! The blow-up of an orbifold singularity is viewed as a deformation of $B+i J$ and so should not affect the B -model. It immediately follows that

B-branes on the orbifold $\mathbb{C}^{d} / G$ and open strings between them are described by the derived category of McKay quiver representations (with relations).

It follows that we have a distinguished set of D-branes on the orbifold associated to the irreducible representations of $G$. These will be associated to the quivers representations $F_{i}$ above. These branes were dubbed fractional branes in [157].

[^31]The BKR result gives a precise recipe for mapping between these two derived categories as follows. Consider the following diagram

where $Z$ is defined as the "fibre product" of $X$ and $\mathbb{C}^{d}$. That is, $Z$ is the subspace of $X \times \mathbb{C}^{d}$ given by points $(x, z)$ such that $\pi(x)=\gamma(z)$. The map $\gamma$ is the quotient by $G$ and $\pi$ is the blow-down. The maps $p$ and $q$ are then the obvious projections on the fibre product. BKR then say that $q_{*} p^{*}$ gives the desired equivalence between $\mathbf{D}(X)$ and the derived category of $G$-equivariant sheaves on $\mathbb{C}^{d}$. Note that we think of sheaves on $X$ as trivially $G$-equivariant. The map $p^{*}$ then typically introduces some non-trivial $G$-action.

As an example, consider the skyscraper sheaf $\mathscr{O}_{x}$ on $X$ for $x$ at a point in $X$ not fixed by the $G$-action. Then $p^{*} \mathscr{O}_{x}$ will be a collection of $|G|$ skyscraper sheaves on $Z$ transforming in the regular representation of $G$. This pushes forward to $\mathbb{C}^{d}$ to give the same collection of skyscraper sheaves. Clearly the global sections of this sheaf on $\mathbb{C}^{d}$ form the regular representation of $G$. Thus the 0 -branes on $X$ correspond to quivers for the regular representation of $G$.

This means that, for 0-branes, the integer $m_{i}$ attached to each node in the quiver representation is equal to the dimension of the corresponding irreducible representation. The location of the 0-brane will be dictated by the matrices associated to the arrows of the quiver. Once we study the stability of these B-branes we will see that the moduli space of such stable quiver representations is equal to $X$ as expected.

It also follows that the 0 -brane is always composed of a nontrivial sum of fractional branes (hence the name). We will see that, at the orbifold point, the 0 -brane is always marginally stable against decay into this set of underlying fractional branes.

### 7.3.2 The Douglas-Moore construction

We arrived at quivers from the mathematical direction of the McKay correspondence. While this is probably the best approach for seeing the appearance of the full derived category of quiver representations, there is a wonderfully direct physics way of seeing why quivers themselves should appear in the context of D-Branes on orbifolds. This is due to Douglas and Moore [156].

Suppose our full ten-dimensional spacetime looks like $\mathbb{R}^{1,3} \times \mathbb{C}^{3} / G$ and we have a 3-brane that fills the space-like directions of $\mathbb{R}^{1,3}$ and so appears as a 0 -brane in the $\mathbb{C}^{3} / G$ directions. The world-volume theory of this 3 -brane yields an $N=1$ supersymmetric field theory in $\mathbb{R}^{1,3}$.

One should note that this theory may, for general D-branes, have anomalies. This is because of the RR flux generated by the D-brane - an issue we have been able to ignore up to this point. One can remove these anomalies by using more background $R R$ fluxes along
the lines of [92]. See also [167] for discussion of when these theories can be anomaly-free. Actually all our discussion of this four-dimensional quantum field theory will essentially be classical, so we will ignore this issue of anomalies.

The idea is that one can analyze this quantum field theory by analyzing a collection of D-branes in $\mathbb{C}^{3}$ corresponding to the preimage of the quotient map, and then imposing $G$-invariance.

Suppose we have $m$ D-branes at the origin of $\mathbb{C}^{3}$ transforming in the representation $V$ of $G$. Let $\rho_{V}(g)$ represent the $m \times m$ matrix representing $g \in G$ in this representation. The field theory will have a $\mathrm{U}(m)$ gauge symmetry. Let $A$ be the corresponding gauge connection. The action of $G$ is given explicitly by

$$
\begin{equation*}
g(A)=\rho_{V}(g) \cdot A \cdot \rho_{V}(g)^{-1} \tag{269}
\end{equation*}
$$

In other words, $A$ transforms in the representation $\operatorname{End}(V)=V \otimes V^{*}$ and the $G$-invariant part of this can be written $\operatorname{Hom}_{G}(V, V)$. By Schur's lemma, the resulting gauge group is $\mathrm{U}\left(m_{1}\right) \times \mathrm{U}\left(m_{2}\right) \times \ldots$, where $V=\bigoplus_{k} m_{k} V_{k}$ as in the last section.

Before dividing by $G$, we have an $N=4$ supersymmetric gauge theory in four dimensions. The $N=4$ gauge superfield contains three scalar fields $Z$ which arise from the components of the ten-dimensional connection pointing in the $\mathbb{C}^{3}$ directions. These fields transform under $G$ similarly to $A$ except that $G$ acts on the $\mathbb{C}^{3}$ directions too. This latter 3-dimensional representation is clearly $Q$ from the previous section. So the $Z$ 's transform in $Q \otimes \operatorname{End}(V)$. In other words, the $G$-invariant subspace of invariant scalar fields is given by $\operatorname{Hom}_{G}(V, Q \otimes V)$. Thus, from (263), the $Z$ 's play the rôle of the matrices associated to the arrows in a quiver representation.

Finally, to obtain complete agreement with the last section, we need to find the commutation relations on the quiver. These arise from the superpotential of the $N=4$ theory [168]. If we write $Z_{\mu}, \mu=1,2,3$, for the three scalar fields (each transforming in the adjoint of $\mathrm{U}(m))$ corresponding to the 3 directions in $\mathbb{C}^{3}$, this superpotential may be written

$$
\begin{equation*}
W=\operatorname{Tr}\left(Z_{1}\left[Z_{2}, Z_{3}\right]\right) \tag{270}
\end{equation*}
$$

The critical points of this superpotential impose precisely the relations $S \wedge S=0$ from section 7.3.1.

To recap, the $N=1$ supersymmetric field theory in $\mathbb{R}^{1,3}$ is described by the McKay quiver representations of section 7.3.1. The integers $m_{i}$ describe the effective gauge group. The homomorphisms associated to the arrows are associated to "bifundamental" scalar fields $Z$ transforming accordingly in this gauge group. Finally, the commutation relations on the quiver representation are given by the superpotential.

This Douglas-Moore construction of a D-brane world-volume is identical mathematically to a problem studied by Sardo Infirri $[169,170]$. In this case one studies translation-invariant $G$-equivariant holomorphic bundles on $Q=\mathbb{C}^{d}$. Let the fibre of a vector bundle be given by a representation $V$ of $G$. Then, for $G$-equivariance, the connection on this bundle transforms
yet again in $\operatorname{Hom}_{G}(V, Q \otimes V)$. Let us write this connection in the form

$$
\begin{equation*}
\sum_{\mu=1}^{d} Z_{\mu} d z_{\mu}-Z_{\mu}^{\dagger} d \bar{z}_{\mu} \tag{271}
\end{equation*}
$$

The (2,0)-part of the curvature is then [169]

$$
\begin{equation*}
\sum_{\mu, \nu}\left(-\frac{\partial Z_{\mu}}{\partial z_{\nu}}+\frac{1}{2}\left[Z_{\mu}, Z_{\nu}\right]\right) d z_{\mu} \wedge d z_{\nu} \tag{272}
\end{equation*}
$$

If we impose translation invariance we demand that the derivatives of $Z_{\mu}$ vanish. The condition that the bundle be holomorphic then amounts to $\left[Z_{\mu}, Z_{\nu}\right]=0$ which again imposes the relations on the quiver representation.

Thus we have three interpretations for the quiver representation:

1. A $G$-equivariant sheaf.
2. The scalar fields in the world-volume of a D-brane on an orbifold.
3. A connection on a translation-invariant holomorphic $G$-equivariant vector bundle.

The fact that these descriptions coincide allow us to prove a fact about the gradings of the fractional branes at the orbifold point. The scalar fields $Z_{\mu}$ in our world-volume theory must arise as open string states in the worldsheet description. That is, they occur as certain Ext's between the D-branes. The Ext-quiver language immediately tells us, of course, that these scalars are actually associated to Ext ${ }^{1}$ 's between the fractional branes. The discussion around equation (176) and section 6.2.3 tells us that a scalar state associated with Ext ${ }^{1}$ is massless if and only if the $\xi$-gradings at the end of the string are equal. Thus we have proven:

Theorem 4 The gradings are equal for all the fractional branes at the orbifold point.
A consequence of this theorem is, of course, that the central charges of fractional Dbranes align to have the same arg at the orbifold point. We will see this explicitly in an example in section 7.3.6.

It is worth noting that the quiver language may be used in more general cases than orbifolds. So long as the gradings of a set of branes are aligned, their Ext-quiver will represent the field content of an $N=1$ theory in four dimensions. This ties the derived category picture into quiver-related work such as $[167,171-173]$ etc. Sadly we do not have the space to expand on this relation further.

### 7.3.3 $\quad \theta$-stability

We have established that the category of topological B-branes for an orbifold $\mathbb{C}^{d} / G$ corresponds to the derived category of quiver representations with relations. Now we want to impose the BPS condition again, that is, we want a stability condition. We already know what the answer is - $\Pi$-stability. However, since we have a relatively simple description of B-branes on an orbifold, we might expect there to be a simpler description of the stability condition for the orbifold for a region of the moduli space of $B+i J$ around the orbifold point (i.e., small blow-ups). In other words, we want something analogous to the way $\mu$ stability was a good description near the large radius limit. The answer is called $\theta$-stability as introduced by King [174]. This was studied in the context of D-branes in [117-119].

Both the Douglas-Moore D-brane world-volume picture and the Sardo Infirri picture of a $G$-equivariant bundle yield the condition for a BPS state. As one might guess, in the $G$ equivariant bundle picture we simply impose the condition that the connection is Hermitian-Yang-Mills. In the D-brane language we are required to minimize the potential arising from the "D-terms" in the action. Either way, the condition for a BPS state becomes ${ }^{43}$

$$
\begin{equation*}
\sum_{i}\left[Z_{\mu}, Z_{\mu}^{\dagger}\right]=0 \tag{273}
\end{equation*}
$$

In section 7.3 .1 we saw that the regular representation of $G$ should correspond to the skyscraper sheaf $\mathscr{O}_{x}$. One can show [169] that the moduli space of $Z$ 's associated to the regular representation that satisfy (273) is indeed given by $\mathbb{C}^{d} / G$ as expected.

Now we want to resolve the orbifold a little. This means that we want to deform our problem in such a way that the moduli space of $Z$ 's associated to the regular representation becomes $X$, a resolution of the orbifold. In the D-brane language one may do this by adding "Fayet-Iliopoulos" terms to the action $[156,175,176]$. One such term may be added for each unbroken $\mathrm{U}(1)$ of the gauge group, i.e., one for each irreducible representation of $G$. Let us call the coefficients of these terms $\zeta_{i}$, where $i$ is an index running over the irreps of $G$. The equivalent deformation is seen in the Sardo Infirri picture as a deformation of a "moment map" [169].

The result is that (273) becomes

$$
\begin{equation*}
\sum_{\mu}\left[Z_{\mu}, Z_{\mu}^{\dagger}\right]=\operatorname{diag}(\underbrace{\zeta_{1}, \zeta_{1}, \ldots, \zeta_{1}}_{\operatorname{dim}\left(V_{1}\right)}, \underbrace{\zeta_{2}, \zeta_{2}, \ldots, \zeta_{2}}_{\operatorname{dim}\left(V_{2}\right)}, \ldots) \tag{274}
\end{equation*}
$$

In the case that $G$ is abelian and $d=3$, Sardo Infirri proved that the resulting moduli space of the 0 -brane is indeed a resolution of $\mathbb{C}^{d} / G$ (see also [175]). The case $d=2$ and an arbitrary $G$ was proven earlier by Kronheimer [177].

Since the blow-up of an orbifold singularity is obtained by a deformation of $B+i J$, this resolution should be produced by closed string marginal operators. To be precise, twisted

[^32]closed string marginal operators. Twisted operators are labeled by conjugacy classes in $G$ [155] which we denote $C$. Let $\phi_{C}$ be a twisted operator present in the topological A-model for closed strings for the conjugacy class $C$ and consider a deformation of $B+i J$ produced by adding a term
\[

$$
\begin{equation*}
a_{C} \int_{\Sigma} d^{2} z \phi_{C} \tag{275}
\end{equation*}
$$

\]

to the action. Turning on the $\zeta_{i}$ 's should be equivalent to turning on some $a_{C}$ 's which implies that $\phi_{C}$ should acquire a nonzero 1-point function in the D-brane background. This onepoint function was computed in $[156,175]$. The result is, at least in a linear approximation for very small blow-ups

$$
\begin{equation*}
a_{C}=\sum_{i} \chi_{i}(C) \zeta_{i} \tag{276}
\end{equation*}
$$

where $\chi_{i}$ 's are the characters of the group $G$.
The operator $\phi_{1}$ associated to the conjugacy class of the identity is the closed string tachyon and is removed by the GSO projection. Thus we require that $a_{1}=0$, i.e.,

$$
\begin{equation*}
\sum_{i} \operatorname{dim}\left(V_{i}\right) \zeta_{i}=0 \tag{277}
\end{equation*}
$$

Thus condition is also imposed by (274) since a commutator must be traceless.
So far we have discussed the effects of nonzero $\zeta_{i}$ 's on the 0-brane, i.e., regular representation of $G$. What about general representations? The set-up is essential identical except that the $Z_{\mu}$ 's now transform in another representation of $G$. Let us consider the representation $V=\bigoplus_{i} m_{i} V_{i}$. Then (274) is modified so that the right-hand side is a diagonal matrix with each $\zeta_{i}$ appearing $m_{i}$ times. In this more general setting the condition (277) can prevent (274) from having a solution.

In D-brane world-volume language this means that we cannot find a solution which makes the contribution of the D-term to the potential equal to zero. At first sight, this would appear to break supersymmetry. In fact, this is not the case as pointed out in [117]. If we simply minimize the potential then a not-so-manifest $N=1$ supersymmetry still exists implying we do have a BPS state. We refer to [117] for more details. Suppose we minimize the potential by setting

$$
\begin{equation*}
\sum_{\mu}\left[Z_{\mu}, Z_{\mu}^{\dagger}\right]=\operatorname{diag}(\underbrace{\theta_{1}, \theta_{1}, \ldots, \theta_{1}}_{m_{1}}, \underbrace{\theta_{2}, \theta_{2}, \ldots, \theta_{2}}_{m_{2}}, \ldots) \tag{278}
\end{equation*}
$$

for some real numbers $\theta_{i}$. The potential is then given by

$$
\begin{equation*}
\sum_{i}\left(\zeta_{i}-\theta_{i}\right)^{2} \tag{279}
\end{equation*}
$$

which is minimized subject to the condition (277) by

$$
\begin{equation*}
\theta_{i}=\zeta_{i}-\frac{\sum_{j} m_{j} \zeta_{j}}{\sum_{j} m_{j}} \tag{280}
\end{equation*}
$$

The equation (278) may be written in a more quiver-friendly way as follows. Let $a$ be an arrow in the quiver with head $h(a)$ and tail $t(a)$. Let $Z_{a}$ be the $m_{h(a)} \times m_{t(a)}$ matrix associated with this arrow in a given quiver representation. Then (278) becomes

$$
\begin{equation*}
\sum_{h(a)=i} Z_{a} Z_{a}^{\dagger}-\sum_{t(a)=i} Z_{a}^{\dagger} Z_{a}=\theta_{i} \mathrm{id} \tag{281}
\end{equation*}
$$

This is exactly the equation studied by King [174]. Fix a representation of the quiver associated to a representation $V=\bigoplus_{i} m_{i} V_{i}$ of $G$. For any representation $W=\bigoplus_{i} n_{i} V_{i}$ of $G$ we define

$$
\begin{equation*}
\theta(W)=\sum_{i} \theta_{i} n_{i} . \tag{282}
\end{equation*}
$$

Thus, by the tracelessness of (278), we see $\theta(V)=0$. We say that the quiver representation is $\theta$-stable if, for any nontrivial quiver subrepresentation associated to a representation $W$ of $G, \theta(W)>0$. King proved the following

Theorem 5 A quiver representation satisfies (281) (with an inner product unique up to obvious automorphisms) if and only if it a direct sum of $\theta$-stable representations.

Thus, very close to the orbifold point we have a stability condition expressed purely in terms of quivers.

### 7.3.4 Periods

For the remainder of these lectures we will focus on a particular example of an orbifold rather than attempt to prove any general statements. The example is $\mathbb{C}^{3} / \mathbb{Z}_{3}$ where $g$ generates $\mathbb{Z}_{3}$ and acts as $g:\left(z_{1}, z_{2}, z_{3}\right) \mapsto\left(\omega z_{1}, \omega z_{2}, \omega z_{3}\right)$ for $\omega=\exp (2 \pi i / 3)$. Much of the analysis of D-branes on orbifolds has been done in this simplest example (e.g., [118, 157, 159, 175]).

Let $V_{i}, i=0, \ldots, 2$ be the one-dimensional irreducible representations of $\mathbb{Z}_{3}$ given by $\rho(g)=\omega^{i}$. A representation $V=\oplus_{i} m_{i} V_{i}$ is then associated to a quiver representation


As is well-known, this orbifold is resolved with an exceptional divisor $E \cong \mathbb{P}^{2}$. In this case, $X$ may be viewed as the total space of the line bundle corresponding to the sheaf $\mathscr{O}_{E}(-3)$.

The computation to compute the periods and thus the central charges can be done in a way very similar to that of the quintic. In this section we perform the computations corresponding to section 7.1.1.

Similarly to the quintic, we have a one-dimensional moduli space of $B+i J$ that can be viewed as a $\mathbb{P}^{1}$. One point on this $\mathbb{P}^{1}$ corresponds to the large radius limit where the
exceptional divisor $E$ is infinitely large. At the other extreme, we have the orbifold point where we have no blow-up. At a third point on the $\mathbb{P}^{1}$, which we denote $P_{0}$, we have the analogue of the "conifold point" where $B+i J$ have a special value that makes the associated conformal field theory singular. Again, as in the quintic, we may use mirror symmetry to analyze this moduli space and the associated periods exactly, as was first done in [35].

The Picard-Fuch's equation in question is given by

$$
\begin{equation*}
\left(z \frac{d}{d z}\right)^{3} \Phi+27 z\left(z \frac{d}{d z}\right)\left(z \frac{d}{d z}+\frac{1}{3}\right)\left(z \frac{d}{d z}+\frac{2}{3}\right) \Phi=0 . \tag{284}
\end{equation*}
$$

Clearly any constant solves this differential equation. Putting $z=\left(3 e^{-\pi i} \psi\right)^{-3}$ we may write a basis for the remaining solutions near $\psi=0$ as:

$$
\begin{equation*}
\varpi_{j}=\frac{1}{2 \pi i} \sum_{n=1}^{\infty} \frac{\Gamma\left(\frac{n}{3}\right) \omega^{n j}}{\Gamma(n+1) \Gamma\left(1-\frac{n}{3}\right)^{2}}(3 \psi)^{n}, \tag{285}
\end{equation*}
$$

where $\omega=\exp (2 \pi i / 3)$. Thus $1, \varpi_{0}$ and $\varpi_{1}$ form a basis for the solutions of the PicardFuchs equation. The analytic continuation to a basis for small $z$ (valid for $|\arg (z)|<\pi)$ is performed similarly to section 7.1.1:

$$
\begin{align*}
\Phi_{0} & =1, \\
\Phi_{1} & =\frac{1}{2 \pi i} \cdot \frac{3}{2 \pi i} \int \frac{\Gamma(3 s) \Gamma(-s)}{\Gamma(1+s)^{2}} z^{s} d s \\
& =\frac{1}{2 \pi i} \log z+O(z) \\
& =t \\
& =\varpi_{0},  \tag{286}\\
\Phi_{2} & =-\frac{1}{4 \pi^{2}} \cdot \frac{-6}{2 \pi i} \int \frac{\Gamma(3 s) \Gamma(-s)^{2}}{\Gamma(s+1)}\left(e^{-\pi i} z\right)^{s} d s \\
& =-\frac{1}{4 \pi^{2}}(\log z-i \pi)^{2}-\frac{5}{12}+O(z) \\
& =t^{2}-t-\frac{1}{6}+O\left(e^{2 \pi i t}\right) \\
& =-\frac{2}{3}\left(\varpi_{0}-\varpi_{1}\right),
\end{align*}
$$

where the mirror map is given by $t=\int_{C} B+i J=\frac{1}{2 \pi i} \log (z)+O(z)$, and $C$ is a $\mathbb{P}^{1}$ hyperplane in $E$. The analogue of figure 1 for the $\mathbb{C}^{3} / \mathbb{Z}_{3}$ orbifold is given in figure 13 . Note that the orbifold point lies at exactly $B=J=0 .{ }^{44}$

Now, using (155) and (191), we can compute the exact value for central charges. For example, consider the 4-branes $\mathscr{O}_{E}(m)$ wrapping the exceptional divisor:

$$
\begin{equation*}
Z\left(\mathscr{O}_{E}(m)\right)=-\left(m+\frac{4}{3}\right) \varpi_{0}+\frac{1}{3} \varpi_{1}+\frac{1}{2} m^{2}+\frac{3}{2} m+\frac{4}{3} . \tag{287}
\end{equation*}
$$

[^33]

Figure 13: Fundamental regions for the moduli space of the $\mathbb{Z}_{3}$-orbifold.

Consider the point $P_{0}$ in the moduli space where $\psi=2 \pi i / 3$ and we have a singular conformal field theory. At $P_{0}$ we have $\varpi_{0}=t=\frac{1}{2}+i J_{0}$ where $J_{0} \approx 0.4628$. From (285) the value of $\varpi_{1}$ at $P_{0}$ will clearly be equal to the value of $\varpi_{0}$ at $\psi=4 \pi i / 3$, namely $-\frac{1}{2}+i J_{0}$. Thus $\varpi_{0}-\varpi_{1}=1$ at $P_{0}$. It follows from (288) that $\mathscr{O}_{E}(-1)$ becomes massless at $P_{0}$. Similarly $\mathscr{O}_{E}(-2)$ becomes massless at the point $\psi=4 \pi i / 3$.

We will therefore assume that the singularity in the conformal field theory at $P_{0}$ is caused by the stable B-brane $\mathscr{O}_{E}(-1)$ becoming massless. This appears to be very similar to the statement that $\mathscr{O}_{C}(-1)$ became massless for the singular conformal field theory for the case of the flop in section 7.2. In the latter case we reset our definitions so that $\mathscr{O}_{C}$ became massless instead. We will do the same here to make the results prettier. In effect we shift $t \mapsto t+1$ so that (287) becomes

$$
\begin{equation*}
Z\left(\mathscr{O}_{E}(m)\right)=-\left(m+\frac{1}{3}\right) \varpi_{0}+\frac{1}{3} \varpi_{1}+\frac{1}{2} m^{2}+\frac{1}{2} m+\frac{1}{3}, \tag{288}
\end{equation*}
$$

and now $\mathscr{O}_{E}$ becomes massless at $P_{0}$.

### 7.3.5 Monodromy

In general orbifolds may exhibit a "quantum symmetry" which acts on a state twisted by $g \in G$ by multiplication by $q(g)$, where

$$
\begin{equation*}
q \in \operatorname{Hom}\left(G, \mathbb{C}^{*}\right) \tag{289}
\end{equation*}
$$

The group $\operatorname{Hom}\left(G, \mathbb{C}^{*}\right)$ is thus the group of quantum symmetries. It is not hard to show that this is isomorphic to the abelianization of $G$, i.e., $G /[G, G]$. Yet another interpretation of $\operatorname{Hom}\left(G, \mathbb{C}^{*}\right)$ is the group of one-dimensional representations of $G$ where the group operation is the tensor product. Given a one-dimensional representation $U$ of $G$ we may act on the set of representations by $U \otimes-$. This gives a symmetry of the McKay quiver and thus shows exactly how the quantum symmetries of an orbifold act on the category of D-branes.

It is manifest from (262) that the extended Dynkin diagram for $E_{8}$ has no symmetries and thus the quantum symmetry group associated to the binary icosahedral group is trivial. That is, there is no quantum symmetry for this orbifold. Thus, we cannot possibly use quantum symmetries as a tool for making general statements about orbifolds but they can be very useful in examples. In particular, if $G$ is abelian, the quantum symmetry group is isomorphic to $G$ and it acts transitively on the nodes of the McKay quiver. In our case, the quantum symmetry group $\mathbb{Z}_{3}$ acts by rotating the McKay quiver by $2 \pi / 3$.

In section 7.1.4 we analyzed the monodromy around the Gepner point and discovered that the quantum $\mathbb{Z}_{5}$ symmetry one might expect in this context was in fact broken in the derived category. In this section we will show that the quantum symmetry of an orbifold, at least in our example, is not broken by the derived category.

Monodromy around the large radius limit corresponds to tensoring by $\mathscr{O}_{X}(D)$ where $D$ is a divisor Poincaré dual to the component of the Kähler form which is being taken to be very large. Thus we require $D$ to intersect $C$ (the $\mathbb{P}^{1}$ hyperplane of $E$ ) in one point. To fit in with the notation used in the case of the quintic we will denote $\mathscr{O}_{X}(D)$ by $\mathscr{O}_{X}(1)$. This notation is also consistent with the fact that $\mathscr{O}_{X}(1) \otimes \mathscr{O}_{E}=\mathscr{O}_{E}(1)$. Note that since the normal bundle of $E$ corresponds to $\mathscr{O}_{E}(-3)$, the analogue of the exact sequence (85) is

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}_{X}(3) \longrightarrow \mathscr{O}_{X} \longrightarrow \mathscr{O}_{E} \longrightarrow 0 . \tag{290}
\end{equation*}
$$

The monodromy associated to the "conifold point" $P_{0}$ must be associated to the fact that $\mathscr{O}_{E}$ becomes massless there. If $\mathcal{K}$ is the Fourier-Mukai transform associated to monodromy around $P_{0}$, then, comparing to (232) we would expect a transform

$$
\begin{equation*}
T_{\mathcal{K}}(\mathscr{F})=\operatorname{Cone}(\operatorname{Hom}(\mathscr{E}, \mathscr{F}) \otimes \mathscr{E} \xrightarrow{r} \mathscr{F}) \tag{291}
\end{equation*}
$$

where $\mathscr{E}$ is now $\mathscr{O}_{E}$. This corresponds to

$$
\begin{equation*}
\mathcal{K}=\left(\mathscr{E}^{\vee} \boxtimes \mathscr{E} \xrightarrow{r} \mathscr{O}_{\Delta X}\right) \tag{292}
\end{equation*}
$$

where $\mathscr{E}^{\vee}$ is the dual of $\mathscr{E}$ defined in the derived category as $\mathbf{R} \mathscr{H} o m\left(\mathscr{E}, \mathscr{O}_{X}\right)$. Thus, if $\mathcal{G}$ is the transform associated to the orbifold point, we have

$$
\begin{align*}
\mathcal{G} & =\left(\mathscr{E}^{\vee}(1) \boxtimes \mathscr{E} \longrightarrow \mathscr{O}_{\Delta X}(1)\right) \\
& =\left(\mathscr{O}_{E}(-1)^{\vee} \boxtimes \mathscr{O}_{E} \longrightarrow \mathscr{O}_{\Delta X}(1)\right) . \tag{293}
\end{align*}
$$

The computation proceeds in a way very similar to section 7.1.4 to yield

$$
\begin{equation*}
\mathcal{G}^{\star 3}=\left(\mathscr{O}_{E}(-1)^{\vee} \boxtimes \Omega_{E}^{2}(2) \longrightarrow \mathscr{O}_{E}(-2)^{\vee} \boxtimes \Omega_{E}(1) \longrightarrow \mathscr{O}_{E}(-3)^{\vee} \boxtimes \mathscr{O}_{E} \longrightarrow \mathscr{O}_{\Delta X}(3)\right) . \tag{294}
\end{equation*}
$$

Using (290) we see that $\mathscr{O}_{E}^{\vee}$ is $\left(\mathscr{O}_{X} \longrightarrow \mathscr{O}_{X}(-3)\right)$ which is $\mathscr{O}_{E}(-3)[-1]$. We may then apply the Beilinson sequence (246) for $E \cong \mathbb{P}^{2}$ to yield

$$
\begin{align*}
\mathcal{G}^{\star 3} & =\operatorname{Cone}\left(\mathscr{O}_{\Delta E}[-1] \longrightarrow \mathscr{O}_{\Delta X}(3)\right) \\
& =\operatorname{Cone}\left(\left(\mathscr{O}_{\Delta X}(3) \longrightarrow \mathscr{O}_{\Delta X}\right) \longrightarrow \mathscr{O}_{\Delta X}(3)\right)  \tag{295}\\
& =\mathscr{O}_{\Delta X}
\end{align*}
$$

Thus $\mathcal{G}^{\star 3}$ is the identity on the nose with no shifts involved. The quantum $\mathbb{Z}_{3}$ symmetry is therefore preserved even at the level of the derived category.

We may apply the BKR map of section 7.3.1 to determine the relationship between quiver representations and coherent sheaves. The McKay equivalence may of course be composed with any autoequivalence of $\mathbf{D}(X)$ and still give an equivalence. This gives a degree of ambiguity to how we may associate the derived category of sheaves to the derived category of quivers. We refer to $[164,179]$ for further details on how to compute the correspondence exactly. Let $\Delta_{m_{0} m_{1} m_{2}}$ denote a quiver representation of the form (283) and consider the fractional branes $F_{0}=\Delta_{100}, F_{1}=\Delta_{010}$ and $F_{2}=\Delta_{001}$. A choice of the McKay correspondence consistent with our conventions is then given by


From the above analysis of $\mathcal{G}$ it is easy to show that

$$
\begin{align*}
& T_{\mathcal{G}}\left(\mathscr{O}_{E}\right)=\Omega_{E}(1)[1] \\
& T_{\mathcal{G}}^{2}\left(\mathscr{O}_{E}\right)=\Omega_{E}^{2}(2)[2]=\mathscr{O}_{E}(-1)[2] . \tag{297}
\end{align*}
$$

Thus $T_{\mathcal{G}}$ indeed corresponds to rotating the McKay quiver clockwise by $2 \pi i / 3$ or, equivalently, by tensoring by the representation $F_{1}$. Perhaps we should emphasize that this picture does not work if we were to assert that $F_{2}=\mathscr{O}_{E}(-1)$ without the shift of 2 as is done in much of the literature.

We may easily generalize the quintic hypersurface and the orbifold computation above to other dimensions (i.e., degree $d$ hypersurfaces in $\mathbb{P}^{d-1}$ and orbifolds $\mathbb{C}^{n} / \mathbb{Z}_{n}$ ). In each case the quantum symmetry of the Landau-Ginzburg orbifold is broken to become a shift left
by two and the quantum symmetry of the geometrical orbifold is preserved. Based on this rather limited set of examples it is tempting to conjecture that this is a general result for all Gepner models and for all orbifolds $\mathbb{C}^{d} / G$. It would be interesting to prove this, or at least study some more examples.

### 7.3.6 Examples of stability

In the case of the quintic we used the large radius limit as our base point for determining stability. In the case of the orbifold, we can use the quantum symmetry to use the orbifold point as the base point. First note that from (286) (with $t$ shifted by 1) and (296) we have

$$
\begin{align*}
& Z\left(F_{0}\right)=\frac{1}{3}\left(1-\varpi_{0}+\varpi_{1}\right) \\
& Z\left(F_{1}\right)=\frac{1}{3}\left(1-\varpi_{0}-2 \varpi_{1}\right)  \tag{298}\\
& Z\left(F_{2}\right)=\frac{1}{3}\left(1+2 \varpi_{0}+\varpi_{1}\right) .
\end{align*}
$$

Thus, at the orbifold point, the $F_{i}$ 's all have central charge $\frac{1}{3}$. This is not surprising as the $\mathbb{Z}_{3}$ quantum symmetry cyclically permutes the $F_{i}$ 's are so their physics must be identical at the orbifold point. In particular, they must all have the same value of $\xi$ - a fact we proved in general at the end of section 7.3.2. We may thus declare at the orbifold point that

$$
\begin{equation*}
\xi\left(F_{0}\right)=\xi\left(F_{1}\right)=\xi\left(F_{2}\right)=0 . \tag{299}
\end{equation*}
$$

Given (285), to a linear approximation in $\psi$ near the orbifold we therefore have

$$
\begin{align*}
\xi\left(\Delta_{m_{0} m_{1} m_{2}}\right) & =-c \frac{\left(-m_{0}-m_{1}+2 m_{2}\right) \operatorname{Re}(\psi)+\left(m_{0}-2 m_{1}+m_{2}\right) \operatorname{Re}(\omega \psi)}{m_{0}+m_{1}+m_{2}}  \tag{300}\\
& =c \frac{\sum_{i} m_{i} \zeta_{i}}{\sum_{i} m_{i}},
\end{align*}
$$

for a positive constant $c$ and we define the $\zeta_{k}$ by

$$
\begin{equation*}
\zeta_{k}=\sqrt{3} \operatorname{Re}\left(e^{\frac{\pi i}{6}(4 k-1)} \psi\right) \tag{301}
\end{equation*}
$$

so that

$$
\begin{align*}
\zeta_{0}+\zeta_{1}+\zeta_{2} & =0 \\
\zeta_{0}+\omega \zeta_{1}+\omega^{2} \zeta_{2} & =\frac{3 \sqrt{3}}{2} e^{\frac{\pi i}{6}} \bar{\psi}  \tag{302}\\
\zeta_{0}+\omega^{2} \zeta_{1}+\omega \zeta_{2} & =\frac{3 \sqrt{3}}{2} e^{-\frac{\pi i}{6}} \psi .
\end{align*}
$$

Clearly this is the analogue of (276) and the $\zeta$ 's we have just introduced here correspond to those of section 7.3.3. Indeed, we may now explicitly show that $\theta$-stability is a limiting form of $\Pi$-stability near the orbifold point. Suppose we have a short exact sequence of quiver representations


Near the orbifold point, the $\xi$ 's of these 3 D-branes will be very close to zero. Thus, by the way central charges add, the $\xi$ of the middle entry in (303) must lie between the $\xi$ 's of the other two. For $\Pi$-stability of the middle entry we draw the distinguished triangle

and look for the condition for $f$ to be tachyonic. From (300) this is precisely

$$
\begin{equation*}
\frac{\sum_{i} n_{i} \zeta_{i}}{\sum_{i} n_{i}}<\frac{\sum_{i} m_{i} \zeta_{i}}{\sum_{i} m_{i}} \tag{305}
\end{equation*}
$$

which, from (280) is equivalent to King's $\theta$-stability statement of section 7.3.3.
This $\theta$-stability formulation allows us to completely classify the stable irreducible Bbranes near the orbifold point. As mentioned in section 6.1.3, the irreducibility for an object $A$ amounts to $\operatorname{Hom}(A, A)=\mathbb{C}$. A quiver representation satisfying this condition is known as a "Schur representation". The problem of finding such representations was discussed in [118].

Determining whether a quiver representation (with relations) is Schur is a purely algebra question but turns out to be fairly awkward. In many cases it is actually more convenient to use the BKR equivalence and rephrase the question in terms of coherent sheaves.

As an example of a non-Schur quiver representation, consider $\Delta_{211}$ with generic maps on the arrows in the quiver. With some effort one can show that the short exact sequence

is split. This immediately implies that $\operatorname{Hom}\left(\Delta_{211}, \Delta_{211}\right) \supset \mathbb{C}^{2}$. This fact becomes more obvious when written in terms of sheaves. $\Delta_{111}$ is a 0 -brane which is generically nowhere near the exceptional divisor $E$, whereas $\Delta_{100}$ is the 4 -brane $\mathscr{O}_{E}$ wrapping $E$. Thus $\Delta_{211}$ is a sum of two quite disjoint D-branes and is obviously reducible. Note that when the maps on the arrows are not generic this quiver presentation might actually be Schur. This would correspond to the 0 -brane being on $E$ leading to a possible irreducible bound state with the 4-brane.

We will not attempt to explicitly provide a complete solution to the classification problem here but some Schur representations of interest are $\Delta_{111}$, and $\Delta_{a b c}$, where $\{a, b, c\}$ is any permutation of $\{0,1, n\}$ for $n \leq 3$.

The fractional branes $F_{k}$ are obviously always stable near the orbifold point since they have no nontrivial subobject in the category of quiver representations. Let us next focus on
$\Delta_{111}$, some of which correspond to 0 -branes. The quiver

with at least one nonzero map between each pair of nodes is stable since there is no injective map from any possible subobject to it. According to the explicit computations in [169,170] such a quiver represents a 0 -brane away from the exceptional divisor $E$ (or orbifold point if we haven't blown up). Indeed, one would expect that the stability of such a 0 -brane should not be affected by orbifold-related matters.

Now consider the following short exact sequence:


This $\Delta_{111}$ is stable against decay to $\Delta_{100}$ by $\theta$-stability if $\zeta_{0}<0$. We also have the sequence

giving a further constraint $\zeta_{2}>0$ on the stability of this 0 -brane. Thus this 0 -brane is stable in $2 \pi / 3$ wedge coming out of the orbifold point. After blowing-up a little into this wedge, this 0 -brane corresponds to a point on the exceptional divisor. Obviously a cyclic permutation of the zero to another edge of the quiver results in similar statements with the $\zeta$ 's permuted accordingly. Thus, all other quivers $\Delta_{111}$ are unstable in the wedge $\zeta_{2}>0, \zeta_{0}<0$ and do not correspond to 0 -branes at all.

The quiver representations with zero maps on the left edge have a close connection to sheaves on $E$ as can be seen as follows. The general representation $\Delta_{a b c}$ falls into the sequence

and thus we iterate

$$
\begin{align*}
& \stackrel{\substack{a \\
0}}{\substack{a-1 \\
0_{c}}}=\text { Cone }\left(\mathscr{O}_{E}\right) \tag{311}
\end{align*}
$$



Figure 14: Some lines of marginal stability for the $\mathbb{Z}_{3}$-orbifold.

Continuing this process yields
explicitly mapping this quiver representation into the derived category of coherent sheaves on $X$ (or $E$ ). This is precisely Beilinson's construction of sheaves on $\mathbb{P}^{2}$ [148] and this correspondence was identified in [118]. Thus, quiver representations with zero maps on the left edge are seen to be associated to D-branes on $E$. We will get a pure sheaf of course only if the cohomology of the complex in (312) is concentrated at one term.

We denote some lines of marginal stability for $\Pi$-stability in figure 14. In each case, the arrow denotes the direction you cross the line to cause the relevant object to decay. Naturally this agrees with $\theta$-stability near the origin. The figure shows the moduli space in the form of the complex $(-i \psi)$-plane. We make this choice so that the picture is aligned with figure 13 , i.e., the large radius limit is upwards. Note that the lines of marginal stability corresponding to $\mathscr{O}_{p}, p \in E$, (i.e., the corresponding $\Delta_{111}$ quivers above) follow the lines of constant $\arg (\psi)$ even when the non-perturbative effects of $\Pi$-stability are taken into account.

Some decays of note are the following:

1. $\mathscr{O}_{C}$ : This sheaf fits into the exact sequence

$$
\begin{equation*}
0 \longrightarrow \mathscr{O}_{E}(-1) \longrightarrow \mathscr{O}_{E} \longrightarrow \mathscr{O}_{C} \longrightarrow 0, \tag{313}
\end{equation*}
$$

and thus decays by $\Pi$-stability in a way essentially identical to the 4 -branes in the quintic as in section 7.1.2. Thus, these 2-branes are stable at large radius but decay before the orbifold point is reached. Note that (313) implies that, in the derived category of quiver representations we have

$$
\begin{equation*}
\mathscr{O}_{C}=\operatorname{Cone}\left(F_{2}[-2] \rightarrow F_{0}\right) . \tag{314}
\end{equation*}
$$

That is, this D-brane is essentially a complex of quivers and cannot be written in terms of a single quiver. In other words, it is not in the abelian category of quiver representations. It is therefore consistent with our picture that it decays before we get close to the orbifold point.
2. $\Delta_{101}$ : Following the logic of section 7.1.3 we can now look for an "exotic" D-brane by taking the "Serre dual" of (314). This gives Cone $\left(F_{0}[-1], F_{2}\right)$, i.e., an extension of $F_{0}$ by $F_{2}$. This is precisely $\Delta_{101}$. As expected from section 7.1.3, these objects should not be stable at large radius but can become stable as we shrink the exceptional divisor down. The line of marginal stability is shown in figure 14 . Note that they do not actually become stable until we shrink down to, or beyond, the orbifold point. These objects generically have nonzero maps along the left edge of the triangle and so are not classified by Beilinson's construction (312).
We see a nice complementarity between the D-branes $\Delta_{101}$ and $\mathscr{O}_{C} . \mathscr{O}_{C}$ is an object in the category of coherent sheaves but is a complex in terms of quivers. $\Delta_{101}$ is an object in the category of quiver representations but becomes an exotic complex Cone $\left(\mathscr{O}_{E}[-1], \mathscr{O}_{E}(-1)[2]\right)$ in the derived category of sheaves.
3. $\Delta_{n 10}$ : This fits into the sequence


The produces a decay as shown in figure 14. The identification (312) together with the short exact sequence

$$
\begin{equation*}
0 \longrightarrow \Omega_{E}(1) \longrightarrow \mathscr{O}_{E}^{\oplus 3} \longrightarrow \mathscr{O}_{E}(1) \longrightarrow 0, \tag{316}
\end{equation*}
$$

may be used to show that

$$
\begin{align*}
\Delta_{210} & \cong \mathscr{O}_{C}(1) \\
\Delta_{310} & \cong \mathscr{O}_{E}(1) \tag{317}
\end{align*}
$$

These D-branes are therefore simply objects both in terms of sheaves and quivers. It is not surprising therefore that they are both stable at large radius limit and near the orbifold point.
On the other hand $\Delta_{110}$ does not correspond to a simple sheaf since the complex in (312) has cohomology in more than one place. In this case we have a sequence

which makes $\Delta_{110}$ only marginally stable at the large radius limit.
4. $\Delta_{0 n 1}$ : This is similar to the $\Delta_{n 10}$ case and again we plot the line of marginal stability in figure 14. This time $\Delta_{011}$ corresponds to the ideal sheaf of a point $\mathscr{I}_{E, p}[1]$ and is again only marginally stable at the large radius limit.

Clearly the study of D-brane stability on orbifolds is a very interesting subject and we have only just begun to scratch the surface. The results above, together with previous analysis in the literature such as $[118,157,159,175]$ provide a good start to the analysis of the subject but much remains to be done.

## 8 Conclusion

We hope that the reader is convinced that the derived category program is essential for understanding D-branes on a Calabi-Yau threefold and thus, presumably, D-branes in any nontrivial spacetime.

The essential ingredient is the extension of the naïve concept of "branes" and "antibranes" to grading branes by arbitrary integers. This inexorably leads one to discuss complexes and soon the whole machinery of the derived category becomes unavoidable. While the mathematics involved in this story might look excessive at first sight, it is hard to imagine how one would understand B-branes without using this language or essentially reinventing something identical.

Given this complexity of B-branes it is perhaps worrying to note that we still made some drastic simplifications in these lectures. The most egregious is probably the assumption that the string coupling is zero and thus the mass of any D-brane is infinite (unless it's zero!). The next step one might therefore wish for is the notion of "quantizing" the derived category story which presumably introduces many many more complications. Clearly our current knowledge of D-brane physics remains relatively poor.

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[^0]:    ${ }^{1}$ The first reference to such objects that the author is aware of is, oddly enough, section 4 of [1].

[^1]:    ${ }^{2}$ There are many conventions for writing $N=(2,2)$ theories. We are following Witten's notation in [9] where $\pm$ refers to left and right-moving - not the sign of the $\mathrm{U}(1)$ charge!
    ${ }^{3}$ Note that, since $\Sigma$ is Kähler, $K^{-1}=\bar{K}$.
    ${ }^{4}$ Wedge products will often be implicit.

[^2]:    ${ }^{5}$ Normal ordering is assumed.

[^3]:    ${ }^{6}$ This apparently backwards convention is used since it renders the notation in some of the sections on algebraic geometry more standard.

[^4]:    ${ }^{7}$ That is, $\int_{Y} e^{3}>0$.

[^5]:    ${ }^{8}$ Special geometry and monodromy considerations alone do not rule out a constant term in this power series.

[^6]:    ${ }^{10}$ A class is basically the same thing as a set but by using this language one avoids Russell's paradox.

[^7]:    ${ }^{11}$ Note that, in our notation, a $p$-brane is a brane with $p$-dimensions in the Calabi-Yau directions and any number of dimensions in the uncompactified part of spacetime.

[^8]:    ${ }^{12}$ In fancy language this makes a presheaf a contravariant functor from the category of open sets on $X$ to the category of abelian groups.

[^9]:    ${ }^{13}$ That is, nowhere zero.

[^10]:    ${ }^{14}$ This is the first Chern class of the line bundle.

[^11]:    ${ }^{15}$ In addition one requires the existence of an object $B \oplus C$ for every pair $B$ and $C$, but this isn't very important in this discussion.
    ${ }^{16}$ In these diagrams the solid lines represent maps which are given, and dotted lines represent maps to be constructed.

[^12]:    ${ }^{19}$ Unfortunately, many references, such as [78], do mean Čech cohomology when they say sheaf cohomology.

[^13]:    ${ }^{20}$ This is not quite right for torsion subgroups but we do not consider such things in these lectures.

[^14]:    ${ }^{21} \mathrm{Or}$, in French, flasque.

[^15]:    ${ }^{22}$ Note we omitted the first term from (101).

[^16]:    ${ }^{23}$ Pedants should remind themselves of the footnote on page 38.

[^17]:    ${ }^{24}$ If $\mathscr{E}$ is associated to a vector bundle $E$ then $\operatorname{Ext}^{1}(\mathscr{E}, \mathscr{E})=H^{1}(X, \operatorname{End}(E))$. See section 15.7.3 of [26] for one way of seeing why this latter group corresponds to deformations of $E$.

[^18]:    ${ }^{25}$ Beware! All sane people define the shift as being to the left but that doesn't include everyone.

[^19]:    ${ }^{26}$ If we kept track of finiteness of complexes, we would assert that it is the bounded derived category.
    ${ }^{27}$ The maximal length of this resolution is given by the dimension of $X$ [74], and thus we know we need go no further than $\mathscr{F}^{-3}$.

[^20]:    ${ }^{28}$ Note that this is not true if $c=0$.

[^21]:    ${ }^{29} N$ spacetime supersymmetries give a $\mathrm{U}(N)$ R-symmetry. One might therefore expect a $\mathrm{U}(2) / \mathrm{U}(1)$ choice of $N=1$ supersymmetries in $N=2$. However, the spectral flow picture of $N=2$ spacetime supersymmetry only sees a $U(1) \times U(1)$ subgroup of the R-symmetry so the parameter only lives in $U(1)$.

[^22]:    ${ }^{30}$ One might also try to derive this relationship directly from (10). See also section 19.3 of [8].

[^23]:    ${ }^{31}$ I thank R. Thomas for discussions on this point.
    ${ }^{32}$ This is often called the "derived Fukaya category" but it's not derived in the sense of complexes etc.

[^24]:    ${ }^{33}$ Here we are following the conventions of [126] but $\mu$ is also often defined to remove the factor of $\operatorname{Vol}(X)$ in (202).

[^25]:    ${ }^{34}$ Note that $\operatorname{dim}(S)$ is the complex dimension of $S$.
    ${ }^{35} \mathscr{O}_{X}[3]$ is the complex with $\mathscr{O}_{X}$ in position -3 and zero elsewhere.

[^26]:    ${ }^{36}$ Technically speaking $p_{1}$ is flat so we don't need to left-derive this functor.

[^27]:    ${ }^{37}$ I thank S. Katz for guiding me in this computation. He in turn thanks A. Bondal for a related conversation.

[^28]:    ${ }^{38}$ In this and subsequent computations we should really keep track of the precise forms of the morphisms. We omit this for brevity. In each case the morphisms form representations of the $\operatorname{PGL}(5, \mathbb{C})$ symmetry acting on the homogeneous coordinates of $\mathbb{P}^{4}$. These representations can be handled conveniently using Young diagrams and Schur functors as in chapter 6 of [147].
    ${ }^{39}$ Do not confuse $\Omega_{X}$ with $\Omega_{X}$, the cotangent bundle of $X$ !

[^29]:    ${ }^{40} \mathrm{~A}$ quick way of proving this is to use the "right adjoint" functor of $i_{*}$ which is written $i^{!}$and whose properties are well-understood [153] in the derived category.

[^30]:    ${ }^{41}$ The notation $f \in \operatorname{Hom}_{G}(A, B)$ means $f g=g f$ for all $g \in G$.

[^31]:    ${ }^{42}$ The only reason why this should fail for $d>3$ is that smooth resolutions need not exist.

[^32]:    ${ }^{43}$ Note that in order to define the Hermitian conjugate $Z_{\mu}^{\dagger}$ we require an inner product on the representation. This is intrinsic in the quantum field theory but is "extra data" in the abstract quiver language.

[^33]:    ${ }^{44}$ There is a false assumption in [35] which shifts $B$ by $\frac{1}{2}$. The correct argument appears in [178].

