Tôhoku Math. Journ.
27 (1975), 135-152.

δ-COMMUTING MAPPINGS AND BETTI NUMBERS

Dedicated to Professor Carl B. Allendoerfer, 1911-1974.

BILL WATSON

(Received May 2, 1972)

The Hodge-de Rham theorem [3] for oriented, compact, Riemannian manifolds says that the classical cohomology groups with real coefficients can be calculated from a knowledge of the linearly independent harmonic differential forms on the manifold. Specifically, let $\mathscr{H}^p(M)$ denote the space of harmonic p-forms on the compact, oriented Riemannian manifold M, and let $H^p(M, R)$ denote the p-th Čech cohomology group with real coefficients. Let $H_d^p(M, R)$ be the de Rham cohomology space; i.e., the quotient vector space,

$$H_d^p(M, R) = \{\mathrm{Ker}\, d: \Lambda^p \longrightarrow \Lambda^{p+1}\}/\{\mathrm{Im}\, d: \Lambda^{p-1} \longrightarrow \Lambda^p\} .$$

THEOREM (Hodge-de Rham).
(a) *The dimension of* $\mathscr{H}^p(M)$ *is finite, and,*
(b) $H^p(M, R) \cong \mathscr{H}^p(M) \cong H_d^p(M, R)$.

On our compact M, it is easy to show that a harmonic form is in the kernels of both the differential operator d and the codifferential operator δ, simultaneously. Therefore,

$$\mathscr{H}^p(M) = \{\mathrm{Ker}\, d: \Lambda^p \longrightarrow \Lambda^{p+1}\} \cap \{\mathrm{Ker}\, \delta: \Lambda^p \longrightarrow \Lambda^{p-1}\}$$

and, since we know that any manifold map $\varphi: M \longrightarrow N$ onto another compact, oriented, Riemannian manifold, N, commutes with d on the p-forms of N ($\varphi^* d_N = d_M \varphi^*$), it is natural to ask which manifold maps will commute with the codifferential. The hope is that we may find a way to transfer information about $\mathscr{H}^p(N)$ over to $\mathscr{H}^p(M)$ via φ^*, and, thereby, relate their cohomology groups.

We report here the complete classification of all C^2 manifold mappings $\varphi: M \longrightarrow N$ between compact, connected, oriented, Riemannian manifolds which satisfy

$$(1) \qquad \varphi^* \delta_N = \delta_M \varphi^*$$

on all of the p-forms of N for a *fixed* $p \geqq 1$. In the case of 1-forms, we find equation (1) to be solved by a rather general class of mappings—

smooth, Riemannian submersions with minimal fibres. For $p \geqq 2$, only a restricted class of mappings—the totally geodesic Riemannian submersions—will solve (1).

The hopes for a new relation between the cohomology groups of M and N are partially realized. Specifically, we find the following inequality on the first Betti numbers of the two manifolds:

$$b_1(N) \leqq b_1(M) . \tag{2}$$

For $p \geqq 2$, it has been known for some time that $b_p(N) \leqq b_p(M)$ for totally geodesic fibre bundle mappings. The main result, then, for $p \geqq 2$, is the total classification of the δ-commuting manifold maps.

In [4], Lichnerowicz reported several theorems on Riemannian locally trivial fibre spaces with minimal fibres. In particular, he found additional conditions which forced these mappings to commute with the codifferential, δ, on the p-forms of N for *all* degrees, *simultaneously*. Our results do not agree with those of Lichnerowicz, for $p \geqq 2$. For $p = 1$, they are independent of his results.

The Betti number inequality (2) has been announced previously [8].

1. Differential operators on tensor-valued forms. We follow the general outlines of Eells and Sampson [1] and Lichnerowicz [5]. The base space will always be a compact, connected, oriented, smooth, real Riemannian manifold of dimension m. If $\Lambda^p(M) \to M$ is the bundle of scalar p-forms of M and $E \to M$ is an arbitrary Riemannian vector bundle over M with group G and fibre dimension n, then the smooth sections of the tensor bundle $\Lambda^p(M) \otimes E \to M$ are called *vector-valued p-forms on M with values in E.*

If we replace the vector bundle E in the above construction by a mixed tensor bundle determined by E, say

$$\begin{array}{ccc} E_s^r = (\otimes_s E^*) & \otimes & (\otimes^r E) \\ \downarrow & & \downarrow \\ M & & M \end{array}$$

we call the smooth sections of the bundle

$$\begin{array}{c} \Lambda^p(M) \otimes E_s^r \\ \downarrow \\ M \end{array}$$

the *tensor-valued p-forms of type (r, s) on M with values in E_s^r.* The set of such tensor-valued p-forms is denoted $A^p_{r,s}(M, E)$. We abbreviate the notation in the case of vector-valued forms to $A^p(M, E)$.

In a similar fashion, using the bundle $T^p(M) \to M$ of covariant p-tensors over M in place of the bundle $\Lambda^p(M) \to M$, we create the smooth sections of

$$\begin{array}{c} T^p(M) \otimes E^r_s \\ \downarrow \\ M \end{array}$$

which are called the *tensor-valued covariant p-tensors of type (r, s) on M with values in E^r_s*. We denote the vector space of such p-tensors by $T^p_{r,s}(M, E)$, and, as before, when s is 0 and r is 1, we denote the space of *vector-valued covariant p-tensors with values* in E by $T^p(M, E)$. Obviously, $A^p_{r,s}(M, E)$ is a vector subspace of $T^p_{r,s}(M, E)$.

Let U be a coordinate neighborhood in a locally finite open covering of M. Locally, in U, a covariant p-tensor Ψ of type (r, s) may be expressed as a tensor field of type (r, s) with covariant p-tensors as coefficients:

$$\Psi_U = \{\psi^{a_1 \cdots a_r}_{b_1 \cdots b_s}\} = \{\psi^{a_1 \cdots a_r}_{b_1 \cdots b_s, k_1 \cdots k_p}\}$$

with $a_i, b_i = 1, \cdots, n$ and $k_j = 1, \cdots, m$.

At this point, we adopt the convention that the indices $\{a, b, c, e\}$ run from 1 to n, while the indices $\{i, j, k, l\}$ vary from 1 to m.

Let $\{g^{ij}\}$ denote the inverse of the Riemannian structure matrix on the neighborhood U, and let $\{h_{ab}\}$ be the Riemannian structure matrix on the fibres of E over U. We shall study extensively the vector space $T^p_{r,0}(M, E)$, and, therefore, we introduce a *local scalar product* there to facilitate calculations.

For $\omega, \eta \in T^p_{r,0}(M, E)$ and $x \in U$, define

$$\langle \omega, \eta \rangle_x = \frac{1}{p!} \omega^{a_1 \cdots a_r}_{i_1 \cdots i_p}(x) \eta^{b_1 \cdots b_r}_{j_1 \cdots j_p}(x) g^{i_1 j_1}(x) \cdots g^{i_p j_p}(x) h_{a_1 b_1}(x) \cdots h_{a_r b_r}(x) .$$

Since, in this report, all manifolds are compact and oriented, it is meaningful to define the *global scalar product* of the two tensor-valued covariant p-tensors of type $(r, 0)$ to be

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle .$$

By means of the connections on E and on M, we now construct a connection by which we can differentiate our tensor-valued p-tensors. Suppose on the coordinate neighborhood U on M, the vector bundle $E \to M$, has a locally defined *connection form* $\pi = \pi_U = \{\pi^a_b\}$. Then π_U is a matrix of differential 1-forms in the local coordinate neighborhood U, and π satisfies the overlap transformation condition:

$$\pi_b^a = \xi_c^a\{\xi^{-1}\}_b^e\pi_e^c + \xi_e^a d\{\xi^{-1}\}_b^e ,$$

with $\xi \in G$, the structural group of the Riemannian vector bundle E.

The *curvature* of the connection π is defined to be

$$\Omega = d\pi + \pi \wedge \pi ,$$

so that, locally in U,

$$\Omega_b^a = d(\pi_b^a) + \pi_c^a \wedge \pi_b^c ,$$

with

$$\pi_b^a = \pi_{b,k}^a dx^k .$$

Let ∇ be the usual torsion-free covariant differentiation operator for the Riemannian connection defined on the manifold M. If $\alpha \in A^p(M, E)$, we define

$$\tilde{\nabla}\alpha = \nabla\alpha + \pi \otimes \alpha \qquad \text{on } U .$$

That is,

$$(\tilde{\nabla}\alpha)^b = \nabla(\alpha^b) + \pi_c^b \otimes \alpha^c \qquad \text{on } U .$$

Clearly, $\tilde{\nabla}\alpha \in T^{p+1}(M, E)$, and $\tilde{\nabla}$ transforms correctly on the overlap of coordinate neighborhoods.

Suppose that $\alpha \in A_{r,0}^p(M, E)$ is a tensor-valued p-form of type $(r, 0)$. On the coordinate neighborhood U, we set,

$$(\tilde{\nabla}\alpha)^{b_1\cdots b_r} = \nabla(\alpha^{b_1\cdots b_r}) + \sum_{\sigma=1}^{r} \pi_c^{b_\sigma} \otimes \alpha^{b_1\cdots b_{\sigma-1}cb_{\sigma+1}\cdots b_r} .$$

Then $\tilde{\nabla}\alpha$ is a tensor-valued covariant $(p+1)$-tensor of type $(r, 0)$. $\tilde{\nabla}$ may also be extended to tensor-valued forms of type (r, s) or to tensor-valued tensors of type (r, s), but, as we shall not need it, we leave it for the reader.

Let $\alpha \in A^p(M, E)$. The *exterior derivative of the vector-valued p-form* α is locally defined in U to be

$$\begin{aligned} \tilde{d}\alpha &= \mathrm{Alt}\,(\tilde{\nabla}\alpha) \\ &= d\alpha + \pi \wedge \alpha , \end{aligned}$$

where d is the ordinary exterior differentiation of scalar p-forms on the base manifold M. Thus, locally in U,

$$(\tilde{d}\alpha)^b = d(\alpha^b) + \pi_c^b \wedge \alpha^c .$$

Clearly, $\tilde{d}: A^p(M, E) \to A^{p+1}(M, E)$.

PROPOSITION 1.1. *In general, $\tilde{d}\tilde{d} \neq 0$. In fact, for $\alpha \in A^p(M, E)$,*

$$\tilde{d}\tilde{d}\alpha = \Omega \wedge \alpha .$$

PROOF.

$$\begin{aligned}\tilde{d}\tilde{d}\alpha &= \tilde{d}(d\alpha + \pi \wedge \alpha) \\ &= -\pi \wedge d\alpha + \pi \wedge d\alpha + (d\pi + \pi \wedge \pi) \wedge \alpha \\ &= \Omega \wedge \alpha .\end{aligned}$$

Corresponding to the formal adjoint, δ, of the exterior differentiation operator d, on M, we define the *codifferential* of a vector-valued p-form. Suppose, locally in U, that the b-th component of the form $\alpha \in A^p(M, E)$ is expressed as

$$\alpha^b = (\alpha^b_{j_1 \cdots j_p}) dx^{j_1} \wedge \cdots \wedge dx^{j_p} .$$

Then,

$$(\tilde{\delta}\alpha)^b_{j_2 \cdots j_p} = -g^{jk}\tilde{\nabla}_j(\alpha^b_{kj_2 \cdots j_p}) .$$

It is clear that $\tilde{\delta}: A^p(M, E) \to A^{p-1}(M, E)$.

PROPOSITION 1.2. *For every $\alpha \in A^p(M, E)$ and $\beta \in A^{p+1}(M, E)$,*

$$(\tilde{d}\alpha, \beta) = (\alpha, \tilde{\delta}\beta) .$$

PROOF. [5].

We define the *generalized Laplacian* operator on vector-valued p-forms to be

$$\tilde{\Delta} = -(\tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d}) .$$

It is straightforward that $\tilde{\Delta}$ is linear and preserves the degree of vector-valued forms. In the same manner as with regular p-forms, a vector-valued p-form $\alpha \in A^p(M, E)$ which satisfies $\tilde{\Delta}\alpha = 0$ is said to be *harmonic*. It can be shown, in the standard manner using the global scalar product on the compact manifold, M, that $\tilde{\Delta}\alpha = 0$ if and only if both $\tilde{d}\alpha = 0$ and $\tilde{\delta}\alpha = 0$.

For tensor-valued p-forms of type $(r, 0)$, we define the differential, codifferential, and Laplacian as before. Specifically, if $\alpha \in A^p_{r,0}(M, E)$, we have, locally in U,

$$(\tilde{d}\alpha)^{b_1 \cdots b_r} = d(\alpha^{b_1 \cdots b_r}) + \sum_{\sigma=1}^{r} \pi^{b_\sigma} \wedge \alpha^{b_1 \cdots b_{\sigma-1} c b_{\sigma+1} \cdots b_r} ,$$

and

$$(\tilde{\delta}\alpha)^{b_1\cdots b_r}_{j_2\cdots j_p} = -g^{ik}\tilde{\nabla}_k(\alpha^{b_1\cdots b_r}_{ij_2\cdots j_p}) ,$$

and

$$\tilde{\Delta} = -(\tilde{d}\tilde{\delta} + \tilde{\delta}\tilde{d}) .$$

Then,

$$\tilde{d}: A^p_{r,0}(M, E) \to A^{p+1}_{r,0}(M, E) ,$$
$$\tilde{\delta}: A^p_{r,0}(M, E) \to A^{p-1}_{r,0}(M, E) ,$$

and

$$\tilde{\Delta}: A^p_{r,0}(M, E) \to A^p_{r,0}(M, E)$$

are all linear operators on tensor-valued p-forms. As before, $\tilde{\delta}$ is the formal adjoint of $\tilde{d}$ with respect to the global scalar product.

We now wish to apply this construction to the situation at hand. Let $\varphi: M \to N$ be a C^2 manifold map between two compact, oriented, smooth Riemannian manifolds of dimension m and n, respectively. The tangent bundle of N is $T(N) \to N$ and we form, in the standard manner, the pull-back bundle $\varphi^{-1}T(N) \to M$. Let U be a coordinate neighborhood of M with the corresponding local basis $\{dx^i\}$ for the smooth 1-forms there. We denote the Riemannian structure tensor of M locally by $\{g_{ij}\}$. Letting $\{dy^a\}$ be a local basis in $\varphi(U) \subseteqq N$, compatible with the $\{dx^i\}$, we may locally express the Riemannian tensor on N, in $\varphi(U)$, as

$$\overline{ds}^2 = h_{ab}dy^a \otimes dy^b .$$

In general, a superior bar will refer to tensors, functions, etc., associated to the target manifold, N. Thus, $\bar{\nabla}$ will denote the Riemannian covariant differentiation operator on tensor fields of N, and $\{\bar{\Gamma}^a_{bc}\}$ will denote the corresponding Christoffel symbols. Then, locally in U,

$$\pi^a_{b,j} = (\bar{\Gamma}^a_{bc} \circ \varphi)\left\{\frac{\partial\varphi^c}{\partial x_j}\right\}$$

and

$$\Omega^a_{b,ij} = (\bar{R}^a_{bce} \circ \varphi)\left\{\frac{\partial\varphi^c}{\partial x_i}\frac{\partial\varphi^e}{\partial x_j}\right\} . \tag{3}$$

We infer from equation (3), that $\tilde{d}\tilde{d} = 0$, for this particular connection which we have constructed, when and only when the Riemannian connection of N is flat.

The differential $\varphi_{*,x}: T_x(M) \to T_{\varphi(x)}(N)$ induces, in an obvious way, a vector-valued 1-form with values in $\varphi^{-1}(T(N))$ which we denote by φ_*. Since we shall be particularly concerned with a study of φ_*, we abbreviate the notation for $A^1(M, \varphi^{-1}T(N))$ to $A^1(M, \varphi)$ for convenience. Locally in U, we have the explicit expression for φ_* as

$$(\varphi_*)^a = \left\{\frac{\partial \varphi^a}{\partial x_i}\right\} dx^i .$$

PROPOSITION 1.3.

(a) *Locally, in U,*

(i) $(\tilde{\nabla}\varphi_*)^a_{ij} = \dfrac{\partial^2\varphi^a}{\partial x_i \partial x_j} + \Gamma^k_{ij}\left\{\dfrac{\partial \varphi^a}{\partial x_k}\right\} - \bar{\Gamma}^a_{bc}\left\{\dfrac{\partial \varphi^b}{\partial x_i}\dfrac{\partial \varphi^c}{\partial x_j}\right\}$

(ii) $(\tilde{\delta}\varphi_*)^a = -g^{ij}\left\{\dfrac{\partial^2\varphi^a}{\partial x_i \partial x_j}\right\} - \Gamma^k_{ij}g^{ij}\left\{\dfrac{\partial \varphi^a}{\partial x_k}\right\} + g^{ij}\bar{\Gamma}^a_{bc}\left\{\dfrac{\partial \varphi^b}{\partial x_i}\dfrac{\partial \varphi^c}{\partial x_j}\right\}.$

(b) $\tilde{d}\varphi_* = 0.$

PROOF. Assertion (a)(i) follows from the definitions, and (a)(ii) is immediate from

$$(\tilde{\delta}\varphi_*)^a = -g^{ij}\tilde{\nabla}_i(\varphi_*)^a_j .$$

Since $\tilde{d} = \mathrm{Alt}\,\tilde{\nabla}$, we see that

$$(d\varphi_*)^a = \frac{\partial^2\varphi^a}{\partial x_i \partial x_j} dx^i \wedge dx^j + \Gamma^k_{ij}\left\{\frac{\partial \varphi^a}{\partial x_k}\right\} dx^i \wedge dx^j - \bar{\Gamma}^a_{bc}\left\{\frac{\partial \varphi^b}{\partial x_i}\frac{\partial \varphi^c}{\partial x_j}\right\} dx^i \wedge dx^j .$$

But every term on the right is symmetric in i and j. Therefore, $\tilde{d}\varphi_* = 0$.

The *fundamental form*, $\beta(\varphi)$, of the mapping $\varphi\colon M \to N$ is the vector-valued 2-tensor $\tilde{\nabla}\varphi_*$ [7]. The justification for this name is the fact that, when φ is an isometric immersion, $\tilde{\nabla}\varphi_*$ is exactly the second fundamental form of the immersion. Based on this fact, the mapping $\varphi\colon M \to N$ is said to be *totally geodesic* if $\beta(\varphi) = 0$, and to be a *harmonic* mapping if $\tilde{\Delta}\varphi_* = 0$. It is easy to see that part (b) of Proposition 1.3 implies that φ is a harmonic mapping if and only if $\tilde{\delta}\varphi_* = 0$. Since $\tilde{\delta} = -\mathrm{Trace}\,\tilde{\nabla}$, totally geodesic must imply harmonicity.

To each mapping $\varphi\colon M \to N$ we associate a canonical tensor-valued p-form of type $(p, 0)$ given by

$$\wedge^p \varphi_* = \varphi_* \wedge \varphi_* \wedge \cdots \wedge \varphi_* ; \quad (p\text{-times}) .$$

Thus, locally, in a coordinate neighborhood U,

$$(\wedge^p \varphi_*)^{a_1 \cdots a_p}_{i_1 \cdots i_p} = \left\{\frac{\partial \varphi^{a_1}}{\partial x_{i_1}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} .$$

Since we shall study several important properties of $\wedge^p \varphi_*$, we shorten the notation of $A^p_{p,0}(M, \varphi^{-1}T(N))$ to $A^p_p(M, \varphi)$ in the remainder. The basic local expressions for the covariant differential, $\tilde{\nabla} \wedge^p \varphi_* \in T^{p+1}_p(M, \varphi)$, and for the codifferential, $\tilde{\delta} \wedge^p \varphi_* \in A_p^{p-1}(M, \varphi)$, of $\wedge^p \varphi_*$ are contained in:

PROPOSITION 1.4. *Let U be a coordinate neighborhood of M. Then,*

(a) $$\tilde{\nabla}_k((\wedge^p \varphi_*)^{a_1 \cdots a_p}_{i_1 \cdots i_p}) = \frac{\partial}{\partial x_k}\left\{\frac{\partial \varphi^{a_1}}{\partial x_{i_1}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} + \sum_{r=1}^{p} \Gamma^j_{ki_r}\left\{\frac{\partial \varphi^{a_1}}{\partial x_{i_1}} \cdots \frac{\partial \varphi^{a_r}}{\partial x_j} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} - \left\{\frac{\partial \varphi^c}{\partial x_k}\right\} \sum_{r=1}^{p} \bar{\Gamma}^{a_r}_{bc}\left\{\frac{\partial \varphi^{a_1}}{\partial x_{i_1}} \cdots \frac{\partial \varphi^b}{\partial x_{i_r}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\}.$$

(b) $$\tilde{\delta}(\wedge^p \varphi_*)^{a_1 \cdots a_p}_{i_2 \cdots i_p} = -g^{jk}\frac{\partial}{\partial x_k}\left\{\frac{\partial \varphi^{a_1}}{\partial x_j}\frac{\partial \varphi^{a_2}}{\partial x_{i_2}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} - g^{jk}\Gamma^1_{kj}\left\{\frac{\partial \varphi^{a_1}}{\partial x_1}\frac{\partial \varphi^{a_2}}{\partial x_{i_2}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} - g^{jk}\sum_{r=2}^{p} \Gamma^1_{ki_r}\left\{\frac{\partial \varphi^{a_1}}{\partial x_j}\frac{\partial \varphi^{a_2}}{\partial x_{i_2}} \cdots \frac{\partial \varphi^{a_r}}{\partial x_1} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} + g^{jk}\frac{\partial \varphi^c}{\partial x_k}\bar{\Gamma}^{a_1}_{bc}\left\{\frac{\partial \varphi^b}{\partial x_j}\frac{\partial \varphi^{a_2}}{\partial x_{i_2}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\} + g^{jk}\frac{\partial \varphi^c}{\partial x_k}\sum_{r=2}^{p} \bar{\Gamma}^{a_r}_{bc}\left\{\frac{\partial \varphi^{a_1}}{\partial x_j}\frac{\partial \varphi^{a_2}}{\partial x_{i_2}} \cdots \frac{\partial \varphi^b}{\partial x_{i_r}} \cdots \frac{\partial \varphi^{a_p}}{\partial x_{i_p}}\right\}.$$

(c) $\tilde{d}(\wedge^p \varphi_*) = 0.$

PROOF. (a) and (b) are direct calculations from the definitions. Assertion (c) follows from the same symmetry argument used in the proof of Proposition 1.3.

Proposition 1.4 (c) implies that $\wedge^p \varphi_*$ is a harmonic tensor-valued p-form if and only if $\tilde{\delta}(\wedge^p \varphi_*) = 0$. When $p = 1$, we saw that totally geodesic mappings were necessarily harmonic mappings. However, any minimal non-totally geodesic immersion is a harmonic mapping without having a zero fundamental form. The same cannot be said for the situation with $\wedge^p \varphi_*$; for, as we shall see later, $\wedge^p \varphi_*$ is harmonic as a tensor-valued p-form for $p \geqq 2$ if and only if φ is a totally geodesic mapping.

We wish now to introduce the mixed trace form of $\tilde{\nabla}\varphi_*$ with φ_* itself. Let $\varphi: M \to N$ continue to be a C^2 mapping and define $\Phi \in A^1_2(M, \varphi)$ locally in a coordinate neighborhood U, by

$$\Phi^{ab}_i = -g^{jk}(\varphi_*)^b_j(\tilde{\nabla}\varphi_*)^a_{ki}.$$

2. Riemannian submersions. In our descriptions of the various properties of Riemannian submersions, we observe the notations of O'Neill [6] and Vilms [7]. A mapping $\varphi: M \to N$ is a *Riemannian submersion* if:

(a) φ has maximal rank, and

(b) φ_*, restricted to $\{\mathrm{Ker}\,\varphi_*\}^\perp$, is a linear isometry.

The submanifolds, $\varphi^{-1}(y)$, $y \in N$, are called the *fibres* of φ. Since we have assumed M to be compact, and since it is well-known that the fibres of φ are closed, regularly imbedded submanifolds of M, they, too, are compact. Thus, φ is a compact, locally trivial, Riemannian fibre space. Those vectors which are in $\mathrm{Ker}\,\varphi_*$ are called *vertical*, while those orthogonal to the fibres are called *horizontal*. In this manner, φ induces an orthogonal decomposition of the tangent bundle of M, which we denote: $T(M) = V \oplus H$. The orthogonal projection maps are written $\mathscr{V}: T(M) \to V$ and $\mathscr{H}: T(M) \to H$. The fact that the vertical distribution is integrable is a consequence of the fact that the fibres are closed submanifolds. In general, H is not integrable.

Important examples of Riemannian submersions (without necessarily requiring M to be compact) are: $S^{2n+1} \to P_n(C)$; $S^{4n+3} \to P_n(Q)$; $M \times N \to N$; the tangent bundle of N; the orthonormal frame bundle of N; Riemannian covering maps; the Hopf mappings, $S^7 \to S^4$ and $S^{15} \to S^8$; and reductive homogeneous coset spaces, $G \to G/H$.

O'Neill [6] defined two tensors, T and A, which essentially characterize Riemannian submersions. The second fundamental form of the fibres induces a skew-symmetric tensor T on the vector fields of M via

$$T_E F = \mathscr{H}\nabla_{\mathscr{V}E}(\mathscr{V}F) + \mathscr{V}\nabla_{\mathscr{V}E}(\mathscr{H}F)\ ,$$

for $E, F \in \mathscr{D}(M)$, the Lie algebra of vector fields on M. In addition, O'Neill constructed the dual tensor A via

$$A_E F = \mathscr{V}\nabla_{\mathscr{H}E}(\mathscr{H}F) + \mathscr{H}\nabla_{\mathscr{H}E}(\mathscr{V}F)\ .$$

A, too, is a skew-symmetric tensor, and both A and T reverse the distributions, V and H.

The main interpretation of the tensor T results from its origins. That is, for V and W, vertical vector fields, the horizontal vector $T_V W$ is identical with the values of the second fundamental form of the fibre submanifolds acting on the vectors fields V and W, which are tangent to the fibres. The dual tensor, A, has an interpretation on $H \times H$ as the horizontal integrability tensor, since a routine calculation shows that when X and Y are horizontal vector fields, then

$$A_X Y = \frac{1}{2}\mathscr{V}[X, Y]\ .$$

Recall, from the previous section, that the fundamental form, $\beta(\varphi)$, of a mapping $\varphi: M \to N$ is the symmetric, vector-valued 2-tensor, $\tilde{\nabla}\varphi_*$. When $\varphi: M \to N$ is a Riemannian submersion, $\beta(\varphi)$ has a particularly straightforward interpretation.

LEMMA 2.1 [7]. *Let $\varphi: M \to N$ be a Riemannian submersion. Then, for $E, F \in \mathscr{D}(M)$,*

(a) $\beta(\varphi)|_{H \times H} = 0$.

(b) $\{(\varphi_*|_H)^{-1}\beta(\varphi)\}(\mathscr{V}E, \mathscr{V}F) = -T_{\mathscr{V}F}(\mathscr{V}E)$.

(c) $\{(\varphi_*|_H)^{-1}\beta(\varphi)\}(\mathscr{V}E, \mathscr{H}F) = -A_{\mathscr{H}F}(\mathscr{V}E)$.

PROPOSITION 2.2.

(a) $\beta(\varphi)|_{V \times V} = 0$ *if and only if the fibres of φ are totally geodesic.*

(b) $\beta(\varphi)|_{V \times H} = 0$ *if and only if the horizontal distribution is integrable.*

(c) $\operatorname{Tr} \beta(\varphi) = \tilde{\delta}\varphi_* = 0$ *if and only if the fibres of φ are minimal.*

COROLLARY. *A Riemannian submersion is a totally geodesic mapping if and only if the fibres are totally geodesic and the horizontal distribution is integrable.*

We now recall a basic theorem of Hermann which Vilms used to characterize totally geodesic Riemannian submersions. In the next three theorems, we assume that the manifold M is complete and connected.

THEOREM 2.3 (Hermann). *Let $\varphi: M \to N$ be a Riemannian submersion. If the fibres of φ are totally geodesic, then φ is a fibre bundle with connection and with structure group, the Lie group of isometries of a fibre.*

PROOF. [2].

THEOREM 2.4 (Vilms). *A totally geodesic Riemannian submersion, which is not a covering map, is a fibre bundle with flat connection.*

PROOF. [7].

THEOREM 2.5 (Vilms). *If M is simply connected and $\varphi: M \to N$ is a totally geodesic Riemannian submersion, then M is a Riemannian product manifold and φ is a product projection mapping.*

PROOF. [7].

Now that we have a characterization of totally geodesic Riemannian submersions, we seek such a global characterization for those Riemannian submersions which have minimal fibres. According to Proposition 2.2 (c), the fibres of a Riemannian submersion $\varphi: M \to N$ are minimal if and only if $\tilde{\delta}\varphi_* = 0$. But in the remarks following Proposition 1.3, we saw that $\tilde{\delta}\varphi_* = 0$ if and only if φ is a harmonic mapping. Therefore,

THEOREM 2.6. *Let $\varphi: M \to N$ be a Riemannian submersion. Then the fibres are minimal if and only if φ is a harmonic mapping.*

We remark that Theorem 2.6 was proven in [1] from local considerations.

3. δ-commuting maps. We now have the machinery to classify those maps which solve the equation

$$\phi^*\delta_N\alpha = \delta_M\phi^*\alpha$$

on all p-forms α of N. Before proving the main theorem however, we collect a few minor properties of such δ-commuting maps.

PROPOSITION 3.1.

(a) *If $\phi: M \to N$ is a constant mapping, then $\delta_M\phi^*\alpha = \phi^*\delta_N\alpha$ for all $\alpha \in \Lambda^p(N)$ and for all $p = 1, 2, \cdots, \dim N$.*

(b) *If $\phi: M \to N$, commutes with δ on p-forms for a fixed p and $\psi: M \to N_2$ is constant, then the map $\xi: M \to N_1 \times N_2$ via $\xi(x) = (\phi(x), \psi(x))$ commutes with δ on p-forms.*

(c) *If $\phi: M \to N_1$ commutes with δ on p-forms and $\psi: N_1 \to N_2$ commutes with δ on p-forms, then $\psi \circ \phi: M \to N_2$ commutes with δ.*

PROOF. (b) and (c) are immediate. For (a), simply note that both sides of the equation are zero.

We are able to give a global characterization of C^2 manifold maps which commute with the codifferential operator on p-forms in terms of tensor-valued differential forms. For this discussion, we fix the integer p, $1 \leqq p \leqq \min\{m, n\}$.

THEOREM 3.2. *Let $\phi: M \to N$ be a surjective C^2 manifold mapping, then $\phi^*\delta_N\alpha = \delta_M\phi^*\alpha$ for all $\alpha \in \Lambda^p(N)$ if and only if ϕ is a Riemannian submersion and*

$$\tilde{\delta}(\wedge^p \phi_*) = 0 .$$

PROOF. Let α be an arbitrary p-form on N and ϕ as in the statement of the theorem. Let $x \in M$. We take sufficiently small coordinate charts U about x and V about $\phi(x)$ letting $\{dy^1, \cdots, dy^n\}$ be local coordinates about $\phi(x)$ compatible with the local coordinates $\{dx^1, \cdots, dx^m\}$ about x. Locally, in V, we may express α as:

$$\alpha = \frac{1}{p!} b_{a_1 \cdots a_p} dy^{a_1} \wedge \cdots \wedge dy^{a_p}$$

and $\phi^*\alpha$ locally in U as:

$$\phi^*\alpha = \frac{1}{p!}\left\{(b_{a_1\cdots a_p}\circ\phi)\frac{\partial\phi^{a_1}}{\partial x_{i_1}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}dx^{i_1}\wedge\cdots\wedge dx^{i_p}\,.$$

We locally calculate the $p-1$ forms $\delta_M\phi^*\alpha$ and $\phi^*\delta_N\alpha$.

$$(4)\quad (\delta_M\phi^*\alpha)_{i_2\cdots i_p} = -g^{jk}\nabla_k\left\{(b_{a_1\cdots a_p}\circ\phi)\left\{\frac{\partial\phi^{a_1}}{\partial x_j}\frac{\partial\phi^{a_2}}{\partial x_{i_p}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}$$
$$= -g^{jk}\left\{\frac{\partial(b_{a_1\cdots a_p})}{\partial y_c}\circ\phi\right\}\left\{\frac{\partial\phi^c}{\partial x_k}\frac{\partial\phi^{a_1}}{\partial x_j}\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}$$
$$- g^{jk}\left\{(b_{a_1\cdots a_p}\circ\phi)\frac{\partial}{\partial x_k}\left\{\frac{\partial\phi^{a_1}}{\partial x_j}\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}$$
$$+ g^{jk}\Gamma^l_{jk}(b_{a_1\cdots a_p}\circ\phi)\left\{\frac{\partial\phi^{a_1}}{\partial x_l}\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}$$
$$+ g^{jk}(b_{a_1\cdots a_p}\circ\phi)\left\{\frac{\partial\phi^{a_1}}{\partial x_j}\right\}\left\{\sum_{\sigma=2}^{p}\left\{\Gamma^l_{i_\sigma k}\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_\sigma}}{\partial x_j}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}.$$

Now, in V,

$$(\delta_N\alpha)_{a_2\cdots a_p} = -h^{ac}\bar\nabla_c(b_{aa_2\cdots a_p})$$
$$= -h^{ac}\left\{\frac{\partial(b_{aa_2\cdots a_p})}{\partial y_c}\right\} + h^{ac}\bar\Gamma^e_{ac}(b_{ea_2\cdots a_p})$$
$$+ h^{ac}\sum_{\sigma=2}^{p}\{\bar\Gamma^e_{a_\alpha c}(b_{aa_2\cdots a_{\sigma-1}ea_{\sigma+1}\cdots a_p})\}\,.$$

Hence, in U,

$$(5)\qquad (\phi^*\delta_N\alpha)_{i_2\cdots i_p} = -(h^{ac}\circ\phi)\left\{\frac{\partial(b_{aa_2\cdots a_p})}{\partial y_c}\right\}\left\{\frac{\partial\phi^{a_2}}{\partial x_i}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}$$
$$+ (h^{ac}\circ\phi)(b_{ea_2\cdots a_p}\circ\phi)\{\bar\Gamma^e_{ac}\circ\phi\}\left\{\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}$$
$$+ (h^{ac}\circ\phi)\left\{\sum_{\sigma=2}^{p}\left\{(b_{aa_2\cdots a_{\sigma-1}ea_{\sigma+1}\cdots a_p}\circ\phi)\{\bar\Gamma^e_{a_\alpha c}\circ\phi\}\right.\right.$$
$$\left.\left.\cdots\left\{\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}\right\}.$$

Since the form α is completely arbitrary, we may compare like expressions in the equation

$$(4) = (5)\qquad \phi^*\delta_N\alpha = \delta_M\phi^*\alpha$$

which contain the term $\partial(b_{aa_2\cdots a_p})/\partial y_c$.

This action yields

$$\left\{\frac{\partial\phi^{a_2}}{\partial x_i}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}(h^{ac}\circ\phi) = \left\{\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\left(g^{jk}\frac{\partial\phi^c}{\partial x_k}\frac{\partial\phi^a}{\partial x_j}\right)$$

for all $a = 1, \cdots, n$ and all $c = 1, \cdots, n$.

Now ϕ is surjective, so some $\partial\phi^{a_\beta}/\partial x_{i_l} \neq 0$. Hence,

$$h^{ac}\circ\phi = g^{jk}\frac{\partial\phi^c}{\partial x_k}\frac{\partial\phi^a}{\partial x_j} \tag{6}$$

for all $a, c = 1, \cdots, n$.

For the surjective map $\phi: M\to N$, (6) is exactly the defining equation for a Riemannian submersion.

Comparison of like expressions which contain the term $(b_{aa_2\cdots a_p}\circ\phi)$ in equations (4) and (5) yields:

$$\begin{aligned}(h^{dc}\circ\phi)\{\bar\Gamma^a_{dc}\circ\phi\}&\left\{\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\\ &+(h^{ac}\circ\phi)\left\{\sum_{\sigma=2}^{p}\left\{\{\bar\Gamma^{a_\sigma}_{a_\sigma c}\circ\phi\}\left\{\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}\right\}\\ &=-g^{jk}\left\{\frac{\partial}{\partial x_k}\left\{\frac{\partial\phi^a}{\partial x_j}\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}\\ &+g^{jk}\Gamma^l_{jk}\left\{\frac{\partial\phi^a}{\partial x_l}\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\\ &+g^{jk}\left\{\frac{\partial\phi^a}{\partial x}\right\}\left\{\sum_{\sigma=2}^{p}\left\{(\Gamma^l_{i_\sigma k})\left\{\frac{\partial\phi^{a_2}}{\partial x_{i_2}}\cdots\frac{\partial\phi^{a_\sigma}}{\partial x_l}\cdots\frac{\partial\phi^{a_p}}{\partial x_{i_p}}\right\}\right\}\right\}.\end{aligned} \tag{7}$$

Substitution of equation (6) into (7) then gives:

$$\tilde\delta(\wedge^p\phi_*)^{aa_2\cdots a_p}_{i_2\cdots i_p} = 0$$

for all $a, a_j = 1, \cdots, n$ and all $i_k = 1, \cdots, m$.

When Theorem 3.2 is specialized to the $p = 1$ case, we find the stronger result:

THEOREM 3.3. *A C^2 manifold mapping $\varphi: M\to N$ commutes with the codifferential δ on the 1-forms of N if and only if φ is a locally trivial Riemannian fibre space with minimal fibres.*

PROOF. Theorem 3.2 implies that φ can only be a Riemannian submersion with $\tilde\delta(\wedge^1\varphi_*) = 0$. As remarked earlier, $\tilde\delta\varphi_* = 0$ is equivalent to $\tilde\Delta\varphi_* = 0$; i.e., φ is a harmonic Riemannian submersion. Now Theorem 2.6 applies.

For the $p \geqq 2$ cases, the possibilities for a manifold mapping $\varphi: M\to N$

commuting with the codifferential, δ, are severely limited. We begin our analysis with several technical lemmas.

LEMMA 3.4.
(a) $\tilde{\delta}(\wedge^2 \varphi_*) = \tilde{\delta}\varphi_* \wedge \varphi_* + \Phi$
(b) $\tilde{\delta}(\wedge^p \varphi_*) = \{\tilde{\delta}\varphi_* \wedge \varphi_* + (p-1)\Phi\} \wedge \{\wedge^{p-2}\varphi_*)$ *for* $p \geqq 3$.

PROOF. A routine calculation.

LEMMA 3.5. *Let* $\varphi: M \to N$ *be a Riemannian submersion. Then the tensor* Φ *is* 0 *if and only if* φ *is a totally geodesic mapping.*

PROOF. We calculate the squared norm of Φ in $A_2^1(M, \varphi)$.

$$\begin{aligned}
\|\Phi\|^2 &= (\Phi, \Phi)\\
&= \int_M g^{jk}g^{rs}g^{it}h_{ac}h_{be}(\tilde{\nabla}_k\varphi_*)_i^a(\tilde{\nabla}_s\varphi_*)_t^c \frac{\partial\varphi^b}{\partial x_j}\frac{\partial\varphi^e}{\partial x_r}\\
&= \int_M g^{jk}g^{rs}g^{it}g_{rj}h_{ac}(\tilde{\nabla}_k\varphi_*)_i^a(\tilde{\nabla}_s\varphi_*)_t^c\\
&= \int_M g^{ks}g^{it}h_{ac}(\tilde{\nabla}_k\varphi_*)_i^a(\tilde{\nabla}_s\varphi_*)_t^c\\
&= \|\tilde{\nabla}\varphi_*\|^2 \text{ in } A_1^2(M, \varphi)\,.
\end{aligned}$$

Thus, $\|\Phi\| = \|\tilde{\nabla}\varphi_*\|$ and the lemma follows.

LEMMA 3.6. *Let* k *be a positive integer and* $\varphi: M \to N$ *be a Riemannian submersion. Then the vector-valued* 1-*form of type* (2, 0) *given by*

$$k\Phi + \tilde{\delta}\varphi_* \wedge \varphi_*$$

is identically zero if and only if φ *is a totally geodesic mapping.*

PROOF. It suffices to show that

$$\|\tilde{\delta}\varphi_*\|^2 = (\Phi, \tilde{\delta}\varphi_* \wedge \varphi_*)\,.$$

For then,

$$\|k\Phi + \tilde{\delta}\varphi_* \wedge \varphi_*\|^2 = k^2\|\Phi\|^2 + 2k\|\tilde{\delta}\varphi_*\|^2 + \|\tilde{\delta}\varphi_* \wedge \varphi_*\|^2\,,$$

and the nullity of $\|\Phi\|$ implies that of the other two norms on the right hand side. We proceed

$$\begin{aligned}
(\Phi, \tilde{\delta}\varphi_* \wedge \varphi_*) &= \int_M g^{it}h_{ac}h_{be}\frac{\partial\varphi^b}{\partial x_j}(\tilde{\nabla}_k\varphi_*)_i^a g^{jk}g^{rs}(\tilde{\nabla}_r\varphi_*)_s^c\frac{\partial\varphi^e}{\partial x_t}\\
&= \int_M g^{it}g^{jk}g^{rs}g_{jt}h_{ac}(\tilde{\nabla}_k\varphi_*)_i^a(\tilde{\nabla}_r\varphi_*)_s^c\\
&= \int_M h_{ac}(\tilde{\delta}\varphi_*)^a(\tilde{\delta}\varphi_*)^c\\
&= \|\tilde{\delta}\varphi_*\|^2\,.
\end{aligned}$$

THEOREM 3.7. *For any $p \geqq 2$, $\varphi: M \to N$ commutes with δ on the p-forms of N if and only if φ is a totally geodesic Riemannian submersion.*

PROOF. First notice that because φ has maximal rank, $\wedge^p \varphi_*$ is never zero. The theorem then follows immediately from Theorem 3.2, and from Lemmas 3.4, 3.5 and 3.6.

COROLLARY 1. *The only C^2 mappings $\varphi: M \to N$ commuting with δ on the p-forms of N for $p \geqq 2$, are the fibre bundle maps with flat connection.*

PROOF. See Theorem 2.4.

COROLLARY 2. *Suppose that $\varphi: M \to N$ commutes with δ on the p-forms of N for $p \geqq 2$ with M, simply connected. Then M is a Riemannian product manifold and φ is a Riemannian product projection mapping.*

PROOF. See Theorem 2.5.

COROLLARY 3. *If $\varphi: M \to N$ commutes with δ on the p-forms of N for $p \geqq 2$, then it also commutes with δ on the 1-forms of N.*

PROOF. Totally geodesic implies harmonic.

Since any C^2 harmonic mapping is smooth (C^∞), by virtue of being the local solution to an elliptic equation [1], we obtain a general smoothness theorem for δ-commuting maps.

THEOREM 3.8. *If $\varphi: M \to N$ is any C^2 manifold map commuting with the codifferential δ on the p-forms of N for any fixed $p \geqq 1$, then φ is C^∞.*

4. **Examples.** As we have seen, when M is simply connected, the only C^2 manifold mappings commuting with the codifferential, δ, on the p-forms of N for $p \geqq 2$ are the product projection mappings. However, the $p = 1$ case is much richer. In fact, the following three mappings commute with δ on 1-forms, but do not commute on higher degree forms.

(a) $\varphi: S^{2n+1} \to P_n(C)$; the classical fibre bundle map over complex projective n-space.

(b) $\varphi: S^7 \to S^4$; the classical Hopf mapping.

(c) $\varphi: G \to G/H$; the canonical fibre bundle map with G, a compact Lie group; H, a closed subgroup of G; and G/H, an oriented homogeneous coset space.

We now examine other non-projection δ-commuting mappings.

THEOREM 4.1. *If $\dim M = \dim N$, then the C^2 manifold mappings which commute with the codifferential on forms of any degree are exactly the Riemannian covering mappings.*

PROOF. Riemannian covering mappings, being local isometries, are obviously totally geodesic Riemannian submersions. Corollary 3 to Theorem 3.5 then applies. Conversely, it is easy to see that the only locally trivial Riemannian fibre spaces with $\dim M = \dim N$ are the Riemannian covering maps. For a proof of this fact, see [9].

THEOREM 4.2. *If* $\dim M = \dim N + 1$, *then a* C^2 *manifold map* $\varphi: M \to N$ *commuting with the codifferential on the* 1*-forms of* N *is a smooth Riemannian fibre bundle mapping with Lie structural group,* $G = I(F_y)$, *the Lie group of isometries of a fibre.*

PROOF. In this case, the dimension of the fibre submanifolds is 1, where the concepts of minimal and totally geodesic coincide. Therefore, φ is a locally trivial Riemannian fibre space with totally geodesic fibres. The theorem of Hermann (Theorem 2.3) then gives the statement of this theorem. Smoothness follows from Theorem 3.8.

Previously, the author [9] characterized all C^3 manifold maps which commute with the Laplacian, Δ, on 0-forms (functions) using much the same methods as in this report. The Laplacian commuters were shown to be exactly the smooth harmonic Riemannian submersions; that is, those Riemannian submersions with minimal fibres. From what we have shown in Section 3, then, if $\varphi: M \to N$ commutes with δ on 1-forms, it must commute with Δ on 0-forms, and conversely. A simple calculation also shows that if $\varphi: M \to N$ commutes with δ on p-forms, for $p \geqq 2$, then φ commutes with the Laplacian Δ on p-forms for all p. It is not known what relation exists between manifold maps which commute with the Laplacian on 1-forms and the δ-commuting mappings.

It is well-known [6] that Riemannian submersions are sectional curvature increasing on horizontal tangent planes. That is, suppose that X and Y are horizontal vector fields on M determined by the Riemannian submersion $\varphi: M \to N$, and that X_* and Y_* are the corresponding φ-related vector fields on N. Then the Riemannian sectional curvatures of the two manifolds satisfy:

$$\{(\bar{K}_{X_* Y_*}) \circ \varphi\} \geqq K_{XY} .$$

In particular, we conclude:

THEOREM 4.3. *Suppose that* M *and* N *are spaces of constant sectional curvature* K *and* $\bar{K}$, *respectively. In order that a* δ*-commuting map* $\varphi: M \to N$ *exist for any form degree,* $p \geqq 1$, *it is necessary that*

$$\bar{K} \geqq K .$$

In addition, we know that totally geodesic Riemannian submersions are sectional curvature *preserving* on horizontal 2-planes. This property is therefore a necessary condition for the existence of a δ-commuting mapping on forms of degree greater than 1.

Utilizing results of [1] on the non-existence of harmonic mappings, we may also rule out certain manifold pairs (M, N) from our search for δ-commuting mappings on 1-forms.

THEOREM 4.4.

(a) *Suppose that the Ricci tensor of the manifold M is everywhere positive semi-definite and that there exists at least one point $x \in M$ such that $[R_{ij}(x)]$ is positive definite. Moreover, suppose that the Riemannian curvature of N is non-positive. Then there can not be any C^2 maps $\varphi: M \to N$ which commute with δ on 1-forms.*

(b) *Suppose M has positive semi-definite Ricci tensor and that the dimension of N is greater than 1. Then if N has everywhere negative Riemannian curvature, there can be no C^2 manifold maps commuting with δ on forms of any degree.*

5. Cohomology. In [9], we related the manifold maps $\varphi: M \to N$ which commute with the Laplacian operator on the p-forms of N with the p-th Betti numbers of the two manifolds, M and N. As we remarked after Theorem 4.2, if a C^2 manifold map commutes with δ on 1-forms, it does not necessarily commute with Δ on 1-forms. However, a similar Betti number result obtains. We remark that the following theorem is trivial when $p \geqq 2$, because of the total geodesic mapping properties, but we choose to include this case to preserve the generality of the proof.

THEOREM 5.1. *Fix the positive integer p. Suppose that there exists a C^2 mapping $\varphi: M \to N$ which commutes with the codifferential, δ, on the p-forms of N. Then,*

$$b_p(N) \leqq b_p(M) .$$

PROOF. As usual, let $\mathscr{H}^p(N)$ denote the real vector space of harmonic p-forms on N. Take $\alpha \in \mathscr{H}^p(N)$. When N is compact, it is well-known that α is harmonic if and only if both $d\alpha = 0$ and $\delta\alpha = 0$. Since the pull-back map φ^* commutes with the d operator for any map $\varphi: M \to N$ and for any form degree p, we conclude that $d_M\varphi^*\alpha = 0$. Since φ^* commutes with the codifferential δ, $\delta_M\varphi^*\alpha = 0$, and $\varphi^*\alpha$ is a harmonic p-form on M. The linearity of φ^* implies

$$\dim\{\varphi^*\mathscr{H}^p(N)\} \leqq \dim\{\mathscr{H}^p(M)\} . \tag{8}$$

Since $\varphi: M \to N$ is a Riemannian submersion, the induced mapping

$\varphi^*: \mathscr{H}^p(N) \to \mathscr{H}^p(M)$ is a linear isometry, and, therefore, has a trivial kernel. We see, then, that

$$(9) \qquad \dim\{\varphi^* \mathscr{H}^p(N)\} = \dim\{\mathscr{H}^p(N)\} .$$

Combining equations (8) and (9) with Hodge's theorem yields

$$b_p(N) = \dim\{\mathscr{H}^p(N)\} \leqq \dim\{\mathscr{H}^p(M)\} = b_p(M) .$$

COROLLARY 1. *Let $\varphi: M \to N$ be a locally trivial C^2 Riemannian fibre space mapping with both M and N compact, connected, oriented Riemannian manifolds and with the fibres of φ minimally immersed in M. Then,*

$$b_1(N) \leqq b_1(M) .$$

COROLLARY 2. *Let $\pi: P \to M$ be a compact principal fibre bundle over M, a compact, oriented manifold with compact Lie structural group, G. Then,*

$$b_1(M) \leqq b_1(P) .$$

PROOF. The fibres of π, being totally geodesic [6], are minimal.

Particular cases of Corollary 2 include the bundle of orthonormal frames over M, compact covering spaces, and homogeneous coset spaces G/H arising from a compact Lie group G.

REFERENCES

[1] J. EELLS, JR. and J. H. SAMPSON, Harmonic mappings of Riemannian manifolds, Amer. J. of Math., 86 (1964), 109-160.

[2] R. HERMANN, A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle, Proc. Amer. Math. Soc., 11 (1960), 236-242.

[3] W. V. D. HODGE, The Theory and Applications of Harmonic Integrals, University Press, Cambridge, 1952.

[4] A. LICHNEROWICZ, Quelques théorèmes de géométrie différentielle globale, Comm. Math. Helv., 22 (1949), 271-301.

[5] A. LICHNEROWICZ, Applications Harmoniques et Variétés Kähleriennes, Sympos. Math. (INDAM, Rome, 1968/69), Vol. 3, Academic Press, London, pp. 341-402.

[6] B. O'NEILL, The fundamental equations of a submersion, Michigan Math. J., 13 (1966), 459-469.

[7] J. VILMS, Totally geodesic maps, J. Differential Geometry, 4 (1970), 73-79.

[8] B. WATSON, The first Betti numbers of certain locally trivial fibre spaces, Bull. Amer. Math. Soc., 78 (1972), 392-393.

[9] B. WATSON, Manifold maps commuting with the Laplacian, J. Differential Geometry, 8 (1973), 89-98.

DEPARTAMENTO DE MATEMÁTICAS
UNIVERSIDAD DE ORIENTE
CUMANÁ, VENEZUELA