$\mathcal{D}$-homothetic transformations for a generalization of contact metric manifolds*

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Abstract

Curvature properties of some generalizations of contact metric manifolds are studied, with special attention to (κ, μ)-nullity conditions in the framework of $\mathcal{S}$-manifolds.

1 Basic definitions

An extensive research about contact geometry is done in recent years. In the present paper we are concerned with a certain generalization of contact metric manifolds in the context of f-manifolds. We recall the precise definitions. Let M be a $(2n+s)$–dimensional manifold. We say that M is equipped with an *f–structure with a parallelizable kernel*, more briefly *$f.pk$–structure*, if there are given on M an f–structure φ, s global vector fields $\xi_1, \ldots, \xi_s$ and 1-forms $\eta_1, \ldots, \eta_s$ on M satisfying the following conditions

$$\varphi(\xi_i) = 0,\ \eta_i \circ \varphi = 0,\ \varphi^2 = -\mathrm{Id} + \sum_{j=1}^{s} \eta_j \otimes \xi_j,\ \eta_i(\xi_j) = \delta^i_j \tag{1.1}$$

for all $i, j \in \{1, \ldots, s\}$; we denote by $\mathcal{D}$ the bundle $\mathrm{Im}\,(\varphi)$, and we set $\overline{\xi} := \xi_1 + \cdots + \xi_s$, $\overline{\eta} := \eta_1 + \cdots + \eta_s$. The structure (φ, ξ_i, η_j) on M is said to be *normal* if and only if $\mathcal{N}_\varphi = 0$, where $\mathcal{N}_\varphi$ is the $(2,1)$-tensor on M given by $\mathcal{N}_\varphi := [\varphi, \varphi] + 2\sum_{i=1}^{s} d\eta_i \otimes \xi_i$.

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On a manifold equipped with an *f.pk*–structure there always exists a *compatible* Riemannian metric g in the sense that for each $X, Y \in \Gamma(TM)$

$$g(X,Y) = g(\varphi(X), \varphi(Y)) + \sum_{j=1}^{s} \eta_j(X)\eta_j(Y). \tag{1.2}$$

However such a metric on M is not unique: we fix one of them; then the structure obtained is called a *metric f.pk–structure.* Let F be the Sasaki form of φ defined by $F(X,Y) := g(X, \varphi Y)$ for $X, Y \in \Gamma(TM)$. It may be observed that $\mathcal{D}$ is the orthogonal complement of the bundle $\ker(\varphi) = \langle \xi_1, \ldots, \xi_s \rangle$.

The metric *f.pk*–manifold $(M, \varphi, \xi_i, \eta_j, g)$ is said to be an *almost $\mathcal{S}$–manifold* if and only if $d\eta_1 = \cdots = d\eta_s = F$. Almost $\mathcal{S}$-manifolds which are normal are called $\mathcal{S}$-manifolds.

The study of f-manifolds was started by D.E. Blair, S.I. Goldberg, K. Yano, J. Vanzura, cf. [1, 11, 14]. Almost $\mathcal{S}$–structures were studied, without being precisely named, by J.L. Cabrerizo, L.M. Fernández and M. Fernández, cf. [4]. Then K. Duggal, A.M. Pastore and S. Ianus, cf. [10], also studied such manifolds and gave them the name "almost $\mathcal{S}$–manifolds". $\mathcal{S}$-manifolds were introduced by D.E. Blair (cf. [1]), who proved that the space of a principal toroidal bundle over a Kähler manifold is an $\mathcal{S}$-manifold. $\mathcal{S}$-structures are a natural generalization of Sasakian structures, but unlike Sasakian manifolds, no $\mathcal{S}$-structure can be realized on a simply connected, compact manifold (cf. [6]). In [8] there is an example of an even dimensional principal toroidal bundle over a Kähler manifold which does not carry any Sasakian structure. On the other hand, there is constructed an $\mathcal{S}$-structure on the even dimensional manifold $U(2)$. It is well known that $U(2)$ does not admit a Kähler structure. We conclude that there exist manifolds such that the best structure we can hope to obtain on them is an $\mathcal{S}$-structure.

On an almost $\mathcal{S}$–manifold $(M, \varphi, \xi_i, \eta_j, g)$ there are defined the (1,1)-tensor fields $h_i := (1/2)L_{\xi_i}\varphi$ for $i = 1, \ldots, s$, cf. [4, (2.5)]. We use extensively the properties of these tensor fields in the present paper. In particular these operators are self adjoint, traceless, anticommute with φ and for each $i, j \in \{1, \ldots, s\}$

$$h_i \xi_j = 0, \quad \eta_i \circ h_j = 0, \tag{1.3}$$

cf. [4]. Moreover the following identities hold, cf. [10],

$$\nabla_X \xi_i = -\varphi X - \varphi h_i X, \quad \nabla_{\xi_i}\varphi = 0, \quad \nabla_{\xi_i}\xi_j = 0 \tag{1.4}$$

where ∇ is the Levi Civita connection of g, $X \in \Gamma(TM)$ and $i, j \in \{1, \ldots, s\}$. We shall sometimes use the following curvature identity related to ∇

$$R_{\xi_i X}\xi_j - \varphi(R_{\xi_i \varphi X}\xi_j) \;=\; 2\left((h_i \circ h_j)X + \varphi^2 X\right) \tag{1.5}$$

which can be immediately obtained combining the first equation on [4, pag. 158] and (1.4).

In 1995 D. Blair, T. Koufogiogos and B.J. Papantoniou, cf. [2], studied contact metric manifolds such that the characteristic vector field belongs to the (κ, μ)–nullity distribution. We generalize this concept for almost $\mathcal{S}$–manifolds as follows.

Definition 1.1. Let M be an almost $\mathcal{S}$-manifold, κ, μ real constants. We say that M verifies the (κ,μ)-*nullity condition* if and only if for each $i \in \{1,\ldots,s\}$, $X, Y \in \Gamma(TM)$ the following identity holds

$$R_{XY}\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y)h_i X - \overline{\eta}(X)h_i Y\right). \tag{1.6}$$

Lemma 1.1. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then we have*

(i). $h_i \circ h_j = h_j \circ h_i$, *for each* $i, j \in \{1,\ldots,s\}$

(ii). $\kappa \le 1$

(iii). *if* $\kappa < 1$ *then, for each* $i \in \{1,\ldots,s\}$, h_i *has eigenvalues* $0, \pm\sqrt{1-\kappa}$.

Proof. From (1.6) it follows that for each $X \in \Gamma(TM)$, $i, j \in \{1,\ldots,s\}$ $R_{\xi_j X}\xi_i - \varphi R_{\xi_j \varphi X}\xi_i = 2\kappa\varphi^2 X$. Using (1.5) we obtain

$$(h_i \circ h_j)X = (\kappa - 1)\varphi^2 X = (h_j \circ h_i)X \tag{1.7}$$

and then (i) is verified. Next, from (1.7) we get

$$h_i^2 = (k-1)\varphi^2 \tag{1.8}$$

$$h_i^2 X = (1-\kappa)X, \quad X \in \Gamma(\mathcal{D}). \tag{1.9}$$

Then, using (1.3), (1.9) we obtain that the eigenvalues of h_i^2 are 0 and $1-\kappa$. Moreover, since h_i is symmetric, $\|h_i X\|^2 = (1-\kappa)\|X\|^2$. Hence $\kappa \le 1$. Finally, let t be a real eigenvalue of h_i and X be an eigenvector corresponding to t. Then $t^2\|X\|^2 = \|h_i X\|^2 = (1-\kappa)\|X\|^2$ and $t = \pm\sqrt{1-\kappa}$. Taking (1.3) into account we get (iii). ∎

Proposition 1.1. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then*

$$h_1 = \cdots = h_s. \tag{1.10}$$

Proof. If $\kappa = 1$ then from (1.8) and the symmetry of each h_i we have $h_1 = \cdots = h_s = 0$. Let now $\kappa < 1$. We fix $x \in M$ and $i \in \{1,\ldots,s\}$. Since h_i is symmetric then we have $\mathcal{D}_x = (\mathcal{D}_+)_x \oplus (\mathcal{D}_-)_x$, where $(\mathcal{D}_+)_x$ is the eigenspace of h_i corresponding to the eigenvalue $\lambda = \sqrt{1-\kappa}$ and $(\mathcal{D}_-)_x$ is the eigenspace of h_i corresponding to the eigenvalue $-\lambda$. If $X \in \mathcal{D}_x$ then we can write $X = X_+ + X_-$, where $X_+ \in (\mathcal{D}_+)_x$, $X_- \in (\mathcal{D}_-)_x$, so that $h_i X = \lambda(X_+ - X_-)$. We fix $j \in \{1,\ldots,s\}$, $j \ne i$. Then from (1.7) we get $h_j X = h_j(X_+ + X_-) = h_j\left(\frac{1}{\lambda}h_i X_+ - \frac{1}{\lambda}h_i X_-\right) = \frac{1}{\lambda}(h_j \circ h_i)(X_+ - X_-) = \lambda(X_+ - X_-) = h_i X$. Taking (1.3) into account we obtain (1.10). ∎

Remark 1.1. Throughout all this paper whenever (1.6) holds we put $h := h_1 = \cdots = h_s$. Then (1.6) becomes

$$R_{XY}\xi_i = \kappa\left(\overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X\right) + \mu\left(\overline{\eta}(Y)hX - \overline{\eta}(X)hY\right). \tag{1.11}$$

Furthermore, using (1.11), the symmetry properties of the curvature tensor and the symmetry of φ^2 and h, we get

$$R_{\xi_i X}Y = \kappa\left(\overline{\eta}(Y)\varphi^2 X - g(X,\varphi^2 Y)\overline{\xi}\right) + \mu\left(g(X,hY)\overline{\xi} - \overline{\eta}(Y)hX\right). \tag{1.12}$$

Remark 1.2. Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition, with $\kappa \neq 1$. We denote by $\mathcal{D}_+$ and $\mathcal{D}_-$ the n-dimensional distributions of the eigenspaces of $\lambda = \sqrt{1-\kappa}$ and $-\lambda$, respectively. We have that $\mathcal{D}_+$ and $\mathcal{D}_-$ are mutually orthogonal. Moreover, since φ anticommutes with h, we have $\varphi(\mathcal{D}_+) = \mathcal{D}_-$ and $\varphi(\mathcal{D}_-) = \mathcal{D}_+$. In other words, $\mathcal{D}_+$ is a Legendrian distribution and $\mathcal{D}_-$ is the conjugate Legendrian distribution of $\mathcal{D}_+$ (cf. [5]).

Proposition 1.2. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition. Then M is an $\mathcal{S}$-manifold if and only if $\kappa = 1$.*

Proof. We observed in the proof of Proposition 1.1 that if $\kappa = 1$ then $h = 0$. It follows that (1.6) reduces to $R_{XY}\xi_i = \overline{\eta}(X)\varphi^2 Y - \overline{\eta}(Y)\varphi^2 X$. From [4, Proposition 3.4 and Theorem 4.3] we get the claim. ∎

Remark 1.3. Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition. If there exists $i \in \{1, \ldots, s\}$ such that ξ_i is a Killing vector field then from [4, Theorem 2.6] we have $h = h_i = 0$. From (1.9) we get $\kappa = 1$ and using Proposition 1.2 we have that M is an $\mathcal{S}$-manifold.

The notion of $\mathcal{D}$-homothetic transformation for contact metric manifolds has been deeply studied (cf. for example [13]). Now we generalize this concept for a metric $f.pk$-manifold (in particular for an almost $\mathcal{S}$-manifold).

Definition 1.2. Let $(\varphi, \xi_i, \eta_j, g)$ be an $f.pk$-structure on a manifold M^{2n+s} and a be a real positive constant. By a *$\mathcal{D}$-homothetic transformation* of constant a we mean a change of the structure tensors in the following way:

$$\tilde{\varphi} = \varphi \quad \tilde{\eta}_i = a\eta_i \quad \tilde{\xi}_i = \frac{1}{a}\xi_i \quad \tilde{g} = ag + a(a-1)\sum_{j=1}^{s} \eta_j \otimes \eta_j \tag{1.13}$$

for each $i \in \{1, \ldots, s\}$.

It is straightforward to prove that if $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$, $i, j \in \{1, \ldots, s\}$, is a structure on the manifold M obtained by a $\mathcal{D}$-homothetic transformation from the $f.pk$-structure $(\varphi, \xi_i, \eta_j, g)$, then $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$ is an (almost) $\mathcal{S}$-structure if and only if $(\varphi, \xi_i, \eta_j, g)$ is an (almost) $\mathcal{S}$-structure.

Lemma 1.2. *Let M^{2n+s} be a manifold and $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$, $i, j \in \{1, \ldots, s\}$, be an almost $\mathcal{S}$-structure on M obtained from the almost $\mathcal{S}$-structure $(\varphi, \xi_i, \eta_j, g)$ by a $\mathcal{D}$-homothetic transformation. Then for each $i \in \{1, \ldots, s\}$, $X, Y \in \Gamma(TM)$ the following identities hold*

$$a\tilde{h}_i = h_i \tag{1.14}$$

$$a\tilde{\nabla}_X \tilde{\xi}_i = \nabla_X \xi_i + (1-a)\varphi X \tag{1.15}$$

$$\eta_i(\tilde{\nabla}_X Y) = X(\eta_i(Y)) - g(Y, \varphi X + \varphi \tilde{h}_i X) \tag{1.16}$$

$$a\tilde{\nabla}_X Y = a\nabla_X Y + (1-a)\Big(\sum_{l=1}^{s} g(\varphi h_l X, Y)\xi_l + a\left(\bar{\eta}(Y)\varphi X + \bar{\eta}(X)\varphi Y\right)\Big) \tag{1.17}$$

where $\tilde{\nabla}$ and ∇ denote the Levi Civita connections of $\tilde{g}$ and g, respectively, and $\tilde{h}_i = \frac{1}{2}L_{\tilde{\xi}_i}\tilde{\varphi}$.

Proof. Identity (1.14) is an immediate consequence of the definitions of h_i, $\tilde{h}_i$. By an easy direct computation, from (1.4), using (1.14), we get (1.15). Since $\eta_i(X) = g(X, \xi_i)$, for each $i \in \{1, \dots, s\}$, $X \in \Gamma(TM)$, using (1.14) we have (1.16). Next, applying the Koszul formulas, cf. [12, pag.160], for $\tilde{\nabla}$, ∇ and $d\eta_1 = \cdots = d\eta_s = F$ we get

$$2\tilde{g}(\tilde{\nabla}_X Y, Z) = 2ag(\nabla_X Y, Z) + a(a-1)\sum_{i=1}^{s}\Big[2g(X, \varphi Z)\eta_i(Y) + 2g(Y, \varphi Z)\eta_i(X) + g\Big((X(\eta_i(Y)) + Y(\eta_i(X)) + \eta_i([X,Y]))\,\xi_i, Z\Big)\Big].$$

Here we substitute the expression of $\tilde{g}$ in (1.13) and then using (1.16) and $d\eta_1 = \cdots = d\eta_s = F$ we obtain

$$\tilde{\nabla}_X Y = \nabla_X Y + (1-a)\sum_{i=1}^{s}\Big[\eta_i(Y)\varphi X + \eta_i(X)\varphi Y - \frac{1}{2}\Big(X(\eta_i(Y)) + Y(\eta_i(X)) + \eta_i([X,Y])\Big)\xi_i + \Big(X(\eta_i(Y)) + g(\varphi X, Y) + g(\varphi\tilde{h}_i X, Y)\Big)\xi_i\Big].$$

Taking (1.14) into account we get (1.17). ∎

Remark 1.4. Under the same hypotheses of Proposition 1.2, from (1.14) and [4, Theorem 2.6] it follows that ξ_i is a Killing vector field if and only if $\tilde{\xi}_i$ is a Killing vector field, $i \in \{1, \dots, s\}$.

Proposition 1.3. *Let M^{2n+s} be a manifold and $(\tilde{\varphi}, \tilde{\xi}_i, \tilde{\eta}_j, \tilde{g})$, $i, j \in \{1, \dots, s\}$, be an almost $\mathcal{S}$-structure on M obtained from the almost $\mathcal{S}$-structure $(\varphi, \xi_i, \eta_j, g)$ by a $\mathcal{D}$-homothetic transformation of constant a. Then for each $i \in \{1, \dots, s\}$, $X, Y \in \Gamma(TM)$ the following identity holds*

$$\begin{aligned} a\tilde{R}_{XY}\tilde{\xi}_i &= R_{XY}\xi_i + \frac{1-a}{a}\sum_{l=1}^{s}\Big(g(h_l Y, h_i X) - g(h_l X, h_i Y)\Big)\xi_l \\ &\quad +(1-a)\Big[\bar{\eta}(X)(h_i Y - \varphi^2 Y) - \bar{\eta}(Y)(h_i X - \varphi^2 X) \\ &\quad +(\nabla_X\varphi)Y - (\nabla_Y\varphi)X\Big] + (1-a)^2\Big(\bar{\eta}(X)\varphi^2 Y - \bar{\eta}(Y)\varphi^2 X\Big). \end{aligned} \tag{1.18}$$

Proof. Using (1.15), (1.17), (1.2), (1.4), (1.3) and the symmetry of each h_i we can straightforwardly obtain (1.18). ∎

2 Properties of the curvature

Let $(M^{2n+s}, \varphi, \xi_i, \eta_j, g)$, $i, j \in \{1, \dots, s\}$, be an almost $\mathcal{S}$-manifold. We consider the $(1,1)$-tensor fields defined by

$$l_{ij}(X) = R_{X\xi_i}\xi_j$$

for each $i, j \in \{1, \dots, s\}$, $X \in \Gamma(TM)$ and put $l_i = l_{ii}$.

Lemma 2.1. *For each $i, j, k \in \{1, \dots, s\}$ the following identities hold*

$$\varphi \circ l_{ij} \circ \varphi - l_{ij} = 2(h_j \circ h_i + \varphi^2) \quad (2.1)$$
$$\eta_k \circ l_{ij} = 0 \quad (2.2)$$
$$l_{ij}(\xi_k) = 0 \quad (2.3)$$
$$\nabla_{\xi_i} h_j = \varphi - \varphi \circ l_{ij} - \varphi \circ h_j \circ h_i + \varphi \circ (h_j - h_i) \quad (2.4)$$
$$\nabla_{\xi_i} h_i = \varphi - \varphi \circ l_i - \varphi \circ h_i^2. \quad (2.5)$$

Proof. Identity (2.1) is a rewriting of [9, (3.4)]; (2.2) and (2.3) are an immediate consequence of (2.1). Next from (1.4) and $\eta_l \circ (\nabla_{\xi_i} h_k) = 0$ we get $(\varphi - \varphi \circ l_{ij} - \varphi \circ h_j \circ h_i)(X) = -\varphi^2 \left((\nabla_{\xi_i} h_j)X\right) - (\varphi \circ h_i - \varphi \circ h_j)(X) = (\nabla_{\xi_i} h_j)(X) + \varphi\left((h_j - h_i)(X)\right)$, for each $X \in \Gamma(TM)$, from which it follows (2.4). Finally, identity (2.5) is (2.4) when $i = j$. ∎

Remark 2.1. In the case when ξ_i is Killing for each $i \in \{1, \dots, s\}$, from [4, Theorem 2.6] we get that (2.4) reduces to $\varphi \circ l_{ij} = \varphi$. Then from (2.2) we have $l_{ij} = -\varphi^2$ so that all the l_{ij}'s coincide.

Remark 2.2. Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition. Then for each $i, j \in \{1, \dots, s\}$ we have

$$l_{ij} = -\kappa\varphi^2 + \mu h. \quad (2.6)$$

It follows that all the l_{ij}'s coincide. We put $l = l_{ij}$

Lemma 2.2. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition. Then for each $X, Y \in \Gamma(TM)$, $i \in \{1, \dots, s\}$, the following identities hold*

$$\nabla_{\xi_i} h = \mu\, h \circ \varphi \quad (2.7)$$
$$l \circ \varphi - \varphi \circ l = 2\mu\, h \circ \varphi \quad (2.8)$$
$$l \circ \varphi + \varphi \circ l = 2\kappa\, \varphi \quad (2.9)$$
$$Q\xi_i = 2n\kappa\, \bar{\xi}. \quad (2.10)$$

Proof. From (2.5), using (2.6), we obtain (2.7). Identities (2.8) and (2.9) follow directly from (2.6) using $h \circ \varphi = -\varphi \circ h$. For the proof of (2.10) we fix $x \in M$ and $\{E_1, \dots, E_{2n+s}\}$ a local φ-basis around x with $E_{2n+1} = \xi_1, \dots, E_{2n+s} = \xi_s$. Then using (1.12) and trace $(h) = 0$ we get $Q\xi_i = \sum_{\alpha=1}^{2n} R_{\xi_i E_\alpha} E_\alpha = \sum_{\alpha=1}^{2n} \kappa g(\varphi^2 E_\alpha, E_\alpha)\bar{\xi} = \kappa \sum_{\alpha=1}^{2n} \delta_{\alpha\alpha}\bar{\xi} = 2n\kappa\bar{\xi}$. ∎

Remark 2.3. Let M be an almost $\mathcal{S}$-manifold. Then from [7, (2.2)] using $(\nabla_{h_i X} F)(Y, Z) = -g((\nabla_{h_i X}\varphi)Y, Z)$, for each $X, Y, Z \in \Gamma(TM)$, we get

$$\begin{aligned}(\nabla_{h_i X}\varphi)Y = {} & \frac{1}{2}\left(\varphi R_{\xi_i \varphi X} Y - R_{\xi_i \varphi X}\varphi Y - \varphi R_{\xi_i X}\varphi Y - R_{\xi_i X} Y\right) \\ & - g(\varphi^2 X - h_i X, Y)\bar{\xi} + \bar{\eta}(Y)(\varphi^2 X - h_i X).\end{aligned} \quad (2.11)$$

Lemma 2.3. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then the following identities hold*

$$\begin{aligned}(\nabla_X\varphi)Y &= g(Y,hX-\varphi^2X)\bar{\xi}-\bar{\eta}(Y)(hX-\varphi^2X) \qquad (2.12)\\ (\nabla_X h)Y-(\nabla_Y h)X &= (1-\kappa)\Big(2g(X,\varphi Y)\bar{\xi}+\bar{\eta}(X)\varphi Y-\bar{\eta}(Y)\varphi X\Big)\\ &\quad +(1-\mu)\left(\bar{\eta}(X)\varphi hY-\bar{\eta}(Y)\varphi hX\right). \qquad (2.13)\end{aligned}$$

Proof. From (2.11) we obtain $(\nabla_{hX}\varphi)Y=\kappa\left(g(X,\varphi^2Y)\bar{\xi}-\bar{\eta}(Y)\varphi^2X\right)-g(\varphi^2X-hX,Y)\bar{\xi}+\bar{\eta}(Y)(\varphi^2X-hX)$. Here we replace X with hX and by a direct computation, taking (1.4), (1.8) into account, we get (2.12). From (2.12), since h and φ^2 are self-adjoint, we have $(\nabla_X(\varphi\circ h))Y-(\nabla_Y(\varphi\circ h))X=\varphi\Big((\nabla_X h)Y-(\nabla_Y h)X\Big)$. It follows that for each $Z\in\Gamma(TM)$

$$\begin{aligned}g(R_{XY}\xi_i,Z) &= g\Big(g(X,hZ-\varphi^2Z)\bar{\xi},Y\Big)-g\Big(\bar{\eta}(X)(hZ-\varphi^2Z),Y\Big)\\ &\quad+g\Big(\varphi((\nabla_Y h)X-(\nabla_X h)Y),Z\Big), \qquad (2.14)\end{aligned}$$

where we use (2.1) of [7] and (2.12). From (2.14) and the symmetry of h and φ^2 it follows that $\varphi((\nabla_Y h)X-(\nabla_X h)Y)=R_{XY}\xi_i-\bar{\eta}(Y)(hX-\varphi^2X)+\bar{\eta}(X)(hY-\varphi^2Y)$. Then, applying φ to both the sides of the last identity, using (1.11) and $\eta_l\Big((\nabla_Y h)X-(\nabla_X h)Y\Big)=2(k-1)g(X,\varphi Y)$, $l\in\{1,\dots,s\}$, we get (2.13). ∎

Theorem 2.1. *Let $\mathcal{Z}=(M^{2n+s},\varphi,\xi_i,\eta_j,g)$ be an almost $\mathcal{S}$-manifold and $(\tilde{\varphi},\tilde{\xi}_i,\tilde{\eta}_j,\tilde{g})$ be an almost $\mathcal{S}$-structure on M obtained by a $\mathcal{D}$-homothetic transformation of constant a. If $\mathcal{Z}$ verifies the (κ,μ)-nullity condition for certain real constants (κ,μ) then $(M,\tilde{\varphi},\tilde{\xi}_i,\tilde{\eta}_j,\tilde{g})$ verifies the $(\tilde{\kappa},\tilde{\mu})$-nullity condition, where*

$$\tilde{\kappa}=\frac{\kappa+a^2-1}{a^2},\qquad \tilde{\mu}=\frac{\mu+2(a-1)}{a}.$$

Proof. From (1.14) and Proposition 1.1 it follows that $\tilde{h}_1=\cdots=\tilde{h}_s$. Then, using (1.18) and (2.12), by a direct calculation we get the claim. ∎

Remark 2.4. In [7] there are studied almost $\mathcal{S}$-manifolds such that $R_{XY}\xi_i=0$ for all $X,Y\in\Gamma(TM)$, $i\in\{1,\dots,s\}$. This is the case when (1.6) is verified for $\kappa=\mu=0$. If we consider $a>0$ and a $\mathcal{D}$-homothetic transformation of constant a on such a manifold, then from Theorem 2.1 we obtain an almost $\mathcal{S}$-manifold verifying the $(\tilde{\kappa},\tilde{\mu})$-nullity condition where $\tilde{\kappa}=\frac{a^2-1}{a^2}$ and $\tilde{\mu}=\frac{2(a-1)}{a}$. This result can be applied for the examples of flat $\mathcal{S}$-manifolds of dimension $2+s$, $s\geq 2$ given in [9] so that we easily obtain examples of $\mathcal{S}$-manifolds of dimension $2+s$ verifying the (κ,μ)-nullity condition with $(\kappa,\mu)\neq(0,0)$ and $(\kappa,\mu)\neq(1,0)$.

Lemma 2.4. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then*

$$\begin{aligned}X,Y\in\Gamma(\mathcal{D}_+) &\Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_+) \qquad (2.15)\\ X,Y\in\Gamma(\mathcal{D}_-) &\Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_-) \qquad (2.16)\\ X\in\Gamma(\mathcal{D}_+),\ \in\Gamma(\mathcal{D}_-) &\Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_-\oplus\ker(\varphi)) \qquad (2.17)\\ X\in\Gamma(\mathcal{D}_-),\ Y\in\Gamma(\mathcal{D}_+) &\Rightarrow \nabla_X Y \in \Gamma(\mathcal{D}_+\oplus\ker(\varphi)) \qquad (2.18)\end{aligned}$$

Proof. From (2.13) we get $g((\nabla_X h)\varphi Z - (\nabla_{\varphi Z} h)X, Y) = 0$, for each $X, Y, Z \in \Gamma(\mathcal{D}_+)$. On the other hand, since h is symmetric, from Remark 1.2 we have $g((\nabla_X h)\varphi Z - (\nabla_{\varphi Z} h)X, Y) = -2\lambda g(\nabla_X(\varphi Z), Y)$.
Then $g(\varphi Z, \nabla_X Y) = -g(\nabla_X(\varphi Z), Y) = 0$, that is $\nabla_X Y$ is normal to $\mathcal{D}_-$. Moreover from (1.4) and Remark 1.2 it follows that, for each $i \in \{1, \ldots, s\}$, $g(\nabla_X Y, \xi_i) = -g(Y, \nabla_X \xi_i) = 0$. Then we have (2.15). The proof of (2.16) is analogous. If $X \in \Gamma(\mathcal{D}_+)$, $Y \in \Gamma(\mathcal{D}_-)$ then from (2.15) and Remark 1.2 we get that, for each $Z \in \Gamma(\mathcal{D}_+)$, $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z) = 0$ and then we have (2.17). Analogously we prove (2.18). ∎

Remark 2.5. It follows from (2.15)-(2.16) that $\mathcal{D}_\pm$ define two orthogonal totally geodesic Legendrian foliations $\mathcal{F}_\pm$ on M.

Example 2.1. Let $\mathfrak{g}$ be a $(2n+s)$-dimensional Lie algebra and let $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi_1, \ldots, \xi_s\}$ be a basis of $\mathfrak{g}$. The Lie bracket is defined as follows:

$$
\begin{gathered}
[X_\alpha, X_\beta] = 0 \text{ for any } \alpha, \beta \in \{1, \ldots, n\}, \\
[Y_\alpha, Y_\beta] = 0 \text{ for any } \alpha \neq 2, \quad [Y_2, Y_\beta] = 2Y_\beta \text{ for any } \beta \neq 2 \\
[X_1, Y_1] = 2\overline{\xi} - 2X_2, \quad [X_1, Y_\beta] = 0 \text{ for any } \beta \geq 2, \\
[X_h, Y_k] = \delta_{hk}\left(2\overline{\xi} - 2X_2\right) \text{ for any } h, k \geq 3, \quad [X_2, Y_\beta] = 2X_\beta \text{ for any } \beta \neq 2, \\
[X_2, Y_2] = 2\overline{\xi}, \quad [X_k, Y_1] = [X_k, Y_2] = 0 \text{ for any } k \geq 3, \\
[\xi_i, \xi_j] = 0, \quad [\xi_i, X_\beta] = 0 \text{ and } [\xi_i, Y_\beta] = 2X_\beta \text{ for any } \beta \in \{1, \ldots, n\},
\end{gathered}
$$

for all $i, j \in \{1, \ldots, s\}$, where $\overline{\xi} = \xi_1 + \ldots + \xi_s$. Let G be the Lie group whose Lie algebra is $\mathfrak{g}$. On G one can define an almost $\mathcal{S}$-structure by defining $\varphi(X_\alpha) = Y_\alpha$, $\varphi(Y_\alpha) = -X_\alpha$, $\varphi(\xi_1) = \cdots = \varphi(\xi_s) = 0$, for all $\alpha \in \{1, \ldots, n\}$, considering the left invariant Riemannian metric g such that $\{X_1, \ldots, X_n, Y_1, \ldots, Y_n, \xi_1, \ldots, \xi_s\}$ is an orthonormal frame and, finally, defining each 1-form η_i as the dual 1-form of the vector field ξ_i with respect to the metric g. Taking into account the previous relations, we have, for all $h, k \in \{3, \ldots, n\}$ and $i, j \in \{1, \ldots, s\}$,

$$
\begin{gathered}
\nabla_{X_1} Y_2 = 0, \quad \nabla_{X_1} Y_k = 0, \quad \nabla_{X_1} \xi_i = -2Y_1, \\
\nabla_{X_2} Y_1 = 0, \quad \nabla_{X_2} Y_k = 0, \quad \nabla_{X_2} \xi_i = -2Y_2, \\
\nabla_{X_k} Y_1 = 0, \quad \nabla_{X_k} Y_2 = 0, \quad \nabla_{X_k} Y_h = 2\delta_{hk}\overline{\xi}, \quad \nabla_{X_k} \xi_i = -2Y_k, \\
\nabla_{Y_1} Y_1 = 2Y_2, \quad \nabla_{Y_1} \xi_i = 0, \\
\nabla_{Y_2} Y_1 = 0, \quad \nabla_{Y_2} Y_2 = 0, \quad \nabla_{Y_2} \xi_i = 0, \\
\nabla_{Y_k} Y_1 = 0, \quad \nabla_{Y_k} Y_2 = -2Y_k, \quad \nabla_{Y_k} Y_h = 2\delta_{hk} Y_2, \quad \nabla_{Y_k} \xi_i = 0,
\end{gathered}
$$

from which we get

$$R\left(X_1X_2\right)\xi_i = R\left(X_1X_k\right)\xi_i = 0,$$
$$R\left(X_1Y_1\right)\xi_i = R\left(X_1Y_2\right)\xi_i = R\left(X_1Y_k\right)\xi_i = 0,$$
$$R\left(X_2X_k\right)\xi_i = R\left(X_2Y_1\right)\xi_i = R\left(X_2Y_2\right)\xi_i = R\left(X_2Y_k\right)\xi_i = 0,$$
$$R\left(X_kX_h\right)\xi_i = R\left(X_kY_1\right)\xi_i = R\left(X_kY_2\right)\xi_i = R\left(X_kY_h\right)\xi_i = 0,$$
$$R\left(Y_1Y_2\right)\xi_i = R\left(Y_1Y_k\right)\xi_i = R\left(Y_2Y_k\right)\xi_i = R\left(Y_hY_k\right)\xi_i = 0,$$
$$R\left(X_1\xi_j\right)\xi_i = 4X_1,\;\; R\left(X_2,\xi_j\right)\xi_i = 4X_2,\;\; R\left(X_k\xi_j\right)\xi_i = 4X_k,$$
$$R\left(Y_1\xi_j\right)\xi_i = -4Y_1,\;\; R\left(Y_2\xi_j\right)\xi_i = -4Y_2,\;\; R\left(Y_k\xi_j\right)\xi_i = -4Y_k,$$
$$R\left(\xi_l\xi_j\right)\xi_i = 0.$$

Moreover, for the tensor fields $h_1,\dots,h_s$ we find that, for each $i\in\{1,\dots,s\}$, $h_i\left(X_\alpha\right) = X_\alpha$, $h_i\left(Y_\alpha\right) = -Y_\alpha$, for all $\alpha\in\{1,\dots,n\}$, and we conclude that G is an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition with $\kappa = 0$ and $\mu = 4$. In this case, $\mathcal{D}_+ = \langle X_1,\dots,X_n\rangle$, $\mathcal{D}_- = \langle Y_1,\dots,Y_n\rangle$ and $\lambda = 1$. Note that each ξ_i is a foliate vector field with respect to the Legendrian foliation $\mathcal{F}_+$ defined by $\mathcal{D}_+$, so that $\mathcal{F}_+$ is strongly flat (cf. [5]). Further note that a family of almost $\mathcal{S}$-manifolds satisfying the $(\kappa(a),\mu(a))$-nullity condition for any $a>0$ is obtained from (G,ϕ,ξ_i,η_j,g) by performing $\mathcal{D}$-homothetic transformations of constant a.

Lemma 2.5. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then for each $X,Y\in\Gamma(TM)$ we have*

$$\begin{aligned}(\nabla_X h)Y &= \Big((1-\kappa)g(X,\varphi Y) + g(X,h\varphi Y)\Big)\bar{\xi} \\ &\quad +\bar{\eta}(Y)h(\varphi X+\varphi hX) - \mu\bar{\eta}(X)\varphi hY\end{aligned} \tag{2.19}$$

Proof. We fix $x\in M$ and a local φ-basis $\{e_1,\dots,e_n,\varphi e_1,\dots,\varphi e_n,\xi_1,\dots,\xi_s\}$ around x such that $\{e_1,\dots,e_n\}$ is a local basis of $\mathcal{D}_+$. If $X\in\Gamma(\mathcal{D}_+)$, $Y\in\Gamma(\mathcal{D}_-)$ from (1.3), (2.17) and Remark 1.2 we have in x

$$(\nabla_X h)Y = -\lambda\nabla_X Y + \lambda\sum_{\alpha=1}^{n} g(\nabla_X Y,\varphi e_\alpha)\varphi e_\alpha = \lambda(1+\lambda)g(X,\varphi Y)\bar{\xi}. \tag{2.20}$$

Moreover, from (1.3) and (2.18) we have $h(\nabla_Y X) = \lambda\sum_{\alpha=1}^{n} g(\nabla_Y X, e_\alpha)e_\alpha$. Then from (1.4) we have

$$(\nabla_Y h)X = \lambda(1-\lambda)g(X,\varphi Y)\bar{\xi}. \tag{2.21}$$

Let $X,Y\in\Gamma(TM)$. We can write $X = X_+ + X_- + \sum_{i=1}^{s}\eta_i(X)\xi_i$, $Y = Y_+ + Y_- + \sum_{i=1}^{s}\eta_i(Y)\xi_i$ because of the decomposition $TM = \mathcal{D}_+\oplus\mathcal{D}_-\oplus\ker(\varphi)$. On the other hand we have $\lambda\left(g(X_+,\varphi Y_-) - g(X_-,\varphi Y_+)\right) = g(hX,\varphi Y)$ and $\lambda^2(g(X_+,\varphi Y_-) + g(X_-,\varphi Y_+)) = g(hX,h\varphi Y)$. Then, from (2.7), (2.20), (2.21), (1.3) and (1.4) we get $(\nabla_X h)Y = \bar{\eta}(X)\mu h\varphi Y + \bar{\eta}(Y)h(\varphi X+\varphi hX) + \lambda^2\left(g(X_+,\varphi Y_-) + g(X_-,\varphi Y_+)\right)\bar{\xi} + \lambda\Big(g(X_+,\varphi Y_-) - g(X_-,\varphi Y_+)\Big)\bar{\xi}$. From the symmetry of h and (1.9) it follows (2.19). ■

Remark 2.6. Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then using (2.12), (2.19) and (1.8) we get, for all $X,Y\in\Gamma(TM)$

$$\begin{aligned}(\nabla_X\varphi h)Y &= \Big[g(X,hY) - (1-\kappa)g(X,\varphi^2 Y)\Big]\bar{\xi} \\ &\quad +\bar{\eta}(Y)\Big[hX - (1-\kappa)\varphi^2 X\Big] + \mu\bar{\eta}(X)hY.\end{aligned} \tag{2.22}$$

Lemma 2.6. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then for each $X,\ Y,\ Z \in \Gamma(\mathcal{D})$ we have*

$$R_{XY}hZ - hR_{XY}Z \;=\; s\Big[\kappa\Big(g(Y,\varphi Z)\varphi hX - g(X,\varphi Z)\varphi hY + g(Z,\varphi hY)\varphi X - g(Z,\varphi hX)\varphi Y\Big) - 2\mu g(X,\varphi Y)\varphi hZ\Big]. \tag{2.23}$$

Proof. Let $X, Y, Z \in \Gamma(TM)$. Then by a direct computation we get

$$(\nabla_X\nabla_Y h)Z = (1-\kappa)\Big[\Big(g(\nabla_X Y,\varphi Z) + g(Y,(\nabla_X\varphi)Z)\Big)\bar{\xi} - \bar{\eta}(Z)\Big((\nabla_X\varphi)Y + \varphi(\nabla_X Y)\Big)\Big] + \Big[(1-\kappa)g(Y,\varphi Z) + g(Y,h\varphi Z)\Big]\nabla_X\bar{\xi} + \Big[g(\nabla_X Y,h\varphi Z) + g(Y,(\nabla_X h\varphi)Z)\Big]\bar{\xi} + g(Z,\nabla_X\bar{\xi})\Big[h\varphi Y - (1-\kappa)\varphi Y\Big] + \bar{\eta}(Z)\Big[(\nabla_X h\varphi)Y + h\varphi(\nabla_X Y)\Big] - \mu\Big[\Big(\bar{\eta}(\nabla_X Y) + g(Y,\nabla_X\bar{\xi})\Big)\varphi hZ - \bar{\eta}(Y)(\nabla_X\varphi h)Z\Big]$$

where we use (2.19), (1.8) and the antisymmetry of $\nabla_X\varphi$. Hence, using the Ricci identity $R_{XY}hZ - hR_{XY}Z = (\nabla_X\nabla_Y h)Z - (\nabla_Y\nabla_X h)Z - (\nabla_{[X,Y]}h)Z$, (2.19), the symmetry of $\nabla_X(h\circ\varphi)$ and (1.4), we obtain

$$R_{XY}hZ - hR_{XY}Z = \mu\Big[\bar{\eta}(Y)(\nabla_X\varphi h)Z - \bar{\eta}(X)(\nabla_Y\varphi h)Z - 2sg(X,\varphi Y)\varphi hZ\Big] + \Big[g(Y,h\varphi Z) + (1-\kappa)g(Y,\varphi Z)\Big]\nabla_X\bar{\xi} - \Big[g(X,h\varphi Z) + (1-\kappa)g(X,\varphi Z)\Big]\nabla_Y\bar{\xi} + g(Z,\nabla_X\bar{\xi})\Big[h\varphi Y - (1-\kappa)\varphi Y\Big] - g(Z,\nabla_Y\bar{\xi})\Big[h\varphi X - (1-\kappa)\varphi X\Big] + \Big[(1-\kappa)g\Big((\nabla_Y\varphi)X - (\nabla_X\varphi)Y, Z\Big) + g\Big((\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X, Z\Big)\Big]\bar{\xi} + \bar{\eta}(Z)\Big[(\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X - (1-\kappa)\Big((\nabla_X\varphi)Y - (\nabla_Y\varphi)X\Big)\Big]. \tag{2.24}$$

If we take $X, Y, Z \in \Gamma(\mathcal{D})$ then from (2.24), using identities (2.22), (2.12) and (1.4), we get (2.23) . ■

Lemma 2.7. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then for each $X, Y, Z \in \Gamma(TM)$ we have*

$$R_{XY}\varphi Z - \varphi R_{XY}Z = \Big[\kappa(\bar{\eta}(Y)g(Z,\varphi X) - \bar{\eta}(X)g(Z,\varphi Y)) + \mu(\bar{\eta}(Y)g(Z,\varphi hX) - \bar{\eta}(X)g(Z,\varphi hY))\Big]\bar{\xi} + s[g(Z,\varphi X + \varphi hX)(hY - \varphi^2 Y) - g(Z,\varphi Y + \varphi hY)(hX - \varphi^2 X) - g(Z,hY - \varphi^2 Y)(\varphi X + \varphi hX) + g(Z,hX - \varphi^2 X)(\varphi Y + \varphi hY)] - \bar{\eta}(Z)[\kappa(\bar{\eta}(Y)\varphi X - \bar{\eta}(X)\varphi Y) + \mu(\bar{\eta}(Y)\varphi hX - \bar{\eta}(X)\varphi hY)].$$

Proof. We proceed fixing a point $x \in M$ and local vector fields $X,\ Y,\ Z$ such that ∇X, ∇Y and ∇Z vanish at x. Applying several times (2.12) and using (1.4) and the symmetry of $\nabla\varphi^2$, we get in x

$$\nabla_X((\nabla_Y\varphi)Z) - \nabla_Y((\nabla_X\varphi)Z) = \Big[g((\nabla_X h)Y - (\nabla_Y h)X, Z) + \bar{\eta}(Y)g(Z,\varphi X + \varphi hX) - \bar{\eta}(X)g(Z,\varphi Y + \varphi hY)\Big]\bar{\xi} + s\Big[g(Z,\varphi X + \varphi hX)(hY - \varphi^2 Y) - g(Z,\varphi Y + \varphi hY)(hX - \varphi^2 X) - g(Z,hY - \varphi^2 Y)(\varphi X + \varphi hX) + g(Z,hX - \varphi^2 X)(\varphi Y + \varphi hY)\Big] - \bar{\eta}(Z)\Big[(\nabla_X h)Y - (\nabla_Y h)X + \bar{\eta}(Y)(\varphi X + \varphi hX) - \bar{\eta}(X)(\varphi Y + \varphi hY)\Big].$$

From the last identity, using $R_{XY}\varphi Z - \varphi R_{XY}Z = \nabla_X(\nabla_Y\varphi)Z - \nabla_Y(\nabla_X\varphi)Z$ and (2.13), we get the claimed identity. ∎

Remark 2.7. In particular, from Lemma 2.7 it follows that for an $\mathcal{S}$-manifold $(M, \varphi, \xi_i, \eta_j, g)$ the following formula holds, for all $X, Y, Z \in \Gamma(TM)$,

$$R_{XY}\varphi Z - \varphi R_{XY}Z = \overline{\eta}(Y)\, g(Z, \varphi X) - \overline{\eta}(X)\, g(Z, \varphi Y)$$
$$+s\left(g(Z, \varphi Y)\,\varphi^2 X - g(Z, \varphi X)\,\varphi^2 Y + g\left(Z, \varphi^2 Y\right)\varphi X - g\left(Z, \varphi^2 X\right)\varphi Y\right)$$
$$-\overline{\eta}(Z)\,(\overline{\eta}(Y)\,\varphi X - \overline{\eta}(X)\,\varphi Y).$$

Theorem 2.2. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition with $\kappa < 1$. Then for each $X_+,\ Y_+,\ Z_+ \in \Gamma(\mathcal{D}_+)$, $X_-,\ Y_-,\ Z_- \in \Gamma(\mathcal{D}_-)$, we have*

$$R_{X_-Y_-}Z_+ = s(\kappa - \mu)\Big(g(\varphi Z_+, X_-)\varphi Y_- - g(\varphi Z_+, Y_-)\varphi X_-\Big) \tag{2.25}$$
$$R_{X_+Y_+}Z_+ = s\Big(2(1+\lambda) - \mu\Big)\Big(g(Z_+, Y_+)X_+ - g(Z_+, X_+)Y_+\Big) \tag{2.26}$$
$$R_{X_+Y_+}Z_- = s(\kappa - \mu)\Big(g(\varphi Z_-, X_+)\varphi Y_+ - g(\varphi Z_-, Y_+)\varphi X_+\Big) \tag{2.27}$$
$$R_{X_+Y_-}Z_- = s\Big(\kappa g(\varphi X_+, Z_-)\varphi Y_- + \mu g(\varphi X_+, Y_-)\varphi Z_-\Big) \tag{2.28}$$
$$R_{X_+Y_-}Z_+ = s\Big(-\kappa g(\varphi Y_-, Z_+)\varphi X_+ - \mu g(\varphi Y_-, X_+)\varphi Z_+\Big) \tag{2.29}$$
$$R_{X_-Y_-}Z_- = s\Big(2(1-\lambda) - \mu\Big)\Big(g(Y_-, Z_-)X_- - g(X_-, Z_-)Y_-\Big) \tag{2.30}$$

Proof. Let $\{e_1, \ldots, e_n, \varphi e_1, \ldots, \varphi e_n, \xi_1, \ldots, \xi_s\}$ be a local φ-basis such that $\{e_1, \ldots, e_n\}$ is a basis of $\mathcal{D}_+$. From Lemma 2.6 we get

$$\lambda R_{Z_+X_-}e_\alpha - hR_{Z_+X_-}e_\alpha = 2s\lambda\Big[\kappa\Big(g(X_-, \varphi e_\alpha)\varphi Z_+ + g(Z_+, \varphi e_\alpha)\varphi X_-\Big) - \mu g(Z_+, \varphi X_-)\varphi e_\alpha\Big]. \tag{2.31}$$

Taking the symmetry of h into account we have $g(\lambda R_{Z_+X_-}e_\alpha - hR_{Z_+X_-}e_\alpha, Y_-) = 2\lambda g(R_{Z_+X_-}e_\alpha, Y_-)$ and then, from (2.31) and Remark 1.2, $g(R_{Z_+X_-}e_\alpha, Y_-) = s\Big(\kappa g(X_-, \varphi e_\alpha)g(\varphi Z_+, Y_-) - \mu g(Z_+, \varphi X_-)g(\varphi e_\alpha, Y_-)\Big)$. It follows that

$$g(R_{X_-Y_-}Z_+, e_\alpha) = s(\kappa - \mu)\Big(g(X_-, \varphi e_\alpha)g(\varphi Z_+, Y_-) - g(Y_-, \varphi e_\alpha)g(\varphi Z_+, X_-)\Big) \tag{2.32}$$

where we use $g(R_{X_-Y_-}Z_+, e_\alpha) = -g(R_{Z_+Y_-}e_\alpha, X_-) + g(R_{Z_+X_-}e_\alpha, Y_-)$. From (2.16) and Remark 1.2 we have $g(R_{X_-Y_-}Z_+, \varphi e_\alpha) = 0$; moreover (1.11) yelds $g(R_{X_-Y_-}Z_+, \xi_i) = 0$. Then $R_{X_-Y_-}Z_+ = \sum_{\beta=1}^{n} g(R_{X_-Y_-}Z_+, e_\beta)e_\beta$. Using (2.32) we get (2.25). Identity (2.26) follows from (2.15), (1.11) Lemma 2.7, (1.2), (2.25) and $(1+\lambda)^2 + \kappa = 2(1+\lambda)$. The other identities follow in a similar way and then are omitted. ∎

Theorem 2.3. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ, μ)-nullity condition with $\kappa < 1$. Then the sectional curvature K of M is determined by*

$$K(X, \xi_i) = \kappa + \mu g(hX, X) = \begin{cases} k + \lambda\mu & \text{if } X \in \mathcal{D}_+ \\ k - \lambda\mu & \text{if } X \in \mathcal{D}_- \end{cases} \tag{2.33}$$

$$K(X, Y) = \begin{cases} s\Big(2(1+\lambda) - \mu\Big) & \text{if } X, Y \in \mathcal{D}_+ \\ s\Big(2(1-\lambda) - \mu\Big) & \text{if } X, Y \in \mathcal{D}_- \\ -s(\kappa + \mu)\Big(g(X, \varphi Y)\Big)^2 & \text{if } X \in \mathcal{D}_+, Y \in \mathcal{D}_- \end{cases} \tag{2.34}$$

where X, Y are orthonormal and in the first two cases of (2.34) *n has to be strictly greater then 1.*

Proof. Identities (2.33) follow directly from (1.11), while identities (2.34) are a consequence of (2.26), (2.30) and (2.28) respectively. ∎

Corollary 2.1. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition with $\kappa < 1$. Then the Ricci operator verifies the following identities*

$$Q = s\Big[\Big(2(1-n)+\mu n\Big)\varphi^2 + \Big(2(n-1)+\mu\Big)h\Big] + 2n\kappa\ \bar{\eta}\otimes\bar{\xi}, \tag{2.35}$$

$$Q\circ\varphi - \varphi\circ Q = 2s\Big(2(n-1)+\mu\Big)h\circ\varphi. \tag{2.36}$$

Proof. Let $\{e_1,\ldots,e_n,\varphi e_1,\ldots,\varphi e_n,\xi_1,\ldots,\xi_s\}$ be a local φ-basis such that $\{e_1,\ldots,e_s\}$ is a basis of $\mathcal{D}_+$ and let $X = X_+ + X_- \in \mathcal{D}_+\oplus\mathcal{D}_-$. From (2.26), (2.28) and (1.11) we get

$$QX_+ = s\Big[\Big(2(1+\lambda)-\mu\Big)(n-1)X_+ - (\kappa+\mu)X_+ + \kappa X_+ + \mu h X_+\Big]. \tag{2.37}$$

On the other hand from (2.29) and (2.30) we obtain

$$QX_- = s\Big[-(\kappa+\mu)X_- + \Big(2(1-\lambda)-\mu\Big)(n-1)X_- + \kappa X_- + \mu h X_-)\Big]. \tag{2.38}$$

Taking (2.37), (2.38) and (2.10) into account we obtain (2.35). Finally, identity (2.36) easily follows from (2.35). ∎

Corollary 2.2. *Let M be an almost $\mathcal{S}$-manifold verifying the (κ,μ)-nullity condition. Then the scalar curvature S of (M,g) is constant and verifies the identity*

$$S = 2ns\Big(2(n-1)-\mu n+\kappa\Big). \tag{2.39}$$

Proof. Let $\{e_1,\ldots,e_n,\varphi e_1,\ldots,\varphi e_n,\xi_1,\ldots,\xi_s\}$ be a local φ-basis such that $\{e_1,\ldots,e_s\}$ is a basis of $\mathcal{D}_+$. Then from (2.26), (2.28) and (1.11) we have

$$\begin{aligned} g(Qe_\beta,e_\beta) &= s\Big(2(1+\lambda)-\mu\Big)(n-1) + s(-\kappa-\mu) + s(\kappa+\lambda\mu)\\ &= s\Big[(n-1)\Big(2(1+\lambda)-\mu\Big) + (\lambda-1)\mu\Big]. \end{aligned} \tag{2.40}$$

Furthermore from (2.29), (2.30) and (1.11) we get

$$g(Q\varphi e_\beta,\varphi e_\beta) = s\Big[(n-1)\Big(2(1-\lambda)-\mu\Big) - (1+\lambda)\mu\Big]; \tag{2.41}$$

then (2.40), (2.41) and (2.10) yeld (2.39). ∎

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