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Results in Mathematics

D-homothetically Deformed K-contact Ricci Almost Solitons

Nagaraja H. Gangadharappa and Ramesh Sharma

Abstract. If a K-contact manifold (M, g) and a D-homothetically deformed K-contact manifold $(M, \bar{g})$ are both Ricci almost solitons with the same associated vector field V, then we show (i) that (M, g) and $(M, \bar{g})$ are both D-homothetically fixed η-Einstein Ricci solitons, and (ii) V preserves ϕ. We also show that, if the associated vector field V of a complete K-contact Ricci almost soliton (M, g, V) is a projective vector field, then V is Killing and (M, g) is compact Sasakian and shrinking. Finally, we show that the divergence of any vector field is invariant under a D-homothetic deformation.

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1. Introduction

Modifying the Ricci soliton equation by allowing the dilation constant λ to become a variable function, Pigola et al. [8] defined a Ricci almost soliton as a Riemannian manifold (M, g) satisfying the condition:

$$L_V g + 2Ric = 2\lambda g, \tag{1.1}$$

where V is a vector field on M, g and Ric denote the metric tensor and its Ricci tensor respectively, L_V denotes the Lie-derivative operator along V, and λ is a smooth function on M. A simple example is the canonical metric g on a Euclidean sphere with V a non-homothetic conformal vector field. For λ constant, (1.1) becomes the Ricci soliton [4]. The Ricci almost soliton is said to be shrinking, steady, and expanding according as λ is positive, zero, and negative respectively; otherwise is indefinite. If the vector field V is the gradient of a smooth function f, up to the addition of a Killing vector field, (M, g, V, λ) is called a gradient Ricci almost soliton, in which case the Eq. (1.1) assumes the form:

$$\nabla\nabla f + Ric = \lambda g. \tag{1.2}$$

A compact Ricci soliton is necessarily gradient [7]. For a Ricci almost soliton with V Killing, g is Einstein and hence λ becomes constant and it becomes the trivial Ricci soliton. We also note for a Ricci almost soliton that V is conformal if and only if g is Einstein.

In [11] Sharma studied a gradient Ricci soliton as a complete K-contact manifold and showed that it is isometric to a compact Einstein Sasakian manifold. Later, Sharma and Ghosh [12] and Ghosh and Sharma [5] studied Sasakian metrics as Ricci solitons and showed that they are either Einstein or η-Einstein D-homothetically fixed. In [13], Sharma showed that a complete Ricci almost soliton whose metric is a K-contact metric and V is an infinitesimal contact transformation, reduces to a Ricci soliton with constant scalar curvature. In this paper we study a K-contact metric g whose D-homothetic deformation to another K-contact metric $\bar{g}$ is a Ricci almost soliton. First, we study the condition that (M, g, V, λ) and $(M, \bar{g}, V, \mu)$ are both Ricci almost solitons and obtain the following rigidity result.

Theorem 1.1. *Let (M, g) be a K-contact manifold and $(M, \bar{g})$ be obtained by a non-identity D-homothetic deformation of (M, g). If (M, g, V, λ) and $(M, \bar{g}, V, \mu)$ are both Ricci almost solitons, then*

(1) *$\mu = \lambda$, and both Ricci almost solitons reduce to D-homothetically fixed η-Einstein expanding Ricci solitons,*
(2) *V preserves ϕ and transforms g, η and ξ according to the equations:* $\bar{g} = ag + a(a-1)\eta \otimes \eta, \quad L_V\xi = 4(n+1)\xi, \quad L_V\eta = (\lambda - 2n)\eta$.

Next, we study the case when $(M, \bar{g}, V, \lambda)$ is a Ricci almost soliton such that V is a projective vector field, and prove the following result.

Theorem 1.2. *Let (M, g) be a complete K-contact manifold such that (M, g, V, λ) is a Ricci almost soliton and V is projective. Then V is Killing and (M, g) is compact Einstein Sasakian.*

It is well known that a D-homothetic deformation preserves many basic properties like being K-contact (in particular, Sasakian). We obtain another invariant under a D-homothetic deformation by proving the following result.

Proposition 1.1. *The divergence of any smooth vector field is invariant under a D-homothetic deformation of a contact metric structure.*

2. Review of K-contact Manifolds and a D-homothetic Deformation

A (2n+1)-dimensional smooth manifold M is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. For a given contact 1-form η, there is a unique vector field ξ called the Reeb vector field such that $d\eta(\xi, .) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric g and a $(1, 1)$-tensor field ϕ such that

$$d\eta(X, Y) = g(X, \phi Y), \quad \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

where X, Y denote arbitrary vector fields on M. Henceforth, X, Y, Z will denote arbitrary vector fields on M. g is called an associated metric of η and (ϕ, η, ξ, g) a contact metric structure. If ξ is Killing, then M is said to be a K-contact manifold. The contact structure on M is said to be normal if the almost complex structure on $M \otimes R$ defined by $J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$, where f is a real function on $M \otimes R$, is integrable. A normal contact metric manifold is called a Sasakian manifold. Sasakian manifolds are K-contact and 3-dimensional K-contact manifolds are Sasakian. For a K-contact manifold,

$$\nabla_X \xi = -\phi X \tag{2.2}$$

$$R(X, \xi)\xi = X - \eta(X)\xi \tag{2.3}$$

$$Q\xi = 2n\xi, \tag{2.4}$$

where ∇, R and Q denote the Levi–Civita connection, curvature tensor and Ricci operator of g respectively. For details we refer to Blair [1].

A contact metric manifold M is said to be η-Einstein in the wider sense, if

$$Ric = \alpha g + \beta \eta \otimes \eta, \tag{2.5}$$

for some smooth functions α and β on M. It is well-known [18] that α and β are constants if M is K-contact, and has dimension greater than 3. Given a contact metric structure (η, ξ, g, φ), let $\bar{\eta} = a\eta, \bar{\xi} = \frac{1}{a}\xi, \bar{\varphi} = \varphi, \bar{g} = ag + a(a-1)\eta \otimes \eta$ for a positive constant a. Then $(\bar{\eta}, \bar{\xi}, \bar{\varphi}, \bar{g})$ is again a contact metric structure. Such a change of structure is called a D-homothetic deformation, and preserves many basic properties like being K-contact (in particular, Sasakian). It is straightforward to verify that, under a D-homothetic deformation, a K-contact η-Einstein manifold transforms to a K-contact η-Einstein manifold such that $\bar{\alpha} = \frac{\alpha+2-2a}{a}$ and $\bar{\beta} = 2n - \bar{\alpha}$. We remark here that the particular value: $\alpha = -2$ remains fixed under a D-homothetic deformation, and as $\alpha + \beta = 2n$, β also remains fixed. This observation motivates us to state the following definition [5].

Definition 2.1. A K-contact η-Einstein manifold with $\alpha = -2$ is said to be D-homothetically fixed.

We also point out that a vector field V on a contact metric manifold is said to be an infinitesimal contact transformation [14] if $L_V\eta = \sigma\eta$ for a smooth function σ, and is strictly so when $\sigma = 0$.

3. Proofs of the Results

Proof of Theorem 1.1. Let (M, g) be a K-contact manifold and $(M, \bar{g})$ be obtained by a D-homothetic deformation of (M, g). Suppose that (M, g, V, μ) and $(M, \bar{g}, V, \lambda)$ are Ricci almost solitons, i.e.

$$L_V g + 2Ric = 2\mu g, \tag{3.1}$$

$$L_V \bar{g} + 2\bar{Ric} = 2\lambda\bar{g}. \tag{3.2}$$

Taking the Lie derivative of $\bar{g} = ag + a(a-1)\eta\otimes\eta$ along V, using Eq. (3.2) and the following formula for the change of Ricci tensor under a D-homothetic deformation [2,15]:

$$\bar{Ric} = Ric - \frac{(2a-1)(a-1)}{a}g + \{2n(a^2-1) + \frac{(2a-1)(a-1)}{a}\}\eta\otimes\eta - \frac{(a-1)}{a}\{g - \eta\otimes\eta\},$$

we obtain

$$\begin{aligned} &2(1-a)Ric(X,Y) + a(a-1)\{(L_V\eta)(X)\eta(Y) + \eta(X)(L_V\eta)(Y)\} \\ &\quad = [2a(\lambda-\mu) + 4(a-1)]g(X,Y) \\ &\qquad +2(a-1)(\lambda a - 2na - 2n - 2))\eta(X)\eta(Y). \end{aligned} \tag{3.3}$$

Now substituting ξ for Y in (3.3) and using (2.4), we have that

$$a(a-1)(L_V\eta)(X) = [2a(\lambda-\mu) + (a-1)(2\lambda a - 4na - a(L_V\eta)\xi)]\eta(X). \tag{3.4}$$

Further, substituting ξ for X in the above equation, gives

$$(a-1)(L_V\eta)\xi = (\lambda-\mu) + (a-1)(\lambda-2n). \tag{3.5}$$

The above two equations lead to

$$(a-1)(L_V\eta)X = [\lambda - \mu + (a-1)(\lambda-2n)]\eta(X). \tag{3.6}$$

At this point, Lie-differentiating $g(\xi,\xi) = 1$ along V and using (1.1) we get

$$2Ric(\xi,\xi) = 2\mu g(\xi,\xi) + 2g(L_V\xi,\xi).$$

The use of (2.4) in the preceding equation provides $\eta(L_V\xi) = -\mu + 2n$. Lie-differentiating $\eta(\xi) = 1$ along V we have $(L_V\eta)(\xi) + \eta(L_V\xi) = 0$. Hence, $(L_V\eta)(\xi) = \mu - 2n$. This, in conjunction with (3.5) shows

$$\lambda = \mu, \tag{3.7}$$

because $a > 0$. Consequently, as $a \neq 1$ by hypothesis, (3.6) assumes the form

$$L_V\eta = (\lambda - 2n)\eta. \tag{3.8}$$

The use of this equation in (3.3) readily shows that

$$Ric = -2g + 2(n+1)\eta \otimes \eta. \tag{3.9}$$

i.e. (M, g) is η-Einstein K-contact and is D-homothetically fixed by Definition 2.1. Hence it is also true for $\bar{g}$. Now, tracing (3.9) gives the scalar curvature

$$r = -2n. \tag{3.10}$$

From (3.8), it also follows from a result of Sharma "If the associated vector field of a K-contact Ricci almost soliton is an infinitesimal contact transformation, then the Ricci almost soliton becomes Ricci soliton and hence λ is constant" that $\lambda = \mu$ is constant, and hence both Ricci almost solitons reduce to Ricci solitons. Let us now recall the following integrability formula [11] for a Ricci soliton:

$$L_V r = -\Delta r + 2R_{ij}R^{ij} - 2\lambda r,$$

where $\Delta r = div(\nabla r)$ and ∇ denotes the gradient of a function. As r is a constant from (3.10), the above formula yields

$$R_{ij}R^{ij} = \lambda r = -2n\lambda. \tag{3.11}$$

On the other hand, computing from (3.9), we get

$$R_{ij}R^{ij} = 4n(n+2),$$

which shows, in view of (3.11), that

$$\lambda = -2(n+2), \tag{3.12}$$

i.e. the soliton is expanding . The soliton Eq. (1.1) along with Eqs. (3.9) and (3.12) implies that

$$L_V g = -4(n+1)(g + \eta \otimes \eta). \tag{3.13}$$

Also, the Eq. (3.8) becomes

$$L_V \eta = -4(n+1)\eta. \tag{3.14}$$

Lie-differentiating the first equation of (2.1) along V, and noting the fact that L_V commutes with the exterior derivative operator d, we get

$$(d(L_V\eta))(X, Y) = (L_V g)(X, \phi Y) + g(X, (L_V\phi)Y).$$

Using (3.13) and (3.14) in the above equation, we find

$$L_V \phi = 0. \tag{3.15}$$

Finally, Lie-differentiating the property $\phi\xi = 0$ along V, and using (3.15) we have

$$\phi(L_V\xi) = 0,$$

which, in turn, implies that

$$L_V\xi = \eta(L_V\xi)\xi.$$

As $\eta(L_V\xi) = 2n - \lambda = 2n + 2(n+2) = 4n + 4$, we obtain

$$L_V\xi = 4(n+1)\xi.$$

This completes the proof.

Proof of Theorem 1.2. As V is a projective vector field, we have [16,17]

$$(L_V\nabla)(X,Y) = p(X)Y + p(Y)X \tag{3.16}$$

where p is an exact 1-form. For $p=0$, V is an affine vector field.
Using the Ricci almost soliton Eq. (1.1) in the commutation formula [17]

$$(L_V\nabla_X g - \nabla_X L_V g - \nabla_{[V,X]} g)(Y,Z) = -g((L_V\nabla)(X,Y),Z) - g((L_V\nabla)(X,Z),Y),$$

and also using (3.16) we derive

$$-2(\nabla_X Ric)(Y,Z) + 2(X\lambda)g(Y,Z) = 2p(X)g(Y,Z) + p(Y)g(X,Z) + p(Z)g(X,Y) \tag{3.17}$$

Substituting ξ for Y and Z in the above equation gives

$$X\lambda = p(X) + p(\xi)\eta(X)$$

and subsequently, substituting ξ for X we obtain

$$\xi\lambda = 2p(\xi).$$

The above two equations yield the relation

$$d\lambda - (\xi\lambda)\eta = p. \tag{3.18}$$

Taking its exterior derivative and then exterior product with η we get $(\xi\lambda)\eta\wedge d\eta = 0$. Noting that $\eta\wedge d\eta$ is nowhere zero by the definition of contact structure, we conclude that $\xi\lambda = 0$ and therefore, $p = d\lambda$. Consequently, Eq. (3.17) reduces to

$$-2(\nabla_X Q)Y = (Y\lambda)X + g(X,Y)\nabla\lambda.$$

Substituting ξ for X and Y in the above equation and using (2.4) we find that $\nabla\lambda = 0$. Hence λ is constant, and so Eq. (3.18) implies that $p=0$, and Eq. (3.17) reduces to $\nabla Ric = 0$. We know from [9] that a second order symmetric parallel tensor on a K-contact manifold is a constant multiple of the metric tensor. So, $Ric = 2ng$, in view of the Eq. (2.4). As $p=0$, V is affine, and so $L_V g$ is parallel and therefore homothetic. But a homothetic vector field on a K-contact manifold is Killing [10]. Hence V is Killing. Equation (1.1) immediately shows that $\lambda = 2n$. Since the K-contact manifold is complete and Einstein (with Einstein constant $2n$), it is compact by Myers' theorem. Applying the following result of Boyer and Galicki [3]: "A compact Einstein K-contact manifold is Sasakian", we conclude that g is Sasakian, completing the proof. □

Proof of Proposition 1.1. We know that

$$\Omega = \eta\wedge(d\eta)^n \tag{3.19}$$

defines the volume form of a contact metric manifold. Under a D-homothetic deformation it changes to

$$\bar{\Omega} = \bar{\eta}\wedge(d\bar{\eta})^n = a^{n+1}\Omega. \tag{3.20}$$

Let us recall the formula $L_V\Omega = (divV)\Omega$, where the divergence div of an arbitrary smooth vector field V is taken with respect to a Riemannian metric associated with the contact structure η on M. Now $L_V\bar{\Omega} = (d\bar{i}vV)\bar{\Omega}$, in conjunction with (3.19) and (3.20) shows that $d\bar{i}vV = divV$, completing the proof. □

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Nagaraja H. Gangadharappa
Department of Mathematics
Bangalore University
Jnana Bharathi Campus
Bengaluru 560 056
India
e-mail: hgnraj@yahoo.com

Ramesh Sharma
Department of Mathematics
University of New Haven
West Haven CT06516
USA
e-mail: rsharma@newhaven.edu

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