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Damage mechanics with long-range interactions: correlation between the seismic *b*-value and the fractal two-point correlation dimension

Ian G. Main

Department of Geology & Geophysics, University of Edinburgh, Grant Institute, West Mains Road, Edinburgh EH9 3JW, *UK*

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SUMMARY

A modified Griffith criterion for a two-dimensional array of aligned elliptical cracks with a long-range interaction potential is presented. In accordance with observation, the pattern of cracks is assumed to be fractal, with a two-point correlation dimension D_C indicating a power-law distribution of crack spacings r, and a power-law exponent D of the crack length distribution. From a simple dislocation theory of the seismic source D is proportional to the seismic b-value if an individual earthquake or acoustic emission is produced by displacement on a specific fault or crack in the population. As a result, the theory is applicable to incremental damage rather than the long-term evolution of crack systems with large displacements. The long-range interaction between cracks is taken to be elastic, implying a positive interaction potential proportional to r^{-1}. Two models are presented for the spatio-temporal evolution of the resulting seismicity due to: (A) progressive alignment of epicentres along an incipient fault plane; and (B) clustering of epicentres around potential nucleation points on an existing fault trace.

The modified Griffith criterion predicts either an increase or a decrease in the potential energy release rate G', depending on the sign of $\partial D_C/\partial D$ and the nature of the concentration of deformation. For model (A), if $\partial D_C/\partial D > 0$ (corresponding to an implied positive correlation between the b-value and D_C), then G' increases in the presence of an interaction potential. In contrast G' increases if $\partial D_C/\partial D < 0$ for model (B). Both results lead to a mechanical weakening effect associated with more concentrated deformation. Such an association of mechanical weakening with concentration of deformation is fundamental to the development of fault systems.

On the other hand if $\partial D_C/\partial D < 0$ (corresponding to an implied negative correlation between the b-value and D_C), G' decreases for model (A) in the presence of an interaction potential, implying a hardening of the material due to the interaction. For model (B) G' decreases when $\partial D_C/\partial D > 0$. The mechanical hardening (lowering G') is associated with geometrically distributed damage in either case. Equivalently this can be seen as a shielding effect, with the zone of damage reducing the local stresses on a particular crack. If there is no correlation the interaction potential has a slight mechanical hardening effect with no strong geometric effect.

These predictions are also consistent with the usual tenets of damage mechanics, in which early crack growth is stable, distributed and is associated with mechanical hardening, and material failure occurs later in the cycle due to localized, unstable crack coalescence, associated with mechanical weakening. The main difference between the theory presented here and standard damage mechanics is that crack coalescence is *organized,* and hence instability can develop at lower crack densities.

Key words: *b*-values, correlation dimension, damage mechanics, fractals, Griffith crack theory.

INTRODUCTION

Natural fracture systems and earthquake populations are both consistent with the idea of a scale-invariant geometry (e.g. Aki 1981; King 1983; Hirata 1989a; Turcotte 1989). Scale invariance in geological systems is an old idea recently given new prominence mainly because of the introduction of a quantitative theory of the geometry of natural systems (Mandelbrot 1982). This *fractal* geometry of fracture systems implies that, although stress is relieved predominantly on the major faults, materials like rocks and the Earth's crust are in fact heterogeneous on all scales. As a consequence, simple fracture theories based on a single expanding crack in an otherwise homogeneous elastic medium are at best only an approximation to the most simple dynamic ruptures, and cannot explain the evolution of damage of a fault or fracture system during one or more earthquake cycles.

Fractal geometry of faults and fractures

It is an observational fact that seismogenic fault populations (e.g. Hirata 1989b) and crack populations producing Acoustic Emissions (AE)—e.g. Main *et al.* (1990a)—are fractal on a wide variety of scales. However, fractal dimensions obtained by different methods should not be compared too literally with each other because they reflect different aspects of the scale invariance. For example, the 'capacity' dimension D_0, estimated by box-counting methods (Feder 1987), measures the space-filling properties of a fracture set with respect to changes in grid-scale (e.g. Hirata 1989a), and the power-law exponent D of the fault length distribution, inferred from the seismic b-value, measures the relative proportion of large and small seismogenic faults (Main & Burton 1984; Turcotte 1989) or cracks producing AE (Main *et al.* 1990a). The correlation dimension D_C is another type of fractal dimension (Grassberger 1983), which measures the spacing or clustering properties of a set of points, and has been applied both to earthquake epicentres (Kagan & Knopoff 1980; Hirata 1989b) and AE hypocentre distributions (Hirata, Satoh & Ito 1987). The seismic b-value and correlation dimension quantify aspects of fractal scaling in an increment of seismic damage on a time-scale much smaller than that of the evolution of crustal-scale faulting. In contrast Davy, Sornette & Sornette (1992) used a multifractal approach to predict scaling relationships between D and different fractal dimensions, and applied this to the fractal properties of laboratory analogue materials representing distributed continental deformation over longer time-scales due to an indenter. The model presented in this paper applies strictly to the case of incremental seismic damage due to the evolution of a set of Griffith cracks, though it's long-term evolution might in principle be determined by integration of the short-term distributions.

If earthquake sources can be regarded as simple scale-invariant dislocations of finite extent (e.g. Kanamori & Anderson 1975), the power-law exponent D is proportional to the seismic b-value (Aki 1981), and can hence be inferred either from earthquake or acoustic emission data (e.g. Meredith, Main & Jones 1990). This depends on independent knowledge of the moment–magnitude relationship for the particular instrument time constant and the typical duration of the seismic rupture (Kanomori & Anderson 1975). The successful prediction of variations in seismic moment–magnitude scaling with source size, notably the saturation of the M_S scale at magnitude 8 or so (e.g. Kanomori 1978), is strong evidence for the applicability of dislocation theory to the scaling between D and the seismic b-value.

Earlier suggestions that the length distribution itself could in general be equated with a capacity dimension, e.g. Aki (1981) are not supported by empirical data (Hirata 1989a,b; Henderson *et al.* 1992). However the specific prediction from this hypothesis that $1 < D < 3$ does appear to be borne out empirically, both for incremental damage during earthquakes and AE during subcritical crack growth (Main *et al.* 1990b), and for the longer term analogue experiments of Davy *et al.* (1992). In the latter model $D = D_0$ only for the specific cases of $D_C = 0$ (no spatial distribution of fault barycentres) or $D_C = D$. There is also some evidence that D is positively correlated to the fractal dimension determined by spectral analysis of scattered seismic waves in different parts of the Earth's lithosphere (Main, Peacock & Meredith 1990b).

Hirata (1989b) showed that there was a weak negative correlation between the two-point correlation dimension D_C of earthquake epicentre spacings and the seismic b-value, for earthquakes with magnitude $M \geq 5$ in the Tohoku region of Japan, over a period of 55 yr (1926–86). A similar result was obtained by Henderson *et al.* (1992) for the Riverside catalogue in southern California, for earthquakes greater than local magnitude 1.3 during the time period 1970–90. In this case the negative correlation was most strongly associated with individual foreshock–mainshock–aftershock sequences on a scale of a few years or so. In contrast b and D_C may be weakly positively correlated, as in the final stages of dynamic failure due to crack coalescence in laboratory creep experiments (Hirata, Satoh & Ito 1987) where acoustic emissions have been located. In either case, the correlation implies a weak ordering of fracture size *and* position. However the ordering has a different signature in laboratory fracture of intact rock when compared to earthquake data on a crustal scale. The cause of this apparent discrepancy must be fully understood before reliable extrapolation of laboratory results to field cases can be carried out with confidence, and is the prime motivation for the present study.

Physical models

An ordering of fracture size and position may be due either to: (1) long-range interactions caused by the elastic stress field; or (2) long-range order produced by short-range interactions combined with the configurational entropy of a self-organized critical process. Rundle & Klein (1989) applied the first hypothesis to the 'non-classical' nucleation and growth of cohesive tensile cracks, and showed that a 'classical' elliptical Griffith crack evolved by coalescence of smaller 'non-classical' cracks which formed ahead of the macrocrack tip analogous to the 'process zone' often postulated for subcritical crack growth. Several workers (Bak & Tang 1989; Sornette & Sornette 1989; Ito & Matsuzaki 1990) have shown, using different approaches, that the earthquake frequency–magnitude distribution, when taken over several cycles, could be explained by self-organized criticality, where fractally distributed 'avalanches' of large events result from the local interaction of smaller elements. Usually this has been illustrated by spring–block–slider models for earthquakes on a fault of the type first proposed by Burridge & Knopoff (1967). The most important property of self-organized criticality in terms of the present work is that it represents a stationary or average state, far from thermodynamic equilibrium, to which the complex dynamical system has *already evolved,* perhaps over millenia (P. Bak, personal communication). Therefore it cannot by definition be applied to the *evolution of damage* within a single earthquake cycle or even a few cycles. If we are interested in determining this evolution, say for predictive purposes, it may be more fruitful to consider a mainshock as a kind of critical phase transition which the system is driven through (e.g. Bruce & Wallace 1989). This approach was applied by Henderson & Main (1992) to a one-dimensional fault model, based on fracture mechanics, which showed systematic variations in the capacity dimension and b-value, depending on the applied stress and the type of local interaction in the model.

In the present paper the problem of how damage evolves in terms of the ordering of crack size and position is addressed using a mean field theory based on weakly interacting Griffith cracks. The main advantage of a mean field theory is the degree of analytic tractability, thereby allowing the correlations between the spacing and length distributions to be explicitly included. The main disadvantage is the lack of applicability near the point of critical, dynamic rupture on the scale of the sample in question, where local stress concentrations lead to strong crack–crack interactions and a much more heterogeneous stress field.

Specifically, the modified Griffith theory of Main (1991) is extended to consider the effect of long-range elastic interactions on a potential energy release rate G' (modified for the case of an ensemble of isolated cracks), and hence a physical connection is determined between material weakening and the concentration of deformation on dominant fault zones. Although primarily aimed at explaining observations of stress corrosion cracking, the theory is completely general, and applies to *any* mechanism of quasi-static subcritical crack growth which produces a power-law distribution of fracture size and spacing. Two processes are described: (A) localization of two-dimensional damage onto an eventual one-dimensional rupture plane; and (B) fracture clustering in two dimensions around potential rupture nucleation points on a pre-existing fault. These two end-member models are schematically illustrated in Fig. 1.

For case A a weak negative correlation between b and D_C is shown to be consistent with distributed deformation and the idea that crack growth and associated acoustic emission in intact specimens in the early stages of damage is associated with a stabilization of the material against macroscopic, dynamic failure. This establishes a direct connection (in an analytic form) between the distributions of crack size and spacing. The mechanical hardening effect is consistent with the usual tenets of damage mechanics, in which sample failure does not occur immediately on formation of the first crack, for example due to the stability of tensile crack growth in a compressive stress field (Ashby

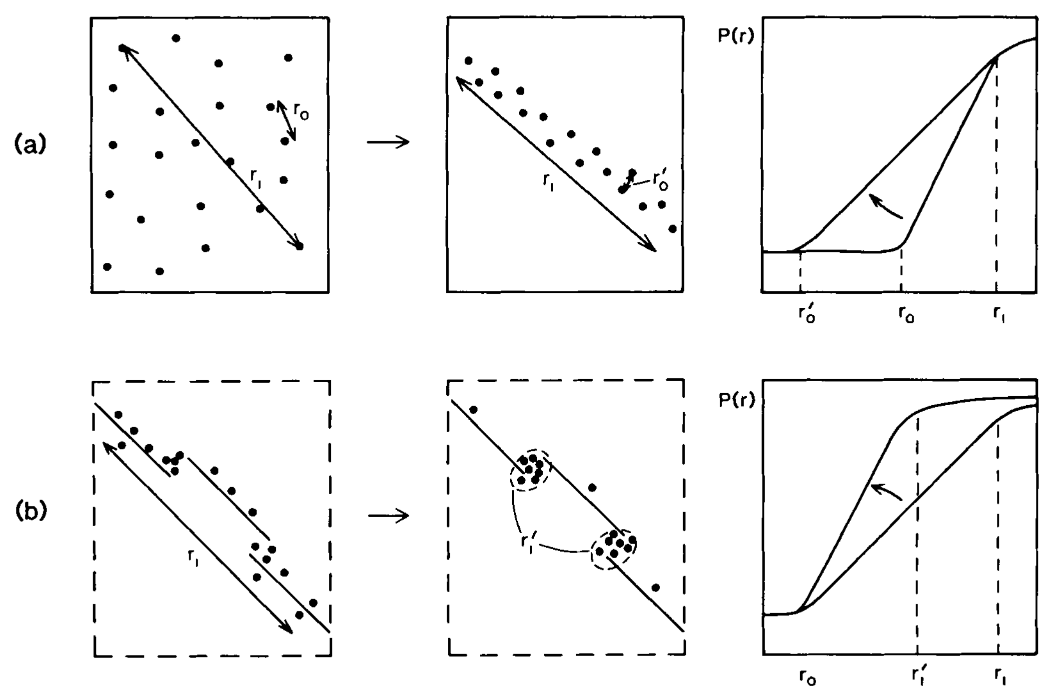

Figure 1. Schematic diagram showing two different types of damage localization: (a) alignment of epicentres on an incipient fault plane (model A); (b) concentration of epicentres on jogs and asperities on a pre-existing fault plane (Model B). The left-hand diagrams show the initial background seismicity, the central diagrams the evolution towards more concentrated activity, and the right-hand diagrams the correlation plot $P(r)$ associated with this change. $P(r)$ is defined in equation (6) of the main text, and the correlation plot is shown with log–log axes. The fractal ranges (r_0, r_1) and (r_1', r_1') are confined to the linear part of the correlation plot $P(r)$. Model (A) has fixed boundary conditions determined by the size of the laboratory sample. Model (B) has more arbitrary boundaries for the epicentre distribution which are chosen by the investigator, and are hence shown as dashed lines.

& Hallam 1986). In contrast the final stages of quasi-static failure is associated with localized, large-scale cracking with a positive correlation between D (and by inference b) and D_C, consistent with crack coalescence as a mechanism for instability. The main difference between the approach described here and standard damage mechanics is that this coalescence is *organized,* and can hence occur at lower crack densities than those modelled by Ashby & Hallam (1986).

On a developed fault (case B) the seismic activity concentrated around jogs and bends on a segmented fault trace shows the opposite correlation. In this case a negative correlation between D (and by implication b) and D_C implies a mechanical weakening effect. Since this correlation also implies increased clustering of activity at potential nucleation points for large-scale rupture, this might form a useful model for earthquake foreshocks.

THEORY

Background

The first attempt to quantify the fracture properties of natural materials was Griffith (1920, 1924), who considered a single two-dimensional elliptical crack of semi-length c and width w embedded in an effectively infinite medium with a tensile stress σ applied at the remote boundary. By analogy with Gibbs' theory for the nucleation of liquid droplets, he considered the effect of a change in the free energy ΔF caused by the introduction of such a crack in an otherwise elastic medium. By minimizing the free energy change, $\partial(\Delta F)/\partial c = 0$, Griffith showed that the minimum condition for dynamic rupture could be stated in the form $G = G_c$, where G is the potential strain energy release rate (per unit crack surface area) stored in the body. A general expression for the free energy is $\Delta F = -B^2\sigma^2c^2w + 4\gamma cw$, where 2γ is the energy needed to separate a unit area of material at the crack tip and B depends on the elastic constants and the geometry of the solid body (Irwin 1948). Explicitly $G = -\partial U/\partial A_c$, where $U = -B^2\sigma^2c^2w$ is the elastic potential strain energy stored in the body, and $A_c = 4cw$ is the crack surface area, so that $G = B^2\sigma^2c$. The critical condition for dynamic failure for a thermodynamically recoverable, elastic process is $G_c = 2\gamma$.

Rundle & Klein (1989) extended the nucleation approach by introducing an elastic interaction potential into a free-energy functional, and identified a fracture process zone introduced by cohesion in the solid. The idea behind a fracture process zone is that it develops in advance of the crack tip during subcritical crack growth, and consists of several smaller microcracks. The macrocrack then grows by cutting a swathe through this weakened material. Although often postulated, such a localized process zone is not always observed at crack tips in actual laboratory experiments on rocks. In fact rocks generally produce a much wider aureole of damage in the early stages of deformation, e.g. Lockner *et al.* (1991). Based on these observations the longer range interaction potential for rocks in the early stages of damage is apparently more important than the more localized cohesion associated with a process zone concentrated ahead of the crack tip.

In an attempt to explain the physical correlation observed by Meredith & Atkinson (1983) between observed seismic b-values and the stress intensity during subcritical crack growth, Main (1991) extended the Griffith theory to a fractal ensemble of isolated elliptical tensile cracks which could all grow because of a chemical contribution to the free energy at the crack tip due to stress corrosion. The theory correctly predicted the observed negative correlation between stress intensity and seismic b-values, as well as the positive correlation between seismic event rates and stress intensity (Main & Meredith 1991). Although primarily aimed at explaining observations of stress corrosion cracking, the theory is completely general, and applies to *any* mechanism of quasi-static subcritical crack growth which produces a power-law distribution of fracture sizes. In this sense *subcritical* refers to displacement on fractures smaller than the size of the sample of interest, so that the deformation is quasi-static, and *critical* implies dynamic rupture leading to fracture or faulting of the whole sample. A characteristic 'sample size' effect is a necessary consequence both of the finite rock samples used in labratory test and also of the finite seismogenic width of the Earth's schizosphere (Scholz 1990).

As in Main (1991) we will simply assume the fracture damage takes the form of a power-law distribution of crack semi-lengths c and constant width w, and apply the modified Griffith criterion for an ensemble of N_T aligned, elliptical cracks. Neglecting crack–crack interactions, the free-energy change associated with this two-dimensional ensemble is

$$\Delta F = N_T w[-B^2\sigma^2\langle c^2\rangle + 4\gamma\langle c\rangle], \tag{1}$$

where $B^2 = \pi/E$, and E is the Young's modulus, for the case of aligned, elliptical tensile cracks (Main 1991). B^2 is taken to be a constant at this stage of the modelling. The total volume of damage is defined for this two-dimensional problem as $V_D = \pi N_T w\langle c^2\rangle$, and the total crack surface area due to the damage is $A_D = 4N_T w\langle c\rangle$. Angular brackets denote expectation values which depend on the range of crack semi-lengths (c_0, c_1) and the probability distribution of semi-lengths in this range.

Inclusion of an interaction potential

We now add the effect of an interaction potential between two cracks of semi-length c_i and c_j, separated by a distance $r_{ij} \gg c_i$ or c_j, so that the cracks can be treated as effective point sources of stress. The elastic interaction between two cracks has been considered by Rundle & Klein (1989) and Rundle (1989), who showed by differentiating the strain energy potential that the stress on a crack at point x due to a displacement ξ on a crack at x' is proportional to $1/|x - x'|^2$ under plane strain conditions. We shall therefore assume the interaction potential between two cracks i and j takes the form $-\alpha' f(\theta, \phi)B^2\sigma^2wc_ic_jr_{ij}^{-1}$, where $0 < \alpha' \ll 1$ represents the strength of the interaction, and r^{-1} decay reflects the interaction potential rather than the interaction stress or force. $f(\theta, \phi)$ represents the directional effect of the solid angles describing the interaction, and reduces to $f(\theta)$ for a two-dimensional array of aligned cracks. This form of interaction is similar to that adopted by Rundle (1989), except that the deformation at the ith crack is represented solely by the displacement at the centre of the crack, rather than the integrated effect over the crack surface area. This is a reasonable assumption for a scale-invariant crack or fault system with small crack length compared to crack spacing. Including this interaction and taking $r_{ij} \gg c_i$ or c_j gives,

$$\Delta F = N_T w\left\{-B^2\sigma^2\left[\langle c^2\rangle - \alpha\left(\frac{N_T-1}{2}\right)\langle c\rangle^2\langle r^{-1}\rangle\right] + 4\gamma\langle c\rangle\right\}. \tag{2}$$

α here also includes the integrated effect of the relative orientation of the crack set c and the crack interaction lines r. The total crack surface area A_D can grow with no change in the free energy when $\partial(\Delta F)/\partial A_D = 0$, with the critical failure criterion

$$G' = G'_C = 2\gamma, \tag{3}$$

where the potential strain energy release rate $G' = -\partial U/\partial A_D$ for the ensemble is given by

$$G' = \frac{B^2\sigma^2}{2}\left\{\left(\frac{\partial\langle c^2\rangle}{\partial\langle c\rangle}\right)_{w,\sigma,N_T,c_0} + 2\alpha\left(\frac{N_T-1}{2}\right)\left[\langle c\rangle\langle r^{-1}\rangle + \frac{\langle c\rangle^2}{2}\left(\frac{\partial\langle r^{-1}\rangle}{\partial\langle c\rangle}\right)_{w,\sigma,N_T,c_0}\right]\right\}. \tag{4}$$

Here G' is defined in terms of the total surface area of damage A_D for a system of N_T isolated elliptical cracks. If stress corrosion is active then $G' = \gamma' < \gamma$ is the quasi-static condition for subcritical crack growth (Main 1991). Chelidze & Guegen (1990) recently derived an analogous modified Griffith model for a fractal crack distribution where A_D is characterized in terms of a fractal distribution of connected cracks using a box-counting algorithm. For a fractal system of isolated cracks the power-law exponent is the relevant parameter (Main 1991).

Clearly the effect of the interaction potential on G' depends on the partial derivative $\partial\langle r^{-1}\rangle/\partial\langle c\rangle$, which in turn depends on the distributions of crack spacing and crack length. Here we will assume a fractal crack population, and hence that the cumulative probability distributions can be expressed in power-law forms, i.e.

$$P_c(C \geq c) = (c/c_0)^{-D_C} \qquad c_0 \leq c \leq c_1, \tag{5}$$

$$P_r(R \leq r) = (r/r_1)^{D_C} \qquad r_0 \leq r \leq r_1. \tag{6}$$

We explicitly restrict our attention to the fractal range specified by the lower (c_0, r_0) and upper (c_1, r_1) fractal limits shown on Fig. 1. With this form of the distribution the normalizing conditions are $P_c(C \geq c_0) = 1$; $P_r(R \leq r_1) = 1$. Cumulative frequency statistics are then related to these probabilities by $N_c(C \geq c) = N_T P_c$; $N_r(R \leq r) = [N_T(N_T - 1)/2]P_r$. D is the power-law exponent of the length distribution of cracks. P_r is known as the 'correlation integral' (Grassberger 1983) and the fractal dimension D_C is the correlation dimension.

Expressions for the energy release rate without an interaction potential are given in Main (1991) and are not repeated here. Instead we concentrate on the second-order term including the derivative $\partial\langle r^{-1}\rangle/\partial\langle c\rangle$ in (4). Again we assume that there is only one largest crack of size c_1 and that c_0 is constant as a result of a minimum energy release rate for subcritical crack growth. However, we now add two criteria which depend on the type of damage being sustained: (A) damage localization along an incipient fault plane due to concentration of deformation which is initially distributed and macroscopically plastic (Fig. 1a); and (B) clustering of activity around nucleation points on a pre-existing fault plane (Fig. 1b).

Case (A) corresponds to localization of damage in the creep experiment described by Hirata *et al.* (1987; Figs 1 and 3), with $r_1 \cong$ constant. The physical reason for this boundary condition is that the largest crack separation is typically limited by the finite sample size. This may either be experimentally controlled in the laboratory, or a consequence of the finite seismogenic width of the Earth's crust. Case (B) is the spatial analogue to the temporal clustering model of earthquakes discussed by Smalley *et al.* (1987; figs 3 and 4). In this case the upper fractal limit r_1 varies because the fractal range is restricted to a range smaller than the sample size in general, and $r_0 \cong$ constant representing the finite lower limit of epicentral resolution (Fig. 1b). These two end-member models are now considered in turn.

Case (A): concentration of damage on an incipient fault plane; r_1 = constant

Here we consider the effect of concentration of deformation, beginning with a rock sample which is originally intact. Initially damage is distributed and plastic, and latterly concentrates on an incipient fault plane (Fig. 1a). As deformation progresses D will decrease as c_1 increases, as larger cracks are produced at higher energy release rates G' (Main 1991). D_C can either increase or decrease respectively as r_0 increases or decreases, depending on the tendency of damage to become more concentrated (lower $\langle r\rangle$ and r_0; model AI) or more distributed (higher $\langle r\rangle$ and r_0; model AII). These trends are illustrated qualitatively in Fig. 2, where N_T is held constant as a result of the definition of G' above. D_C will tend to decrease if epicentres are initially spaced randomly in two dimensions, and subsequently line up in plane along a one-dimensional fault break. Fig. 3 shows an example of this type of behaviour during a creep experiment, after Hirata *et al.* (1987).

From equation (4), we can now solve explicitly for G' for the fractal distribution of crack lengths and spacings given in equations (5) and (6). From the chain rule, holding all other variables constant,

$$\frac{\partial\langle r^{-1}\rangle}{\partial\langle c\rangle} = \left(\frac{\partial\langle r^{-1}\rangle}{\partial D_C}\right)\left(\frac{\partial D_C}{\partial D}\right)\left(\frac{\partial D}{\partial\langle c\rangle}\right). \tag{7}$$

Thus the effect of the interaction potential depends on the sign of the correlation between D and D_C via the term $\partial D_C/\partial D$. From (5) and (6)

$$\langle c\rangle = c_0\left(\frac{D}{D-1}\right)\left[\frac{1-(c_1/c_0)^{1-D}}{1-(c_1/c_0)^{-D}}\right], \tag{8}$$

and

$$\langle r^{-1}\rangle = r_1^{-1}\left(\frac{D_C}{D_C-1}\right)\left[\frac{1-(r_1/r_0)^{1-D_C}}{1-(r_1/r_0)^{-D_C}}\right], \tag{9}$$

except when D or $D_C = 1$, when logarithmic forms result from integrating the density distributions (Main 1991). Note the symmetry of these two equations due to the power-law probability distributions used to define the expectation values. In order to solve them uniquely an extra constraint needs to be applied to determine c_1 as a function of D and

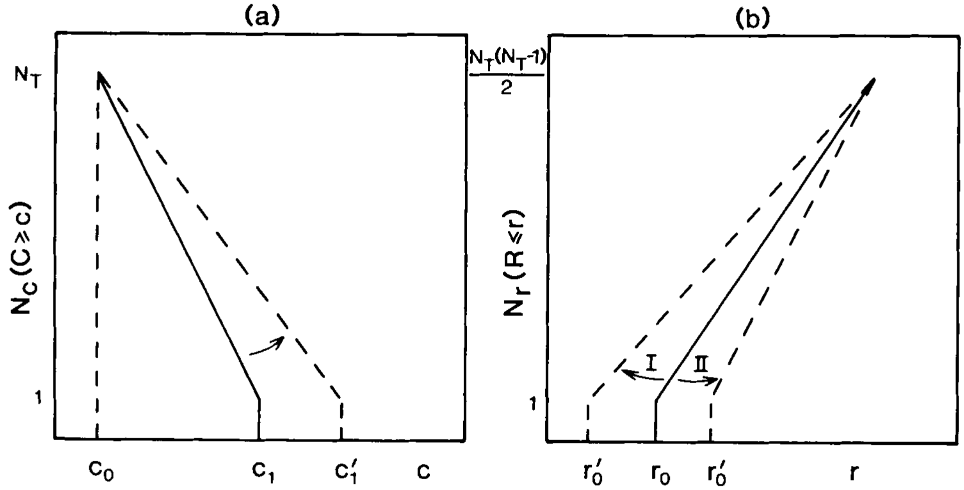

Figure 2. Model (A). Schematic diagram of the effect of (a) changing length distribution exponent D and (b) correlation dimension D_C on changing frequency–length and frequency–spacing statistics for constant c_0 and r_1. Plot (a) shows the cumulative frequency–length distribution N_c and (b) shows the frequency–spacing distribution N_r defined in the main text. Both diagrams are drawn as log–log plots, so the slopes of the straight lines correspond to D and D_C. As time goes on D decreases as more large fractures are formed. D_C may either decrease (Model AI), or increase (Model AII) as the minimum crack spacing r_0 decreases or increases respectively.

r_0 as a function of D_C. Here we assume there is only one largest crack of size $c_1 \pm \delta c/2$ and one largest crack separation $r_1 \pm \delta r/2$. For example if the size of the discrete bin in the frequency–length distribution $\delta c = c_0$, and $c_1/c_0 > 10^2$ then

$$c_1/c_0 = (DN_T)^{[1/(D+1)]}, \tag{10}$$

holds to an error of less than 1 per cent for $D \geq 1$ (see Appendix). Similarly if there is one largest crack separation $r_1 \pm \delta r$, and $r_1/r_0 > 10^2$, then to the same accuracy for $D_C \geq 1$,

$$r_1/r_0 = [D_C(\delta r/r_1)N_T(N_T-1)/2]^{[1/(D_C+1)]}. \tag{11}$$

Again since δr is arbitrary we may choose $\delta r = 2r_1/(N_T - 1)$, whence

$$r_1/r_0 = (D_C N_T)^{[1/(D_C+1)]}, \tag{12}$$

again preserving symmetry for simplicity.

N_T is a constant of the order of 100 or so for a reasonable estimate of the seismic b-value for most earthquake catalogues (e.g. Hirata 1989b), and we would normally expect the fractal range to be at least two orders of magnitude, so these approximations are not unreasonable. For a two-point correlation function based on an epicentral plot $1 \leq D_C \leq 2$, and G' is stable with respect to increasing N_T for $D \geq 1$ (Main 1991). The predicted range for D is $1 < D < 3$ (Main *et al.* 1990b). In practice the observed ranges inferred from the seismic b-value are usually tighter than this (e.g. Hirata 1989b). The terms $\partial\langle r^{-1}\rangle/\partial D_C$ and $\partial\langle c\rangle/\partial D$ in (7) for these ranges can then be calculated for different values of N_T from equations (8)–(10) and (12). Note that as $N_T \to \infty$, $c_1/c_0 \to \infty$ and $r_1/r_0 \to \infty$, and so $\langle c\rangle \to c_0[D/(D-1]$ for $D > 1$, and $\langle r^{-1}\rangle \to r_1^{-1}[D_C/(D_C - 1]$ for $D_C > 1$. Thus the mean crack length is finite as long as $D > 1$, but becomes infinite for $D \leq 1$ as $N_T \to \infty$.

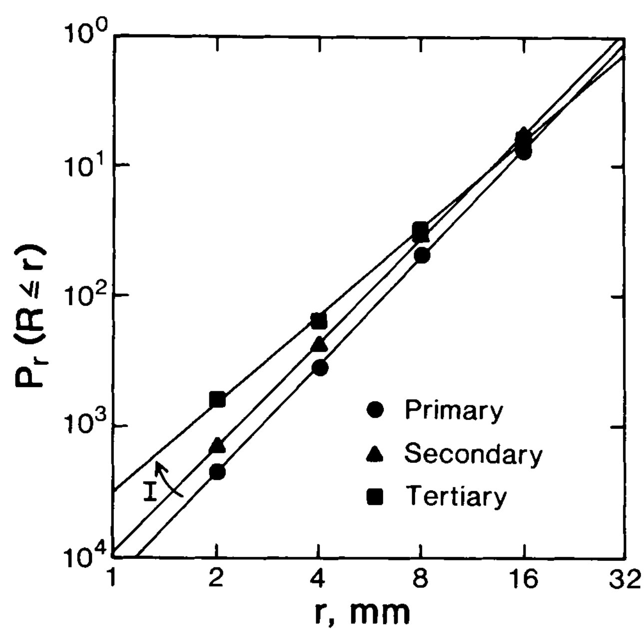

Figure 3. An example of the behaviour shown in Fig. 2, Model (AI), after Hirata *et al.* (1987). D_C decreases here as an incipient fault plane develops via stages of primary, secondary and tertiary creep. During the tertiary creep stage the b-value also dropped as deformation localized. The correlation integral P_r is shown, where N_r on Fig. 2(b) corresponds to $[N_T(N_T-1)/2]P_r$.

Case (B): clustering of epicentres around nucleation points on a pre-existing fault; r_0 = constant

A similar exercise to the above can be carried out allowing r_1 to vary and keeping r_0 constant, with one minimum crack spacing of size $r_0 \pm \delta r$. This is the spatial analogue of the temporal fractal clustering model of Smalley *et al.* (1987). An explicit solution for this case is given in the Appendix, of the form

$$\langle r^{-1}\rangle = r_0^{-1}\left(\frac{r_1}{r_0}\right)^{-1}\left(\frac{D_C}{D_C-1}\right)\left[\frac{1-(r_1/r_0)^{1-D_C}}{1-(r_1/r_0)^{-D_C}}\right], \tag{13}$$

with

$$r_1/r_0 = (D_C N_T)[1/(D_C - 1)]. \tag{14}$$

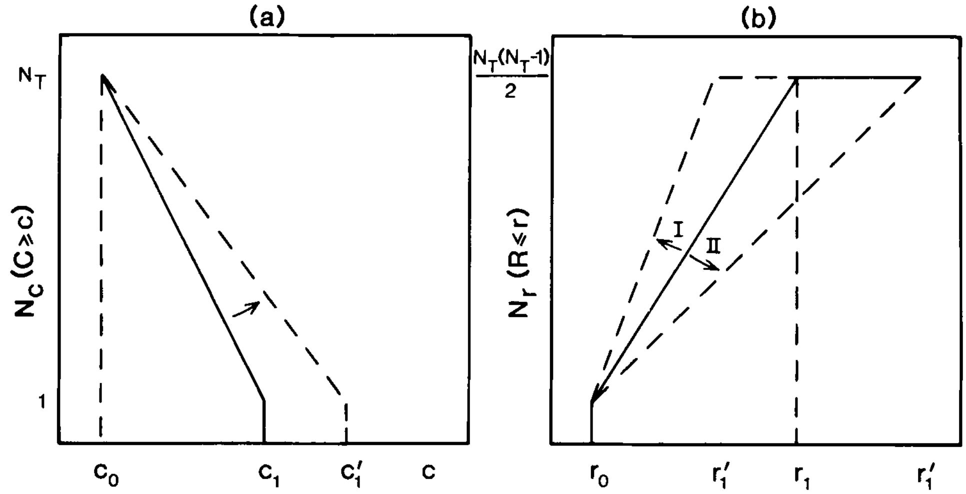

Figure 4. Model (B). Schematic diagram of the effect of (a) changing length distribution exponent D and (b) correlation dimension D_C on changing frequency–length and frequency–spacing statistics for constant r_0. Plot (a) shows the cumulative frequency–length distribution N_c and (b) shows the frequency–spacing distribution N_r defined in the main text. Both diagrams are drawn as log–log plots, so the slopes of the straight lines correspond to D and D_C. As time goes on D decreases as more large fractures are formed. D_C may either increase (Model BI), or decrease (model BII) as the maximum crack spacing r_1 decreases or increases respectively.

This results in the model shown in Fig. 4. Note that at high crack separations the distribution is not strictly fractal when the epicentres begin to cluster around the eventual nucleation point. Thus the spatial order during concentration of deformation is on a smaller scale relative to the 'sample size' than for model (A). As a result greater clustering is associated with *decreasing* $\langle r \rangle$ and r_1 and *increasing* D_C as deformation progresses. This is exactly the opposite behaviour to case (AI). The common indication of localization of damage in models (AI) and (BI) is this reduction in mean crack spacing $\langle r \rangle$, consistent with crack coalescence as a mechanism for failure.

PREDICTIONS OF THE THEORY AND COMPARISON WITH OBSERVATION

Model (A): concentration of deformation and material weakening

Fig. 5 shows $\langle r^{-1} \rangle$ as a function of D_C and $\langle c \rangle$ as a function of D for the ranges predicted above, and for different values of N_T. Note that for $D > 2$ the curve for $\langle c \rangle$ reaches the asymptotic limit $N_T \rightarrow \infty$ very quickly with increasing N_T. Thus damage with $D > 2$ tends to be more stable with respect to increasing numbers of cracks. This is consistent with experimental observations of $D > 2$ inferred from seismic b-values for acoustic emissions in the early strain hardening phase of cataclastic damage (Main *et al.* 1990a). In this phase the number of cracks producing acoustic emission is increasing exponentially, but the damage remains stable due to mechanical hardening.

Fig. 5 shows that both $\partial\langle r^{-1} \rangle / \partial D_C$ and $\partial \langle c \rangle / \partial D$ are negative, so that from (7) the sign of $\partial\langle r^{-1} \rangle / \partial D_C$ is the same as the sign of $\partial D_C / \partial D$. Thus if the seismic b-value is negatively correlated to the correlation dimension then $\partial D_C / \partial D < 0$, and the potential energy release rate G' is decreased by the presence of an elastic interaction potential (from equations 4 and 7). If the correlation is positive, then $\partial D_C / \partial D > 0$ and G' is increased by the presence of the elastic interaction potential. If there is no correlation $\partial D_C / \partial D = 0$ then there is a small increase in G' due to the term $\langle r^{-1} \rangle \langle c \rangle$ in (4). In the above we have assumed that the early phase of damage is associated with large separations r and small crack lengths c, so that in general this term will be small.

Thus $\partial D_C / \partial D > 0$ corresponds to the material being brought nearer the critical failure criterion $G' = G_c'$. Since D is negatively correlated to G' (Main 1991) this weakening is also associated with a reduction in D, or a concentration of deformation on the larger fractures. This prediction is consistent with the positive correlation between b and D_C observed in the tertiary creep stage reported by Hirata *et al.* (1987). It is also consistent with empirical observation that D is negatively correlated to the stress intensity K for a system dominated by a single macrocrack (Main *et al.* 1990a), since G is proportional to K^2. Thus the geometric effect of the mechanical weakening is a progressive concentration of deformation. For example geologists often infer that material weakening in the form of strain softening has occurred when deformation becomes concentrated into narrow bands (e.g. Ramsay & Huber 1987), though Hobbs, Muhlhaus & Ord (1990) have cautioned against a general association of the two. Wesnousky (1988) describes geological evidence for the evolution of concentrated deformation on long, linear features with weak mechanical properties, such as the San Andreas fault.

In contrast $\partial D_C / \partial D < 0$ corresponds to a hardening effect as the material is moved further away from the critical failure criterion. In this case D increases as G' decreases, leading to deformation being distributed in large numbes of smaller fractures. This is consistent with the observation of a wide aureole of damage observed during subcritical crack growth in the laboratory at low stress intensities (e.g. Main *et al.* 1990a; Lockner *et al.* 1991). Thus the mechanical hardening effect of decreased G' is associated with a geometry corresponding to more distributed deformation.

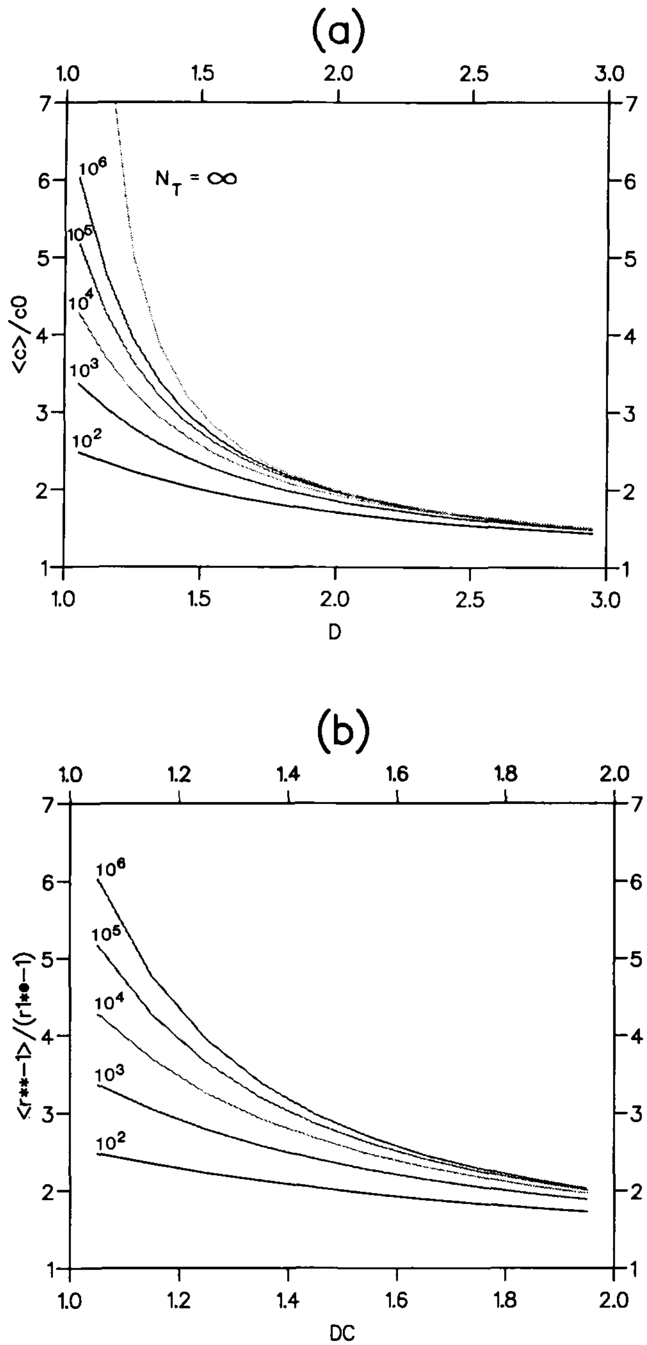

Figure 5. Functional form of (a) $\langle c\rangle/c_0=f(D)$ and (b) $\langle r^{-1}\rangle/r_1^{-1}=f(D_C)$ predicted from equations (5), (6) and (8)–(12) for different values of N_T for the case of constant r_1 (model A). The slope of both distributions is negative, so that (a) $\partial\langle c\rangle/\partial D<0$, and (b) $\partial\langle r^{-1}\rangle/\partial D_C<0$ for this case.

Equivalently this can be seen as a shielding effect, with the presence of a zone of damage reducing the local stresses on a particular crack.

Model (B): clustering of epicentres and incipient nucleation

We now consider the case where a fault has already developed, with microseismic activity on and around jogs and asperities on the fault. Thus, although deformation is concentrated on a 1-D fault break on a large scale, the small-scale deformation is clustered in 2-D around jogs and asperities along the fault trace (Fig. 1b). Fig. 6 shows the calculated dependence of $\langle r^{-1}\rangle$ on D_C for constant r_0, showing the opposite behaviour to Fig. 5. As a result the sign of $\partial\langle r^{-1}\rangle/\partial D_C$ is the opposite to the sign of $\partial D_C/\partial D$, and the results of the above section are reversed. In this case material weakening (increased G'/G'_C) is associated with a negative correlation between b and D_C, this time due to a concentration of deformation around the eventual nucleation point. This is consistent with the results from the southern California earthquake catalogue, where the seismicity preceding each large mainshock is associated with low seismic b-values and high correlation dimensions (Henderson *et al.* 1992). (The individual mainshocks show no clear relationship to each other, and form too sparse a data set to be analysed separately in terms of their spacing and size distribution.) Both the model and these observations are consistent with the idea that foreshocks consist of a progressively larger sequence of earthquakes concentrated around the nucleation point of a mainshock.

The common theme in both models (A) and (B) is that material weakening (strengthening) is associated with more concentrated (distributed) deformation, the difference being the predicted sign of the correlation between b and D_C.

APPLICABILITY OF A MEAN FIELD THEORY

As stated in the Introduction, the present paper is based on a mean field theory approach which ultimately breaks down when the mean crack length becomes comparable to the mean crack spacing $\langle r\rangle\approx\langle c\rangle$. The resulting localized stress concentrations limit the validity of the mean field theory approach to the early stages of damage, $\langle r\rangle\gg\langle c\rangle$, or to time intervals where the effect of transient stress concentrations is smeared out. In a separate paper, Henderson & Main (1992) developed a crack damage model

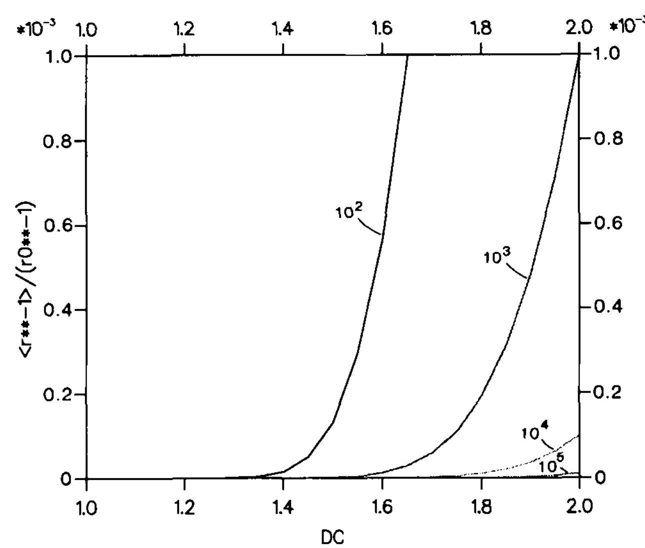

Figure 6. Functional form of $\langle r^{-1}\rangle/r_1^{-1}=f(D_C)$ predicted from equations (13) and (14) for different values of N_T for the case of constant r_0 (model B). The slope of the distribution is positive, so that $\partial\langle r^{-1}\rangle/\partial D_C>0$ for this case.

with an initially random strength distribution on a one-dimensional fault with local hardening (stopping crack growth) or softening (promoting crack growth by stress concentration) effects specified *a priori*. The model also has the advantage of not requiring a fractal distribution of cracks to be specified in advance, as in the present work. Nevertheless a fractal distribution of cracks does evolve, thereby allowing a comparison of the length distribution exponent D and a fractal capacity dimension D_0 determined by a box-counting algorithm. D_0 usually correlates well with the correlation dimension, and for a set of points, $D_0 \geq D_C$ (Grassberger & Procaccia 1983).

If the local rule is material hardening (e.g. due to crack-opening dilatancy), then a positive correlation between D and D_0 results: $\partial D/\partial D_0 > 0$ (Henderson & Main 1992). Similarly when the local rule is a softening effect (e.g. due to cooperative stress concentration between neighbouring crack tips) the two are negatively correlated: $\partial D/\partial D_0 < 0$. This is consistent with the predictiosn of the mean field theory described above for the case of a pre-existing fault. Thus the mean field theory preserves the general trend of the more complete approach, but will probably underestimate the magnitude of any non-linear trends due to localized stress concentration.

DISCUSSION

We have seen that the inclusion of an elastic interaction potential in the free energy of an ensemble of N_T isolated cracks can have the effect of increasing or decreasing the modified potential energy release rate $G' = -\partial U/\partial A_D$. The sign of the change in G' brought about by such long-range crack interactions is shown to depend on the correlation between the seismic b-value and the two-point correlation dimension D_C. Two cases have been described: (A) concentration of deformation on an incipient fault plane at large length-scales; and (B) the clustering properties of earthquake epicentres on a small scale around jogs and asperities. Case (A) can be seen as a gradual loss of one of the degrees of freedom of the system on a large scale. G' is increased with $\partial D/\partial D_C$ is *positive* for case (A), and when $\partial D/\partial D_C$ is *negative* for case (B). Both correspond to more localized deformation, thereby associating material weakening (increased G' relative to G'_C) with greater concentration of deformation (decreased $\langle r \rangle$). Great care must therefore be taken in associating any empirically observed correlation of D (or the seismic b-value) and D_C with increased or decreased G'. The type of concentration can be determined by comparison with Figs 1, 2 and 4, to establish whether the correlation $\partial D/\partial D_C$ is due to long-range localization effects (Figs 1a and 2) or short-range clustering properties (Figs 1b and 4). Thus it is important for authors of empirical studies to publish the whole distribution for the correlation dimension, not just the linear portion, and to show epicentral plots of snapshots corresponding to high and low D_C. This will allow comparison with the end members of Figs 2 and 4 to see which process is dominant in any particular case. For example it is clear from Figs 1 and 2 of Hirata *et al.* (1987) that long-range localization and a gradual loss of one of the degrees of freedom are the dominant processes in this laboratory creep experiment. In contrast the results of Henderson *et al.* (1992) for the San Andreas fault system are dominated by short-range clustering effects with behaviour more similar to Fig. 4. Hirata (1989b) shows no examples which would allow comparison of the temporal evolution of b and D_C to be assessed.

In addition to these material weakening effects being associated with concentrated deformation, material strengthening effects (decreased G' relative to G'_C) are predicted with more distributed deformation (increased $\langle r \rangle$). In a single earthquake cycle we would expect to see both strain hardening and strain softening effects prior to the dynamic failure of the sample, corresponding to material strengthening or weakening respectively. For example dilatant microcracking in relatively impermeable crystalline rock will tend to stabilize the system against failure, due to dilatant hardening (Scholz 1990). In contrast we might expect strain softening prior to dynamic failure because of quasi-static pre-seismic slip on a growing shear crack (Stuart 1979). Main & Meredith (1989) have suggested that the final nucleation phase of earthquakes is due to an increase in stress intensity (and hence G'/G'_C) during stress reduction, consistent with the slip-weakening model of Stuart (1979), and with the short-range clustering model described above. This final phase is associated with a short-term drop in b-value, in field examples (e.g. Smith 1981), during the compressional failure of intact laboratory samples in compression (Meredith *et al.* 1990; Lockner *et al.* 1991), and during the final stages of tertiary creep (Hirata *et al.* 1987).

Because of the importance of the distinction between short-range and long-range concentration of deformation, further discussion should probably await a more systematic and complete study of global data sets, where edge effects such as the size of the sample area and the epicentral resolution (Knopoff & Kagan 1980) can be corrected for in an identical way before being compared and interpreted with respect to Figs 2 and 4.

CONCLUSION

The effective energy release rate G' is either increased or decreased relative to its critical value G'_C by the presence of an interaction potential, depending on the correlation between the seismic b-value and the correlation dimension. Increased G'/G'_C corresponds to material weakening in the sense of the sample being brought nearer to the dynamic failure criterion. This is associated with greater concentration of deformation in both models considered in the present paper. This in turn justifies the common assumption by geologists that strain softening is associated with more concentrated deformation. In model (A) increased G'/G'_C occurs due to concentration of deformation on an incipient fault plane, resulting in a gradual loss of one of the degrees of freedom of the crack system. This results in a positive correlation at long distances between b and D_C. In case (B) a fault already exists and increased G'/G'_C occurs due to concentration of deformation on jogs and asperities which may nucleate larger shocks. This results in a negative correlation at short distance sbetween b and D_C.

Model (A) is a useful analogue for the evolution of localized damage on dominant fault planes, both in laboratory creep experiments and for the concentration of global deformation on narrow fault systems. Model (B) describes the clustering properties of earthquake epicentres on a fully developed fault system, and predicts the same correlations between b and D_C observed in numerical experiments which include the short-range stress concentrations not considered in the mean field theory presented here. Model (B) is also a useful analogue for the process of mainshock nucleation by progressively larger foreshocks clustering at short range.

For the case of material strengthening, both models are consistent with more distributed deformation, and the opposite correlations to those described above hold.

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APPENDIX

From equation (5) of the main text, the probability density function for a crack of semi-length c is

$$p_c(c) = -\frac{\partial P_c}{\partial c} = a\left(\frac{c}{c_0}\right)^{-D-1}, \tag{A1}$$

where a is a constant determined by the normalizing condition

$$\int_{c_0}^{c_1} p_c(c)\, dc = 1, \tag{A2}$$

so that

$$a = \frac{D}{c_0[1-(c_1/c_0)^{-D}]}. \tag{A3}$$

The number density distribution is then

$$n_c = -\frac{\partial N_c}{\partial c} = N_T p_c, \tag{A4}$$

since $N_c = N_T P_c$. The maximum crack size c_1 is then defined using

$$F_c(c_1) = \int_{c_1-\delta c/2}^{c_1+\delta c/2} n_c(c)\, dc = 1, \tag{A5}$$

so that there is one crack in the largest bin of the discrete frequency–length distribution F_c. After integration

$$F_c(c_1) = \left[\frac{N_T(c_1/c_0)^{-D}}{1-(c_1/c_0)^{-D}}\right] \times\left[\left(1-\frac{\delta c}{2c_1}\right)^{-D} - \left(1+\frac{\delta c}{2c_1}\right)^{-D}\right]. \tag{A6}$$

Using the binomial expansion

$$F_c(c_1) = \left[\frac{N_T(c_1/c_0)^{-D}}{1-(c_1/c_0)^{-D}}\right] \times\left[\left(1+\frac{D\delta c}{2c_1}+\text{H.O.T.}\right) - \left(1-\frac{D\delta c}{2c_1}+\text{H.O.T.}\right)\right]. \tag{A7}$$

Thus, for $\delta c < 10^{-2}c_1$, the higher order terms can be neglected to an accuracy of order 10^{-4}, whence

$$F_c(c_1) = \left[\frac{N_T(c_1/c_0)^{-D}}{1-(c_1/c_0)^{-D}}\right]\frac{D\delta c}{c_1} = 1. \tag{A8}$$

For $c_0 < 10^{-2}c_1$ the denominator inside the square brackets is unity within an error of order 10^{2D}, or less than 1 per cent for $D \geq 1$. If we let $\delta c = c_0$, then

$$c_1/c_0 = (DN_T)^{[1/(D+1)]}, \tag{A9}$$

to an accuracy of less than 1 per cent. This is the same as equation (10) of the main text.

An exactly similar exercise can be carried out for $N_r = N_T(N_T-1)/2P_r$ to derive equation (11) of the main text from the correlation integral P_r.

For the case of constant r_0 with one crack of size $r_0 \pm \delta r$, a similar exercise to the above can be carried out, producing the following equivalent to (A8):

$$F_r(r_0) = \left[\frac{N_T(r_1/r_0)^{-D_c}}{1-(r_1/r_0)^{-D_c}}\right]\frac{D_c\delta r}{r_0} = 1. \tag{A10}$$

For large r_1/r_0 it follows that

$$r_1/r_0 = [D_C(\delta r/r_1)N_T(N_T-1)/2]^{[1/(D_c-1)]}. \tag{A11}$$

Again choosing $\delta r = 2r_1/(N_T-1)$ we obtain equation (14) of the main text.