TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 358, Number 3, Pages 1165–1185
S 0002-9947(05)03818-3
Article electronically published on April 22, 2005

DAMPED WAVE EQUATION WITH A CRITICAL NONLINEARITY

NAKAO HAYASHI, ELENA I. KAIKINA, AND PAVEL I. NAUMKIN

Abstract. We study large time asymptotics of small solutions to the Cauchy problem for nonlinear damped wave equations with a critical nonlinearity

$$\begin{cases} \partial_t^2 u + \partial_t u - \Delta u + \lambda u^{1+\frac{2}{n}} = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = \varepsilon u_0(x), \partial_t u(0,x) = \varepsilon u_1(x), x \in \mathbf{R}^n, \end{cases}$$

where $\varepsilon > 0$, and space dimensions $n = 1, 2, 3$. Assume that the initial data

$$u_0 \in \mathbf{H}^{\delta,0} \cap \mathbf{H}^{0,\delta}, \ u_1 \in \mathbf{H}^{\delta-1,0} \cap \mathbf{H}^{-1,\delta},$$

where $\delta > \frac{n}{2}$, weighted Sobolev spaces are

$$\mathbf{H}^{l,m} = \left\{ \phi \in \mathbf{L}^2; \left\| \langle x \rangle^m \langle i\partial_x \rangle^l \phi(x) \right\|_{\mathbf{L}^2} < \infty \right\},$$

$\langle x \rangle = \sqrt{1+x^2}$. Also we suppose that

$$\lambda \theta^{\frac{2}{n}} > 0, \int u_0(x)\, dx > 0,$$

where

$$\theta = \int (u_0(x) + u_1(x))\, dx.$$

Then we prove that there exists a positive ε_0 such that the Cauchy problem above has a unique global solution $u \in \mathbf{C}([0,\infty); \mathbf{H}^{\delta,0})$ satisfying the time decay property

$$\left\| u(t) - \varepsilon\theta G(t,x) e^{-\varphi(t)} \right\|_{\mathbf{L}^p} \le C \varepsilon^{1+\frac{2}{n}} g^{-1-\frac{n}{2}}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $1 \le p \le \infty$, where $\varepsilon \in (0, \varepsilon_0]$.

1. Introduction

We study the large time asymptotics of solutions to the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \mathcal{L}u + \lambda \mathcal{N}(u) = 0, \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = \varepsilon u_0(x), \ \partial_t u(0,x) = \varepsilon u_1(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$, $\varepsilon > 0$, the spatial dimension $n = 1, 2, 3$, and the critical nonlinearity $\mathcal{N}(u)$ is defined by

$$\mathcal{N}(u) = u^{1+\frac{2}{n}}.$$

Recently much attention was drawn to nonlinear wave equations with dissipative terms. We mention here some recent works concerning global existence and

Received by the editors April 1, 2003 and, in revised form, April 22, 2004.
2000 *Mathematics Subject Classification.* Primary 35Q55; Secondary 35B40.
Key words and phrases. Damped wave equation, large time asymptotics.
The second and the third authors were supported in part by CONACYT.

nonexistence of solutions. The blow-up results were proved in [17], [18] for the case $\mathcal{N}(u) = |u|^{\rho}$, $1 < \rho < 1 + \frac{2}{n}$, $\lambda < 0$, when the initial data $u_0 \in \mathbf{H}^1$, $u_1 \in \mathbf{L}^2$ have a compact support and are such that $\int u_0(x)\,dx > 0$, $\int u_1(x)\,dx > 0$. This blow-up result was extended to the critical and subcritical cases $1 < \rho \leq 1 + \frac{2}{n}$ in [19]. In [10] Li and Zhou obtained an upper bound of the life-span of solutions to (1.1) with $\mathcal{N}(u) = |u|^{\rho}$, $1 < \rho < 1 + \frac{2}{n}$, $\lambda < 0$ with certain small initial data. In [14], the blow-up result was obtained for problem (1.1) with $\mathcal{N}(u) = |u|^{\rho-1}u$, $\lambda < 0$ in the space dimension $n = 3$ under the conditions $1 < \rho \leq 1 + \frac{2}{n}$ and $u_0(x) = 0$, $u_1(x) \geq 0$, $\int u_1(x)\,dx > 0$. Note that similar behavior was first discovered in [1] for the nonlinear heat equation $u_t - \Delta u = u^{\rho}$ in the critical and subcritical cases $1 < \rho \leq 1 + \frac{2}{n}$. We mention here some works regarding the nonlinear heat equation in the critical case (i.e. equation (1.1) with $\mathcal{L} = \partial_t - \Delta$ and $\mathcal{N}(u) = u^{1+\frac{2}{n}}$). If $\lambda < 0$, there are blow-up results for positive solutions (see [3], [9]). For any space dimension and $\lambda > 0$, it was shown that positive solutions have an additional time decay compared to the linear heat equation; more precisely, it was proved that (see [2], [4], [5])

$$\|u(t)\|_{\mathbf{L}^{\infty}} \leq C(1+t)^{-\frac{n}{2}}(1+\log(1+t))^{-\frac{n}{2}}.$$

From the heuristic point of view the term with the second time derivative u_{tt} in the nonlinear damped wave equation (1.1) has an additional time decay, hence it cannot affect essentially the large time asymptotic behavior of solutions to (1.1). Therefore we can also expect an additional time decay for solutions of the nonlinear damped wave equation (1.1). Below we state more precisely the conditions which guarantee global existence and decay of solutions to (1.1).

From the previous works [8], [11] we know that under the condition $\langle\xi\rangle^{\delta}\widehat{u}_0(\xi)$, $\langle\xi\rangle^{\delta-1}\widehat{u}_1(\xi) \in \mathbf{L}^2$ with $\delta > \frac{n}{2}$, the Fourier transform of a solution to the linearized problem corresponding to (1.1) decays exponentially in time and behaves like a solution of the linear wave equation in the high-frequency part $|\xi| \geq \frac{1}{2}$, and in the low-frequency part $|\xi| \leq \frac{1}{2}$ it is similar to a solution of the linear heat equation. These facts were used to prove large time decay estimates and global existence of solutions to (1.1) for the super critical cases $\rho > 1 + \frac{2}{n}$. In [18], Todorova and Yordanov proved global existence and large time decay estimates of solutions to the Cauchy problem for the damped wave equation (1.1) with nonlinearity $\mathcal{N}(u) = |u|^{\rho}$, where $1 + \frac{2}{n} < \rho \leq \frac{n}{n-2}$, in the case of sufficiently small initial data having a compact support. When the initial data are in the usual Sobolev space $\partial^{\alpha}u_0 \in \mathbf{L}^1 \cap \mathbf{L}^{\infty}$, $|\alpha| \leq 1$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^{\infty}$, problem (1.1) was considered in [13]. By making use of the fundamental solution of the linear problem, global existence of small solutions and large time decay estimates $\|u\|_{\mathbf{L}^q} \leq Ct^{-\frac{n}{2}\left(1-\frac{1}{q}\right)}$, $1 \leq q \leq \infty$, for space dimension $n = 3$, was proved. Later these requirements on the initial data were relaxed in [16] as follows $u_0 \in \mathbf{L}^1$, $\partial^{\alpha}u_0 \in \mathbf{L}^2$, $|\alpha| \leq 1$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^2$, under the additional assumptions on ρ and q such that $\rho \leq 5$, $q \leq 6$ for the space dimension $n = 3$ and $q < \infty$ for the two-dimensional case $n = 2$.

Applying energy type estimates obtained in [11] and [8] it was proved in [7] that solutions of the nonlinear damped wave equation (1.1) in the super critical cases $1 + \frac{4}{n} < \rho \leq \frac{n}{n-2}$, if $n = 3$ and $1 + \frac{4}{n} < \rho < \infty$, if $n = 1, 2$, with arbitrary initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{L}^1$, $u_1 \in \mathbf{L}^2 \cap \mathbf{L}^1$ (i.e. without smallness assumption on the initial data) have the same large time asymptotics as that for the linear heat equation

$\mathcal{L} = \partial_t - \Delta$, that is,

$$\|u(t) - MG(t)\|_{\mathbf{L}^p} = o\left(t^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}\right)$$

as $t \to \infty$, where $2 \le p < \frac{2n}{n-2}$ for $n = 3$, $2 \le p < \infty$ for $n = 2$ and $2 \le p \le \infty$ for $n = 1$; here $G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel and M is a constant. Recently in [15], sharp $\mathbf{L}^p$-time decay estimates of solutions to the nonlinear damped wave equation (1.1) were obtained in the subcritical case $1 < \rho < 1 + \frac{2}{n}$ under the condition that the initial data decay exponentially at infinity without any restriction on the size, where $2 \le p \le \frac{2}{n-2}$ for the space dimension $n \ge 3$, $2 \le p < \infty$ for $n = 2$ and $2 \le p \le \infty$ for $n = 1$. As far as we know, there are no results on the large time asymptotic behavior for the critical damped nonlinear wave equations.

For the case of higher dimensions $4 \le n \le 5$, global existence and $\mathbf{L}^q$ - time decay estimates for $\rho \le q \le \frac{\rho}{\rho-1}$ were obtained via Fourier analysis in paper [12] (see Theorem 1.3), when the power of the nonlinearity ρ is such that $1+\frac{2}{n} < \rho \le \frac{n+2}{n-2}$ and the initial data are small enough and satisfy $u_0, \partial^\alpha u_0 \in \mathbf{L}^\rho \cap \mathbf{L}^{\frac{\rho}{\rho-1}}, \partial^\beta u_0 \in \mathbf{L}^2$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^{\frac{\rho}{\rho-1}}$, $\partial^\alpha u_1 \in \mathbf{L}^2$, $|\alpha| \le 1$, $|\beta| \le 2$. Thus we see that due to the hyperbolic character of the equation some regularity assumptions on the initial data are needed to be able to treat the case of higher space dimensions. However, the nonlinear term under consideration does not possess enough regularity. This is one of the reasons why we restrict our attention below to the case $n \le 3$. Another reason is that in order to get an additional time decay of solutions we translate the original equation to another one containing time derivative of a solution (see (3.6) below), which require more regularity properties of the solution and prevents us from considering higher-dimensional cases.

Define by

$$\mathbf{H}^{l,m} = \left\{\phi \in \mathbf{L}^2; \left\|\langle x\rangle^m \langle i\partial_x\rangle^l \phi(x)\right\|_{\mathbf{L}^2} < \infty\right\}$$

the weighted Sobolev space, $\langle\xi\rangle = \sqrt{1+\xi^2}$. Denote

$$\theta = \int (u_0(x) + u_1(x))\, dx,\ \mu = \frac{\lambda}{4\pi} (\varepsilon\theta)^{\frac{2}{n}} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}.$$

Define $g(t) = 1 + \mu \log\langle t\rangle$ and let $G(t,x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ be the heat kernel.

Our main result is the following.

Theorem 1.1. *Let the initial data u_0, u_1 be such that*

$$u_0 \in \mathbf{H}^{\delta,0} \cap \mathbf{H}^{0,\delta},\ u_1 \in \mathbf{H}^{\delta-1,0} \cap \mathbf{H}^{-1,\delta},$$

where $\delta > \frac{n}{2}$. Also we assume

$$\lambda\theta^{\frac{2}{n}} > 0,\ \int u_0(x)\, dx > 0.$$

Then there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \le \varepsilon_0$ the Cauchy problem (1.1) has a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^{\delta,0}\right)$ satisfying the asymptotic property

$$\left\|u(t) - \varepsilon\theta G(t,x)\, e^{-\varphi(t)}\right\|_{\mathbf{L}^p} \le C\varepsilon^{1+\frac{2}{n}} g^{-1-\frac{n}{2}}(t) \langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $1 \le p \le \infty$, *where*

$$\left| e^{\frac{2}{n}\varphi(t)} - g(t) \right| \le C\varepsilon^{\frac{2}{n}} \log g(t),$$

for all $t > 0$.

Remark 1.2. The nonlinearity $u^{1+\frac{2}{n}}$ can be replaced by $|u|^{\frac{2}{n}} u$ or $|u|^{1+\frac{2}{n}}$ if we assume $\lambda > 0$ or $\lambda\theta > 0$ instead of $\lambda\theta^{\frac{2}{n}} > 0$, respectively. In these cases $\mu = \frac{\lambda}{4\pi} (\varepsilon |\theta|)^{\frac{2}{n}} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}$ for $|u|^{\frac{2}{n}} u$ and $\mu = \frac{\lambda}{4\pi} \varepsilon^{\frac{2}{n}} |\theta|^{1+\frac{2}{n}} \theta^{-1} \left(\frac{n}{n+2}\right)^{\frac{n}{2}}$. We note that our conditions always keep $\mu > 0$.

We denote by $\mathcal{F}$ the Fourier transformation

$$\widehat{u}(\xi) \equiv \mathcal{F}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i\xi x} u(x)\, dx$$

and by $\mathcal{F}^{-1}$ the inverse Fourier transformation

$$\check{u}(x) \equiv \mathcal{F}^{-1}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi x} u(\xi)\, d\xi.$$

In what follows we denote by C different positive constants. The rest of the paper is organized as follows. In Section 2 we obtain some preliminary estimates of the Green operator solving the linearized Cauchy problem corresponding to (1.1). Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary lemmas

The solution of the linear Cauchy problem

$$\begin{cases} \mathcal{L}u = f(t,x),\ x \in \mathbf{R}^n,\ t > 0, \\ u(0,x) = \varepsilon u_0(x),\ \partial_t u(0,x) = \varepsilon u_1(x),\ x \in \mathbf{R}^n, \end{cases} \tag{2.1}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$, $\varepsilon > 0$, can be written by the Duhamel formula

$$u(t) = \varepsilon \mathcal{G}_0(t) u_0 + \varepsilon \mathcal{G}_1(t) u_1 + \int_0^t \mathcal{G}_1(t-\tau) f(\tau) d\tau,$$

where

$$\mathcal{G}_j(t) = e^{-\frac{t}{2}} \mathcal{F}^{-1} L_j(t,\xi) \mathcal{F},$$

$j = 0, 1$, and

$$\begin{aligned} L_0(t,\xi) &= \cos\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right) + \frac{1}{2} L_1(t,\xi), \\ L_1(t,\xi) &= \frac{\sin\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)}{\sqrt{|\xi|^2 - \frac{1}{4}}} \end{aligned}$$

(we take the principal value of the square root). Also we define the operators

$$\mathcal{G}_j'(t) = \mathcal{F}^{-1} \frac{\partial}{\partial t} \left(e^{-\frac{t}{2}} L_j(t,\xi) \right) \mathcal{F}.$$

Note that the symbols $L_0(t,\xi)$ and $L_1(t,\xi)$ are smooth and bounded: $L_j(t,\xi) \in \mathbf{C}^\infty(\mathbf{R}^n)$, $j = 0, 1$; moreover, the symbol $L_1(t,\xi)$ decays as $\frac{1}{|\xi|}$ for $|\xi| \to \infty$, this means the gain of regularity concerning the initial datum u_1. We first collect some preliminary estimates for the Green operators $\mathcal{G}_0(t)$, $\mathcal{G}_1(t)$.

Lemma 2.1. *The estimates*

$$\|(-\Delta)^{\alpha}\mathcal{G}_j(t)\phi\|_{\mathbf{L}^2} \le C\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2},$$
$$\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{1}{2}}\mathcal{G}_j'(t)\phi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{-1}\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2},$$

and

$$\||\cdot|^{\alpha}\mathcal{G}_j(t)\phi\|_{\mathbf{L}^2} \le C\left\||\cdot|^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2} + C\langle t\rangle^{\frac{\alpha}{2}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}$$

are true for all $t \ge 0$, $j = 0, 1$, where $\alpha \ge 0$, provided that the right-hand sides are finite.

Proof. Since

$$|L_j(t,\xi)| \le Ce^{\frac{t}{2}}\langle\xi\rangle^{-j},$$

for all $t > 0$, $\xi \in \mathbf{R}^n$, $j = 0, 1$, we have

$$\begin{aligned}\|(-\Delta)^{\alpha}\mathcal{G}_j(t)\phi\|_{\mathbf{L}^2} &= Ce^{-\frac{t}{2}}\left\||\xi|^{2\alpha}L_j(t,\xi)\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\ &\le C\left\||\xi|^{2\alpha}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2} \le C\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}.\end{aligned}$$

Thus the first estimate of the lemma is true. For the second estimate we note that

$$\frac{\partial}{\partial t}\left(e^{-\frac{t}{2}}L_0(t,\xi)\right) = \frac{2|\xi|^2}{\sqrt{1-4|\xi|^2}}e^{-\frac{t}{2}}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right)$$

and

$$\begin{aligned}&\frac{\partial}{\partial t}\left(e^{-\frac{t}{2}}L_1(t,\xi)\right) = e^{-\frac{t}{2}-\frac{t}{2}\sqrt{1-4|\xi|^2}}\\ &-\frac{4|\xi|^2}{\sqrt{1-4|\xi|^2}\left(1+\sqrt{1-4|\xi|^2}\right)}e^{-\frac{t}{2}}\sinh\left(\frac{t}{2}\sqrt{1-4|\xi|^2}\right).\end{aligned}$$

Therefore, we have the estimate

$$\left|\frac{\partial}{\partial t}\left(e^{-\frac{t}{2}}L_j(t,\xi)\right)\right| \le C\langle t\rangle^{-1}\langle\xi\rangle^{1-j}$$

for all $t > 0$, $\xi \in \mathbf{R}^n$, $j = 0, 1$, hence

$$\begin{aligned}\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{1}{2}}\mathcal{G}_j'(t)\phi\right\|_{\mathbf{L}^2} &= C\left\||\xi|^{2\alpha}\langle\xi\rangle^{-1}\frac{\partial}{\partial t}\left(e^{-\frac{t}{2}}L_j(t,\xi)\right)\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\ &\le C\langle t\rangle^{-1}\left\||\xi|^{2\alpha}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\ &\le C\langle t\rangle^{-1}\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}.\end{aligned}$$

To prove the last estimate we note that

$$\left|(-\Delta)^k L_j(t,\xi)\right| \le \langle t\rangle^k e^{\frac{t}{2}}\langle\xi\rangle^{-j}$$

for all $t>0$, $\xi\in\mathbf{R}^n$, $j=0,1$, $k\geq 0$; therefore, by the Leibnitz rule we obtain

$$\begin{aligned}
&\left\||\cdot|^{\alpha}\mathcal{G}_j(t)\phi\right\|_{\mathbf{L}^2}=Ce^{-\frac{t}{2}}\left\|(-\Delta)^{\frac{\alpha}{2}}L_j(t,\xi)\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
\leq\;& Ce^{-\frac{t}{2}}\left\|\langle\xi\rangle^{j}L_j(t,\xi)\right\|_{\mathbf{L}^\infty}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
&+Ce^{-\frac{t}{2}}\sum_{k=0}^{[\alpha]}\left\|(-\Delta)^{[\alpha]+1}\langle\xi\rangle^{j}L_j(t,\xi)\right\|_{\mathbf{L}^\infty}^{\frac{\alpha-k}{2([\alpha]+1)}}\left\|\langle\xi\rangle^{j}L_j(t,\xi)\right\|_{\mathbf{L}^\infty}^{1-\frac{\alpha-k}{2([\alpha]+1)}}\\
&\times\left\|(-\Delta)^{\frac{k}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
\leq\;& C\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}+C\sum_{k=0}^{[\alpha]}\langle t\rangle^{\frac{\alpha-k}{2}}\left\|(-\Delta)^{\frac{k}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
\leq\;& C\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}+C\langle t\rangle^{\frac{\alpha}{2}}\left\|\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
\leq\;& C\left\||\cdot|^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}+C\langle t\rangle^{\frac{\alpha}{2}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}.
\end{aligned}$$

This completes the proof of Lemma 2.1. □

Denote by $G(t,x)\equiv(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$ the heat kernel. The following lemma says that the asymptotic behavior of solutions to the linear Cauchy problem (2.1) is similar to that for the heat equation.

Lemma 2.2. *The estimates*

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{\alpha}{2}}\left(\mathcal{G}_j(t)\phi-G(t,x)\widehat{\phi}(0)\right)\right\|_{\mathbf{L}^2}\\
\leq\;& Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}\\
&+Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}}\\
&+Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}},
\end{aligned}$$

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}}\left(\mathcal{G}_j'(t)\phi-\Delta G(t,x)\widehat{\phi}(0)\right)\right\|_{\mathbf{L}^2}\\
\leq\;& Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1}\left\|(-\Delta)^{\alpha}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}\\
&+Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}}\\
&+Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}}
\end{aligned}$$

and

$$\begin{aligned}
&\left\||x|^{\delta}\left(\mathcal{G}_j(t)\phi-G(t,x)\widehat{\phi}(0)\right)\right\|_{\mathbf{L}^2_x}\\
\leq\;& C\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}\\
&+Ct^{\frac{\delta-\gamma}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}}\\
&+Ct^{\frac{\delta-\gamma}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}}\left\|\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}}
\end{aligned}$$

are true for all $t \geq 1$, $j = 0, 1$, *where* $\delta > \frac{n}{2}$, $\alpha \geq 0$, $0 < \gamma < \min\left(1, \delta - \frac{n}{2}\right)$, *provided that the right-hand sides are finite.*

Proof. By the Schwartz inequality with $\gamma \geq 0$, $\delta - \gamma > \frac{n}{2}$ we obtain, choosing $\sigma = \left\| \langle \cdot \rangle^{\delta} \phi \right\|_{\mathbf{L}^2}^{\frac{1}{\delta}} \|\phi\|_{\mathbf{L}^2}^{-\frac{1}{\delta}} > 0$,

$$\begin{aligned} \left\| |\cdot|^{\gamma} \phi \right\|_{\mathbf{L}^1} &\leq \left(\int_{\mathbf{R}^n} \left(\sigma^2 + x^2\right)^{\delta - \gamma} |x|^{2\gamma} |\phi(x)|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} \left(\sigma^2 + x^2\right)^{\gamma - \delta} dx \right)^{\frac{1}{2}} \\ &\leq C \sigma^{\frac{n}{2} + \gamma - \delta} \left(\int_{\mathbf{R}^n} \left(\sigma^2 + x^2\right)^{\delta} |\phi(x)|^2 dx \right)^{\frac{1}{2}} \\ &\leq C \sigma^{\frac{n}{2} + \gamma} \|\phi\|_{\mathbf{L}^2} + C \sigma^{\frac{n}{2} - \delta + \gamma} \left\| \langle \cdot \rangle^{\delta} \phi \right\|_{\mathbf{L}^2} \leq C \left\| \langle \cdot \rangle^{\delta} \phi \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \|\phi\|_{\mathbf{L}^2}^{1 - \frac{n+2\gamma}{2\delta}} . \qquad (2.2) \end{aligned}$$

Hence we get

$$\begin{aligned} &\left\| |\xi|^{-\gamma} \left(\widehat{\phi}(\xi) - \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^\infty} \leq C \left\| (-\Delta)^{\frac{\gamma}{2}} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^\infty} \leq \left\| |\cdot|^{\gamma} \phi \right\|_{\mathbf{L}^1} \\ \leq \; & C \left\| \langle \cdot \rangle^{\delta} \phi \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \|\phi\|_{\mathbf{L}^2}^{1 - \frac{n+2\gamma}{2\delta}} . \end{aligned}$$

Taking into account the estimates

$$\begin{aligned} \left\| |\xi|^{\alpha} \left(e^{-\frac{t}{2}} L_j(t, \xi) - e^{-t|\xi|^2} \right) \right\|_{\mathbf{L}^2(|\xi| \leq 1)} &\leq C t^{-\frac{\alpha}{2} - \frac{n}{4} - 1}, \\ \left\| \langle \xi \rangle^{j} L_j(t, \xi) \right\|_{\mathbf{L}^\infty(|\xi| \geq 1)} &\leq C, \end{aligned}$$

and

$$\left\| |\xi|^{\alpha + \gamma} e^{-t|\xi|^2} \right\|_{\mathbf{L}^2(|\xi| \leq 1)} \leq C t^{-\frac{\alpha + \gamma}{2} - \frac{n}{4}}, \; \left\| |\xi|^{\alpha} e^{-t|\xi|^2} \right\|_{\mathbf{L}^2(|\xi| \geq 1)} \leq C e^{-\frac{t}{2}}$$

for all $t \geq 1$, we get

$$\begin{aligned} &\left\| (-\Delta)^{\frac{\alpha}{2}} \left(\mathcal{G}_j(t) \phi - G(t, x) \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} \\ = \; & C \left\| |\xi|^{\alpha} \left(e^{-\frac{t}{2}} L_j(t, \xi) \widehat{\phi}(\xi) - e^{-t|\xi|^2} \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} \\ \leq \; & C \left\| |\xi|^{\alpha} \left(e^{-\frac{t}{2}} L_j(t, \xi) - e^{-t|\xi|^2} \right) \right\|_{\mathbf{L}^2(|\xi| \leq 1)} \left\| \langle \xi \rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^\infty(|\xi| \leq 1)} \\ & + C \left\| |\xi|^{\alpha + \gamma} e^{-t|\xi|^2} \right\|_{\mathbf{L}^2(|\xi| \leq 1)} \left\| |\xi|^{-\gamma} \left(\langle \xi \rangle^{-j} \widehat{\phi}(\xi) - \langle 0 \rangle^{-j} \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^\infty(|\xi| \leq 1)} \\ & + C e^{-\frac{t}{2}} \left\| \langle \xi \rangle^{j} L_j(t, \xi) \right\|_{\mathbf{L}^\infty(|\xi| \geq 1)} \left\| |\xi|^{\alpha} \langle \xi \rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2(|\xi| \geq 1)} \\ & + C \left\| \langle \xi \rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^\infty(|\xi| \leq 1)} \left\| |\xi|^{\alpha} e^{-t|\xi|^2} \right\|_{\mathbf{L}^2(|\xi| \geq 1)} \\ \leq \; & C t^{-\frac{\alpha + \gamma}{2} - \frac{n}{4}} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{1 - \frac{n}{2\delta}} \\ & + C t^{-\frac{\alpha + \gamma}{2} - \frac{n}{4}} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{1 - \frac{n+2\gamma}{2\delta}} \\ & + C t^{-\frac{\alpha + \gamma}{2} - \frac{n}{4}} \left\| (-\Delta)^{\alpha} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2} . \end{aligned}$$

Thus the first estimate of the lemma is valid.

Since

$$\left\| |\xi|^{\alpha} \left(\frac{\partial}{\partial t} \left(e^{-\frac{t}{2}} L_j(t,\xi) \right) + |\xi|^2 e^{-t|\xi|^2} \right) \right\|_{\mathbf{L}^2(|\xi|\le 1)} \le Ct^{-\frac{\alpha}{2}-\frac{n}{4}-2}$$

and

$$\left\| \langle\xi\rangle^{j-1} \frac{\partial}{\partial t} \left(e^{-\frac{t}{2}} L_j(t,\xi) \right) \right\|_{\mathbf{L}^\infty(|\xi|\ge 1)} \le Ce^{-\frac{t}{4}}$$

for all $t \ge 1$, we get

$$\begin{aligned}
& \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} \left(\mathcal{G}_j'(t)\phi - \widehat{\phi}(0)\Delta G(t,x) \right) \right\|_{\mathbf{L}^2} \\
= {} & C \left\| |\xi|^{\alpha} \langle\xi\rangle^{-1} \left(\frac{\partial}{\partial t} \left(e^{-\frac{t}{2}} L_j(t,\xi) \right) \widehat{\phi}(\xi) + |\xi|^2 e^{-t|\xi|^2} \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} \\
\le {} & C \left\| |\xi|^{\alpha} \left(\frac{\partial}{\partial t} \left(e^{-\frac{t}{2}} L_j(t,\xi) \right) + |\xi|^2 e^{-t|\xi|^2} \right) \right\|_{\mathbf{L}^2(|\xi|\le 1)} \left\| \langle\xi\rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^\infty(|\xi|\le 1)} \\
& + C \left\| |\xi|^{\alpha+2+\gamma} e^{-t|\xi|^2} \right\|_{\mathbf{L}^2(|\xi|\le 1)} \left\| |\xi|^{-\gamma} \left(\langle\xi\rangle^{-j} \widehat{\phi}(\xi) - \langle 0\rangle^{-j} \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^\infty(|\xi|\le 1)} \\
& + Ce^{-\frac{t}{2}} \left\| \langle\xi\rangle^{j-1} \frac{\partial}{\partial t} \left(e^{-\frac{t}{2}} L_j(t,\xi) \right) \right\|_{\mathbf{L}^\infty(|\xi|\ge 1)} \left\| |\xi|^{\alpha} \langle\xi\rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2(|\xi|\ge 1)} \\
& + C \left\| \langle\xi\rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^\infty(|\xi|\le 1)} \left\| |\xi|^{\alpha+2} e^{-t|\xi|^2} \right\|_{\mathbf{L}^2(|\xi|\ge 1)} \\
\le {} & Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| \langle\cdot\rangle^{\delta} \langle\Delta\rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle\Delta\rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \\
& + Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| \langle\cdot\rangle^{\delta} \langle\Delta\rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle\Delta\rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}} \\
& + Ct^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}.
\end{aligned}$$

Whence the second estimate of the lemma follows. To prove the last estimate we write

$$\begin{aligned}
& \left\| |x|^{\delta} \left(\mathcal{G}_j(t)\phi - G(t,x)\widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2} \\
\le {} & C \left\| (-\Delta)^{\frac{\delta}{2}} \left(\langle\xi\rangle^{j} e^{-\frac{t}{2}} L_j(t,\xi) - e^{-t|\xi|^2} \right) \langle\xi\rangle^{-j} \widehat{\phi}(\xi) \right\|_{\mathbf{L}^2} \\
& + C \left\| (-\Delta)^{\frac{\delta}{2}} e^{-t|\xi|^2} \left(\langle\xi\rangle^{-j} \widehat{\phi}(\xi) - \langle 0\rangle^{-j} \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2}.
\end{aligned} \tag{2.3}$$

Denote $\chi_1(\xi) \in \mathbf{C}^\infty(\mathbf{R}^n)$ such that $\chi_1(\xi) = 1$ for $|\xi| \le 1$ and $\chi_1(\xi) = 0$ for $|\xi| \ge 2$, also we define $\chi_2(\xi) = 1 - \chi_1(\xi)$. Note that there exists a smooth and rapidly decaying kernel

$$K(t,x) = \mathcal{F}^{-1} \left(\left(\langle\xi\rangle^{j} e^{-\frac{t}{2}} L_j(t,\xi) - e^{-t|\xi|^2} \right) \chi_1(\xi) \right),$$

so that by the Young inequality we have

$$
\begin{aligned}
& \left\| |\cdot|^{\delta} \mathcal{F}^{-1} \left(\left(\langle \xi \rangle^{j} e^{-\frac{t}{2}} L_j (t, \xi) - e^{-t|\xi|^2} \right) \chi_1 (\xi) \right) \langle \xi \rangle^{-j} \widehat{\phi} (\xi) \right\|_{\mathbf{L}^2} \\
= & \left\| |x|^{\delta} \int_{\mathbf{R}^n} K (t, x - y) \langle \Delta \rangle^{-\frac{j}{2}} \phi (y) \, dy \right\|_{\mathbf{L}^2} \\
\leq & C \left\| \int_{\mathbf{R}^n} |x - y|^{\delta} |K (t, x - y)| \left| \langle \Delta \rangle^{-\frac{j}{2}} \phi (y) \right| dy \right\|_{\mathbf{L}^2} \\
& + C \left\| \int_{\mathbf{R}^n} |K (t, x - y)| \, |y|^{\delta} \left| \langle \Delta \rangle^{-\frac{j}{2}} \phi (y) \right| dy \right\|_{\mathbf{L}^2} \\
\leq & C \left\| |x|^{\delta} K (t, x) \right\|_{\mathbf{L}^2} \left\| \langle \Delta \rangle^{-\frac{j}{2}} \phi (x) \right\|_{\mathbf{L}^1} \\
& + C \| K (t, x) \|_{\mathbf{L}^1} \left\| |x|^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi (x) \right\|_{\mathbf{L}^2} . \qquad (2.4)
\end{aligned}
$$

By the estimate

$$
(-\Delta)^{k} \left(\left(\langle \xi \rangle^{j} e^{-\frac{t}{2}} L_j (t, \xi) - e^{-t|\xi|^2} \right) \chi_1 (\xi) \right) \leq C \langle t \rangle^{k-1} e^{-Ct|\xi|^2} \chi_1 (\xi)
$$

for all $t > 0$, $|\xi| \leq 2$, $j = 0, 1$, $k \geq 0$, we have

$$
\begin{aligned}
& \left\| |x|^{2k} K (t, x) \right\|_{\mathbf{L}^2} \leq C \left\| \langle \cdot \rangle^{2k} K \right\|_{\mathbf{L}^2} \\
\leq & C \left\| \langle \Delta \rangle^{k} \left(\left(\langle \xi \rangle^{j} e^{-\frac{t}{2}} L_j (t, \xi) - e^{-t|\xi|^2} \right) \chi_1 (\xi) \right) \right\|_{\mathbf{L}^2} \leq C \langle t \rangle^{k-1-\frac{n}{4}} ;
\end{aligned}
$$

hence via estimate (2.2) with $\gamma = 0$,

$$
\left\| |x|^{\delta} K (t, x) \right\|_{\mathbf{L}^2} \leq C \langle t \rangle^{\frac{\delta}{2} - 1 - \frac{n}{4}}
$$

and

$$
\| K (t, x) \|_{\mathbf{L}^1} \leq C \left\| \langle \cdot \rangle^{\delta} K \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \| K \|_{\mathbf{L}^2}^{1 - \frac{n}{2\delta}} \leq C \langle t \rangle^{-1} .
$$

Therefore (2.4) yields

$$
\begin{aligned}
& \left\| |\cdot|^{\delta} \mathcal{F}^{-1} \left(\left(\langle \xi \rangle^{j} e^{-\frac{t}{2}} L_j (t, \xi) - e^{-t|\xi|^2} \right) \chi_1 (\xi) \right) \langle \xi \rangle^{-j} \widehat{\phi} (\xi) \right\|_{\mathbf{L}^2} \\
\leq & C \langle t \rangle^{\frac{\delta}{2} - \frac{n}{4} - 1} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{1 - \frac{n}{2\delta}} \\
& + C \langle t \rangle^{-1} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2} . \qquad (2.5)
\end{aligned}
$$

We have

$$
(-\Delta)^{k} \left(\left(\langle \xi \rangle^{j} e^{-\frac{t}{2}} L_j (t, \xi) - e^{-t|\xi|^2} \right) \chi_2 (\xi) \right) \leq C e^{-\frac{t}{4}}
$$

for all $t>0$, $\xi\in\mathbf{R}^n$, $j=0,1$, $k\geq 0$; therefore, by the Leibnitz rule we obtain

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{\delta}{2}}\left(\left(\langle\xi\rangle^{j}e^{-\frac{t}{2}}L_j(t,\xi)-e^{-t|\xi|^2}\right)\chi_2(\xi)\right)\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
\leq\;& C\left\|\left(\langle\xi\rangle^{j}e^{-\frac{t}{2}}L_j(t,\xi)-e^{-t|\xi|^2}\right)\chi_2(\xi)\right\|_{\mathbf{L}^\infty}\left\|(-\Delta)^{\frac{\delta}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
&+C\sum_{k=0}^{[\delta]}\left\|(-\Delta)^{[\delta]+1}\left(\left(\langle\xi\rangle^{j}e^{-\frac{t}{2}}L_j(t,\xi)-e^{-t|\xi|^2}\right)\chi_2(\xi)\right)\right\|_{\mathbf{L}^\infty}^{\frac{\delta-k}{2([\delta]+1)}}\\
&\times\left\|\left(\langle\xi\rangle^{j}e^{-\frac{t}{2}}L_j(t,\xi)-e^{-t|\xi|^2}\right)\chi_2(\xi)\right\|_{\mathbf{L}^\infty}^{1-\frac{\delta-k}{2([\delta]+1)}}\left\|(-\Delta)^{\frac{k}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\\
\leq\;& Ce^{-\frac{t}{4}}\left\|\langle\Delta\rangle^{\frac{\delta}{2}}\langle\xi\rangle^{-j}\widehat{\phi}(\xi)\right\|_{\mathbf{L}^2}\leq C\langle t\rangle^{-1}\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac{j}{2}}\phi\right\|_{\mathbf{L}^2}.
\end{aligned}\tag{2.6}$$

In the same manner, using the heat kernel,

$$G(t,x)=\mathcal{F}^{-1}\left(e^{-t|\xi|^2}\right)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{x^2}{4t}},$$

and $\mathcal{G}_2(t)=\mathcal{F}^{-1}e^{-t|\xi|^2}\mathcal{F}$, we write

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{\delta}{2}}e^{-t|\xi|^2}\left(\langle\xi\rangle^{-j}\widehat{\phi}(\xi)-\langle 0\rangle^{-j}\widehat{\phi}(0)\right)\right\|_{\mathbf{L}^2_\xi}\\
=\;&\left\||x|^{\delta}\left(\mathcal{G}_2(t)\langle\Delta\rangle^{-\frac{j}{2}}\phi-G(t,x)\langle 0\rangle^{-j}\widehat{\phi}(0)\right)\right\|_{\mathbf{L}^2_x}.
\end{aligned}$$

Taking $\gamma\in[0,1]$, $\gamma<\delta-\frac{n}{2}$, changing the dependent variables $x=\xi\sqrt{t}$ and $y=\eta\sqrt{t}$ we write

$$\begin{aligned}
&|x|^{\delta}\left(\mathcal{G}_2(t)\langle\Delta\rangle^{-\frac{j}{2}}\phi-G(t,x)\langle 0\rangle^{-j}\widehat{\phi}(0)\right)\\
=\;&(4\pi t)^{-\frac{n}{2}}\int_{|y|\leq\sqrt{t}}\frac{|x|^{\delta}}{|y|^{\gamma}}\left(e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{x^2}{4t}}\right)|y|^{\gamma}\left(\langle\Delta\rangle^{-\frac{j}{2}}\phi\right)(y)\,dy\\
&+(4\pi t)^{-\frac{n}{2}}\int_{|y|\geq\sqrt{t}}\frac{|x|^{\delta}}{|y|^{\delta}}\left(e^{-\frac{(x-y)^2}{4t}}-e^{-\frac{x^2}{4t}}\right)|y|^{\delta}\left(\langle\Delta\rangle^{-\frac{j}{2}}\phi\right)(y)\,dy\\
=\;&(4\pi)^{-\frac{n}{2}}t^{\frac{\delta}{2}}\int_{|\eta|\leq 1}\frac{|\xi|^{\delta}}{|\eta|^{\gamma}}\left(e^{-\frac{(\xi-\eta)^2}{4}}-e^{-\frac{\xi^2}{4}}\right)|\eta|^{\gamma}\left(\langle\Delta\rangle^{-\frac{j}{2}}\phi\right)\left(\eta\sqrt{t}\right)d\eta\\
&+(4\pi)^{-\frac{n}{2}}t^{\frac{\delta}{2}}\int_{|\eta|\geq 1}\frac{|\xi|^{\delta}}{|\eta|^{\delta}}\left(e^{-\frac{(\xi-\eta)^2}{4}}-e^{-\frac{\xi^2}{4}}\right)|\eta|^{\delta}\left(\langle\Delta\rangle^{-\frac{j}{2}}\phi\right)\left(\eta\sqrt{t}\right)d\eta.
\end{aligned}\tag{2.7}$$

Applying the inequality

$$\begin{aligned}
\frac{|\xi|^{\delta}}{|\eta|^{\delta}}\left|e^{-\frac{(\xi-\eta)^2}{4}}-e^{-\frac{\xi^2}{4}}\right|\;&\leq\; C\frac{|\xi-\eta|^{\delta}+|\eta|^{\delta}}{|\eta|^{\delta}}e^{-\frac{(\xi-\eta)^2}{4}}+C\frac{|\xi|^{\delta}}{|\eta|^{\delta}}e^{-\frac{\xi^2}{4}}\\
&\leq\; Ce^{-C(\xi-\eta)^2}+Ce^{-C\xi^2}
\end{aligned}$$

for all $|\eta|\geq 1$ and the estimate

$$\frac{|\xi|^{\delta}}{|\eta|^{\gamma}}\left|e^{-\frac{(\xi-\eta)^2}{4}}-e^{-\frac{\xi^2}{4}}\right|\leq Ce^{-C(\xi-\eta)^2}+Ce^{-C\xi^2}$$

for all $|\eta| \le 1$, in view of (2.2) we obtain from (2.7)

$$\begin{aligned}
&\left\| |x|^{\delta} \left(\mathcal{G}_2 (t) \langle \Delta \rangle^{-\frac{j}{2}} \phi - G(t,x) \langle 0 \rangle^{-j} \widehat{\phi}(0) \right) \right\|_{\mathbf{L}^2_x} \\
\le\ & Ct^{\frac{\delta}{2}+\frac{n}{4}} \left\| \int_{|\eta|\le 1} \frac{|\xi|^{\delta}}{|\eta|^{\gamma}} \left| e^{-\frac{(\xi-\eta)^2}{4}} - e^{-\frac{\xi^2}{4}} \right| |\eta|^{\gamma} \left(\langle \Delta \rangle^{-\frac{j}{2}} \phi \right) \left(\eta \sqrt{t} \right) d\eta \right\|_{\mathbf{L}^2_\xi} \\
&+ Ct^{\frac{\delta}{2}+\frac{n}{4}} \left\| \int_{|\eta|\ge 1} \frac{|\xi|^{\delta}}{|\eta|^{\delta}} \left| e^{-\frac{(\xi-\eta)^2}{4}} - e^{-\frac{\xi^2}{4}} \right| |\eta|^{\delta} \left(\langle \Delta \rangle^{-\frac{j}{2}} \phi \right) \left(\eta \sqrt{t} \right) d\eta \right\|_{\mathbf{L}^2_\xi} \\
\le\ & Ct^{\frac{\delta}{2}+\frac{n}{4}} \left\| \int_{|\eta|\le 1} \left(e^{-C(\xi-\eta)^2} + e^{-C\xi^2} \right) |\eta|^{\gamma} \left(\langle \Delta \rangle^{-\frac{j}{2}} \phi \right) \left(\eta \sqrt{t} \right) d\eta \right\|_{\mathbf{L}^2_\xi} \\
&+ Ct^{\frac{\delta}{2}+\frac{n}{4}} \left\| \int_{|\eta|\ge 1} \left(e^{-C(\xi-\eta)^2} + e^{-C\xi^2} \right) |\eta|^{\delta} \left(\langle \Delta \rangle^{-\frac{j}{2}} \phi \right) \left(\eta \sqrt{t} \right) d\eta \right\|_{\mathbf{L}^2_\xi} \\
\le\ & Ct^{\frac{\delta}{2}+\frac{n}{4}} \left\| |\eta|^{\gamma} \phi \left(\eta \sqrt{t} \right) \right\|_{\mathbf{L}^1_\eta} + Ct^{\frac{\delta}{2}+\frac{n}{4}} \left\| |\eta|^{\delta} \left(\langle \Delta \rangle^{-\frac{j}{2}} \phi \right) \left(\eta \sqrt{t} \right) \right\|_{\mathbf{L}^2_\eta} \\
\le\ & Ct^{\frac{\delta-\gamma}{2}-\frac{n}{4}} \left\| |\cdot|^{\gamma} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^1} + C \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2} \\
\le\ & Ct^{\frac{\delta-\gamma}{2}-\frac{n}{4}} \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}+\frac{\gamma}{\delta}} \left\| \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}-\frac{\gamma}{\delta}} \\
&+ C \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{j}{2}} \phi \right\|_{\mathbf{L}^2} \qquad (2.8)
\end{aligned}$$

for all $t \ge 1$. Substitution of (2.5), (2.6) and (2.8) into (2.3) yields the third estimate of the lemma. Lemma 2.2 is proved. □

We let

$$g(t) = 1 + \kappa \log \langle t \rangle$$

with some $\kappa > 0$ (in the proof of the theorem we put $\kappa = \mu$) and we define two norms

$$\begin{aligned}
\|\phi\|_{\mathbf{X}} =\ & \sup_{t>0} \sup_{0\le\alpha\le\delta} \langle t \rangle^{\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \phi(t) \right\|_{\mathbf{L}^2} \\
&+ \sup_{t>0} \sup_{0\le\alpha\le\delta} \langle t \rangle^{\frac{n}{4}+\frac{\alpha}{2}+\frac{1}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \partial_t \phi(t) \right\|_{\mathbf{L}^2} \\
&+ \sup_{t>0} \langle t \rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \phi(t) \right\|_{\mathbf{L}^2}
\end{aligned}$$

and

$$\begin{aligned}
\|\phi\|_{\mathbf{Y}} =\ & \sup_{t>0} \sup_{0\le\alpha\le\delta} \langle t \rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle \Delta \rangle^{-\frac{1}{2}} \phi(t) \right\|_{\mathbf{L}^2} \\
&+ \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} \phi(t) \right\|_{\mathbf{L}^2}.
\end{aligned}$$

Next lemma will be necessary for estimating the nonlinear term in the proof of the theorem.

Lemma 2.3. *Let the function $f(t,x)$ have a zero mean value $\widehat{f}(t,0)=0$. Then the inequality*

$$\left\| g(t)\int_0^t g^{-1}(\tau)\,\mathcal{G}_1(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{X}} \le C\,\|f\|_{\mathbf{Y}}$$

is valid, provided that the right-hand side is finite.

Proof. By Lemma 2.1 we get

$$\begin{aligned} &\left\|(-\Delta)^{\frac{\alpha}{2}}\int_0^t g^{-1}(\tau)\,\mathcal{G}_1(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^2} \\ \le\ & C\int_0^t g^{-1}(\tau)\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2} d\tau \le C\,\|f\|_{\mathbf{Y}}, \end{aligned}$$

$$\begin{aligned} &\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac12}\,\partial_t\int_0^t g^{-1}(\tau)\,\mathcal{G}_1(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^2} \\ =\ & \left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac12}\int_0^t g^{-1}(\tau)\,\mathcal{G}_1'(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^2} \\ \le\ & C\int_0^t g^{-1}(\tau)\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2} d\tau \le C\,\|f\|_{\mathbf{Y}} \end{aligned}$$

and

$$\begin{aligned} &\left\||\cdot|^{\alpha}\int_0^t g^{-1}(\tau)\,\mathcal{G}_1(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^2} \\ \le\ & C\int_0^t g^{-1}(\tau)\,\langle t-\tau\rangle^{\frac{\alpha}{2}}\left\|\langle\cdot\rangle^{\alpha}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2} d\tau \le C\,\|f\|_{\mathbf{Y}} \end{aligned}$$

for all $t\in[0,1]$, where $\alpha\in[0,\delta]$. We now consider $t\ge1$. In view of Lemma 2.2 we obtain

$$\begin{aligned} &\left\|(-\Delta)^{\frac{\alpha}{2}}\int_0^t g^{-1}(\tau)\,\mathcal{G}_1(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^2} \\ \le\ & \int_0^{\frac{t}{2}} g^{-1}(\tau)\left\|(-\Delta)^{\frac{\alpha}{2}}\,\mathcal{G}_1(t-\tau)\,f(\tau)\right\|_{\mathbf{L}^2} d\tau \\ & +\int_{\frac{t}{2}}^t g^{-1}(\tau)\left\|(-\Delta)^{\frac{\alpha}{2}}\,\mathcal{G}_1(t-\tau)\,f(\tau)\right\|_{\mathbf{L}^2} d\tau \\ \le\ & C\int_0^{\frac{t}{2}} g^{-1}(\tau)\,(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2} d\tau \\ & +C\int_0^{\frac{t}{2}} g^{-1}(\tau)\,(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\Bigg(\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}}\left\|\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \\ & +\left\|\langle\cdot\rangle^{\delta}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}}\left\|\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}}\Bigg) d\tau \\ & +C\int_{\frac{t}{2}}^t g^{-1}(\tau)\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac12} f(\tau)\right\|_{\mathbf{L}^2} d\tau, \end{aligned}$$

whence

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \int_0^t g^{-1}(\tau)\, \mathcal{G}_1(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
\leq\ & C \int_0^{\frac{t}{2}} g^{-1}(\tau)\,(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}}\, \tau^{\frac{\gamma}{2}-1} d\tau \\
&\times \sup_{t>0} \langle t\rangle^{1+\frac{n}{4}} \left(\langle t\rangle^{\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right. \\
&\left. + \left\| \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} + \langle t\rangle^{-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right) \\
&+ C \int_{\frac{t}{2}}^{t} g^{-1}(\tau)\, \tau^{-1-\frac{n}{4}-\frac{\alpha}{2}} d\tau \sup_{t>0} \langle t\rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \\
\leq\ & C t^{-\frac{n}{4}-\frac{\alpha}{2}} g^{-1}(t)\, \|f\|_{\mathbf{Y}}\,.
\end{aligned}$$

Similarly by virtue of the second estimate of Lemma 2.2 we have

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} \partial_t \int_0^t g^{-1}(\tau)\, \mathcal{G}_1(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
\leq\ & C \int_0^{\frac{t}{2}} g^{-1}(\tau)\,(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau \\
&+ C \int_0^{\frac{t}{2}} g^{-1}(\tau)\,(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1} \left(\left\| \langle\cdot\rangle^{\delta} \langle\Delta\rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle\Delta\rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \right. \\
&\left. + \left\| \langle\cdot\rangle^{\delta} \langle\Delta\rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle\Delta\rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}} \right) d\tau \\
&+ C \int_{\frac{t}{2}}^{t} g^{-1}(\tau)\, \langle t-\tau\rangle^{-1} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau,
\end{aligned}$$

whence

$$\begin{aligned}
&\left\| (-\Delta)^{\frac{\alpha}{2}} \partial_t \int_0^t g^{-1}(\tau)\, \mathcal{G}_1(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{L}^2} \\
\leq\ & C \int_0^{\frac{t}{2}} g^{-1}(\tau)\,(t-\tau)^{-\frac{\alpha+\gamma}{2}-\frac{n}{4}-1}\, \tau^{\frac{\gamma}{2}-1} d\tau \\
&\times \sup_{t>0} \langle t\rangle^{1+\frac{n}{4}} \left(\langle t\rangle^{\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right. \\
&\left. + \left\| \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} + \langle t\rangle^{-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right) \\
&+ C \int_{\frac{t}{2}}^{t} g^{-1}(\tau)\, \langle t-\tau\rangle^{-1} \tau^{-1-\frac{n}{4}-\frac{\alpha}{2}} d\tau \sup_{t>0} \langle t\rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\| (-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \\
\leq\ & C t^{-\frac{n}{4}-\frac{\alpha+1}{2}} g^{-1}(t)\, \|f\|_{\mathbf{Y}}\,.
\end{aligned}$$

Finally, for all $t \geq 1$, applying the third estimate of Lemma 2.2 we get

$$\left\| |\cdot|^{\delta} \int_0^t g^{-1}(\tau) \mathcal{G}_1(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^2} \leq C \int_0^{\frac{t}{2}} g^{-1}(\tau) \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} d\tau$$
$$+ C \int_0^{\frac{t}{2}} g^{-1}(\tau) (t-\tau)^{\frac{\delta-\gamma}{2} - \frac{n}{4}} \left(\left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}} \right.$$
$$\left. + \left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{\frac{n+2\gamma}{2\delta}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2}^{1-\frac{n+2\gamma}{2\delta}} \right) d\tau$$
$$+ C \int_{\frac{t}{2}}^t g^{-1}(\tau) \left(\left\| \langle \cdot \rangle^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} + \langle t-\tau \rangle^{\frac{\delta}{2}} \left\| \langle \Delta \rangle^{-\frac{1}{2}} f(\tau) \right\|_{\mathbf{L}^2} \right) d\tau,$$

whence

$$\begin{aligned} & \left\| |\cdot|^{\delta} \int_0^t g^{-1}(\tau) \mathcal{G}_1(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^2} \\ \leq \; & \left(\int_0^t g^{-1}(\tau) \tau^{\frac{\delta}{2} - \frac{n}{4} - 1} d\tau + \int_0^{\frac{t}{2}} g^{-1}(\tau) (t-\tau)^{\frac{\delta-\gamma}{2} - \frac{n}{4}} \tau^{\frac{\gamma}{2} - 1} d\tau \right) \\ & \times \sup_{t>0} \langle t \rangle^{1+\frac{n}{4}} \left(\left\| \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} + \langle t \rangle^{-\frac{\delta}{2}} \left\| |\cdot|^{\delta} \langle \Delta \rangle^{-\frac{1}{2}} f(t) \right\|_{\mathbf{L}^2} \right) \\ \leq \; & C \langle t \rangle^{\frac{\delta}{2} - \frac{n}{4}} g^{-1}(t) \|f\|_{\mathbf{Y}}. \end{aligned}$$

This completes the proof of Lemma 2.3. □

Consider the Cauchy problem

$$\left\{ \begin{array}{c} \frac{d}{dt}\left(h'(t)\left(e^t - \beta\right)\right) = \frac{2\lambda}{n\varepsilon\theta} e^t \int \mathcal{N}(v(t,x))\, dx \\ + \frac{n+2}{2h(t)} \left(h'(t)\right)^2 \left(e^t - \beta\right) - \beta h'(t), \\ h(0) = 1, h'(0) = 0, \end{array} \right. \tag{2.9}$$

where $v(t,x)$ is an auxiliary given function. Denote

$$g(t) = 1 + \mu \log \langle t \rangle, \ \mu = \frac{\lambda}{2n\pi} (\varepsilon\theta)^{\frac{2}{n}} \left(\frac{n}{n+2} \right)^{\frac{n}{2}} > 0$$

and define $v_0(t) = \varepsilon \sum_{j=0}^{1} \mathcal{G}_j(t) u_j$.

Lemma 2.4. *Suppose that*

$$\|v\|_{\mathbf{X}} \leq C\varepsilon, \ \ \|v(t) - v_0(t)\|_{\mathbf{L}^p} \leq C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $1 \leq p \leq \infty$, *then there exists a unique solution* $h(t) \in \mathbf{C}^1((0,\infty))$ *of the Cauchy problem* (2.9) *such that*

$$|h(t) - g(t)| \leq C\varepsilon^{\frac{2}{n}} \log g(t), \ \ |h'(t)| \leq C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-1} \tag{2.10}$$

for all $t > 0$.

Proof. Integration of (2.9) with respect to time yields

$$\begin{aligned} h'(t) &= \frac{2\lambda}{n\varepsilon\theta\left(e^t - \beta\right)} \int_0^t d\tau e^{\tau} \int \mathcal{N}(v(\tau,x))\, dx \\ &\quad + \frac{n+2}{2\left(e^t - \beta\right)} \int_0^t d\tau \frac{\left(e^{\tau} - \beta\right)}{h(\tau)} \left(h'(\tau)\right)^2 + \frac{\beta(1 - h(t))}{e^t - \beta}, \ h(0) = 1. \end{aligned} \tag{2.11}$$

Integration by parts gives us

$$\int_0^t d\tau e^{\tau} \int \mathcal{N}(v(\tau,x))\,dx = e^t \int \mathcal{N}(v(t,x))\,dx - \int \mathcal{N}(v(0,x))\,dx - \int_0^t d\tau e^{\tau} \int \partial_\tau \mathcal{N}(v(\tau,x))\,dx. \tag{2.12}$$

Therefore by virtue of (2.11) and (2.12) we have

$$\begin{cases} h'(t) = \frac{2\lambda}{n\varepsilon\theta} \int \mathcal{N}(v(t,x))\,dx + Q(t), \\ h(0) = 1, \end{cases} \tag{2.13}$$

where

$$\begin{aligned} Q(t) &= \frac{2\lambda}{n\varepsilon\theta(e^t-\beta)} \left(\beta \int \mathcal{N}(v(t,x))\,dx - \int \mathcal{N}(v(0,x))\,dx \right. \\ &\quad \left. - \int_0^t d\tau e^{\tau} \int \partial_\tau \mathcal{N}(v(\tau,x))\,dx \right) \\ &\quad + \frac{n+2}{2(e^t-\beta)} \int_0^t d\tau \frac{(e^{\tau}-\beta)}{h(\tau)} (h'(\tau))^2 + \frac{\beta(1-h(t))}{e^t-\beta}. \end{aligned}$$

We solve the Cauchy problem (2.13) by the successive approximations. Denote $h_0(t) = g(t)$ and define $h_{m+1}(t)$, $m \geq 0$ as a solution of the linearized Cauchy problem

$$\begin{cases} h'_{m+1}(t) = \frac{2\lambda}{n\varepsilon\theta} \int \mathcal{N}(v(t,x))\,dx + Q_m(t), \\ h_{m+1}(0) = 1, \end{cases} \tag{2.14}$$

where

$$\begin{aligned} Q_m(t) &= \frac{2\lambda}{n\varepsilon\theta(e^t-\beta)} \left(\beta \int \mathcal{N}(v(t,x))\,dx - \int \mathcal{N}(v(0,x))\,dx \right. \\ &\quad \left. - \int_0^t d\tau e^{\tau} \int \partial_\tau \mathcal{N}(v(\tau,x))\,dx \right) \\ &\quad + \frac{n+2}{2(e^t-\beta)} \int_0^t d\tau \frac{(e^{\tau}-\beta)}{h_m(\tau)} (h'_m(\tau))^2 + \frac{\beta(1-h_m(t))}{e^t-\beta}. \end{aligned}$$

We prove that for all $m \geq 0$,

$$|h_m(t) - g(t)| \leq C\varepsilon^{\frac{2}{n}} \log g(t), \quad |h'_m(t)| \leq C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-1}. \tag{2.15}$$

For $m = 0$ estimates (2.15) are valid. By induction we suppose that (2.15) is true for some $m \geq 0$. Then in view of the inequality $\|v\|_{\mathbf{X}} \leq C\varepsilon$, we see that $Q_m(t)$ has a better time decay

$$|Q_m(t)| \leq C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-\frac{3}{2}}$$

for all $t > 0$. Hence in view of (2.14)

$$\begin{aligned} &\left| h_{m+1}(t) - 1 - \frac{2\lambda}{\varepsilon n\theta} \int_0^t \int \mathcal{N}(v(\tau,x))\,dx d\tau \right| \leq C\varepsilon^{\frac{2}{n}}, \\ &\left| h'_{m+1}(t) \right| \leq C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-\frac{3}{2}} + \varepsilon^{-1} \left| \int \mathcal{N}(v(\tau,x))\,dx d\tau \right|. \end{aligned} \tag{2.16}$$

We write

$$\begin{aligned}&\int \mathcal{N}(v(t,x))\,dx - (\varepsilon\theta)^{1+\frac{2}{n}} \int \mathcal{N}(G(t,x))\,dx \\ (2.17)\quad =\ & \int (\mathcal{N}(v(t,x)) - \mathcal{N}(v_0))\,dx + \int \left(\mathcal{N}(v_0) - \mathcal{N}\left(\varepsilon\theta G\left(xt^{-\frac{1}{2}}\right)\right)\right)dx,\end{aligned}$$

where $G(t,x) \equiv (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$. By the condition

$$\|v(t) - v_0(t)\|_{\mathbf{L}^p} \le C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

we obtain

$$\left|\int (\mathcal{N}(v(t,x)) - \mathcal{N}(v_0))\,dx\right| \le C\varepsilon^{1+\frac{4}{n}} \langle t\rangle^{-1} g^{-1}(t) \tag{2.18}$$

and via Lemmas 2.1 and 2.2 we have

$$\left|\int (\mathcal{N}(v_0) - \mathcal{N}(\varepsilon\theta G(t,x)))\,dx\right| \le C\varepsilon^{1+\frac{2}{n}} \langle t\rangle^{-1-\gamma}. \tag{2.19}$$

A direct calculation shows

$$\frac{2\lambda}{\varepsilon n\theta} \int \mathcal{N}(\varepsilon\theta G(t,x))\,dx = \frac{2\lambda(\varepsilon\theta)^{\frac{2}{n}}}{n(4\pi t)^{1+\frac{n}{2}}} \int e^{-\left(1+\frac{2}{n}\right)\frac{|x|^2}{4t}}\,dx = \frac{\varkappa}{t}, \tag{2.20}$$

where $\varkappa = \frac{\lambda}{2n\pi}(\varepsilon\theta)^{\frac{2}{n}}\left(\frac{n}{n+2}\right)^{\frac{n}{2}}$. Therefore by virtue of (2.17)–(2.20) we get

$$\int \mathcal{N}(v(t,x))\,dx = g(t) + C\varepsilon^{1+\frac{2}{n}} \log g(t),$$

whence by (2.16) the estimates (2.15) follow with m replaced by $m+1$. In the same manner we estimate the differences

$$\begin{aligned}|h_{m+1}(t) - h_m(t)| &\le \frac{1}{2}|h_m(t) - h_{m-1}(t)| \text{ and} \\ \left|h'_{m+1}(t) - h'_m(t)\right| &\le \frac{1}{2}\left|h'_m(t) - h'_{m-1}(t)\right|.\end{aligned}$$

Hence there exists a unique solution $h(t) \in \mathbf{C}^1((0,\infty))$ of the Cauchy problem (2.9) satisfying estimates (2.10) for all $t > 0$. Lemma 2.4 is proved. □

3. Proof of Theorem 1.1

Following the method of paper [4] we make a change of the dependent variable $u(t,x) = e^{-\varphi(t)} v(t,x)$ in the damped wave equation (1.1) to get

$$\mathcal{L}v = f, \tag{3.1}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$ and

$$f = -\lambda e^{-\frac{2}{n}\varphi} \mathcal{N}(v) + 2\varphi' v_t + \left(\varphi'' - (\varphi')^2 + \varphi'\right) v.$$

Now we choose $\varphi(t)$ by the condition $\int f(t,x)\,dx=0$, i.e. $\varphi(t)$ is a solution of the following equation

$$
\begin{aligned}
&-\lambda e^{-\frac{2}{n}\varphi(t)}\int \mathcal{N}(v(t,x))\,dx+2\varphi'(t)\int v_t(t,x)\,dx\\
&+\left(\varphi''(t)-(\varphi'(t))^2+\varphi'(t)\right)\int v(t,x)\,dx=0,
\end{aligned}
\tag{3.2}
$$

and also we suppose that $\varphi(0)=\varphi'(0)=0$. Integrating (3.1) with respect to x and using (3.2) we obtain

$$
\frac{d}{dt}\int (v_t(t,x)+v(t,x))\,dx=0
$$

which implies

$$
\begin{aligned}
\int (v_t(t,x)+v(t,x))\,dx &= \int (v_t(0,x)+v(0,x))\,dx\\
&= \varepsilon\int (u_0(x)+u_1(x))\,dx=\varepsilon\theta,
\end{aligned}
\tag{3.3}
$$

since $u(0,x)=e^{-\varphi(0)}v(0,x)$ and $u_t(0,x)=-\varphi'(0)\,e^{-\varphi(0)}v(0,x)+e^{-\varphi(0)}v_t(0,x)$. By (3.3) we have

$$
e^{-t}\frac{d}{dt}\left(e^t\int v(t,x)\,dx\right)=\varepsilon\theta,
$$

whence it follows that

$$
\int v(t,x)\,dx=\varepsilon\theta\left(1-\beta e^{-t}\right),
$$

where $\beta=\frac{1}{\theta}\int u_1(x)\,dx$. By virtue of (3.3) and (3.2) we get

$$
\begin{aligned}
&\varphi''(t)\left(1-\beta e^{-t}\right)+\left(1+\beta e^{-t}\right)\varphi'(t)\\
=\;&\frac{\lambda}{\varepsilon\theta}e^{-\frac{2}{n}\varphi(t)}\int \mathcal{N}(v(t,x))\,dx+(\varphi'(t))^2\left(1-\beta e^{-t}\right).
\end{aligned}
\tag{3.4}
$$

We put $h(t)=e^{\frac{2}{n}\varphi(t)}$, then multiplying (3.4) by $e^{t+\frac{2}{n}\varphi(t)}$ we find

$$
\begin{aligned}
\frac{d}{dt}\left(h'(t)\left(e^t-\beta\right)\right) &= \frac{2\lambda}{n\varepsilon\theta}e^t\int \mathcal{N}(v(t,x))\,dx\\
&\quad+\frac{n+2}{2h(t)}(h'(t))^2\left(e^t-\beta\right)-\beta h'(t),
\end{aligned}
\tag{3.5}
$$

with initial conditions $h(0)=1$, $h'(0)=0$. Thus we obtain the following system of equations for $(v(t,x),h(t))$

$$
\begin{cases}
\mathcal{L}v=f,\\
\frac{d}{dt}\left(h'(t)\left(e^t-\beta\right)\right)=\frac{2\lambda}{n\varepsilon\theta}e^t\int \mathcal{N}(v(t,x))\,dx\\
\quad+\frac{n+2}{2h(t)}(h'(t))^2\left(e^t-\beta\right)-\beta h'(t),\\
v(0,x)=\varepsilon u_0(x),\ v_t(0,x)=\varepsilon u_1(x),\\
h(0)=1,h'(0)=0,
\end{cases}
\tag{3.6}
$$

where

$$\begin{aligned} f &= -\frac{\lambda}{h}\mathcal{N}(v) + \varphi'(t)\,v + 2\varphi'(t)\,v_t + \left(\varphi''(t) - (\varphi'(t))^2\right)v \\ &= -\frac{\lambda}{h}\mathcal{N}(v) - \frac{nh'(t)}{h(t)}\left(\frac{e^{-t}\beta}{\theta - e^{-t}\beta}v + v_t\right) \\ &\quad + \frac{\lambda v}{\varepsilon h(t)(\theta - \beta e^{-t})}\int \mathcal{N}(v(t,x))\,dx. \end{aligned}$$

We find a solution $(v(t,x), h(t))$ of the Cauchy problem (3.6) using the successive approximations method in the function space

$$\mathbf{Z} = \left\{(v,h) \in \mathbf{X} \times \mathbf{C}^1[0,\infty) ; \|(v,h)\|_{\mathbf{Z}} < \infty\right\},$$

where the norm

$$\begin{aligned} \|(v,h)\|_{\mathbf{Z}} &\equiv \|v\|_{\mathbf{X}} + \sup_{t>0}\sup_{1\le p\le\infty} g(t)\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{p}\right)}\|v(t) - v_0(t)\|_{\mathbf{L}^p} \\ &\quad + \sup_{t>0}(\log g(t))^{-1}|h(t) - g(t)| + \sup_{t>0}\langle t\rangle |h'(t)|, \end{aligned}$$

where

$$\begin{aligned} \|\phi\|_{\mathbf{X}} &= \sup_{t>0}\sup_{0\le\alpha\le\delta}\langle t\rangle^{\frac{n}{4}+\frac{\alpha}{2}}\left\|(-\Delta)^{\frac{\alpha}{2}}\phi(t)\right\|_{\mathbf{L}^2} \\ &\quad + \sup_{t>0}\sup_{0\le\alpha\le\delta}\langle t\rangle^{\frac{n}{4}+\frac{\alpha+1}{2}}\left\|(-\Delta)^{\frac{\alpha}{2}}\langle\Delta\rangle^{-\frac{1}{2}}\partial_t\phi(t)\right\|_{\mathbf{L}^2} \\ &\quad + \sup_{t>0}\langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}}\left\||\cdot|^{\delta}\phi(t)\right\|_{\mathbf{L}^2} \end{aligned}$$

and

$$v_0(t) = \varepsilon\sum_{j=0}^{1}\mathcal{G}_j(t)\,u_j,$$

$$g(t) = 1 + \mu\log\langle t\rangle,\ \ \mu = \frac{\lambda}{2n\pi}(\varepsilon\theta)^{\frac{2}{n}}\left(\frac{n}{n+2}\right)^{\frac{n}{2}}.$$

We now define $v_0(t) = \varepsilon\sum_{j=0}^{1}\mathcal{G}_j(t)\,u_j$, $h_0(t) = g(t)$, and for $(v_{m+1}(t), h_{m+1}(t))$, $m \ge 0$, we consider the linearized system of equations corresponding to (3.6)

$$\text{(3.7)}\qquad \begin{cases} \mathcal{L}v_{m+1} = f_m, \\ \frac{d}{dt}\left(h'_{m+1}(t)(e^t - \beta)\right) = \frac{2\lambda}{n\varepsilon\theta}e^t\int\mathcal{N}(v_m(t,x))\,dx \\ \quad + \frac{n+2}{2h_{m+1}(t)}\left(h'_{m+1}(t)\right)^2(e^t - \beta) - \beta h'_{m+1}(t), \\ v_{m+1}(0,x) = \varepsilon u_0(x),\ \ \partial_t v_{m+1}(0,x) = \varepsilon u_1(x), \\ h_{m+1}(0) = 1, h'_{m+1}(0) = 0, \end{cases}$$

where for $m \ge 1$,

$$\begin{aligned} f_m &= -\frac{\lambda}{h_{m+1}}\mathcal{N}(v_m) - \frac{nh'_{m+1}(t)}{h_{m+1}(t)}\left(\frac{e^{-t}\beta}{\theta - e^{-t}\beta}v_m + \partial_t v_m\right) \\ &\quad + \frac{\lambda v_m}{\varepsilon h_{m+1}(t)(\theta - \beta e^{-t})}\int\mathcal{N}(v_m(t,x))\,dx. \end{aligned}$$

We now prove that for all $m \ge 0$,

$$\text{(3.8)}\qquad \|v_m\|_{\mathbf{X}} \le C\varepsilon,\ \ \|v_m(t) - v_0(t)\|_{\mathbf{L}^p} \le C\varepsilon^{1+\frac{2}{n}}g^{-1}(t)\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)},$$

and

$$|h_m(t) - g(t)| \leq C\varepsilon^{\frac{2}{n}} \log g(t), \quad |\partial_t h_m(t)| \leq C\varepsilon^{\frac{2}{n}} \langle t \rangle^{-1} \tag{3.9}$$

for all $t > 0$, $1 \leq p \leq \infty$. By Lemmas 2.1 and 2.2 we see that (3.8) and (3.9) are valid for $m = 0$. We assume by induction that (3.8) and (3.9) are true for some m. By the definition of $h_m(t) = e^{\frac{2}{n}\varphi_m(t)}$, it follows that

$$\int f_m(t,x)\, dx = 0$$

and

$$\int v_{m+1}(t,x)\, dx = \varepsilon\theta\left(1 - \beta e^{-t}\right)$$

for all $t > 0$. We write equation $\mathcal{L}v_{m+1} = f_m$ in the integral form $v_{m+1} = v_0 + \int_0^t \mathcal{G}_1(t-\tau) f_m(\tau)\, d\tau$ and apply Lemma 2.3 to get

$$\|(v_{m+1} - v_0)\, g\|_{\mathbf{X}} \leq C\|f_m g\|_{\mathbf{Y}} \leq C\varepsilon^{1+\frac{2}{n}}, \tag{3.10}$$

where

$$\begin{aligned} \|f\|_{\mathbf{Y}} &= \sup_{t>0} \sup_{0\leq\alpha\leq\delta} \langle t\rangle^{1+\frac{n}{4}+\frac{\alpha}{2}} \left\|(-\Delta)^{\frac{\alpha}{2}} \langle\Delta\rangle^{-\frac{1}{2}} f(t)\right\|_{\mathbf{L}^2} \\ &\quad + \sup_{t>0} \langle t\rangle^{1+\frac{n}{4}-\frac{\delta}{2}} \left\||\cdot|^{\delta} \langle\Delta\rangle^{-\frac{1}{2}} f(t)\right\|_{\mathbf{L}^2}. \end{aligned}$$

Whence in view of the Sobolev embedding theorem it follows that

$$\|v_{m+1}(t) - v_0(t)\|_{\mathbf{L}^p} \leq C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}, \tag{3.11}$$

for all $t > 0$, $1 \leq p \leq \infty$. We also find by Lemma 2.4 that

$$|h_{m+1}(t) - g(t)| \leq C\varepsilon^{\frac{2}{n}} \log g(t), \quad \left|h'_{m+1}(t)\right| \leq C\varepsilon^{\frac{2}{n}} \langle t\rangle^{-1}. \tag{3.12}$$

Therefore the estimates (3.8) and (3.9) are valid for any m.

For the difference $w_m = v_{m+1} - v_m$ we get from (3.7)

$$\begin{cases} \mathcal{L}w_m = f_{m+1} - f_m, \\ w_m(0,x) = 0,\ \partial_t w_m(0,x) = 0. \end{cases}$$

Since

$$\int \left(f_{m+1}(t,x) - f_m(t,x)\right) dx = 0,$$

applying Lemma 2.3 we obtain

$$\|w_m\|_{\mathbf{X}} \leq \frac{1}{2}\|w_{m-1}\|_{\mathbf{X}}$$

and by Lemma 2.4

$$\sup_{t>0} (g(t))^{-1} |h_{m+1}(t) - h_m(t)| \leq \frac{1}{2}\|w_m\|_{\mathbf{X}}.$$

These estimates imply that there exists a unique solution $(v,h) \in \mathbf{Z}$ of the Cauchy problem (3.6) satisfying in view of (3.11) and (3.12) the estimates

$$\|v(t) - v_0(t)\|_{\mathbf{L}^p} \leq C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $1 \leq p \leq \infty$ and

$$\left|e^{\frac{2}{n}\varphi(t)} - g(t)\right| \leq C\varepsilon^{\frac{2}{n}} \log g(t)$$

for all $t > 0$. Since $u = e^{-\varphi} v$, we have

$$\begin{aligned} & \left\| u(t) e^{\varphi(t)} - \varepsilon\theta G(t,x) \right\|_{\mathbf{L}^p} \\ \leq \; & C \left\| u(t) e^{\varphi(t)} - v_0(t) \right\|_{\mathbf{L}^p} + C \left\| v_0(t) - \varepsilon\theta G(t,x) \right\|_{\mathbf{L}^p} \\ \leq \; & C\varepsilon^{1+\frac{2}{n}} g^{-1}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}, \end{aligned}$$

hence

$$\left\| u(t) - \varepsilon\theta G(t,x) e^{-\varphi(t)} \right\|_{\mathbf{L}^p} \leq C\varepsilon^{1+\frac{2}{n}} g^{-1-\frac{n}{2}}(t) \langle t \rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)}$$

for all $t > 0$, $1 \leq p \leq \infty$. This completes the proof of Theorem 1.1.

Acknowledgement

The works of E.I.K. and P.I.N. are partially supported by CONACYT. We are grateful to an unknown referee for many useful suggestions and comments.

References

[1] H. Fujita, *On the blowing-up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. of Tokyo, Sect. I, **13** (1966), 109-124. MR0214914 (35:5761)

[2] V.A. Galaktionov, S.P. Kurdyumov and A.A. Samarskii, *On asymptotic eigenfunctions of the Cauchy problem for a nonlinear parabolic equation,* Math. USSR Sbornik, **54** (1986), 421-455.

[3] K. Hayakawa, *On nonexistence of global solutions of some semi-linear parabolic equations*, Proc. Japan Acad., **49** (1973), 503-505. MR0338569 (49:3333)

[4] N. Hayashi, E.I. Kaikina and P.I. Naumkin, *Large time behavior of solutions to the dissipative nonlinear Schrödinger equation,* Proceedings of the Royal Soc. Edingburgh, **130**-A (2000), 1029-1043. MR1800091 (2001j:35253)

[5] N. Hayashi, E.I. Kaikina and P.I. Naumkin, *Global existence and time decay of small solutions to the Landau-Ginzburg type equations*, Journal d'Analyse Mathematique **90** (2003), 141-173. MR2001068 (2004g:35116)

[6] R. Ikehata and M. Ohta, *Critical exponents for semilinear dissipative wave equations in* $\mathbf{R}^N$, J. Math. Anal. Appl. **269** (2002), 87-97. MR1907875 (2003i:35193)

[7] G. Karch, *Selfsimilar profiles in large time asymptotics of solutions to damped wave equations*, Studia Math., **143** (2) (2000), 175-197. MR1813366 (2001k:35209)

[8] S. Kawashima, M. Nakao, K. Ono, *On the decay property of solutions to the Cauchy problem of the semilinear wave equation with a dissipative term*, J. Math. Soc. Japan, **47** (4) (1995), 617-653. MR1348752 (96g:35126)

[9] K. Kobayashi, T. Sirao and H. Tanaka, *On the growing up problem for semilinear heat equations*, J. Math. Soc. Japan, **29** (1977), 407-424. MR0450783 (56:9076)

[10] T.T. Li and Y. Zhou, *Breakdown of solutions to* $\Box u + u_t = |u|^{1+\alpha}$*, Discrete Contin. Dynam. Systems*, **1** (4) (1995), 503-520. MR1357291 (96m:35222)

[11] A. Matsumura, *On the asymptotic behavior of solutions of semi-linear wave equations*, Publ. Res. Inst. Math. Sci., Kyoto Univ., **12** (1976), 169-189. MR0420031 (54:8048)

[12] T. Narazaki, $\mathbf{L}^p - \mathbf{L}^q$ *estimates for the damped wave equations and their applications to semi-linear problem*, J. Math. Soc. Japan, **56** (2004), 585–626. MR2048476

[13] K. Nishihara, $\mathbf{L}^p - \mathbf{L}^q$ *estimates of solutions to the damped wave equation in 3-dimensional space and their application,* Math. Z, **244** (2003), 631-649. MR1992029

[14] K. Nishihara, $\mathbf{L}^p - \mathbf{L}^q$ *estimates for the 3-D damped wave equation and their application to the semilinear problem,* Seminor Notes of Math. Sci. **6** (2003), 69-83, Ibaraki University.

[15] K. Nishitani and H. Zhao, Decay properties of solutions to the Cauchy problem for the damped wave equation with absorption, preprint (2004).

[16] K. Ono, *Global existence and asymptotic behavior of small solutions for semilinear dissipative wave equations*, Discrete and Continuous Dynamical. Systems, 9 (2003), 651-662. MR1974531 (2004b:35237)

[17] G. Todorova and B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, C.R. Acad. Sci. Paris, Série I, **330** (2000) 557-562. MR1760438 (2001a:35025)

[18] G. Todorova and B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Diff. Equations, **174** (2001), 464-489. MR1846744 (2002k:35218)

[19] Q.S. Zhang, *A blow-up result for a nonlinear wave equation with damping: The critical case*, C.R. Acad. Sci. Paris, Série I, **333** (2001), 109-114. MR1847355 (2003d:35189)

Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka, 560-0043, Japan

E-mail address: nhayashi@math.wani.osaka-u.ac.jp

Departamento de Ciencias Básicas, Instituto Tecnológico de Morelia, Morelia CP 58120, Michoacán, Mexico

E-mail address: ekaikina@matmor.unam.mx

Instituto de Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico

E-mail address: pavelni@matmor.unam.mx