

## DANES' DROP THEOREM IN LOCALLY CONVEX SPACES

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ABSTRACT. Danes' Drop Theorem is generalized to locally convex spaces.

### 1. INTRODUCTION

Suppose that  $(E, \|\cdot\|)$  is a Banach space and  $B$  is the closed unit ball of  $E$ . By a drop  $D(x, B)$  determined by a point  $x$  with  $\|x\| > 1$ , we shall mean the convex hull of the set  $\{x\} \cup B$ . If a nonempty closed set  $A$  of Banach space  $E$  with a positive distance from the unit ball  $B$  is given, then there exists  $a \in A$  such that  $D(a, B) \cap A = \{a\}$ , which is the so-called Danes' Drop Theorem [1]. The drop theorem has found many applications in various situations (see, for instance, [2], [5], [6], [7], [8] and [9]) and it is equivalent to Ekeland's variational principle.

Modifying the concept underlying Danes' Drop Theorem, Rolewicz [3] and Giles and Kutzarova [4] introduced the notion of the drop and weak drop properties and many papers have appeared (see, for instance, [10], [11], [12] and [15]).

This note generalizes Danes' Drop Theorem to locally convex spaces and it is done by substituting "sequentially closed bounded convex set  $C$ " in the space for "the closed unit ball  $B$ " of the Banach space and " $A$  is strongly Minkowski separated from  $C$ " for " $A$  is a positive distance from  $B$ ".

### 2. MINKOWSKI SEPARATION OF SETS

**Definition 1.** Two nonempty subsets  $A, B$  of a locally convex space  $E$  are said to be Minkowski separated (resp., strongly Minkowski separated) if there exist a continuous Minkowski gauge  $p$  on  $E$  and a point  $x_0$  in  $E$  such that either  $p(x) > p(y)$  for all  $x \in A_{x_0} \equiv A + x_0$  and  $y \in B_{x_0} \equiv B + x_0$  or  $p(x) < p(y)$  for all  $x \in A_{x_0}$  and  $y \in B_{x_0}$  (resp., either  $\inf\{p(x); x \in A_{x_0}\} > \sup\{p(y); y \in B_{x_0}\}$  or  $\sup\{p(x); x \in A_{x_0}\} < \inf\{p(y); y \in B_{x_0}\}$ ).

We replace the Minkowski gauge  $p$  by a continuous linear functional in the above definition to obtain the common concept of separation sets. Clearly, two separated (resp., strongly separated) sets are Minkowski separation (resp., strong Minkowski separation) sets, if either of them is bounded.

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**Proposition 2.** *Suppose that  $A$  is a convex set in a normed linear space  $E$ , which is a positive distance from  $B \subset E$ , if either of  $A, B$  is bounded; then  $A, B$  are strongly Minkowski separated.*

*Proof.* Let  $d = d(A, B) > 0$  and let  $S = \{x \in E; d(A, x) \leq d/2\}$ . Then  $S$ , with nonempty interior, is a positive distance from  $B$ . Without loss of generality we assume that  $0 \in \text{int } S$  and let  $P_s$  be the Minkowski gauge of  $S$ . It suffices to show

$$R = \inf\{P_s(x) - P_s(y); x \in B, y \in S\} > 0.$$

Note  $d(S, B) \geq d/2$ .

Suppose to the contrary, that  $R = 0$ . Then we can choose sequences  $\{x_n\}, \{y_n\}$ , from  $B$  and  $S$ , resp., such that  $P_s(x_n) - P_s(y_n) \rightarrow 0$ ; since  $P_s(x_n) > 1, P_s(y_n) < 1$ , we get  $P_s(x_n) \rightarrow 1$ . Let  $k_n = P_s(x_n)^{-1}$ ; then  $P_s(k_n x_n) = 1, k_n x_n \in S$ .

(i) If  $B$  is bounded, then  $\|x_n - k_n x_n\| = (1 - k_n)\|x_n\| \rightarrow 0$ . This contradicts the hypothesis that  $d(S, B) \geq d/2$ .

(ii) If  $A$  is bounded, then  $S$  is bounded also. Thus, there exists a positive constant  $k \geq 1$  such that  $k^{-1}P_s(x) \leq \|x\| \leq kP_s(x)$  for all  $x$  in  $E$ . Therefore  $\{x_n\}$  is a bounded sequence. Hence

$$\frac{d}{2} \leq d(S, B) \leq \|x_n - k_n x_n\| \rightarrow 0,$$

a contradiction which completes the proof.

### 3. DANES' DROP THEOREM IN LOCALLY CONVEX SPACES

**Theorem 3.** *Given a sequentially closed bounded convex set  $C$  in a sequentially complete locally convex space  $(E, \tau)$ . For every sequentially closed set  $A$ , which is strongly Minkowski separated from  $C$ , there exists a point  $z \in A$  such that  $D(z, C) \cap A = \{z\}$ , where  $D(z, C) = \text{co}(C \cup z)$  and  $\text{co}$  stands for the convex hull operator.*

*Proof.* Without loss of generality we assume that  $0 \in C$ . Fix any  $u_0 \in A$ . Let  $G = \text{s-clco}(C \cup -C \cup \pm u_0)$  (s-cl denotes  $\tau$ -sequential closure operator) and let  $E_1 = \text{span } G$ . Next, let  $p$  be the Minkowski gauge by  $G$ ; then it is a norm on  $E_1$ .

First, we show that  $(E_1, p)$  is a Banach space. It suffices to show that the unit ball  $G$  of  $(E_1, p)$  is complete relative to  $p$ . Suppose that  $\{x_n\}$  is a  $\tau$ -Cauchy sequence since  $G$  is bounded and  $p$  is generated by  $G$ , and which implies  $\tau < \tau_p$  on  $E_1$  where  $\tau_p$  denotes the topology generated by the norm  $p$ . Since  $C$  is  $\tau$ -sequentially complete,  $x_n$  must be  $\tau$ -convergent to some point  $x_0$  of  $G$ . Given a positive number  $\varepsilon$ , there is an integer  $k$  such that  $p(x_m - x_n) > \varepsilon$  whenever  $m, n \geq k$ , or equivalently  $x_m - x_n \in \varepsilon G$ , whenever  $m, n \geq k$ , because  $G$  is  $\tau$ -sequentially closed,  $x_m - x_0 \in \varepsilon G$ , that is,  $P(x_m - x_0) \leq \varepsilon$  for all  $m \geq k$ . Therefore the sequence  $\{x_n\}$  converges to  $x_0$  relative to the norm topology  $\tau_p$ . Thus,  $G$  is complete relative to  $p$ .

Since  $C$  is bounded, convex and since  $A$  is strongly Minkowski separated from  $C$ , Proposition 2 implies that there exist a point  $x_0 \in E$  and a  $\tau$ -continuous Minkowski gauge  $p_1$  on  $E_1$  such that

$$p_1(x) \leq \alpha < \alpha + \varepsilon \leq p_1(y)$$

whenever  $x \in C + x_0, y \in A + x_0$  for some fixed  $\alpha, \varepsilon > 0$ .

Without loss of generality we can assume that  $x_0 = 0$  and write  $(\varepsilon \leq)d = \inf\{p_1(y) - p_1(x); x \in C, y \in A\}$ . It is easy to see that  $C$  is closed, bounded and

convex relative to the norm  $p_1$  of  $E_1$  and  $A \cap D(u_0, C)$  is also nonempty (it contains  $u_0$ , for example), closed and bounded. Now, define the function  $f$  on  $E_1$ , by

$$f(x) = \begin{cases} p_1(x), & x \in A \cap D(u_0, C), \\ \infty, & \text{otherwise;} \end{cases}$$

then  $f$  is a norm ( $p$ ) lower semicontinuous proper function on  $E_1$  since  $p_1$  is  $\tau$ -continuous on  $E$ . Choose  $\lambda > 0$  such that  $\text{diam } D(u_0, C) < d/\lambda$ , where the diameter of  $D(u_0, C)$  is in norm  $p$ . Use Ekeland's variational principle (see, for instance, [13] and [14]) to obtain a point  $z \in A \cap D(u_0, C)$  such that

$$f(x) + \lambda p(x - z) > f(z) \quad \text{for all } x (\neq z) \text{ in } E_1.$$

We claim that  $\{z\} = D(z, C) \cap A$ . Suppose that  $y \in D(z, C) \cap A$  with  $y \neq z$ . Then there exists  $0 < \mu < 1$  and  $v \in C$  such that  $y = (1 - \mu)z + \mu v$ , so that  $p_1(y) \leq (1 - \mu)p_1(z) + \mu p_1(v)$  and  $\mu d \leq \mu[p_1(z) - p_1(v)] \leq p_1(z) - p_1(y)$ . Hence

$$\begin{aligned} p_1(z) &= f(z) < f(y) + \lambda p(y - z) = f(y) + \lambda p(\mu(v - z)) \\ &= p_1(y) + \lambda \mu p(v - z) \leq p_1(y) + \lambda \mu \text{diam } D(u_0, C) \leq p_1(y) + \mu d \\ &\leq p_1(y) + (p_1(z) - p_1(y)) = p_1(z), \end{aligned}$$

a contradiction.

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