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Citation: Journal of Mathematical Physics **56**, 071505 (2015); doi: 10.1063/1.4926439
View online: http://dx.doi.org/10.1063/1.4926439
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JOURNAL OF MATHEMATICAL PHYSICS **56**, 071505 (2015)

Dark-bright soliton solutions with nontrivial polarization interactions for the three-component defocusing nonlinear Schrödinger equation with nonzero boundary conditions

Barbara Prinari,[1,2] Federica Vitale,[2] and Gino Biondini[3]
[1]*Department of Mathematics, University of Colorado Colorado Springs, Colorado Springs, Colorado 80918, USA*
[2]*Department of Mathematics and Physics, University of Salento and Sezione INFN, 73100 Lecce, Italy*
[3]*Department of Mathematics and Department of Physics, State University of New York, Buffalo, New York 14260, USA*

(Received 29 March 2015; accepted 25 June 2015; published online 17 July 2015)

We derive novel dark-bright soliton solutions for the three-component defocusing nonlinear Schrödinger equation with nonzero boundary conditions. The solutions are obtained within the framework of a recently developed inverse scattering transform for the underlying nonlinear integrable partial differential equation, and unlike dark-bright solitons in the two component (Manakov) system in the same dispersion regime, their interactions display non-trivial polarization shift for the two bright components. © *2015 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4926439]

I. INTRODUCTION

Multi-component defocusing nonlinear Schrödinger (NLS) systems have important applications in nonlinear optical media and in such as repulsive Bose-Einstein condensates (BECs). Recall that repulsive, cigar-shaped single-component BECs can be modeled by the scalar defocusing NLS equation.[29] Similarly, multi-component repulsive BECs are modeled by a defocusing vector NLS (VNLS) equation. Mathematically, these systems have been the subject of intense study for over 40 years, and renewed interest is motivated by recent experiments in multi-component BECs, which have highlighted a variety of dark-dark and dark-bright soliton solutions.[4,10,11] Moreover, multi-component VNLS equations also arise in nonlinear optics,[16,19,21] and the same kind of dark and dark-bright soliton solutions appears in optical media with Kerr nonlinearity and normal dispersion.[13,18,20,24,25]

To model dark-bright soliton interactions, one must consider solutions with non-zero boundary conditions (NZBC). In this paper, we are concerned with the 3-component defocusing VNLS equation

$$i\mathbf{q}_t + \mathbf{q}_{xx} - 2(\|\mathbf{q}\|^2 - q_o^2)\mathbf{q} = \mathbf{0}, \tag{1.1}$$

where $\mathbf{q}(x,t) = (q_1, q_2, q_3)^T$, and with the NZBC

$$\lim_{x\to\pm\infty} \mathbf{q}(x,t) = \mathbf{q}_\pm\,, \tag{1.2}$$

with $\|\mathbf{q}_\pm\| = q_o > 0$. The term proportional to q_o in Eq. (1.1) makes the boundary conditions $\mathbf{q}_\pm$ independent of time but can be removed by a simple gauge transformation.

Equation (1.1) is of course a completely integrable system, so its initial value problem can be solved by means of an appropriate inverse scattering transform (IST).[9] The IST for the case of zero boundary conditions (ZBC) was developed in Ref. 15 for the 2-component case [in fact, for this reason, the 2-component VNLS is often referred to as the Manakov system], and is easily extended to the multi-component case.[1] However, the IST for NLS systems with NZBC is much more challenging, and the analysis becomes increasingly more complicated with coupling and more so with the increase of the number of components. As a matter of fact, the IST for the scalar defocusing NLS with NZBC was first developed in 1973,[29] but the IST for the defocusing Manakov

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system with NZBC remained an open problem until 2006[22] (see also Refs. 5 and 6). A general formulation of the IST for the multi-component defocusing case was then introduced in Ref. 23 and subsequently further developed in Ref. 14.

In this work, we employ the machinery of Refs. 14 and 23 to derive novel dark-bright soliton solutions for three-component equation (1.1) and study the resulting soliton interactions. The solutions presented in this paper have one dark and two bright components, and, unlike dark-bright solitons in the Manakov system,[7] they exhibit nontrivial polarization interactions in the bright components, i.e., energy (and phase) exchanges between the bright components of the interacting solitons which are similar to those in focusing 2-component nonlinear Schrödinger systems.[15] To the best of our knowledge, this is the first time that non-trivial soliton polarization interactions have been reported in a defocusing system. Since such multicomponent NLS systems are relevant model equations in BECs and in nonlinear optics, the results of this work will have applications to both physical contexts.

Importantly, the nontrivial interaction of vector solitons in focusing coupled nonlinear Schrödinger systems has been shown to provide a mechanism for computation with solitons.[12,26] Jakubowsky *et al.*[12] expressed the energy redistributions as linear fractional transformations and described how vector-soliton logic gates could be developed. Later, Steiglitz[27] explicitly constructed such logic gates based on the shape changing collision properties and hence pointed out the possibility of designing an all optical computer equivalent to a Turing machine, at least in a mathematical sense. The results of the present work suggest the same kind of applications is possible in multicomponent nonlinear Schrödinger systems in the defocusing dispersion regime, thus opening up the opportunity for new theoretical and experimental investigations.

The structure of this work is the following: In Section II, we briefly recall the essential elements of the IST formalism developed in Refs. 14 and 23 (referring the reader to those works for more details). In Section III, we discuss the symmetries of the eigenfunctions and scattering data, which were not examined in Ref. 23. In Section IV, we give an overview of the discrete spectrum, and in Section V, we systematically analyze all the possible cases. In Section VI, we then use the results of Secs. II–V to write exact expressions for general soliton solutions, and we use these expressions to study the soliton interactions, including the interaction-induced polarization shift. Section VII contains some concluding remarks.

II. DIRECT AND INVERSE SCATTERING FOR THE 3-COMPONENT VNLS EQUATION WITH NZBC

VNLS equation (1.1) admits the following 4×4 Lax pair:

$$\varphi_x = X\,\varphi\,, \qquad \varphi_t = T\,\varphi, \tag{2.1}$$

where

$$X(x,t,k) = -ikJ + Q\,, \tag{2.2a}$$

$$T(x,t,k) = 2ik^2 J - iJ(Q_x - Q^2 + q_o^2) - 2kQ\,, \tag{2.2b}$$

and with

$$J = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & -I \end{pmatrix}, \qquad Q(x,t) = \begin{pmatrix} 0 & \mathbf{q}^\dagger \\ \mathbf{q} & O \end{pmatrix}, \tag{2.2c}$$

I and O denoting identity and zero matrix of appropriate size. That is, VNLS equation (1.1) is the zero curvature condition

$$X_t - T_x + [X,T] = 0\,, \tag{2.3}$$

where $[A,B] = AB - BA$ is the matrix commutator. In turn, Eq. (2.3) is the compatibility condition $\varphi_{tx} = \varphi_{xt}$ of overdetermined linear system (2.1).

For simplicity, we consider the case in which the asymptotic vectors $\mathbf{q}_\pm$ at $x \to \pm\infty$ are parallel. In this case, thanks to the $U(N)$ invariance of Eq. (1.1), without loss of generality they can be

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chosen in the form

$$\mathbf{q}_\pm = (0,0,q_\pm)^T\,, \qquad q_\pm = q_o\,\mathrm{e}^{i\theta_\pm}\,, \tag{2.4}$$

and $\theta_\pm$ arbitrary real constants.

As in the scalar case, the scattering problem (i.e., the first equation in Lax pair (2.1)) is self-adjoint, but the continuous spectrum $\mathbb{R}\setminus(-q_o,q_o)$ exhibits a gap, and the eigenfunctions are expressed in terms of $\lambda = (k^2-q_o^2)^{1/2}$.[29] To deal effectively with the branching of λ, one introduces a Riemann surface by gluing two copies of the complex k-plane in which λ takes on either value of the square root. One then defines a uniformization variable $z = k+\lambda$, such that the first/second copy of the complex k-plane is mapped into the upper/lower half of the z-plane.[8] The original variables are $k = \frac{1}{2}(z+q_o^2/z)$ and $\lambda = \frac{1}{2}(z-q_o^2/z)$. Expressing all functional dependence on k and λ in the IST in terms of z then eliminates the branching.

For all $z\in\mathbb{R}$, one defines the Jost solutions $\varphi_\pm(x,t,z)$ as the simultaneous solutions of both parts of the Lax pair with the free-particle asymptotic behavior,

$$\varphi_\pm(x,t,z) = E_\pm \mathrm{e}^{i\Theta} + o(1), \qquad x\to\pm\infty, \tag{2.5}$$

with $\Theta(x,t,z) = \mathrm{diag}(\theta_1,\ldots,\theta_4) = \Lambda x - \Omega t$, where

$$i\Lambda(z) = i\mathrm{diag}(-\lambda,k,k,\lambda),$$
$$-i\Omega(z) = -i\mathrm{diag}(-2k\lambda,k^2+\lambda^2,k^2+\lambda^2,2k\lambda)$$

are, respectively, the eigenvalue matrices of $X_\pm = \lim_{x\to\pm\infty} X$ and $T_\pm = \lim_{x\to\pm\infty} T$, and

$$E_\pm(z) = I + J\,Q_\pm/(iz) \tag{2.6}$$

is the corresponding eigenvector matrix, with $Q_\pm = \lim_{x\to\pm\infty} Q$. For future reference, it is convenient to provide the explicit expression of θ_j above as functions of the uniform variable z,

$$\theta_4(x,t,z) = -\theta_1(x,t,z) = \frac{1}{2}\left[(z-q_o^2/z)x - (z^2-q_o^4/z^2)t)\right], \tag{2.7a}$$

$$\theta_2(x,t,z) = \theta_3(x,t,z) = \frac{1}{2}\left[(z+q_o^2/z)x - (z^2+q_o^4/z^2)t\right]. \tag{2.7b}$$

Note that

$$\det\varphi_\pm = \gamma\,\mathrm{e}^{2i[kx-(k^2+\lambda^2)t]}, \qquad \gamma(z) = \det E_\pm = 1 - q_o^2/z^2\,, \tag{2.8}$$

so for $z\in\mathbb{R}\setminus\{\pm q_o\}$, one can introduce the scattering matrix $A(z) = (a_{ij})$ via

$$\varphi_-(x,t,z) = \varphi_+(x,t,z)A(z)\,, \tag{2.9}$$

with $\det A(z) = 1$. Also, since the Jost solutions solve the t-part of the Lax pair as well, all entries of the scattering matrix $A(z)$, which will enter in the definition of the scattering data, are time-independent.

Unlike the scalar case,[8,29] only two of the columns of each of $\varphi_\pm$ admit analytic continuation onto the complex z-plane, specifically the first and the last columns $\varphi_{\pm,1}$ and $\varphi_{\pm,4}$. This is major obstruction in the development of the IST, since the solution of the inverse problem requires complete sets of analytical (or, more generally, meromorphic) eigenfunctions. This problem was circumvented in Ref. 23, where a fundamental set of meromorphic eigenfunctions $\Xi^\pm(x,t,z)$ in each half plane was constructed using an appropriate extension of the scattering problem to higher-dimensional tensors. [Hereafter, subscripts $\pm$ denote normalization as $x\to\pm\infty$, whereas superscripts $\pm$ denote analyticity or meromorphicity in the upper/lower half of the complex z-plane. Also, a subscript j in a matrix will be used to refer to the jth column of the matrix.]

For $z\in\mathbb{R}$, the meromorphic eigenfunctions can be written in terms of the Jost eigenfunctions as follows:

$$\Xi^+(x,t,z) = \varphi_-(x,t,z)\alpha(z) = \varphi_+(x,t,z)\beta(z), \tag{2.10a}$$

$$\Xi^-(x,t,z) = \varphi_-(x,t,z)\tilde\beta(z) = \varphi_+(x,t,z)\tilde\alpha(z), \tag{2.10b}$$

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where α and $\tilde{\alpha}$ are the upper triangular matrices, while β and $\tilde{\beta}$ are the lower triangular ones. (In particular, the diagonal entries of α and $\tilde{\beta}$ are all unity.) Then Eq. (2.9) yields triangular decompositions of the scattering matrix $A(z) = \beta(z)\alpha^{-1}(z) = \tilde{\alpha}(z)\tilde{\beta}^{-1}(z)$, similarly to the N-wave interactions.[17] In turn, these decompositions allow one to express the entries of $\alpha, \beta, \tilde{\alpha}, \tilde{\beta}$ in terms of the minors of A, denoted as

$$A_{\binom{i_1,\ldots,i_p}{k_1,\ldots,k_p}} = \det\begin{pmatrix} a_{i_1k_1} & \cdots & a_{i_1k_p} \\ \vdots & \ddots & \vdots \\ a_{i_pk_1} & \cdots & a_{i_pk_p} \end{pmatrix},$$

where $1 \leqslant i_1 < \cdots < i_p \leqslant 4$ and $1 \leqslant k_1 < \cdots < k_p \leqslant 4$. In particular, the upper and lower principal minors of A are, respectively, determinants of the form

$$A_{[1,\ldots,p]} = A_{\binom{1,\ldots,p}{1,\ldots,p}}, \qquad A_{[p,\ldots,N]} = A_{\binom{p,\ldots,4}{p,\ldots,4}}$$

for $1 \leqslant p \leqslant 4$. Importantly, Eqs. (2.10) allow one to obtain the analyticity properties of the minors of A. The upper principal minors $A_{[1]}(z), A_{[1,2]}(z), A_{[1,2,3]}(z)$ are analytic in the upper half plane (UHP), while the lower principal minors $A_{[2,3,4]}(z), A_{[3,4]}(z), A_{[4]}(z)$ are analytic in the lower half plane (LHP). In addition, the following non-principal minors are also analytic:

$$A_{\binom{1,2}{1,3}}(z), A_{\binom{1,3}{1,2}}(z): \ \operatorname{Im} z > 0; \tag{2.11a}$$

$$A_{\binom{2,4}{3,4}}(z), A_{\binom{3,4}{2,4}}(z): \ \operatorname{Im} z < 0. \tag{2.11b}$$

Using these results, one can write a fundamental set of analytic eigenfunctions in either half plane as

$$\chi^{\pm}(x,t,z) = \Xi^{\pm}(x,t,z)D^{\pm}(z), \tag{2.12}$$

where

$$D^{+}(z) = \operatorname{diag}(1, A_{[1]}(z), A_{[1,2]}(z), A_{[1,2,3]}(z)),$$
$$D^{-}(z) = \operatorname{diag}(A_{[2,3,4]}(z), A_{[3,4]}(z), A_{[4]}(z), 1).$$

Note $\varphi_{-,1}(x,t,z) = \chi_1^{+}(x,t,z)$ and $\varphi_{+,4}(x,t,z) = \chi_4^{+}(x,t,z)$ are analytic in the UHP of z, while $\varphi_{+,1}(x,t,z) = \chi_1^{-}(x,t,z)$ and $\varphi_{-,4}(x,t,z) = \chi_4^{-}(x,t,z)$ are analytic in the LHP.

The inverse problem is formulated in terms of a Riemann-Hilbert problem (RHP) for the sectionally meromorphic matrix $M(x,t,z) = M^{\pm}$ for $\operatorname{Im} z \gtrless 0$, with

$$M^{+} = \left(\frac{\varphi_{-,1}}{A_{[1]}}, \frac{\chi_2^{+}}{A_{[1,2]}} - \frac{A_{\binom{1,3}{1,2}}\chi_3^{+}}{A_{[1,2]}A_{[1,2,3]}}, \frac{\chi_3^{+}}{A_{[1,2,3]}}, \varphi_{+,4}\right)\mathrm{e}^{-i\Theta}, \tag{2.13a}$$

$$M^{-} = \left(\varphi_{+,1}, \frac{\chi_2^{-}}{A_{[2,3,4]}}, \frac{\chi_3^{-}}{A_{[3,4]}} - \frac{A_{\binom{2,4}{3,4}}\chi_2^{-}}{A_{[3,4]}A_{[2,3,4]}}, \frac{\varphi_{-,4}}{A_{[4]}}\right)\mathrm{e}^{-i\Theta}. \tag{2.13b}$$

Indeed, manipulating scattering relation (2.9), one obtains the jump condition

$$M^{+} = M^{-}(I - \mathrm{e}^{-iK\Theta}L\mathrm{e}^{iK\Theta}), \quad z \in \mathbb{R}, \tag{2.14}$$

where $K = \operatorname{diag}(-1,1,1,-1)$ and $L(z)$ is explicitly determined in terms of the reflection coefficients of the problem: $\rho_1(z) = a_{21}(z)/a_{11(z)}$, $\rho_2(z) = a_{31}(z)/a_{11}(z)$, and $\rho_3(z) = a_{41}(z)/a_{11}(z)$, $a_{ij}(z)$ denoting the corresponding entries of the scattering matrix A. Of course in the reflectionless case, $L(z) \equiv 0$ for all $z \in \mathbb{R}$. Moreover, we see already that the zeros of the upper/lower principal minors of the scattering matrix $A_{[1]}, A_{[1,2]}, A_{[1,2,3]}$ and $A_{[2,3,4]}, A_{[3,4]}, A_{[4]}$ play the role of discrete eigenvalues of the scattering problem.

Since $M^{\pm}(x,t,z) \to I$ as $z \to \infty$, the use of standard Cauchy projectors will reduce RHP (2.14) to a system of linear integral equations. In addition, if a nontrivial discrete spectrum is present, as usual one must supplement the system with appropriate algebraic equations, obtained by computing the residues of $M^{\pm}(x,t,z)$ at the discrete eigenvalues. Finally, computing the asymptotic behavior

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of the solution of the RHP as $z \to \infty$ and comparing with the asymptotic behavior obtained from the direct problem allow one to write down a reconstruction formula for the solution of VNLS equation (1.1),

$$q_j(x,t) = -i \lim_{z\to\infty} z M_{j+1,1}(x,t,z), \qquad j = 1,2,3. \tag{2.15}$$

As usual, in the reflectionless case $[\rho_j(z) \equiv 0]$, the solution to RHP (2.14) reduces to a linear algebraic system and one obtains the pure soliton solutions.

III. SYMMETRIES

The richness of the 3-component VNLS equation compared to the Manakov system comes from the discrete spectrum and symmetries. In turn, these features result in a larger variety of soliton solutions, as we discuss next.

Similarly to the scalar case,[8,29] the Lax pair admits two involutions: $z \mapsto z^*$, mapping the UHP into the LHP and vice versa, and $z \mapsto q_o^2/z$, mapping the exterior of the circle $C_o = \{z \in \mathbb{C} : |z| = q_o\}$ into the interior and vice versa. The behavior of the analytic eigenfunctions under these symmetries is much more involved than in the scalar case, however, and it is obtained by first noting that, for $z \in \mathbb{R}$,

$$\varphi_\pm(x,t,z) = J[\varphi_\pm^\dagger(x,t,z)]^{-1} C = \varphi_\pm(x,t,q_o^2/z)\Pi_\pm, \tag{3.1}$$

with $\Pi_\pm(z) = \mathrm{diag}(0,1,1,0) - iJQ_\pm^*/z$ and $C(z) = \mathrm{diag}(\gamma(z),-1,-1,-\gamma(z))$, with $\gamma(z)$ defined in (2.8). One then expresses the non-analytic Jost eigenfunctions in Eq. (3.1) in terms of the columns of $\chi^\pm$ via Eqs. (2.10) and (2.12) and uses the Schwarz reflection principle to lift the resulting relations off the real axis.

For the purpose of describing the symmetries among Jost eigenfunctions and auxiliary eigenfunctions, it is convenient to introduce the notion of a "generalized cross product." For all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{C}^4$,

$$L[\mathbf{u},\mathbf{v},\mathbf{w}] = \begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \end{vmatrix}, \tag{3.2}$$

where $\{\mathbf{e}_1,\ldots,\mathbf{e}_4\}$ is the standard basis for $\mathbb{C}^4$. Like the usual cross product in three dimensions, $L[\cdot]$ is multilinear and totally antisymmetric.

Then, we have the following results.

First symmetry: $z \mapsto z^*$. The analytic columns of $\varphi_\pm(x,t,z)$ obey the following symmetry relations:

$$\varphi_{+,1}^*(x,t,z^*) = -\frac{e^{-2i\theta_2(x,t,z)}}{A_{[1,2]}(z)A_{[1,2,3]}(z)}\mathbf{J}\, L[\chi_2^+(x,t,z),\chi_3^+(x,t,z),\varphi_{+,4}(x,t,z)], \qquad \mathrm{Im}\, z \ge 0, \tag{3.3a}$$

$$\varphi_{-,1}^*(x,t,z^*) = -\frac{e^{-2i\theta_2(x,t,z)}}{A_{[4]}(z)A_{[3,4]}(z)}\mathbf{J}\, L[\chi_2^-(x,t,z),\chi_3^-(x,t,z),\varphi_{-,4}(x,t,z)], \qquad \mathrm{Im}\, z \le 0, \tag{3.3b}$$

$$\varphi_{+,4}^*(x,t,z^*) = -\frac{e^{-2i\theta_2(x,t,z)}}{A_{[3,4]}(z)A_{[2,3,4]}(z)}\mathbf{J}\, L[\varphi_{+,1}(x,t,z),\chi_2^-(x,t,z),\chi_3^-(x,t,z)], \qquad \mathrm{Im}\, z \le 0, \tag{3.3c}$$

$$\varphi_{-,4}^*(x,t,z^*) = -\frac{e^{-2i\theta_2(x,t,z)}}{A_{[1]}(z)A_{[1,2]}(z)}\mathbf{J}\, L[\varphi_{-,1}(x,t,z),\chi_2^+(x,t,z),\chi_3^+(x,t,z)], \qquad \mathrm{Im}\, z \ge 0. \tag{3.3d}$$

The auxiliary eigenfunctions obey the following symmetry relations:

$$[\chi_2^+(x,t,z^*)]^* = -\frac{e^{-2i\theta_2(x,t,z)}}{A_{[4]}(z)\gamma(z)}\mathbf{J}L[\varphi_{+,1}(x,t,z),\chi_3^-(x,t,z),\varphi_{-,4}(x,t,z)], \qquad \mathrm{Im}\, z \le 0, \tag{3.4a}$$

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$$[\chi_3^+(x,t,z^*)]^* = \frac{e^{-2i\theta_2(x,t,z)}}{A_{[2,3,4]}(z)\gamma(z)}\mathbf{J}L[\varphi_{+,1}(x,t,z),\chi_2^-(x,t,z),\varphi_{-,4}(x,t,z)], \qquad \operatorname{Im} z \le 0, \quad (3.4b)$$

$$[\chi_2^-(x,t,z^*)]^* = -\frac{e^{-2i\theta_2(x,t,z)}}{A_{[1,2,3]}(z)\gamma(z)}\mathbf{J}L[\varphi_{-,1}(x,t,z),\chi_3^+(x,t,z),\varphi_{+,4}(x,t,z)], \qquad \operatorname{Im} z \ge 0, \quad (3.4c)$$

$$[\chi_3^-(x,t,z^*)]^* = \frac{e^{-2i\theta_2(x,t,z)}}{A_{[1]}(z)\gamma(z)}\mathbf{J}L[\varphi_{-,1}(x,t,z),\chi_2^+(x,t,z),\varphi_{+,4}(x,t,z)], \qquad \operatorname{Im} z \ge 0. \quad (3.4d)$$

Second symmetry: $z \mapsto q_o^2/z$. The Jost eigenfunctions satisfy the following in/out symmetry relations:

$$\varphi_{\pm,1}(x,t,z) = \frac{iq_\pm^*}{z}\varphi_{\pm,4}(x,t,\hat{z}^*), \quad \operatorname{Im} z \lessgtr 0, \quad (3.5a)$$

$$\varphi_{\pm,4}(x,t,z) = -\frac{iq_\pm}{z}\varphi_{\pm,1}(x,t,\hat{z}^*), \quad \operatorname{Im} z \gtrless 0, \quad (3.5b)$$

where $\hat{z} = q_o^2/z^*$. Finally, the auxiliary eigenfunctions satisfy the following in/out symmetry relations:

$$\chi_2^+(x,t,\hat{z}^*) = \frac{e^{i\Delta\theta}}{A_{[3,4]}(z)}\left[A_{[4]}(z)\chi_2^-(x,t,z) + A_{\left(\substack{3,4\\2,4}\right)}(z)\chi_3^-(x,t,z)\right], \qquad \operatorname{Im} z \le 0, \quad (3.6a)$$

$$\chi_3^+(x,t,\hat{z}^*) = \frac{e^{i\Delta\theta}}{A_{[3,4]}(z)}\left[A_{[2,3,4]}(z)\chi_3^-(x,t,z) - A_{\left(\substack{2,4\\3,4}\right)}(z)\chi_2^-(x,t,z)\right], \qquad \operatorname{Im} z \le 0, \quad (3.6b)$$

$$\chi_2^-(x,t,\hat{z}^*) = \frac{e^{-i\Delta\theta}}{A_{[1,2]}(z)}\left[A_{[1,2,3]}(z)\chi_2^+(x,t,z) - A_{\left(\substack{1,3\\1,2}\right)}(z)\chi_3^+(x,t,z)\right], \qquad \operatorname{Im} z \ge 0, \quad (3.6c)$$

$$\chi_3^-(x,t,\hat{z}^*) = \frac{e^{-i\Delta\theta}}{A_{[1,2]}(z)}\left[A_{[1]}(z)\chi_3^+(x,t,z) + A_{\left(\substack{1,2\\1,3}\right)}(z)\chi_2^+(x,t,z)\right], \qquad \operatorname{Im} z \ge 0. \quad (3.6d)$$

For future reference, we also note the following symmetries for the phases θ_j in (2.7):

$$\theta_1(x,t,z_o) = \theta_4(x,t,\hat{z}_o^*), \qquad \theta_j(x,t,z_o) = \theta_j(x,t,\hat{z}_o^*), \quad j = 2,3. \quad (3.7)$$

Symmetries of the scattering coefficients and the meromorphic matrices $M^\pm(x,t,z)$. The above symmetries of the eigenfunctions induce corresponding symmetries for the scattering data. In particular, for $\operatorname{Im} z \gtrless 0$, one has

$$A_{[1]}(z) = A^*_{[2,3,4]}(z^*) = e^{i\Delta\theta}A_{[4]}(q_o^2/z) = e^{i\Delta\theta}A^*_{[1,2,3]}(q_o^2/z^*), \quad (3.8a)$$

$$A_{[1,2]}(z) = A^*_{[3,4]}(z^*), \quad A_{[1,2,3]}(z) = A^*_{[4]}(z^*) = e^{i\Delta\theta}A_{[2,3,4]}(q_o^2/z), \quad (3.8b)$$

$$A_{\left(\substack{1,2\\1,3}\right)}(z) = -A^*_{\left(\substack{3,4\\2,4}\right)}(z^*) = e^{i\Delta\theta}A_{\left(\substack{2,4\\3,4}\right)}(q_o^2/z), \quad (3.8c)$$

$$A_{\left(\substack{1,3\\1,2}\right)}(z) = -A^*_{\left(\substack{2,4\\3,4}\right)}(z^*). \quad (3.8d)$$

Moreover, the principal and analytic non-principal minors are related by

$$e^{i\Delta\theta}A_{[1,2]}(z)A^*_{[1,2]}(q_o^2/z^*) = A_{[1]}(z)A_{[1,2,3]}(z) + A_{\left(\substack{1,2\\1,3}\right)}(z)A_{\left(\substack{1,3\\1,2}\right)}(z), \qquad \operatorname{Im} z \geqslant 0, \quad (3.8e)$$

$$e^{-i\Delta\theta}A_{[3,4]}(z)A^*_{[3,4]}(q_o^2/z^*) = A_{[4]}(z)A_{[2,3,4]}(z) + A_{\left(\substack{2,4\\3,4}\right)}(z)A_{\left(\substack{3,4\\2,4}\right)}(z), \qquad \operatorname{Im} z \leqslant 0. \quad (3.8f)$$

Importantly, the above symmetries involving the analytic extra-minors show that once the value of one the minors is specified, the values of all other minors are uniquely determined. Moreover, symmetries (3.5) and (3.6) for the Jost eigenfunctions and auxiliary eigenfunctions, together with the above symmetries for the principal minors, imply the following symmetry relations for the columns of $M^\pm$ in (2.13):

$$M_1^-(z^*) = \frac{iq_+^*}{z^*}M_4^+(\hat{z}), \qquad M_1^+(z) = \frac{iq_+^*}{z}M_4^-(\hat{z}^*), \qquad M_j^+(z) = M_j^-(\hat{z}^*), \quad j = 2,3, \quad (3.9)$$

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which hold at any $z \in \mathbb{C}$ where the eigenfunctions are analytic. All the above symmetries play a crucial role in the characterization of the discrete spectrum. Indeed, it is the presence of $L[\cdot]$ and the non-principal analytic minors in Eqs. (3.3), (3.4), and (3.6) as well as in $M^{\pm}(x,t,z)$ [cf. (2.13)] that make the discrete spectrum and the corresponding soliton solutions of the 3-component case much richer and more complex than those of the Manakov system.

IV. DISCRETE SPECTRUM: OVERVIEW AND SUMMARY OF RESULTS

It is straightforward to show that the fundamental analytic eigenfunctions $\chi^{\pm}(x,t,z)$ introduced in (2.12) satisfy the relations

$$\det \chi^{+}(x,t,z) = A_{[1]}(z)A_{[1,2]}(z)A_{[1,2,3]}(z)\gamma(z)\mathrm{e}^{2i\theta_2(x,t,z)}, \tag{4.1a}$$

$$\det \chi^{-}(x,t,z) = A_{[4]}(z)A_{[3,4]}(z)A_{[2,3,4]}(z)\gamma(z)\mathrm{e}^{2i\theta_2(x,t,z)}. \tag{4.1b}$$

Also recall that the meromorphic matrices $M^{\pm}(x,t,z)$ for the inverse problem are defined in (2.13). Equations (2.13) and (4.1) show that the zeros of the upper/lower principal minors of the scattering matrix $A_{[1]}, A_{[1,2]}, A_{[1,2,3]}$ and $A_{[2,3,4]}, A_{[3,4]}, A_{[4]}$ are z-values at which the columns of the fundamental analytic eigenfunctions become linearly dependent and hence play the role of discrete eigenvalues of the scattering problem.

Since the latter is self-adjoint, bound states can only occur for $k \in \mathbb{R}$ [i.e., $|z| = q_o$]. As in the scalar case,[29] they correspond to values of $z = \zeta_o$ on the upper semicircle of radius q_o [with a counterpart ζ_o^* on the lower semicircle] where two of the Jost functions are proportional to each other, i.e., $\varphi_{-,1}(\zeta_o) = b_o\varphi_{+,4}(\zeta_o)$ [and correspondingly $\varphi_{-,4}(\zeta_o^*) = \bar{b}_o\varphi_{+,1}(\zeta_o^*)$], and these give rise to dark solitons. On the other hand, the analytic principal minors in Eq. (4.1) can have zeros for $|z| \neq q_o$. This is not a contradiction, as such solutions correspond to linear combinations involving the auxiliary eigenfunctions in addition to the Jost ones and hence do not lead to bound states for the eigenfunctions.[6,22] As in the Manakov system,[22] such zeros yield dark-bright solitons.[13,20,24,25]

When dealing with zeros of the principal minors off the circle, the symmetries discussed in Section III imply that discrete eigenvalues appear in symmetric quartets,

$$Z_n = \{z_n, z_n^*, \hat{z}_n, \hat{z}_n^*\}\,, \tag{4.2}$$

where we recall $\hat{z} = q_o^2/z^*$. Assuming the eigenvalues are simple, it is enough to specify the combination of zeros among the minors at the two point of the quartet on either half plane to completely determine the combination of zeros at each of the two points in the other half plane. Without loss of generality, we can therefore restrict ourselves to analyzing zeros in the upper half plane. This leaves seven possible combinations of zeros of the principal minors, as listed in Table I.

All seven combinations were analyzed in Ref. 14 in the special case when all the non-principal analytic minors are identically zero. Such cases were shown to correspond to straightforward analogues of the dark-bright soliton solutions of the defocusing Manakov system with NZBC. Explicitly, when the non-principal analytic minors are identically zero in each of the admissible cases

TABLE I. The possible combinations of zeros for the principal minors in the UHP. An asterisk denotes an arbitrary non-zero value. The rightmost column shows the results of the analysis when the extra minors are identically zero.[14] The corresponding results when the extra minors are non-zero will be given in Table III.

	$A_{[1]}(z_o)$	$A_{[1,2]}(z_o)$	$A_{[1,2,3]}b(z_o)$	Summary with extra minors zero
I	0	$*$	$*$	Admissible [symmetric to II]
II	$*$	$*$	0	Admissible [symmetric to I]
III	$*$	0	$*$	Inadmissible [contradiction]
IV	0	0	$*$	Admissible [symmetric to V]
V	$*$	0	0	Admissible [symmetric to IV]
VI	0	$*$	0	Inadmissible [contradiction]
VII	0	0	0	Inadmissible [contradiction]

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TABLE II. The possible combinations of zeros for the extra analytic minor $A_{\binom{1,2}{1,3}}(z)$ in the UHP.

	a	b	c	d
$A_{\binom{1,2}{1,3}}(z_o)$	0	*	*	0
$A_{\binom{1,2}{1,3}}(\hat{z}_o)$	*	0	*	0

in Table I, each eigenvalue quartet yields a dark-bright soliton in which the bright component is aligned exclusively with either the first or the second component of $\mathbf{q}(x,t)$, while the dark part is along the third component of $\mathbf{q}(x,t)$.[14] In this case, the soliton interactions are analogous to the ones in the 2-component system and do not exhibit a polarization shift.[7]

In this paper, we discuss the novel situation in which some of the non-principal analytic minors are non-zero. Hereafter, we will refer to such minors as the "extra minors" for brevity. We will see that, in some of these cases, each eigenvalue quartet leads to a dark-bright soliton in which the bright part has a non-zero projection along both of the first two components of $\mathbf{q}(x,t)$. We will also see that, as a result, the interactions of these solitons produce a non-trivial polarization shift.

As in the case when the extra minors are zero, once the values of all the analytic minors (including both the principal minors and the extra minors) at one point of the eigenvalue quartet have been specified, the values of all these minors at the remaining three points of the quartet are completely determined by symmetries (3.8). When the extra minors are not assumed to be identically zero, for each of the seven cases in Table I, there are four sub-cases, depending on whether each of the extra minors vanishes at exactly one of the points [$z_o, \hat{z}_o$ in UHP, or $z_o^*, \hat{z}_o^*$ in LHP] of a quartet of eigenvalues (cases a and b) at both points (case d), or at neither of the two points (case c). These sub-cases will be then labeled according to Table II.

In order to completely specify the inverse problem for the meromorphic matrices $M^{\pm}(x,t,z)$ in (2.13), one must obtain appropriate residue conditions for $M^{\pm}(x,t,z)$ at each point of the discrete spectrum. For this purpose, one must determine the relations among the fundamental analytic eigenfunctions at each point of the discrete spectrum. In turn, to this end, one needs to know the value of the analytic minors at *all four points of an eigenvalue quartet*. It is easy to see that, when the extra minors are identically zero, it is sufficient to give the values of the three principal minors at a single point in the quartet, since the values of the analytic minors at all other points of the quartet are completely determined by the symmetries. On the other hand, *when the extra minors are not identically zero, one must specify the value of one extra minor at both points of the quartet in the same half plane* — or, equivalently, the value of the two extra minors at the same point of the quartet. This is because, unlike the symmetries of $A_{[1]}$, $A_{[1,2,3]}$, $A_{[2,3,4]}$, and $A_{[4]}$ (whose values are completely determined from one another), symmetry (3.8e) for $A_{[1,2]}$ involves the two extra minors evaluated at a single point, or equivalently a single extra minor evaluated at both points in the same half plane. In other words, while the values of the principal minors $A_{[1]}$, $A_{[1,2,3]}$, $A_{[2,3,4]}$, and $A_{[4]}$ at the four points of an eigenvalue quartet are the same for all subcases of any given case, this is not necessarily true for $A_{[1,2]}$ and $A_{[3,4]}$. Such caveat must be taken into account in the analysis of the various subcases.

In Sections V and VI, we will analyze all the four subcases a–d in Table II for each of the seven cases I–VII in Table I. In each case we will establish the relations among the eigenfunctions at each point of a quartet of discrete eigenvalues, and we will solve the inverse problem in the reflectionless case to determine the corresponding soliton solutions. Similarly to what happens when the extra minors are identically zero, cases I and II are symmetric to each other, as are cases IV and V. Therefore, we will present the results for only one of each, namely, case I and case IV. Also, the same arguments as in Ref. 14 can be used to establish that all of the cases VIa–d and VIIa–d also lead to a contradiction when the extra minors are not identically zero. Hence, in the following, we will only discuss the various subcases of I, III, and IV. We will assume that all zeros of both principal and extra minors are simple. The results of Sections V and VI are summarized in Table III, which give the analogue of Table I in the case when the extra minors are not all identically zero.

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TABLE III. Summary of the admissible cases and corresponding solutions when the extra minors are not identically zero.

Cases	Summary with extra minors nonzero
Ia	Admissible [novel solution]
Ib	Admissible [standard solution, bright part aligned with q_1]
Ic	Admissible [novel solution, same as Ia]
Id	Admissible [standard solution, same as Ib]
IIIa	Inadmissible [contradiction]
IIIb	Inadmissible [contradiction]
IIIc	Admissible [constant solution]
IIId	Inadmissible [contradiction]
IVa (generic)	Admissible [standard solution, bright part aligned with q_2]
IVa (exceptional)	Admissible [standard solution, same as IVa generic]
IVb (generic)	Admissible [novel solution, same as Ia]
IVb (exceptional)	Admissible [standard solution, same as Ib]
IVc	Inadmissible [contradiction]
IVd (generic)	Admissible [standard solution, same as IVa]
IVd (exceptional)	Inadmissible [contradiction]
VI	Inadmissible [contradiction]
VII	Inadmissible [contradiction]

V. DISCRETE SPECTRUM: ANALYSIS WHEN THE EXTRA MINORS ARE NON-ZERO

A. Case I

For case I, bilinear symmetry (3.8e) implies that, while $A_{[1,2]}(z_o) \neq 0$ for all subcases of case I, $A_{[1,2]}(\hat{z}_o) = 0$ in subcases Ia, Ib, and Id, but $A_{[1,2]}(\hat{z}_o) \neq 0$ in subcase Ic. Specifically, for all of Ia, Ib, and Id, the combination of the zeros of the principal minors at all four points of a quartet of eigenvalues is given by the following diagram:

$$\begin{array}{cccc} & A_{[1]} & A_{[1,2]} & A_{[1,2,3]} \\ z_o & 0 & * & * \\ \hat{z}_o & * & 0 & 0 \\ \hline z_o^* & 0 & * & * \\ \hat{z}_o^* & * & 0 & 0 \\ & A_{[2,3,4]} & A_{[3,4]} & A_{[4]} \end{array}$$

Relations (3.3) and (3.4) are independent of the extra minors and therefore can be analyzed all together for all subcases of a given case with the same combination of zeros.

Analyzing (3.3a), (3.3d), (3.4c), and (3.4d) at z_o, on account of the zeros in the denominators, yields [omitting the x, t dependence for brevity]

$$L[\chi_2^+(z_o), \chi_3^+(z_o), \varphi_{+,4}(z_o)] \neq 0, \tag{5.1a}$$

$$L[\varphi_{-,1}(z_o), \chi_2^+(z_o), \chi_3^+(z_o)] = 0 \qquad \text{(simple zero)}, \tag{5.1b}$$

$$L[\varphi_{-,1}(z_o), \chi_3^+(z_o), \varphi_{+,4}(z_o)] \neq 0, \tag{5.1c}$$

$$L[\varphi_{-,1}(z_o), \chi_2^+(z_o), \varphi_{+,4}(z_o)] = 0. \tag{5.1d}$$

The first equation implies $\chi_2^+(z_o)$, $\chi_3^+(z_o)$, $\varphi_{+,4}(z_o)$ are linearly independent, and, in particular, $\chi_2^+(z_o) \neq 0$, $\chi_3^+(z_o) \neq 0$; similarly, the third one implies that $\varphi_{-,1}(z_o)$, $\chi_3^+(z_o)$, and $\varphi_{+,4}(z_o)$ are linearly independent as well. On the other hand, (5.1b) implies that $\varphi_{-,1}(z_o)$, $\chi_2^+(z_o)$, and $\chi_3^+(z_o)$ are linearly dependent, and we know the coefficient of $\varphi_{-,1}(z_o)$ in the linear combination needs to be nonzero [otherwise, we would have that $\chi_2^+(z_o), \chi_3^+(z_o)$ are linearly dependent, contradicting (5.1a)], so there exist complex constants c_1, c_2,

$$\varphi_{-,1}(z_o) = c_1\chi_2^+(z_o) + c_2\chi_3^+(z_o), \qquad c_1^2 + c_2^2 \neq 0. \tag{5.2}$$

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Substituting (5.2) into (5.1d), we obtain

$$c_2 L[\chi_3^+(z_o)\,,\chi_2^+(z_o)\,,\varphi_{+,4}(z_o)] = 0\,,$$

and taking into account (5.1a), we get $c_2 = 0$, so (5.2) reduces to

$$\varphi_{-,1}(z_o) = c_1 \chi_2^+(z_o)\,, \tag{5.3}$$

which provides the explicit relationship between the eigenfunctions at z_o.

Similarly, (3.3b), (3.3c), (3.4a), and (3.4b) at z_o^* yield

$$L[\chi_2^-(z_o^*)\,,\chi_3^-(z_o^*)\,,\varphi_{-4}(z_o^*)] \neq 0\,, \tag{5.4a}$$
$$L[\varphi_{+1}(z_o^*)\,,\chi_2^-(z_o^*)\,,\chi_3^-(z_o^*)] = 0\,, \tag{5.4b}$$
$$L[\varphi_{+1}(z_o^*)\,,\chi_3^-(z_o^*)\,,\varphi_{-4}(z_o^*)] \neq 0\,, \tag{5.4c}$$
$$L[\varphi_{+1}(z_o^*)\,,\chi_2^-(z_o^*)\,,\varphi_{-4}(z_o^*)] = 0\,, \tag{5.4d}$$

and as before, one can easily show that they imply

$$\chi_2^-(z_o^*) = d_1 \varphi_{+,1}(z_o^*)\,, \tag{5.5}$$

d_1 being an arbitrary complex constant.

A detailed analysis of (3.3) and (3.4) at the other two points in the quartet of eigenvalues $\hat{z}_o, \hat{z}_o^*$ yields the following results:

$$\left.\frac{L[\chi_2^+(z)\,,\chi_3^+(z)\,,\varphi_{+,4}(z)]}{A_{[1,2]}(z)}\right|_{z=\hat{z}_o} = 0 \qquad \text{(simple zero)}, \tag{5.6a}$$
$$L[\varphi_{-,1}(\hat{z}_o)\,,\chi_2^+(\hat{z}_o)\,,\chi_3^+(\hat{z}_o)] = 0 \qquad \text{(simple zero)}, \tag{5.6b}$$
$$L[\varphi_{-,1}(\hat{z}_o)\,,\chi_3^+(\hat{z}_o)\,,\varphi_{+,4}(\hat{z}_o)] = 0 \qquad [\text{double zero if } \chi_2^-(\hat{z}_o^*) = 0], \tag{5.6c}$$
$$L[\varphi_{-,1}(\hat{z}_o)\,,\chi_2^+(\hat{z}_o)\,,\varphi_{+,4}(\hat{z}_o)] = 0 \quad \text{iff} \quad \chi_3^-(\hat{z}_o^*) = 0, \tag{5.6d}$$
$$\left.\frac{L[\chi_2^-(z)\,,\chi_3^-(z)\,,\varphi_{-,4}(z)]}{A_{[3,4]}(z)}\right|_{z=\hat{z}_o^*} = 0 \qquad \text{(simple zero)}, \tag{5.6e}$$
$$L[\varphi_{+,1}(\hat{z}_o^*)\,,\chi_3^-(\hat{z}_o^*)\,,\varphi_{-,4}(\hat{z}_o^*)] = 0 \qquad [\text{double zero if } \chi_2^+(\hat{z}_o) = 0], \tag{5.6f}$$
$$L[\varphi_{+,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\varphi_{-4}(\hat{z}_o^*)] = 0 \quad \text{iff} \quad \chi_3^+(\hat{z}_o) = 0, \tag{5.6g}$$
$$L[\varphi_{+,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\chi_3^-(\hat{z}_o^*)] = 0 \qquad \text{(simple zero)}. \tag{5.6h}$$

To proceed further in determining the relationships among the eigenfunctions at $\hat{z}_o$ and $\hat{z}_o^*$, however, one must consider each subcase separately.

Case Ia. Including the extra minors, the combination of zeros in this case is given by

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	$*$	$*$	0	$*$
$\hat{z}_o$	$*$	0	0	$*$	0
z_o^*	0	$*$	$*$	$*$	0
$\hat{z}_o^*$	$*$	0	0	0	$*$
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

If we now evaluate Eqs. (3.6) at each point of the given quartet of eigenvalues, the symmetries provide

$$\chi_3^-(\hat{z}_o^*) = 0\,, \qquad \hat{\chi}_3^-(z) = \frac{\chi_3^-(z)}{A_{[3,4]}(z)} \qquad \text{finite at } \hat{z}_o^*\,, \tag{5.7a}$$

$$\chi_2^+(\hat{z}_o) = -\frac{A_{[1]}(\hat{z}_o)}{A_{\binom{1,2}{1,3}}(\hat{z}_o)}\chi_3^+(\hat{z}_o) \qquad \Rightarrow \qquad \chi_2^+(\hat{z}_o)\chi_3^+(\hat{z}_o) \neq 0\,, \tag{5.7b}$$

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[the latter follows from the former and (3.3d) at $\hat{z}_o$, since $\varphi_{-,4}(z) \neq 0$ for all $z \in \mathbb{C}^-$] (5.7b)

$$\chi_2^+(\hat{z}_o) = e^{i\Delta\theta} \frac{A_{[4]}(z_o^*)}{A_{[3,4]}(z_o^*)} \chi_2^-(z_o^*), \tag{5.7c}$$

$$\chi_2^+(z_o) = e^{i\Delta\theta} \left[\frac{A'_{[4]}(\hat{z}_o^*)}{A'_{[3,4]}(\hat{z}_o^*)} \chi_2^-(\hat{z}_o^*) + A_{\binom{3,4}{2,4}}(\hat{z}_o^*) \hat{\chi}_3^-(\hat{z}_o^*) \right], \tag{5.7d}$$

$$\chi_3^+(z_o) = e^{i\Delta\theta} \left[A_{[2,3,4]}(\hat{z}_o^*) \hat{\chi}_3^-(\hat{z}_o^*) - \frac{A'_{\binom{2,4}{3,4}}(\hat{z}_o^*)}{A'_{[3,4]}(\hat{z}_o^*)} \chi_2^-(\hat{z}_o^*) \right], \tag{5.7e}$$

$$\chi_2^-(\hat{z}_o^*) = \frac{e^{-i\Delta\theta}}{A_{[1,2]}(z_o)} \left[A_{[1,2,3]}(z_o) \chi_2^+(z_o) - A_{\binom{1,3}{1,2}}(z_o) \chi_3^+(z_o) \right]. \tag{5.7f}$$

Then, in order to determine how the eigenfunctions are related at $\hat{z}_o^*$, consider symmetry (3.5) on Equation (5.3) for $\varphi_{-,1}(z_o)$ and use (5.7d) to obtain

$$\varphi_{-,4}(\hat{z}_o^*) = -i \frac{z_o}{q_+^*} c_1 \frac{A'_{[4]}(\hat{z}_o^*)}{A'_{[3,4]}(\hat{z}_o^*)} \chi_2^-(\hat{z}_o^*) - i \frac{z_o}{q_+^*} c_1 A_{\binom{3,4}{2,4}}(\hat{z}_o^*) \hat{\chi}_3^-(\hat{z}_o^*).$$

Note that the above is consistent with all the L-combinations at $\hat{z}_o^*$ resulting from (5.6), and moreover, the latter imply that (i) $\hat{\chi}_3(\hat{z}_o^*) \neq 0$ [i.e., $\chi_3^-(z)$ cannot have a double zero at $\hat{z}_o^*$]; (ii) both coefficients in the linear combination for $\varphi_{-,4}(z)$ need to be nonzero.

Similarly, we can apply symmetry (3.5) on (5.5) and use (3.6b) evaluated at z_o^* to obtain

$$-e^{-i\Delta\theta} \frac{A_{[3,4]}(z_o^*)}{A_{\binom{2,4}{3,4}}(z_o^*)} \chi_3^+(\hat{z}_o) = d_1 \frac{i q_+^*}{z_o^*} \varphi_{+4}(\hat{z}_o),$$

and again the above is consistent with all the L-combinations at $\hat{z}_o$ resulting from (5.6). In particular, since both $\chi_2^+(z)$ and $\varphi_{+,4}(z)$ are proportional to $\chi_3^+(z)$ at $\hat{z}_o$, it follows that (5.6c) has indeed a double zero at $\hat{z}_o$.

Ultimately, aside from the equations that relate values of $\chi_2^\pm$ and $\chi_3^\pm$ at different points via the minors, the proportionality of the eigenfunctions is summarized by

$$\varphi_{-,1}(z_o) = c_1 \chi_2^+(z_o), \tag{5.8a}$$

$$\chi_2^-(z_o^*) = d_1 \varphi_{+,1}(z_o^*), \tag{5.8b}$$

$$\chi_3^+(\hat{z}_o) = e_1 \varphi_{+,4}(\hat{z}_o), \tag{5.8c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1 \chi_2^-(\hat{z}_o^*) + f_2 \hat{\chi}_3^-(\hat{z}_o^*), \tag{5.8d}$$

with

$$e_1 = -i \frac{q_-^*}{z_o^*} \frac{A_{\binom{2,4}{3,4}}(z_o^*)}{A_{[3,4]}(z_o^*)} d_1, \qquad f_1 = -i \frac{z_o}{q_+^*} c_1 \frac{A'_{[4]}(\hat{z}_o^*)}{A'_{[3,4]}(\hat{z}_o^*)} \neq 0, \qquad f_2 = -i \frac{z_o}{q_+^*} c_1 A_{\binom{3,4}{2,4}}(\hat{z}_o^*) \neq 0. \tag{5.9a}$$

Recall now that the meromorphic eigenfunctions that are used in the inverse problem are given by (2.13). Then, taking into account the combination of zeros for the analytic minors in case Ia, the pole structure in the inverse problem is as follows.

- At z_o, only M_1^+ has a pole.
- At $\hat{z}_o$, both M_2^+ and M_3^+ have a pole.
  Note that in M_2^+, both terms in the linear combination have a pole, and in fact at $\hat{z}_o$ using (3.6d), one finds

$$\chi_2^+(\hat{z}_o) - \frac{A'_{\binom{1,3}{1,2}}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)} \chi_3^+(\hat{z}_o) = - \frac{A_{[1]}(\hat{z}_o) A'_{[1,2,3]}(\hat{z}_o) + A'_{\binom{1,3}{1,2}}(\hat{z}_o) A_{\binom{1,2}{1,3}}(\hat{z}_o)}{A_{\binom{1,2}{1,3}}(\hat{z}_o) A'_{[1,2,3]}(\hat{z}_o)} \chi_3^+(\hat{z}_o),$$

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i.e., taking into account (3.6c) at $\hat{z}_o$ and (3.8e),

$$\chi_2^+(\hat{z}_o) - \frac{A'_{\left(\substack{1,3\\1,2}\right)}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)} \chi_3^+(\hat{z}_o) = e^{i\Delta\theta} \frac{A'_{[1,2]}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)} \chi_2^-(z_o^*) \neq 0. \tag{5.10}$$

- At z_o^* both M_2^- and M_3^- have a pole (in M_3^- the pole comes from the second term in the linear combination).
- At $\hat{z}_o^*$ only M_4^- has a pole.
  Note that in M_3^-, the first term is $\hat{\chi}_3^-(\hat{z}_o^*)$, which is finite, and the second term is finite as well since both $A_{\left(\substack{2,4\\3,4}\right)}(z)$ and $A_{[3,4]}(z)$ have a simple zero at $\hat{z}_o^*$, while $A_{[2,3,4]}(\hat{z}_o^*) \neq 0$, $\chi_2^-(\hat{z}_o^*) \neq 0$.

For the corresponding residues, we find

$$\mathrm{Res}_{z_o} M_1^+ = \tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))} + \tilde{c}_2 M_3^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))},$$

$$\mathrm{Res}_{\hat{z}_o} M_3^+ = \tilde{e}_1 M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))} \equiv -i\frac{z_o^*}{q_+^*} \tilde{e}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$

$$\mathrm{Res}_{\hat{z}_o} M_2^+ = -\frac{A_{[3,4]}(z_o^*)}{A_{\left(\substack{2,4\\3,4}\right)}(z_o^*)} \tilde{e}_1 M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))} \equiv \frac{q_o^2}{(z_o^*)^2} \tilde{d}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$

$$\mathrm{Res}_{z_o^*} M_2^- = \tilde{d}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$

$$\mathrm{Res}_{z_o^*} M_3^- = -\frac{A_{\left(\substack{2,4\\3,4}\right)}(z_o^*)}{A_{[3,4]}(z_o^*)} \tilde{d}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$

$$\mathrm{Res}_{\hat{z}_o^*} M_4^- = -i\frac{z_o}{q_-^*} c_1 \frac{A^*_{[3,4]}(z_o^*)}{A'_{[4]}(\hat{z}_o^*)} M_2^-(\hat{z}_o^*) e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))} + \tilde{f}_2 M_3^-(\hat{z}_o^*) e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))},$$

where in the second and third residue conditions, we have used second symmetry relation (3.5a) at z_o^*; moreover, the phases $\theta_j(z)$ are as defined in (2.7) and

$$\tilde{c}_1 = \frac{c_1 A_{[1,2]}(z_o)}{A'_{[1]}(z_o)}, \qquad \tilde{c}_2 = \tilde{c}_1 \frac{A_{\left(\substack{3,4\\2,4}\right)}(\hat{z}_o^*) e^{i\Delta\theta}}{A_{[1,2]}(z_o)}, \tag{5.11a}$$

$$\tilde{e}_1 = \frac{e_1}{A'_{[1,2,3]}(\hat{z}_o)}, \qquad \tilde{d}_1 = \frac{d_1}{A'_{[2,3,4]}(z_o^*)}, \qquad \tilde{f}_2 = \frac{f_2}{A'_{[4]}(\hat{z}_o^*)}. \tag{5.11b}$$

The next step is to determine the symmetries in the norming constants appearing above. We already have f_1, f_2 expressed in terms of c_1 and of one of the extra minors, and e_1 expressed in terms of d_1 and one of the extra minors [cf. (5.9)]. Now we can relate norming constants in UHP and LHP using the first symmetry.

First, we consider (3.4c) at z_o and get

$$\begin{aligned}[\chi_2^-(z_o^*)]^* &= -\frac{e^{-2i\theta_2(z_o)}}{A_{[1,2,3]}(z_o)\gamma(z_o)} \mathbf{J}L[\varphi_{-,1}(z_o), \chi_3^+(z_o), \varphi_{+,4}(z_o)] \\ &= -c_1 \frac{e^{-2i\theta_2(z_o)}}{A_{[1,2,3]}(z_o)\gamma(z_o)} \mathbf{J}L[\chi_2^+(z_o), \chi_3^+(z_o), \varphi_{+,4}(z_o)],\end{aligned}$$

where in the second equality, we have used (5.8a). Then we consider (3.3a) at z_o, multiply by d_1^*, and get from (5.8b),

$$[\chi_2^-(z_o^*)]^* = d_1^* \varphi_{+1}^*(z_o^*) = -\frac{e^{-2i\theta_2(z_o)}}{A_{[1,2,3]}(z_o) A_{[1,2]}(z_o)} d_1^* \mathbf{J}L[\chi_2^+(z_o), \chi_3^+(z_o), \varphi_{+,4}(z_o)].$$

Since the LHSs of the two equations above are equal, so are the RHSs, and this gives

$$d_1^* = c_1 \frac{A_{[1,2]}(z_o)}{\gamma(z_o)}.$$

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Taking the complex conjugate of (5.8d) and using (3.4b) at $\hat{z}_o^*$, we then obtain

$$\left[\chi_3^+(\hat{z}_o)\right]^* = \frac{e^{-2i\theta_2(\hat{z}_o^*)}}{A_{[2,3,4]}(\hat{z}_o^*)\gamma(\hat{z}_o^*)}\mathbf{J}L[\varphi_{-,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\varphi_{-,4}(\hat{z}_o^*)]$$
$$= f_2\frac{e^{-2i\theta_2(\hat{z}_o^*)}}{A_{[2,3,4]}(\hat{z}_o^*)\gamma(\hat{z}_o^*)}\mathbf{J}L[\varphi_{-,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\hat{\chi}_3^-(\hat{z}_o^*)]\,.$$

On the other hand, using (3.3c) at $\hat{z}_o^*$, we also have

$$\left[\chi_3^+(\hat{z}_o)\right]^* = e_1^*\varphi_{+,4}^*(\hat{z}_o) = -e_1^*\frac{e^{-2i\theta_2(\hat{z}_o^*)}}{A_{[2,3,4]}(\hat{z}_o^*)}\mathbf{J}L[\varphi_{-,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\hat{\chi}_3^-(\hat{z}_o^*)]\,,$$

and hence comparing the RHSs, we finally find

$$f_2 = -e_1^*\gamma(\hat{z}_o^*)\,.$$

Summarizing, using symmetries (3.8) among the analytic minors and the above equations, one can relate all other norming constants to c_1,

$$e_1 = i\frac{q_+^*}{\gamma^*(z_o)z_o^*}A^*_{\binom{3,4}{2,4}}(\hat{z}_o^*)c_1^*\,,\qquad d_1 = \frac{1}{\gamma^*(z_o)}c_1^*A^*_{[1,2]}(z_o)\,, \tag{5.12a}$$

$$f_2 = -i\frac{z_o}{q_+^*}c_1A_{\binom{3,4}{2,4}}(\hat{z}_o^*)\,, \tag{5.12b}$$

and write all residue coefficients in terms of two independent complex norming constants $\tilde{c}_1$ and $\tilde{c}_2$,

$$\mathrm{Res}_{z_o}M_1^+ = \tilde{c}_1M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))} + \tilde{c}_2M_3^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.13a}$$

$$\mathrm{Res}_{\hat{z}_o}M_3^+ = -i\frac{q_+^*q_0^2}{(z_o^*)^3\gamma^*(z_o)}\tilde{c}_2^*M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}$$
$$\equiv -\frac{q_o^2}{(z_o^*)^2\gamma^*(z_o)}\tilde{c}_2^*M_1^-(z_o^*)e^{i(\theta_2(z_o^*)-\theta_1(z_o^*))}\,, \tag{5.13b}$$

$$\mathrm{Res}_{\hat{z}_o}M_2^+ = -i\frac{q_+^*q_0^2}{(z_o^*)^3\gamma^*(z_o)}\tilde{c}_1^*M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}$$
$$\equiv -\frac{q_o^2}{(z_o^*)^2\gamma^*(z_o)}\tilde{c}_1^*M_1^-(z_o^*)e^{i(\theta_2(z_o^*)-\theta_1(z_o^*))}\,, \tag{5.13c}$$

$$\mathrm{Res}_{z_o^*}M_2^- = \frac{\tilde{c}_1^*}{\gamma^*(z_o)}M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.13d}$$

$$\mathrm{Res}_{z_o^*}M_3^- = \frac{\tilde{c}_2^*}{\gamma^*(z_o)}M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.13e}$$

$$\mathrm{Res}_{\hat{z}_o^*}M_4^- = i\frac{q_+}{z_o}\tilde{c}_1M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))} + i\frac{q_+}{z_o}\tilde{c}_2M_3^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,. \tag{5.13f}$$

Note that in (5.13f), we have used (3.9) to replace $M_2^+(z_o) = M_2^-(\hat{z}_o^*)$, $M_3^+(z_o) = M_3^-(\hat{z}_o^*)$.

Case Ib. This case corresponds to the following combinations of zeros of the minors:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	*	*	*	0
$\hat{z}_o$	*	0	0	0	*
z_o^*	0	*	*	0	*
$\hat{z}_o^*$	*	0	0	*	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

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Proceeding as in the previous case, we can show that

$$\chi_3^+(\hat{z}_o) = 0\,, \qquad \hat{\chi}_3^+(z) = \frac{\chi_3^+(z)}{A_{[1,2]}(z)} \qquad \text{finite at } \hat{z}_o\,, \tag{5.14a}$$

$$\chi_3^-(\hat{z}_o^*) = \frac{A_{\binom{2,4}{3,4}}(\hat{z}_o^*)}{A_{[2,3,4]}(\hat{z}_o^*)}\chi_2^-(\hat{z}_o^*) \qquad \Rightarrow \qquad \chi_3^-(\hat{z}_o^*), \chi_2^-(\hat{z}_o^*) \neq 0 \tag{5.14b}$$

and establish the following proportionality relations among the Jost and auxiliary eigenfunctions:

$$\varphi_{-,1}(z_o) = c_1\chi_2^+(z_o)\,, \tag{5.15a}$$

$$\chi_2^-(z_o^*) = d_1\varphi_{+,1}(z_o^*)\,, \tag{5.15b}$$

$$\chi_2^+(\hat{z}_o) = e_1\hat{\chi}_3^+(\hat{z}_o) + e_2\varphi_{+,4}(\hat{z}_o)\,, \tag{5.15c}$$

$$\varphi_{-4}(\hat{z}_o^*) = f_1\chi_2^-(\hat{z}_o^*)\,. \tag{5.15d}$$

Looking at Eqs. (2.13), the pole structure of $M^\pm$ is determined as follows: at z_o, only M_1^+ has a pole; at $\hat{z}_o$, only M_2^+ has a pole (from both terms in the linear combination); at z_o^*, only M_2^- has a pole; at $\hat{z}_o^*$, only M_4^- has a pole. On the other hand, M_3^+ is finite in $\hat{z}_o$; M_3^- is finite in z_o^*; M_3^- is finite at $\hat{z}_o^*$ using (5.14b).

Once the proportionality relationships of eigenfunctions (5.15) are taken into account, the residue conditions are the following:

$$\mathrm{Res}_{z_o}M_1^+ = \tilde{c}_1 M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \qquad \tilde{c}_1 = c_1\frac{A_{[1,2]}(z_o)}{A'_{[1]}(z_o)}\,,$$

$$\mathrm{Res}_{\hat{z}_o}M_2^+ = \tilde{e}_1 M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))} + \tilde{e}_2 M_3^+(\hat{z}_o)\,,$$

$$\tilde{e}_1 = \frac{e_2}{A'_{[1,2]}(\hat{z}_o)}\,, \qquad \tilde{e}_2 = \frac{A'_{[1,2,3]}(\hat{z}_o)}{A'_{[1,2]}(\hat{z}_o)}\left[\frac{e_1}{A'_{[1,2]}(\hat{z}_o)} - \frac{A_{\binom{1,3}{1,2}}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)}\right],$$

$$\mathrm{Res}_{z_o^*}M_2^- = \tilde{d}_1 M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \qquad \tilde{d}_1 = \frac{d_1}{A'_{[2,3,4]}(z_o^*)}\,,$$

$$\mathrm{Res}_{\hat{z}_o^*}M_4^- = \tilde{f}_1 M_2^-(\hat{z}_o^*)e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))}\,, \qquad \tilde{f}_1 = f_1\frac{A_{[2,3,4]}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)}\,.$$

Taking into account symmetries (3.8b) to relate all other norming constants to $\tilde{c}_1$, we find the following residue conditions:

$$\mathrm{Res}_{z_o}M_1^+ = \tilde{c}_1 M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.16a}$$

$$\mathrm{Res}_{\hat{z}_o}M_2^+ = \frac{\tilde{c}_1^*}{\gamma(\hat{z}_o)}M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.16b}$$

$$\mathrm{Res}_{z_o^*}M_2^- = \frac{\tilde{c}_1^*}{\gamma(z_o^*)}M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.16c}$$

$$\mathrm{Res}_{\hat{z}_o^*}M_4^- = \frac{iq_+}{z_o}\tilde{c}_1 M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.16d}$$

with $\tilde{c}_1 = c_1 A_{[1,2]}(z_o)/A'_{[1]}(z_o)$, an arbitrary complex constant. Note that in (5.16b) and (5.16d), we have used (3.9) to replace $M_4^+(\hat{z}_o)$ and $M_2^-(\hat{z}_o^*)$. Importantly, unlike case Ia, here there is only one degree of freedom in the choice of the complex norming constant, and as a result, the bright component in the corresponding soliton solution will be aligned exclusively along the first component of $\mathbf{q}(x,t)$ (see Sec. VI).

Case Id. It is worthwhile noticing that even though in this case all the analytic extra minors are zero at both points of the discrete spectrum in the half-plane where they are defined, this is not equivalent to the situation when the extra minors are identically zero, since one might get contributions in the residues from ratios of derivatives. This explains why all sub-cases d need to be revisited when the extra minors are not identically zero.

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The combination of zeros for the analytic minors in this case is given by

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	*	*	0	0
$\hat{z}_o$	*	0	0	0	0
z_o^*	0	*	*	0	0
$\hat{z}_o^*$	*	0	0	0	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

Using (3.6) into (5.6) we find that

$$\chi_3^+(\hat{z}_o) = 0\,, \qquad \hat{\chi}_3^+(z) = \frac{\chi_3^+(z)}{A_{[1,2]}(z)} \qquad \text{finite at } \hat{z}_o\,, \tag{5.17}$$

$$\chi_3^-(\hat{z}_o^*) = 0\,, \qquad \hat{\chi}_3^-(z) = \frac{\chi_3^-(z)}{A_{[3,4]}(z)} \qquad \text{finite at } \hat{z}_o^*\,, \tag{5.18}$$

while $\chi_2^+(z_o) \neq 0$ and $\chi_2^-(\hat{z}_o^*) \neq 0$.

The following proportionality relations among the Jost and auxiliary eigenfunctions hold

$$\varphi_{-,1}(z_o) = c_1\chi_2^+(z_o)\,, \tag{5.19a}$$

$$\chi_2^-(z_o^*) = d_1\varphi_{+,1}(z_o^*)\,, \tag{5.19b}$$

$$\chi_2^+(\hat{z}_o) = e_1\varphi_{+,4}(\hat{z}_o)\,, \tag{5.19c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1\chi_2^-(\hat{z}_o^*)\,. \tag{5.19d}$$

Looking at Eqs. (2.13), we see that the pole structure is as follows: at z_o, only M_1^+ has a pole; at $\hat{z}_o$, only M_2^+ has a pole (coming from the first term in the linear combination); at z_o^*, only M_2^- has a pole; at $\hat{z}_o^*$, only M_4^- has a pole. Moreover, M_3^+ is finite at $\hat{z}_o$ and M_3^- is finite at $\hat{z}_o^*$.

Once the proportionality relationships of eigenfunctions (5.19) are taken into account, we can write the following residue conditions:

$$\mathrm{Res}_{z_o} M_1^+ = \tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \qquad \tilde{c}_1 = c_1\frac{A_{[1,2]}(z_o)}{A'_{[1]}(z_o)}\,,$$

$$\mathrm{Res}_{\hat{z}_o} M_2^+ = \tilde{e}_1 M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}\,, \qquad \tilde{e}_1 = \frac{e_1}{A'_{[1,2]}(\hat{z}_o)}\,,$$

$$\mathrm{Res}_{z_o^*} M_2^- = \tilde{d}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \qquad \tilde{d}_1 = \frac{d_1}{A'_{[2,3,4]}(z_o^*)}\,,$$

$$\mathrm{Res}_{\hat{z}_o^*} M_4^- = \tilde{f}_1 M_2^-(\hat{z}_o^*) e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))}\,, \qquad \tilde{f}_1 = f_1\frac{A_{[2,3,4]}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)}\,.$$

The symmetries in the eigenfunctions, norming constants, and analytic minors finally yield

$$\mathrm{Res}_{z_o} M_1^+ = \tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.20a}$$

$$\mathrm{Res}_{\hat{z}_o} M_2^+ = \frac{\tilde{c}_1^*}{\gamma(\hat{z}_o)} M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.20b}$$

$$\mathrm{Res}_{z_o^*} M_2^- = \frac{\tilde{c}_1^*}{\gamma(z_o^*)} M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.20c}$$

$$\mathrm{Res}_{\hat{z}_o^*} M_4^- = \frac{iq_+}{z_o}\tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.20d}$$

with $\tilde{c}_1$ arbitrary complex constant. Note that above we have used (3.9) to replace $M_4^+(\hat{z}_o)$ and $M_2^-(\hat{z}_o^*)$.

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Since the residue conditions are the same of case Ib [cf. (5.16)], this case will therefore lead to the same solution.

Case Ic. In this case, all extra-minors are nonzero at all points of a given quartet of eigenvalues, and as a consequence, the combination of zeros for the principal minors in case Ic has to be modified so as to be compatible with symmetry (3.8e). Therefore, said combination of zeros is as follows:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	*	*	*	*
$\hat{z}_o$	*	*	0	*	*
z_o^*	0	*	*	*	*
$\hat{z}_o^*$	*	*	0	*	*
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

Eqs. (5.6) that follow from (3.3) and (3.4) will be modified accordingly, and the proportionality of the eigenfunctions is summarized by

$$\varphi_{-,1}(z_o) = c_1 \chi_2^+(z_o)\,, \tag{5.21a}$$

$$\chi_3^+(\hat{z}_o) = d_1 \varphi_{+,4}(\hat{z}_o)\,, \tag{5.21b}$$

$$\chi_2^-(z_o^*) = e_1 \varphi_{+,1}(z_o^*)\,, \tag{5.21c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1 \chi_3^-(\hat{z}_o^*)\,. \tag{5.21d}$$

Looking at Eqs. (2.13), the pole structure of $M^\pm$ is the following: M_1^+ has a pole at z_o; M_2^+ has a pole at $\hat{z}_o$ (from the second term in the linear combination); M_3^+ has a pole at $\hat{z}_o$; M_2^- has a pole at z_o^*; M_3^- has a pole at z_o^* (from the second term in the linear combination); M_4^- has a pole at $\hat{z}_o^*$.

Once the proportionality relationships of the eigenfunctions are taken into account, the residue conditions have the following form:

$$\mathrm{Res}_{z_o} M_1^+ = \left[\tilde{c}_1 M_2^+(z_o) + \tilde{c}_2 M_3^+(z_o)\right] e^{i(\theta_2(z_o)-\theta_1(z_o))}\,,$$

$$\tilde{c}_1 = c_1 \frac{A_{[1,2]}(z_o)}{A'_{[1]}(z_o)}\,, \quad \tilde{c}_2 = \tilde{c}_1 \frac{A_{\binom{1,3}{1,2}}(z_o)}{A_{[1,2]}(z_o)}\,,$$

$$\mathrm{Res}_{\hat{z}_o} M_3^+ = \tilde{d}_1 M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}\,, \quad \tilde{d}_1 = \frac{d_1}{A'_{[1,2,3]}(\hat{z}_o)}\,,$$

$$\mathrm{Res}_{\hat{z}_o} M_2^+ = \tilde{d}_2 M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}\,, \quad \tilde{d}_2 = -\tilde{d}_1 \frac{A_{\binom{1,3}{1,2}}(\hat{z}_o)}{A_{[1,2]}(\hat{z}_o)}\,,$$

$$\mathrm{Res}_{z_o^*} M_2^- = \tilde{e}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \quad \tilde{e}_1 = \frac{e_1}{A'_{[2,3,4]}(z_o^*)}\,,$$

$$\mathrm{Res}_{z_o^*} M_3^- = \tilde{e}_2 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \quad \tilde{e}_2 = -\tilde{e}_1 \frac{A_{\binom{2,4}{3,4}}(z_o^*)}{A_{[3,4]}(z_o^*)}\,,$$

$$\mathrm{Res}_{\hat{z}_o^*} M_4^- = \left[\tilde{f}_1 M_2^-(\hat{z}_o^*) + \tilde{f}_2 M_3^-(\hat{z}_o^*)\right] e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))}\,,$$

$$\tilde{f}_2 = f_1 \frac{A_{[3,4]}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)}\,, \quad \tilde{f}_1 = \tilde{f}_2 \frac{A_{\binom{2,4}{3,4}}(\hat{z}_o^*)}{A_{[3,4]}(\hat{z}_o^*)}\,.$$

The symmetries in the eigenfunctions and analytic minors then yield the following residue relationships:

$$\mathrm{Res}_{z_o} M_1^+ = \tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))} + \tilde{c}_2 M_3^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.22a}$$

$$\mathrm{Res}_{\hat{z}_o} M_2^+ = -i \frac{q_+^* q_0^2}{(z_o^*)^3 \gamma^*(z_o)} \tilde{c}_1^* M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}$$

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$$\equiv -\frac{q_o^2}{(z_o^*)^2\gamma^*(z_o)}\tilde{c}_1^* M_1^-(z_o^*)e^{i(\theta_2(z_o^*)-\theta_1(z_o^*))}, \tag{5.22b}$$

$$\text{Res}_{\hat{z}_o} M_3^+ = -i\frac{q_+^* q_0^2}{(z_o^*)^3\gamma^*(z_o)}\tilde{c}_2^* M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}$$

$$\equiv -\frac{q_o^2}{(z_o^*)^2\gamma^*(z_o)}\tilde{c}_2^* M_1^-(z_o^*)e^{i(\theta_2(z_o^*)-\theta_1(z_o^*))}, \tag{5.22c}$$

$$\text{Res}_{z_o^*} M_2^- = \frac{\tilde{c}_1^*}{\gamma^*(z_o)} M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}, \tag{5.22d}$$

$$\text{Res}_{z_o^*} M_3^- = \frac{\tilde{c}_2^*}{\gamma^*(z_o)} M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}, \tag{5.22e}$$

$$\text{Res}_{\hat{z}_o^*} M_4^- = i\frac{q_+}{z_o}\tilde{c}_1 M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))} + i\frac{q_+}{z_o}\tilde{c}_2 M_3^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}, \tag{5.22f}$$

with $\tilde{c}_1$ and $\tilde{c}_2$ arbitrary, independent complex constants [note that in the second and third equations, we have used symmetry relation (3.5a) at z_o^*, and in the last equation, $M_2^-(\hat{z}_o^*) = M_2^+(z_o)$ and $M_3^-(\hat{z}_o^*) = M_3^+(z_o)$ from (3.9)].

Since the residue conditions are the same of case Ia, this case will therefore lead to the same solution.

B. Discrete spectrum: Cases III and IV

1. Case III

This case corresponds to $A_{[1,2]}(z)$ in the UHP [and $A_{[3,4]}(z)$, by symmetry, in the LHP] only having zeros. Assume $A_{[1,2]}(z_o) = 0$ [and $A_{[1,2]}(\hat{z}_o) \neq 0$ so that we are dealing with a fundamental solution]. Symmetry (3.8e) then implies that none of the extra minors can vanish at any of the quartet points where they are defined, namely,

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	*	0	*	*	*
$\hat{z}_o$	*	*	*	*	*
z_o^*	*	0	*	*	*
$\hat{z}_o^*$	*	*	*	*	*
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

and also that

$$A_{[1]}(z_o)A_{[1,2,3]}(z_o) = -A_{\binom{1,2}{1,3}}(z_o)A_{\binom{1,3}{1,2}}(z_o), \tag{5.23a}$$

$$A_{[4]}(z_o^*)A_{[2,3,4]}(z_o^*) = -A_{\binom{2,4}{3,4}}(z_o^*)A_{\binom{3,4}{2,4}}(z_o^*). \tag{5.23b}$$

In fact, the above relationships also hold at $\hat{z}_o$ and $\hat{z}_o^*$, respectively. Essentially, this means that only case IIIc is allowed, which also explains why one would get a contradiction in the case when the extra minors are all identically zero [cf. Table I].

From (3.6c) and (3.6d) at z_o, we have

$$\begin{pmatrix} A_{[1,2,3]}(z_o) & -A_{\binom{1,3}{1,2}}(z_o) \\ A_{\binom{1,2}{1,3}}(z_o) & A_{[1]}(z_o) \end{pmatrix}\begin{pmatrix} \chi_2^+(z_o) \\ \chi_3^+(z_o) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and since the determinant of the coefficient matrix is zero [cf. (5.23a); incidentally, this is consistent with the fact that χ_2^+ and χ_3^- cannot both vanish at a point, or (3.3) would imply one of the Jost

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eigenfunctions is zero at that point], it means $\chi_2^+(z_o)$ and $\chi_3^+(z_o)$ are proportional,

$$\chi_2^+(z_o) = \frac{A_{\binom{1,3}{1,2}}(z_o)}{A_{[1,2,3]}(z_o)}\chi_3^+(z_o) \equiv -\frac{A_{[1]}(z_o)}{A_{\binom{1,2}{1,3}}(z_o)}\chi_3^+(z_o)\,. \tag{5.24a}$$

Similarly, from (3.6a) and (3.6b) at z_o^* follows

$$\begin{pmatrix} A_{[4]}(z_o^*) & A_{\binom{3,4}{2,4}}(z_o^*) \\ -A_{\binom{2,4}{3,4}}(z_o^*) & A_{[2,3,4]}(z_o^*) \end{pmatrix}\begin{pmatrix} \chi_2^-(z_o^*) \\ \chi_3^-(z_o^*) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

and therefore

$$\chi_2^-(z_o^*) = -\frac{A_{\binom{3,4}{2,4}}(z_o^*)}{A_{[4]}(z_o^*)}\chi_3^-(z_o^*) \equiv \frac{A_{[2,3,4]}(z_o^*)}{A_{\binom{2,4}{3,4}}(z_o^*)}\chi_3^-(z_o^*)\,. \tag{5.24b}$$

Then, looking at (2.13), we find that all columns of M^+ and M^- are analytic: specifically, from (5.24), it follows that the columns M_2^+ and M_3^- have no poles, and this corresponds to the trivial solution $\mathbf{q}(x,t) = \mathbf{q}_+$.

2. Case IV

Case IV requires some extra care in light of the fact that the conditions $A_{[1]}(z_o) = A_{[1,2]}(z_o) = 0$ and $A_{[1,2,3]}(z_o) \neq 0$ corresponding to it in Table I are compatible, at least in some of the cases a-d, with either one or the other combination of zeros for the principal minors,

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$
z_o	0	0	$*$
$\hat{z}_o$	$*$	0	0
z_o^*	0	0	$*$
$\hat{z}_o^*$	$*$	0	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$
z_o	0	0	$*$
$\hat{z}_o$	$*$	$*$	0
z_o^*	0	0	$*$
$\hat{z}_o^*$	$*$	$*$	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$

depending on whether $A_{[1,2]}(\hat{z}_o) = 0$ or $A_{[1,2]}(\hat{z}_o) \neq 0$. We will refer to the combination of zeros on the left as "exceptional" and the one of the right as the "generic" case. Looking at bilinear symmetry (3.8e), it is obvious that case c is incompatible with both of them, and therefore, both VIc generic and VIc exceptional can be ruled out. Similarly, IVd exceptional can be ruled out since it would lead to a double zero on the LHS of (3.8e), while the RHS would only have a simple zero. Hence, below we will analyze IVa and IVb both in the generic and in the exceptional situation, and IVd generic.

It is important to notice that in each of the two admissible exceptional cases, the condition that the RHS of (3.8e) has a double zero results in the following constraints for the derivatives of the extra minors:

$$A_{\binom{1,3}{1,2}}(z_o)A'_{\binom{1,2}{1,3}}(z_o) = -A'_{[1]}(z_o)A_{[1,2,3]}(z_o)\,, \qquad A'_{\binom{1,3}{1,2}}(\hat{z}_o)A_{\binom{1,2}{1,3}}(\hat{z}_o) = -A'_{[1,2,3]}(\hat{z}_o)A_{[1]}(\hat{z}_o)\,, \tag{5.25}$$

which follows by differentiating (3.8e) and evaluating it at z_o and $\hat{z}_o$, respectively. The analogous relations for the minors analytic in the LHP can be obtained by symmetry.

Case IV exceptional. Let us start with the two admissible exceptional cases. A detailed analysis of (3.3) and (3.4) yields the following:

- at z_o,

$$L[\chi_2^+(z_o)\,,\chi_3^+(z_o)\,,\varphi_{+,4}(z_o)] = 0 \qquad \text{(simple zero)}, \tag{5.26a}$$

$$L[\varphi_{-,1}(z_o)\,,\chi_2^+(z_o)\,,\varphi_{+,4}(z_o)] = 0 \qquad [\text{simple zero if } \chi_3^-(z_o^*) \neq 0], \tag{5.26b}$$

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$$\left.\frac{L[\varphi_{-,1}(z)\,,\chi_2^+(z)\,,\chi_3^+(z)]}{A_{[1,2]}(z)}\right|_{z_o} = 0 \qquad \text{(simple zero)}, \tag{5.26c}$$

$$L[\varphi_{-,1}(z_o)\,,\chi_3^+(z_o)\,,\varphi_{+,4}(z_o)] = 0 \quad \text{iff} \quad \chi_2^-(z_o^*) = 0; \tag{5.26d}$$

• at $\hat{z}_o$,

$$L[\varphi_{-,1}(\hat{z}_o)\,,\chi_2^+(\hat{z}_o)\,,\chi_3^+(\hat{z}_o)] = 0 \qquad \text{(simple zero)}, \tag{5.26e}$$

$$L[\varphi_{-,1}(\hat{z}_o)\,,\chi_3^+(\hat{z}_o)\,,\varphi_{+,4}(\hat{z}_o)] = 0 \qquad [\text{simple zero if } \chi_2^-(\hat{z}_o^*) \neq 0], \tag{5.26f}$$

$$\left.\frac{L[\chi_2^+(z)\,,\chi_3^+(z)\,,\varphi_{+,4}(z)]}{A_{[1,2]}(z)}\right|_{\hat{z}_o} = 0 \qquad \text{(simple zero)}, \tag{5.26g}$$

$$L[\varphi_{-,1}(\hat{z}_o)\,,\chi_2^+(\hat{z}_o)\,,\varphi_{+,4}(\hat{z}_o)] = 0 \quad \text{iff} \quad \chi_3^-(\hat{z}_o^*) = 0; \tag{5.26h}$$

• at z_o^*,

$$L[\chi_2^-(z_o^*)\,,\chi_3^-(z_o^*)\,,\varphi_{-,4}(z_o^*)] = 0 \qquad \text{(simple zero)}, \tag{5.26i}$$

$$L[\varphi_{+,1}(z_o^*)\,,\chi_2^-(z_o^*)\,,\varphi_{-,4}(z_o^*)] = 0 \qquad [\text{simple zero if } \chi_3^+(z_o) \neq 0], \tag{5.26j}$$

$$\left.\frac{L[\varphi_{+,1}(z)\,,\chi_2^-(z)\,,\chi_3^-(z)]}{A_{[3,4]}(z)}\right|_{z_o^*} = 0 \qquad \text{(simple zero)}, \tag{5.26k}$$

$$L[\varphi_{+,1}(z_o^*)\,,\chi_3^-(z_o^*)\,,\varphi_{-,4}(z_o^*)] = 0 \quad \text{iff} \quad \chi_2^+(z_o) = 0; \tag{5.26l}$$

• at $\hat{z}_o^*$,

$$L[\varphi_{+,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\chi_3^-(\hat{z}_o^*)] = 0 \qquad \text{(simple zero)}, \tag{5.26m}$$

$$L[\varphi_{+,1}(\hat{z}_o^*)\,,\chi_3^-(\hat{z}_o^*)\,,\varphi_{-,4}(\hat{z}_o^*)] = 0 \qquad [\text{simple zero if } \chi_2^+(\hat{z}_o) \neq 0], \tag{5.26n}$$

$$\left.\frac{L[\chi_2^-(z)\,,\chi_3^-(z)\,,\varphi_{-,4}(z)]}{A_{[3,4]}(z)}\right|_{\hat{z}_o^*} = 0 \qquad \text{(simple zero)}, \tag{5.26o}$$

$$L[\varphi_{+,1}(\hat{z}_o^*)\,,\chi_2^-(\hat{z}_o^*)\,,\varphi_{-,4}(\hat{z}_o^*)] = 0 \quad \text{iff} \quad \chi_3^+(\hat{z}_o) = 0. \tag{5.26p}$$

Case IVa exceptional. In this case, the specific structure for the extra minors is the following:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	0	$*$	0	$*$
$\hat{z}_o$	$*$	0	0	$*$	0
z_o^*	0	0	$*$	$*$	0
$\hat{z}_o^*$	$*$	0	0	0	$*$
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

We then consider (3.6) to determine the behavior of the auxiliary eigenfunctions. On one hand, we have

$$\chi_2^-(z_o^*) = 0\,, \qquad \hat{\chi}_2^-(z) = \frac{\chi_2^-(z)}{A_{[3,4]}(z)} \qquad \text{finite at } z_o^*\,, \tag{5.27a}$$

$$\chi_3^-(\hat{z}_o^*) = 0\,, \qquad \hat{\chi}_3^-(z) = \frac{\chi_3^-(z)}{A_{[3,4]}(z)} \qquad \text{finite at } \hat{z}_o^*\,. \tag{5.27b}$$

Also, from (3.6c) at z_o and at $\hat{z}_o$, it follows

$$\chi_2^+(z_o) = \frac{A_{\binom{1,3}{1,2}}(z_o)}{A_{[1,2,3]}(z_o)}\chi_3^+(z_o) \quad \Rightarrow \quad \chi_2^+(z_o) \neq 0\,, \chi_3^+(z_o) \neq 0\,, \tag{5.28a}$$

$$\chi_2^+(\hat{z}_o) = \frac{A'_{\binom{1,3}{1,2}}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)}\chi_3^+(\hat{z}_o) \quad \Rightarrow \quad \chi_2^+(\hat{z}_o) \neq 0\,, \chi_3^+(\hat{z}_o) \neq 0\,. \tag{5.28b}$$

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Moreover, taking into account that $\chi_3^-(\hat{z}_o^*) = 0$, (3.6d) evaluated at z_o and at $\hat{z}_o$, respectively, gives

$$\chi_2^+(z_o) = -\frac{A'_{[1]}(z_o)}{A'_{\binom{1,2}{1,3}}(z_o)}\chi_3^+(z_o)\,, \qquad \chi_2^+(\hat{z}_o) = -\frac{A_{[1]}(\hat{z}_o)}{A_{\binom{1,2}{1,3}}(\hat{z}_o)}\chi_3^+(\hat{z}_o)\,. \tag{5.28c}$$

Note that comparing the last four equations, we obtain consistency relations that are exactly (5.25).

Then, the L-combinations in (5.26) yield for the proportionality relations among the eigenfunctions,

$$\varphi_{-,1}(z_o) = c_1\chi_3^+(z_o)\,, \tag{5.29a}$$

$$\chi_3^+(\hat{z}_o) = d_1\varphi_{+,4}(\hat{z}_o)\,, \tag{5.29b}$$

$$\chi_3^-(z_o^*) = e_1\hat{\chi}_2^-(z_o^*) + e_2\varphi_{+,1}(z_o^*)\,, \qquad e_1e_2 \neq 0, \tag{5.29c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1\hat{\chi}_3^-(\hat{z}_o^*) + f_2\chi_2^-(\hat{z}_o^*)\,, \qquad f_1f_2 \neq 0\,. \tag{5.29d}$$

The pole structure of $M^\pm$ in (2.13) is the following: M_1^+ has a pole at z_o; M_3^+ has a pole at $\hat{z}_o$; M_3^- has a pole at z_o^* (from both terms in the linear combination); M_4^- has a pole at $\hat{z}_o^*$. Note, in particular, that M_2^- is finite at z_o^*; M_3^- is finite at $\hat{z}_o^*$; M_2^+ is finite at z_o using (5.28a); M_2^+ is finite at $\hat{z}_o$ using (5.28b) and (3.6c) at $\hat{z}_o$, that is,

$$\chi_2^+(\hat{z}_o) - \frac{A'_{\binom{1,3}{1,2}}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)}\chi_3^+(\hat{z}_o) = \chi_2^+(\hat{z}_o) + \frac{A_{[1]}(\hat{z}_o)}{A_{\binom{1,2}{1,3}}(\hat{z}_o)}\chi_3^+(\hat{z}_o) = 0\,.$$

This allows us to determine the following residue conditions:

$$\mathrm{Res}_{z_o} M_1^+ = \tilde{c}_1 M_3^+(z_o)e^{i(\theta_3(z_o)-\theta_1(z_o))}\,, \qquad \tilde{c}_1 = c_1\frac{A_{[1,2,3]}(z_o)}{A'_{[1]}(z_o)},$$

$$\mathrm{Res}_{\hat{z}_o} M_3^+ = \tilde{d}_1 M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_3(\hat{z}_o))}\,, \qquad \tilde{d}_1 = \frac{d_1}{A'_{[1,2,3]}(\hat{z}_o)},$$

$$\mathrm{Res}_{z_o^*} M_3^- = \tilde{e}_1 M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))} + \tilde{e}_2 M_2^+(z_o^*)\,,$$

$$\tilde{e}_1 = \frac{e_2}{A'_{[3,4]}(z_o^*)}\,, \quad \tilde{e}_2 = \frac{A'_{[2,3,4]}(z_o^*)}{A'_{[3,4]}(z_o^*)}\left[\frac{e_1}{A'_{[3,4]}(z_o^*)} - \frac{A_{\binom{2,4}{3,4}}(z_o^*)}{A'_{[2,3,4]}(z_o^*)}\right],$$

$$\mathrm{Res}_{\hat{z}_o^*} M_4^- = \tilde{f}_1 M_3^-(\hat{z}_o^*)e^{i(\theta_3(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))} + \tilde{f}_2 M_2^-(\hat{z}_o^*)e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))}\,,$$

$$\tilde{f}_1 = \frac{f_1}{A'_{[4]}(\hat{z}_o^*)}\,, \quad \tilde{f}_2 = f_2\frac{A_{[2,3,4]}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)} + f_1\frac{A_{\binom{2,4}{3,4}}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)A'_{[3,4]}(\hat{z}_o^*)}\,,$$

which can then be all expressed in terms of the single norming constant $\tilde{c}_1$,

$$\mathrm{Res}_{z_o} M_1^+ = \tilde{c}_1 M_3^+(z_o)e^{i(\theta_3(z_o)-\theta_1(z_o))}\,, \tag{5.30a}$$

$$\mathrm{Res}_{\hat{z}_o} M_3^+ = \frac{\tilde{c}_1^*}{\gamma(\hat{z}_o)} M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))}\,, \tag{5.30b}$$

$$\mathrm{Res}_{z_o^*} M_3^- = \frac{\tilde{c}_1^*}{\gamma(z_o^*)} M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))}\,, \tag{5.30c}$$

$$\mathrm{Res}_{\hat{z}_o^*} M_4^- = \frac{iq_+}{z_o}\tilde{c}_1 M_3^+(z_o)e^{i(\theta_3(z_o)-\theta_1(z_o))}\,. \tag{5.30d}$$

Note that in (5.30b) and in (5.30d), we have used symmetry relations (3.9).

As before, there is only one degree of freedom in the choice of the complex norming constant, and as a result, as we will see in Sec. VI, the bright component in the corresponding soliton solution will be aligned exclusively along the second component of $\mathbf{q}(x,t)$.

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Case IVb exceptional. In this case, the specific structure for the extra minors is as follows:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	0	$*$	$*$	0
$\hat{z}_o$	$*$	0	0	0	$*$
z_o^*	0	0	$*$	0	$*$
$\hat{z}_o^*$	$*$	0	0	$*$	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

Now consider (3.6) to determine the behavior of the auxiliary eigenfunctions. On one hand, we have

$$\chi_2^+(z_o) = 0\,, \qquad \hat{\chi}_2^+(z) = \frac{\chi_2^+(z)}{A_{[1,2]}(z)} \qquad \text{finite at } z_o\,, \tag{5.31a}$$

$$\chi_3^+(\hat{z}_o) = 0\,, \qquad \hat{\chi}_3^+(z) = \frac{\chi_3^+(z)}{A_{[1,2]}(z)} \qquad \text{finite at } \hat{z}_o\,. \tag{5.31b}$$

Also, (3.6a) evaluated at z_o^* and at $\hat{z}_o^*$ yields

$$\chi_2^-(z_o^*) = -\frac{A_{\binom{3,4}{2,4}}(z_o^*)}{A_{[4]}(z_o^*)}\chi_3^-(z_o^*) \quad \Rightarrow \quad \chi_2^-(z_o^*) \neq 0\,, \chi_3^-(z_o^*) \neq 0\,, \tag{5.32a}$$

$$\chi_2^-(\hat{z}_o^*) = -\frac{A'_{\binom{3,4}{2,4}}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)}\chi_3^-(\hat{z}_o^*) \quad \Rightarrow \quad \chi_2^-(\hat{z}_o^*) \neq 0\,, \chi_3^-(\hat{z}_o^*) \neq 0\,. \tag{5.32b}$$

Moreover, taking into account that $\chi_3^+(\hat{z}_o) = 0$, (3.6b) evaluated at z_o^* and at $\hat{z}_o^*$ gives, respectively,

$$\chi_2^-(z_o^*) = \frac{A'_{[2,3,4]}(z_o^*)}{A'_{\binom{2,4}{3,4}}(z_o^*)}\chi_3^-(z_o^*)\,, \qquad \chi_2^-(\hat{z}_o^*) = \frac{A_{[2,3,4]}(\hat{z}_o^*)}{A_{\binom{2,4}{3,4}}(\hat{z}_o^*)}\chi_3^-(\hat{z}_o^*)\,. \tag{5.32c}$$

Note that comparing the last four equations, we obtain compatibility conditions for the minors,

$$A_{\binom{3,4}{2,4}}(z_o^*)A'_{\binom{2,4}{3,4}}(z_o^*) = -A'_{[2,3,4]}(z_o^*)A_{[4]}(z_o^*)\,, \qquad A'_{\binom{3,4}{2,4}}(\hat{z}_o^*)A_{\binom{2,4}{3,4}}(\hat{z}_o^*) = -A'_{[4]}(\hat{z}_o^*)A_{[2,3,4]}(\hat{z}_o^*)\,,$$

which are the analog of (5.25) in the LHP.

Ultimately, L-combinations (5.26) give the following proportionality relations among the eigenfunctions:

$$\varphi_{-,1}(z_o) = c_1\hat{\chi}_2^+(z_o) + c_2\chi_3^+(z_o)\,, \qquad c_1c_2 \neq 0\,, \tag{5.33a}$$

$$\chi_2^+(\hat{z}_o) = d_1\hat{\chi}_3^+(\hat{z}_o) + d_2\varphi_{+,4}(\hat{z}_o)\,, \qquad d_1d_2 \neq 0\,, \tag{5.33b}$$

$$\chi_2^-(z_o^*) = e_1\varphi_{+,1}(z_o^*)\,, \tag{5.33c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1\chi_2^-(\hat{z}_o^*)\,. \tag{5.33d}$$

For the pole structure of $M^\pm$ in (2.13), we have the following: M_1^+ has a pole at z_o; M_2^+ has a pole at $\hat{z}_o$; M_2^- has a pole at z_o^*; M_4^- has a pole at $\hat{z}_o^*$. Note, in particular, that M_3^+ is finite at $\hat{z}_o$, and M_3^- is finite both at z_o^* and at $\hat{z}_o^*$. At $\hat{z}_o^*$, this follows directly from linear combination (5.32b). At z_o^*, we note that

$$\chi_3^-(z_o^*) - \frac{A'_{\binom{2,4}{3,4}}}{A'_{[2,3,4]}(z_o^*)}\chi_2^-(z_o^*) = \chi_3^-(z_o^*) + \frac{A_{[4]}(z_o^*)}{A_{\binom{3,4}{2,4}}(z_o^*)}\chi_2^-(z_o^*)$$

and the latter is zero because of (5.32a) and (3.6b) at z_o^*.

Taking into account the symmetries in the norming constants and the eigenfunctions, we obtain the same residue conditions as in cases Ib and Id, namely,

$$\operatorname{Res}_{z_o} M_1^+ = \tilde{c}_1 M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.34a}$$

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$$\text{Res}_{\hat{z}_o} M_2^+ = \frac{\tilde{c}_1^*}{\gamma(\hat{z}_o)} M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.34b}$$

$$\text{Res}_{z_o^*} M_2^- = \frac{\tilde{c}_1^*}{\gamma(z_o^*)} M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.34c}$$

$$\text{Res}_{\hat{z}_o^*} M_4^- = \frac{iq_+}{z_o} \tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.34d}$$

where $\tilde{c}_1 = c_1/A'_{[1]}(z_o)$ and we have used symmetry relations (3.9). Since the residue conditions are the same, this will lead to the same solution as in cases Ib and Id.

Let us now consider the combination of zeros of case IV in the generic situation, i.e., when $A_{[1,2]}(\hat{z}_o) \neq 0$.

Case IVa generic. The structure of the zeros of the analytic minors is as follows:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	0	$*$	0	$*$
$\hat{z}_o$	$*$	$*$	0	$*$	0
z_o^*	0	0	$*$	$*$	0
$\hat{z}_o^*$	$*$	$*$	0	0	$*$
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

The usual analysis of (3.6) allows one to determine for the behavior of the auxiliary eigenfunctions,

$$\chi_2^-(z_o^*) = 0\,, \qquad \hat{\chi}_2^-(z) = \frac{\chi_2^-(z)}{A_{[3,4]}(z)} \qquad \text{finite at } z_o^*\,,$$

$$\chi_2^+(z_o) = \frac{A_{\binom{1,3}{1,2}}(z_o)}{A_{[1,2,3]}(z_o)} \chi_3^+(z_o)\,,$$

$$\chi_2^+(z_o) = e^{i\Delta\theta} \frac{A_{\binom{3,4}{2,4}}(\hat{z}_o^*)}{A_{[3,4]}(\hat{z}_o^*)} \chi_3^-(\hat{z}_o^*)\,,$$

$$\chi_3^+(z_o) = e^{i\Delta\theta} \frac{A_{[2,3,4]}(\hat{z}_o^*)}{A_{[3,4]}(\hat{z}_o^*)} \chi_3^-(\hat{z}_o^*)\,,$$

and all together they also imply $\chi_3^-(\hat{z}_o^*) \neq 0$ [otherwise from (3.3a) at z_o would imply $\varphi_{+,1}(z_o^*) = 0$]. Eqs. (3.3) and (3.4) then give the proportionality among the eigenfunctions, for which we find

$$\varphi_{-,1}(z_o) = c_1 \chi_3^+(z_o)\,, \tag{5.35a}$$

$$\chi_3^+(\hat{z}_o) = d_1 \varphi_{+,4}(\hat{z}_o)\,, \tag{5.35b}$$

$$\chi_3^-(z_o^*) = e_1 \varphi_{+,1}(z_o^*) + e_2 \hat{\chi}_2^-(z_o^*)\,, \tag{5.35c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1 \chi_3^-(\hat{z}_o^*)\,, \tag{5.35d}$$

where the constants are related as follows:

$$f_1 = -\frac{iz_o}{q_+^*} c_1 \frac{A_{[2,3,4]}(\hat{z}_o^*)}{A_{[3,4]}(\hat{z}_o^*)}\,, \qquad d_1 = \frac{iq_-^*}{z_o^* \gamma(z_o^*)} A_{[4]}(z_o^*) c_1^*\,,$$

$$e_1 = -\frac{iz_o^*}{q_-^*} \frac{A'_{[3,4]}(z_o^*)}{A'_{[2,3,4]}(z_o^*)} d_1\,, \qquad e_2 = A_{\binom{2,4}{3,4}}(z_o^*) \frac{A'_{[3,4]}(z_o^*)}{A'_{[2,3,4]}(z_o^*)}\,.$$

Looking at the pole structure, one finds that at z_o, only M_1^+ has a pole; at $\hat{z}_o$, only M_3^+ has a pole; at z_o^*, only M_3^- has a pole (from both terms in the linear combination in the second of (2.13)); at $\hat{z}_o^*$, only M_4^- has a pole.

The residue conditions at said poles can be determined as usual from the above linear combinations to be

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$$\text{Res}_{z_o} M_1^+ = \tilde{c}_1 M_3^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}, \qquad \tilde{c}_1 = c_1 \frac{A_{[1,2,3]}(z_o)}{A'_{[1]}(z_o)}, \tag{5.36a}$$

$$\text{Res}_{\hat{z}_o} M_3^+ = \tilde{d}_1 M_4^+(\hat{z}_o) e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}, \qquad \tilde{d}_1 = \frac{d_1}{A'_{[1,2,3]}(\hat{z}_o)}, \tag{5.36b}$$

$$\text{Res}_{z_o^*} M_3^- = \tilde{e}_1 M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}, \qquad \tilde{e}_1 = \frac{-iq_- d_1}{\hat{z}_o A'_{[2,3,4]}(z_o^*)}, \tag{5.36c}$$

$$\text{Res}_{\hat{z}_o^*} M_4^- = \tilde{f}_1 M_3^-(\hat{z}_o^*) e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))}, \qquad \tilde{f}_1 = f_1 \frac{A_{[3,4]}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)}. \tag{5.36d}$$

Once all the norming constants are expressed in terms of $\tilde{c}_1$ using the above symmetries, the residue conditions turn out to be the same as in case IVa exceptional [cf. (5.30)], and hence the two cases lead to the same solution.

Case IVb generic. The structure of the zeros of the analytic minors is as follows:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	0	$*$	$*$	0
$\hat{z}_o$	$*$	$*$	0	0	$*$
z_o^*	0	0	$*$	0	$*$
$\hat{z}_o^*$	$*$	$*$	0	$*$	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

The usual analysis of (3.6) yields

$$\chi_2^+(z_o) = 0, \qquad \hat{\chi}_2^+(z) = \frac{\chi_2^+(z)}{A_{[1,2]}(z)} \qquad \text{finite at } z_o, \tag{5.37a}$$

$$\chi_2^-(z_o^*) = -\frac{A_{\binom{3,4}{2,4}}(z_o^*)}{A_{[4]}(z_o^*)} \chi_3^-(z_o^*), \tag{5.37b}$$

$$\chi_2^-(z_o^*) = -e^{-i\Delta\theta} \frac{A_{\binom{1,3}{1,2}}(\hat{z}_o)}{A_{[1,2]}(\hat{z}_o)} \chi_3^+(\hat{z}_o), \tag{5.37c}$$

$$\chi_3^-(z_o^*) = e^{-i\Delta\theta} \frac{A_{[1]}(\hat{z}_o)}{A_{[1,2]}(\hat{z}_o)} \chi_3^+(\hat{z}_o). \tag{5.37d}$$

Note that $\chi_2^-(z_o^*)$ and $\chi_3^-(z_o^*)$ cannot be both zero, otherwise (3.3b) would give $\varphi_{-,1}(z_o) = 0$; then as a consequence of the above symmetries for the auxiliary eigenfunctions, it follows that $\chi_3^+(\hat{z}_o) \neq 0$, $\chi_2^-(z_o^*) \neq 0$, and $\chi_3^-(z_o^*) \neq 0$. Then, from (3.3) and (3.6), one finds

$$\varphi_{-,1}(z_o) = c_1 \hat{\chi}_2^+(z_o) + c_2 \chi_3^+(z_o), \tag{5.38a}$$

$$\chi_2^-(z_o^*) = d_1 \varphi_{+,1}(z_o^*), \tag{5.38b}$$

$$\chi_3^+(\hat{z}_o) = e_1 \varphi_{+,4}(\hat{z}_o), \tag{5.38c}$$

$$\varphi_{-,4}(\hat{z}_o^*) = f_1 \chi_3^-(\hat{z}_o^*), \tag{5.38d}$$

where the constants satisfy the following symmetries:

$$c_1 = \frac{iq_+^*}{z_o} f_1 A_{\binom{1,2}{1,3}}(z_o), \qquad c_2 = \frac{iq_+^*}{z_o} f_1 \frac{A'_{[1]}(z_o)}{A'_{[1,2]}(z_o)}, \tag{5.39a}$$

$$d_1 = \frac{iz_o^*}{q_-^*} \frac{A_{\binom{1,3}{1,2}}(\hat{z}_o)}{A_{[1,2]}(\hat{z}_o)} e_1, \qquad e_1 = -\frac{f_1^* A_{[1,2]}(\hat{z}_o)}{\gamma(\hat{z}_o)}. \tag{5.39b}$$

Now observe that the pole structure of $M^\pm$ is as follows: at z_o, only M_1^+ has a pole [M_2^+ does not, because $\chi_2^+(z_o) = 0$ and $A_{\binom{1,3}{1,2}}(z_o) = 0$]; at $\hat{z}_o$, both M_2^+ and M_3^+ have a pole; at z_o^*, both M_2^- and M_3^-

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have a pole; at $\hat{z}_o^*$, only M_4^- has a pole. Taking into account that from (2.13) and (5.37a), one has

$$\hat{\chi}_2^+(z_o)e^{-i\theta_2(z_o)} = M_2^+(z_o) + \frac{A'_{\binom{1,3}{1,2}}(z_o)}{A'_{[1,2]}(z_o)} M_3^+(z_o)\,,$$
$$\chi_3^-(\hat{z}_o^*)e^{-i\theta_2(\hat{z}_o^*)} = A_{[3,4]}(\hat{z}_o^*)M_3^-(\hat{z}_o^*) + A_{\binom{2,4}{3,4}}(\hat{z}_o^*)M_2^-(\hat{z}_o^*)\,;$$

the residues can be written as follows:

$$\operatorname{Res}_{z_o} M_1^+ = \tilde{c}_1 M_2^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))} + \tilde{c}_2 M_3^+(z_o)e^{i(\theta_2(z_o)-\theta_1(z_o))}\,, \tag{5.40a}$$

$$\tilde{c}_1 = \frac{c_1}{A'_{[1]}(z_o)}\,, \qquad \tilde{c}_2 = c_1\frac{A'_{\binom{1,3}{1,2}}(z_o)}{A'_{[1]}(z_o)A'_{[1,2]}(z_o)} + c_2\frac{A_{[1,2,3](z_o)}}{A'_{[1]}(z_o)} \equiv \frac{iq_-^*}{z_o}\frac{A^*_{[1,2]}(\hat{z}_o)}{A'_{[1]}(z_o)}f_1\,,$$

$$\operatorname{Res}_{\hat{z}_o} M_2^+ = -\frac{A_{\binom{1,3}{1,2}}(\hat{z}_o)}{A'_{[1,2,3]}(\hat{z}_o)A_{[1,2]}(\hat{z}_o)} e_1 M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}\,, \tag{5.40b}$$

$$\operatorname{Res}_{\hat{z}_o} M_3^+ = \frac{e_1}{A'_{[1,2,3]}(\hat{z}_o)} M_4^+(\hat{z}_o)e^{i(\theta_4(\hat{z}_o)-\theta_2(\hat{z}_o))}\,, \tag{5.40c}$$

$$\operatorname{Res}_{z_o^*} M_2^- = \frac{d_1}{A'_{[2,3,4]}(z_o^*)} M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.40d}$$

$$\operatorname{Res}_{z_o^*} M_3^- = -d_1\frac{e^{-i\Delta\theta}A^*_{[3,4]}(\hat{z}_o^*)}{A_{\binom{3,4}{2,4}}(z_o^*)A'_{[2,3,4]}(z_o^*)} M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}\,, \tag{5.40e}$$

$$\operatorname{Res}_{\hat{z}_o^*} M_4^- = \frac{f_1 A_{[3,4]}(\hat{z}_o^*)}{A'_{[4]}(\hat{z}_o^*)} M_3^-(\hat{z}_o^*)e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))} + \frac{f_1 A_{\binom{2,4}{3,4}(\hat{z}_o^*)}}{A'_{[4]}(\hat{z}_o^*)} M_2^-(\hat{z}_o^*)e^{i(\theta_2(\hat{z}_o^*)-\theta_4(\hat{z}_o^*))}, \tag{5.40f}$$

where in the expression for $\tilde{c}_2$ above, we have used both the symmetries relating c_1, c_2 to f_1 and the derivative of (3.8e) at z_o. Once all the norming constants are expressed in terms of f_1, the residue conditions take the same form as in case Ic and hence will lead to the same solution.

Case IVd generic. In this case, the specific structure for the extra minors is the following:

	$A_{[1]}$	$A_{[1,2]}$	$A_{[1,2,3]}$	$A_{\binom{1,2}{1,3}}$	$A_{\binom{1,3}{1,2}}$
z_o	0	0	$*$	0	0
$\hat{z}_o$	$*$	$*$	0	0	0
z_o^*	0	0	$*$	0	0
$\hat{z}_o^*$	$*$	$*$	0	0	0
	$A_{[2,3,4]}$	$A_{[3,4]}$	$A_{[4]}$	$A_{\binom{2,4}{3,4}}$	$A_{\binom{3,4}{2,4}}$

We then consider (3.6) to determine the behavior of the auxiliary eigenfunctions. On one hand, we have

$$\chi_2^+(z_o) = 0\,, \qquad \hat{\chi}_2^+(z) = \frac{\chi_2^+(z)}{A_{[1,2]}(z)} \qquad \text{finite at } z_o\,, \tag{5.41a}$$

$$\chi_2^-(z_o^*) = 0\,, \qquad \hat{\chi}_2^-(z) = \frac{\chi_2^-(z)}{A_{[3,4]}(z)} \qquad \text{finite at } z_o^*\,. \tag{5.41b}$$

Also, from (3.6b) at $\hat{z}_o^*$ and (3.6d) at $\hat{z}_o$, it follows

$$\chi_3^+(z_o) = e^{i\Delta\theta}\frac{A_{[2,3,4]}(\hat{z}_o^*)}{A_{[3,4]}(\hat{z}_o^*)}\chi_3^-(\hat{z}_o^*)) \quad\Rightarrow\quad \chi_3^+(z_o) \neq 0\,, \chi_3^-(\hat{z}_o^*) \neq 0\,, \tag{5.42a}$$

$$\chi_3^-(z_o^*) = e^{-i\Delta\theta}\frac{A_{[1]}(\hat{z}_o)}{A_{[1,2]}(\hat{z}_o)}\chi_3^+(\hat{z}_o) \quad\Rightarrow\quad \chi_3^-(z_o^*) \neq 0\,, \chi_3^+(\hat{z}_o) \neq 0\,. \tag{5.42b}$$

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On the other hand, the proportionality relations among the eigenfunctions are

$$\varphi_{-,1}(z_o) = c_1\chi_3^+(z_o)\,, \tag{5.43a}$$
$$\chi_3^+(\hat z_o) = d_1\varphi_{+,4}(\hat z_o)\,, \tag{5.43b}$$
$$\chi_3^-(z_o^*) = e_1\varphi_{+,1}(z_o^*)\,, \tag{5.43c}$$
$$\varphi_{-,4}(\hat z_o^*) = f_1\chi_3^-(\hat z_o^*)\,. \tag{5.43d}$$

The pole structure of $M^\pm$ in (2.13) is the following: M_1^+ has a pole at z_o; M_3^+ has a pole at $\hat z_o$; M_3^- has a pole at z_o^* (from the first term in the linear combination); M_4^- has a pole at $\hat z_o^*$. Note, in particular, that M_2^+ is finite at z_o and $\hat z_o$; M_3^- is finite at $\hat z_o^*$. This allows us to determine the following residue conditions:

$$\mathrm{Res}_{z_o}M_1^+ = \tilde c_1 M_3^+(z_o)e^{i(\theta_3(z_o)-\theta_1(z_o))}\,, \qquad \tilde c_1 = c_1\frac{A_{[1,2,3]}(z_o)}{A'_{[1]}(z_o)}\,,$$
$$\mathrm{Res}_{\hat z_o}M_3^+ = \tilde d_1 M_4^+(\hat z_o)e^{i(\theta_4(\hat z_o)-\theta_3(\hat z_o))}\,, \qquad \tilde d_1 = \frac{d_1}{A'_{[1,2,3]}(\hat z_o)}\,,$$
$$\mathrm{Res}_{z_o^*}M_3^- = \tilde e_1 M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))}\,, \qquad \tilde e_1 = \frac{e_1}{A'_{[3,4]}(z_o^*)}\,,$$
$$\mathrm{Res}_{\hat z_o^*}M_4^- = \tilde f_1 M_3^-(\hat z_o^*)e^{i(\theta_3(\hat z_o^*)-\theta_4(\hat z_o^*))}\,, \qquad \tilde f_1 = \frac{f_1\,A_{[3,4]}(\hat z_o^*)}{A'_{[4]}(\hat z_o^*)}\,,$$

which can then be all expressed in terms of the single norming constant $\tilde c_1$,

$$\mathrm{Res}_{z_o}M_1^+ = \tilde c_1 M_3^+(z_o)e^{i(\theta_3(z_o)-\theta_1(z_o))}\,, \tag{5.44a}$$
$$\mathrm{Res}_{\hat z_o}M_3^+ = \frac{\tilde c_1^*}{\gamma(\hat z_o)}M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))}\,, \tag{5.44b}$$
$$\mathrm{Res}_{z_o^*}M_3^- = \frac{\tilde c_1^*}{\gamma(z_o^*)}M_1^-(z_o^*)e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))}\,, \tag{5.44c}$$
$$\mathrm{Res}_{\hat z_o^*}M_4^- = \frac{iq_+}{z_o}\tilde c_1 M_3^+(z_o)e^{i(\theta_3(z_o)-\theta_1(z_o))}\,. \tag{5.44d}$$

Note that we have used symmetry relations (3.9). These residue conditions are the same of case IVa exceptional, and this case will therefore lead to the same one-soliton solution.

VI. SOLITON SOLUTIONS AND POLARIZATION SHIFT

In this section, we explore the different possibilities for reflectionless solutions in the inverse problem, construct explicit soliton solutions for each distinct case (the solutions corresponding to symmetric cases can be obtained in a similar fashion), and discuss the 2-soliton interactions.

A. One-soliton solutions

As is well-known, in the reflectionless case the solution of inverse problem reduces to an algebraic system of equations for the meromorphic eigenfunctions $M^\pm$. Specifically, with only one quartet of discrete eigenvalues $z_o, \hat z_o, z_o^*, \hat z_o^*$, the solution of RHP (2.14) is simply given by

$$M(z) = E_+(z) + \frac{\mathrm{Res}_{z_o}M^+}{z-z_o} + \frac{\mathrm{Res}_{z_o^*}M^-}{z-z_o^*} + \frac{\mathrm{Res}_{\hat z_o}M^+}{z-\hat z_o} + \frac{\mathrm{Res}_{\hat z_o^*}M^-}{z-\hat z_o^*}, \tag{6.1}$$

and the above residues have been evaluated in Sec. V for all cases.

Case Ia. As shown in Sec. III, in this case, the eigenfunctions involved in the residues have been reduced to the following three: $M_2^+(z_o)$, $M_3^+(z_o)$, $M_1^-(z_o^*)$ [after dropping $M_4^+(\hat z_o)$, $M_2^-(\hat z_o^*)$, and

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$M_3^-(\hat{z}_o^*)$ by a careful use of the symmetries]. The residues of the matrices $M^\pm$ are then as follows:

$$\operatorname{Res}_{z_o} M^+ = \left[\tilde{c}_1 M_2^+(z_o) + \tilde{c}_2 M_3^+(z_o), 0, 0, 0\right] e^{i(\theta_2(z_o)-\theta_1(z_o))},$$
$$\operatorname{Res}_{z_o^*} M^- = \frac{1}{\gamma^*(z_o)}\left[0, \tilde{c}_1^* M_1^-(z_o^*), \tilde{c}_2^* M_1^-(z_o^*), 0\right] e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$
$$\operatorname{Res}_{\hat{z}_o} M^+ = -\frac{q_o^2}{(z_o^*)^2\gamma^*(z_o)}\left[0, \tilde{c}_1^* M_1^-(z_o^*), \tilde{c}_2^* M_1^-(z_o^*), 0\right] e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$
$$\operatorname{Res}_{\hat{z}_o^*} M^- = \frac{iq_+}{z_o}\left[0, 0, 0, \tilde{c}_1 M_2^+(z_o) + \tilde{c}_2 M_3^+(z_o)\right] e^{i(\theta_2(z_o)-\theta_1(z_o))}.$$

We can then evaluate the appropriate columns of (6.1) as follows:

- 2nd column at z_o yields

$$M_2^+(z_o) = E_+^{(2)}(z_o) + \frac{\tilde{c}_1^*}{\gamma^*(z_o)}\left[\frac{1}{z_o - z_o^*} - \frac{q_o^2}{(z_o^*)^2}\frac{1}{z_o - \hat{z}_o}\right] M_1^-(z_o^*)\, e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}, \tag{6.2a}$$

- 3rd column at z_o yields

$$M_3^+(z_o) = E_+^{(3)}(z_o) + \frac{\tilde{c}_2^*}{\gamma^*(z_o)}\left[\frac{1}{z_o - z_o^*} - \frac{q_o^2}{(z_o^*)^2}\frac{1}{z_o - \hat{z}_o}\right] M_1^-(z_o^*)\, e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}, \tag{6.2b}$$

- 1st column at z_o^* yields

$$M_1^-(z_o^*) = E_+^{(1)}(z_o^*) + \frac{1}{z_o^* - z_o}\tilde{c}_1 M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))} + \frac{1}{z_o^* - z_o}\tilde{c}_2 M_3^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}, \tag{6.2c}$$

which is a 3×3 system for the unknowns $M_2^+(z_o), M_3^+(z_o), M_1^-(z_o^*)$.

The system is easily solved, and recalling that $q_{+,1} = q_{+,2} = 0$ and $q_{+,3} = q_+ \equiv q_o e^{i\theta_+}$, reconstruction formula (2.15) then gives

$$q_j(x,t) = \frac{-i\mathbf{c}_{o,j}}{\|\mathbf{c}_o\|}\sin\alpha\sqrt{q_o^2 - |z_o|^2}\, e^{i(\xi_o x - (\xi_o^2 - \nu_o^2)t)}\operatorname{sech}(\nu_o(x - 2\xi_o t - x_o)), \quad j = 1,2,$$
$$q_3(x,t) = q_+ e^{i\alpha}\left[\cos\alpha - i\sin\alpha\tanh(\nu_o(x - 2\xi_o t - x_o))\right], \tag{6.3}$$

where we introduced $z_o = \xi_o + i\nu_o \equiv |z_o|e^{i\alpha}$ with $|z_o| < q_o$ and $\nu_o > 0$; $\mathbf{c}_o = (\tilde{c}_1, \tilde{c}_2)^T$ is an arbitrary two-component, complex norming constant, and

$$e^{\nu_o x_o} = \frac{|z_o|\|\mathbf{c}_o\|}{2\nu_o\sqrt{q_o^2 - |z_o|^2}}. \tag{6.4}$$

Case Ib. As shown in Sec. III, in this case, the eigenfunctions involved in the residues have been reduced to the following two: $M_2^+(z_o)$, $M_1^-(z_o^*)$. The residues of the matrices $M^\pm$ are as follows [here, $\gamma(z_o^*) = \gamma^*(z_o)$]:

$$\operatorname{Res}_{z_o} M^+ = \left[\tilde{c}_1 M_2^+(z_o), 0, 0, 0\right] e^{i(\theta_2(z_o)-\theta_1(z_o))},$$
$$\operatorname{Res}_{z_o^*} M^- = \frac{1}{\gamma^*(z_o)}\left[0, \tilde{c}_1^* M_1^-(z_o^*), 0, 0\right] e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$
$$\operatorname{Res}_{\hat{z}_o} M^+ = \frac{1}{\gamma(\hat{z}_o)}\left[0, \tilde{c}_1^* M_1^-(z_o^*), 0, 0\right] e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))},$$
$$\operatorname{Res}_{\hat{z}_o^*} M^- = \frac{iq_+}{z_o}\left[0, 0, 0, \tilde{c}_1 M_2^+(z_o)\right] e^{i(\theta_2(z_o)-\theta_1(z_o))}.$$

Evaluating the appropriate columns of (6.1) then yields the following:

- 2nd column at z_o yields

$$M_2^+(z_o) = E_+^{(2)}(z_o) + \tilde{c}_1^* \frac{|z_o|^2}{(z_o - z_o^*)(|z_o|^2 - q_o^2)} M_1^-(z_o^*)\, e^{i(\theta_1(z_o^*)-\theta_2(z_o^*))}, \tag{6.5a}$$

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- 1st column at z_o^* yields

$$M_1^-(z_o^*) = E_+^{(1)}(z_o^*) + \frac{\tilde{c}_1}{(z_o^* - z_o)} M_2^+(z_o) e^{i(\theta_2(z_o)-\theta_1(z_o))}. \tag{6.5b}$$

The above is a 2×2 system for the unknowns $M_2^+(z_o)$ and $M_1^-(z_o^*)$, which is readily solved and via reconstruction formula (2.15) yields

$$q_1(x,t) = \frac{-i\tilde{c}_1}{|\tilde{c}_1|} \sin\alpha \sqrt{q_o^2 - |z_o|^2} e^{i(\xi_o x - (\xi_o^2 - \nu_o^2)t)} \operatorname{sech}\left(\nu_o(x - 2\xi_o t - x_o)\right), \tag{6.6}$$

$$q_2(x,t) = 0, \tag{6.7}$$

$$q_3(x,t) = q_+ e^{i\alpha} \left[\cos\alpha - i \sin\alpha \tanh\left(\nu_o(x - 2\xi_o t - x_o)\right)\right]. \tag{6.8}$$

Case IVa. As shown in Sec. III, in this case, the eigenfunctions involved in the residues have been reduced to the following two: $M_3^+(z_o)$, $M_1^-(z_o^*)$. The residues of the matrices $M^\pm$ are as follows [here, $\gamma(z_o^*) = \gamma^*(z_o)$]:

$$\operatorname{Res}_{z_o} M^+ = \left[\tilde{c}_1 M_3^+(z_o), 0, 0, 0\right] e^{i(\theta_3(z_o)-\theta_1(z_o))},$$

$$\operatorname{Res}_{z_o^*} M^- = \frac{1}{\gamma^*(z_o)} \left[0, 0, \tilde{c}_1^* M_1^-(z_o^*), 0\right] e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))},$$

$$\operatorname{Res}_{\hat{z}_o} M^+ = \frac{1}{\gamma(\hat{z}_o)} \left[0, 0, \tilde{c}_1^* M_1^-(z_o^*), 0\right] e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))},$$

$$\operatorname{Res}_{\hat{z}_o^*} M^- = \frac{iq_+}{z_o} \left[0, 0, 0, \tilde{c}_1 M_3^+(z_o)\right] e^{i(\theta_3(z_o)-\theta_1(z_o))}.$$

We are then going to evaluate the appropriate columns of (6.1),

- 3nd column at z_o yields

$$M_3^+(z_o) = E_+^{(3)}(z_o) + \tilde{c}_1^* \frac{|z_o|^2}{(z_o - z_o^*)(|z_o|^2 - q_o^2)} M_1^-(z_o^*) e^{i(\theta_1(z_o^*)-\theta_3(z_o^*))}, \tag{6.9a}$$

- 1st column at z_o^* yields

$$M_1^-(z_o^*) = E_+^{(1)}(z_o^*) + \frac{\tilde{c}_1}{(z_o^* - z_o)} M_3^+(z_o) e^{i(\theta_3(z_o)-\theta_1(z_o))}, \tag{6.9b}$$

which provides a 2×2 system for the unknowns $M_3^+(z_o)$ and $M_1^-(z_o^*)$. The system can be easily solved, and reconstruction formula (2.15) then gives a dark-bright soliton whose bright part is aligned with the second component of the vector $\mathbf{q}_+$, i.e.,

$$q_1(x,t) = 0, \tag{6.10a}$$

$$q_2(x,t) = \frac{-i\tilde{c}_1}{|\tilde{c}_1|} \sin\alpha \sqrt{q_o^2 - |z_o|^2} e^{i(\xi_o x - (\xi_o^2 - \nu_o^2)t)} \operatorname{sech}\left(\nu_o(x - 2\xi_o t - x_o)\right), \tag{6.10b}$$

$$q_3(x,t) = q_+ e^{i\alpha} \left[\cos\alpha - i \sin\alpha \tanh\left(\nu_o(x - 2\xi_o t - x_o)\right)\right]. \tag{6.10c}$$

B. Two-soliton solutions

We are now ready to investigate soliton solutions corresponding to two quartets of eigenvalues Z_1 and Z_2 as in (4.2), with combination of zeros as in case Ia. Let $z_j = \xi_j + i\nu_j \equiv |z_j| e^{i\alpha_j}$ for $j = 1,2$ denote the two eigenvalues in the interior of the circle of radius q_o in the UHP, and assume wlog $\xi_1 < \xi_2$. With two quartets of discrete eigenvalues, the solution of RHP (2.14) is given by

$$M(z) = E_+(z) + \sum_{j=1,2} \frac{\operatorname{Res}_{z_j} M^+}{z - z_j} + \sum_{j=1,2} \frac{\operatorname{Res}_{z_j^*} M^-}{z - z_j^*} + \sum_{j=1,2} \frac{\operatorname{Res}_{\hat{z}_j} M^+}{z - \hat{z}_j} + \sum_{j=1,2} \frac{\operatorname{Res}_{\hat{z}_j^*} M^-}{z - \hat{z}_j^*}, \tag{6.11}$$

and the above residues have been evaluated in Sec. V for all cases.

Below we present the solution of associated linear system corresponding to the novel case when both quartets of eigenvalues are of type ia). First, let the two complex vector norming

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constants for soliton 1 and 2 be given by $\mathbf{c}_j = (c_{1j}, c_{2j})^T$ for $j = 1,2$, respectively, and introduce

$$S_j = \nu_j(x - 2\xi_j t)\,, \qquad \Psi_j = \xi_j x - \left(\xi_j^2 - \nu_j^2\right) t, \qquad j = 1,2\,. \tag{6.12}$$

For future reference, we note that

$$-S_2 = -\frac{\nu_2}{\nu_1} S_1 + 2\nu_2(\xi_2 - \xi_1)t\,, \qquad -S_1 = -\frac{\nu_1}{\nu_2} S_2 - 2\nu_1(\xi_2 - \xi_1)t\,. \tag{6.13}$$

Introduce also a short-hand notation for

$$\Gamma = \frac{|z_1 - z_2|^2}{(z_1 - z_1^*)(z_2^* - z_2)|z_1 - z_2^*|^2},$$
$$\Delta = |\mathbf{c}_1^\dagger \mathbf{c}_2|^2 |g|^2 - \|\mathbf{c}_1\|^2 \|\mathbf{c}_2\|^2 h_1 h_2\,,$$

with

$$g = \frac{z_1 z_2^*}{(z_2^* - z_1)\left(q_o^2 - z_1 z_2^*\right)}\,, \qquad h_j = \frac{|z_j|^2}{2\nu_j\left(q_o^2 - |z_j|^2\right)}\,, \quad j = 1,2\,,$$

and for

$$\begin{aligned} D = 1 &+ \|\mathbf{c}_1\|^2 \frac{|z_1|^2}{|z_1 - z_1^*|^2 (q_o^2 - |z_1|^2)} e^{-2S_1} + \|\mathbf{c}_2\|^2 \frac{|z_2|^2}{|z_2 - z_2^*|^2 (q_o^2 - |z_2|^2)} e^{-2S_2} \\ &- 2e^{-S_1 - S_2}\,\mathrm{Re}\left(\mathbf{c}_2^\dagger \mathbf{c}_1 \frac{z_1 z_2^*}{(z_1 - z_2^*)^2 (q_o^2 - z_1 z_2^*)} e^{i(\Psi_1 - \Psi_2)}\right) - \Delta\Gamma e^{-2S_1 - 2S_2}\,, \end{aligned}$$

which is the denominator of linear algebraic system (6.11).

It is important to point out that $\Delta \leqslant 0$ and $-\Delta\Gamma \geqslant 0$, as the term in square bracket in the definition of Δ above is non-negative, due to the obvious chains of inequalities,

$$\begin{aligned} &\|\mathbf{c}_1\|^2 \|\mathbf{c}_2\|^2 \geqslant |\mathbf{c}_1^\dagger \mathbf{c}_2|^2\,, \\ &|z_1 - z_2^*|^2 \geqslant (z_1^* - z_1)(z_2 - z_2^*)\,, \\ &(q_o^2 - z_1^* z_2)(q_o^2 - z_1 z_2^*) \geqslant (q_o^2 - |z_1|^2)(q_o^2 - |z_2|^2)\,. \end{aligned}$$

The two-soliton solution one obtains from (2.15) after solving the linear system has the form

$$\begin{aligned} i\mathbf{q}(x,t) = i\mathbf{q}_+ & \\ &+ \begin{pmatrix} \mathbf{0} \\ \dfrac{iq_+}{z_1^*} \end{pmatrix} \frac{1}{D} \Bigg[i\|\mathbf{c}_1\|^2 h_1 e^{-2S_1} + i\|\mathbf{c}_2\|^2 \frac{z_1^*}{z_2^*} h_2 e^{-2S_2} + \mathbf{c}_2^\dagger \mathbf{c}_1 \frac{z_1^*}{z_2^*} g\, e^{-S_1 - S_2 + i(\Psi_1 - \Psi_2)} \\ &\quad - \mathbf{c}_1^\dagger \mathbf{c}_2\, g^* e^{-S_1 - S_2 - i(\Psi_1 - \Psi_2)} + \frac{|z_1 - z_2|^2 (z_1 z_2 - z_1^* z_2^*)}{z_2^* |z_1 - z_2^*|^2 (z_1^* - z_1)(z_2^* - z_2)} \Delta e^{-2S_1 - 2S_2} \Bigg] \\ &+ \begin{pmatrix} \mathbf{c}_1 \\ 0 \end{pmatrix} \frac{e^{-S_1 + i\Psi_1}}{D} \Bigg[1 - i\mathbf{c}_1^\dagger \mathbf{c}_2\, g^* \frac{(z_2 - z_1)}{2\nu_1 (z_2 - z_1^*)} e^{-S_1 - S_2 - i(\Psi_1 - \Psi_2)} \\ &\qquad + \|\mathbf{c}_2\|^2 h_2 \frac{(z_2 - z_1)}{2\nu_2 (z_2^* - z_1)} e^{-2S_2} \Bigg] \\ &+ \begin{pmatrix} \mathbf{c}_2 \\ 0 \end{pmatrix} \frac{e^{-S_2 + i\Psi_2}}{D} \Bigg[1 + i\mathbf{c}_2^\dagger \mathbf{c}_1\, g \frac{(z_1 - z_2)}{2\nu_2 (z_1 - z_2^*)} e^{-S_1 - S_2 + i(\Psi_1 - \Psi_2)} \\ &\qquad + \|\mathbf{c}_1\|^2 h_1 \frac{(z_1 - z_2)}{2\nu_1 (z_1^* - z_2)} e^{-2S_1} \Bigg]\,. \end{aligned} \tag{6.14}$$

Note the solution is completely determined by the complex soliton eigenvalues z_n and the associated complex vector norming constants $\mathbf{c}_n = (c_{n,1}, c_{n,2})^T$, $n = 1,2$. An example of such a two-soliton solution describing the interaction of two dark-bright solitons is shown in Fig. 1. Note the nontrivial redistribution of energy between the two components of the bright part of both solitons.

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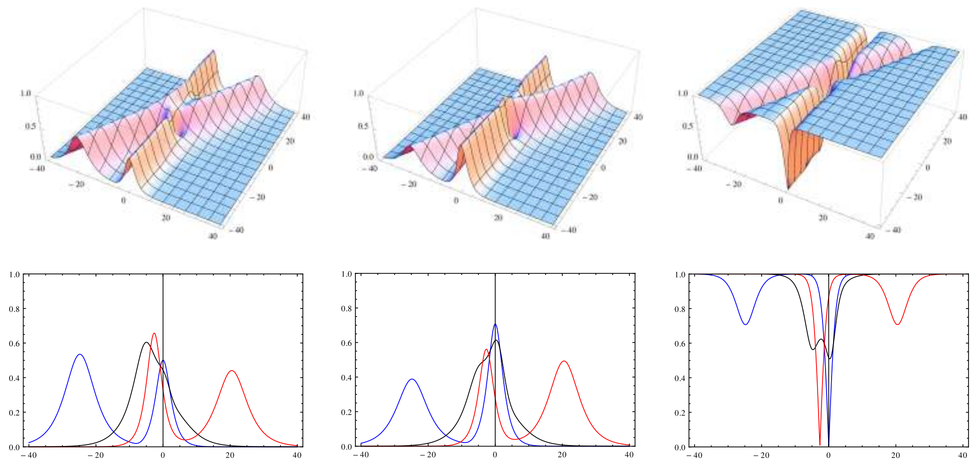

FIG. 1. A two-soliton solution of the 3-component defocusing VNLS equation with NZBC obtained with discrete eigenvalues $z_1 = i/2$ and $z_2 = (1+i)/4$, and norming constants $\mathbf{c}_1 = (1, 1-i)^T$ and $\mathbf{c}_2 = (i, 1+i/2)^T$. The left, center, and right plots show the absolute value of each of the three components of $\mathbf{q}(x,t)$, respectively. The plots on the top show the value of the solution as a function of x and t. The plots on the bottom show the profile of each of the three components at $t = -40$ (blue), $t = 0$ (black), and $t = 40$ (red).

C. Soliton interactions and polarization shift

We now characterize the soliton interactions, including the polarization shift by investigating the long-time asymptotic behavior of two-soliton solution (6.14). If we now look at the dominant terms of each combination, both when S_1 is fixed and $t \to \pm\infty$, and when S_2 is fixed and $t \to \pm\infty$, we obtain the following.

- S_1 fixed and $t \to -\infty$ [corresponding to $e^{-S_2} \to 0$, cf. (6.13)],

$$\begin{pmatrix} q_1(x,t) \\ q_2(x,t) \end{pmatrix} \sim -i\mathbf{p}_1^- \sin\alpha_1 \sqrt{q_o^2 - |z_1|^2} e^{i\Psi_1} \mathrm{sech}(S_1 - x_1^-), \tag{6.15a}$$

$$q_3(x,t) \sim q_+ e^{i\alpha_1}[\cos\alpha_1 - i\sin\alpha_1 \tanh(S_1 - x_1^-)], \tag{6.15b}$$

where

$$\mathbf{p}_1^- = \frac{\mathbf{c}_1}{\|\mathbf{c}_1\|}, \qquad e^{x_1^-} = \frac{\|\mathbf{c}_1\| |z_1|}{2\nu_1\sqrt{q_o^2 - |z_1|^2}}. \tag{6.16}$$

- S_2 fixed and $t \to +\infty$ [corresponding to $e^{-S_1} \to 0$],

$$\begin{pmatrix} q_1(x,t) \\ q_2(x,t) \end{pmatrix} \sim -i\mathbf{p}_2^+ \sin\alpha_2 \sqrt{q_o^2 - |z_2|^2} e^{i\Psi_1} \mathrm{sech}(S_2 - x_2^+), \tag{6.17a}$$

$$q_3(x,t) \sim q_+ e^{i\alpha_2}[\cos\alpha_2 - i\sin\alpha_2 \tanh[S_2 - x_2^+]], \tag{6.17b}$$

where

$$\mathbf{p}_2^+ = \frac{\mathbf{c}_2}{\|\mathbf{c}_2\|}, \qquad e^{x_2^+} = \frac{\|\mathbf{c}_2\| |z_2|}{2\nu_2\sqrt{q_o^2 - |z_2|^2}}. \tag{6.18}$$

- S_1 fixed and $t \to \infty$ [corresponding to $e^{-S_2} \to \infty$],

$$\begin{pmatrix} q_1(x,t) \\ q_2(x,t) \end{pmatrix} \sim -i\mathbf{p}_1^+ \sin\alpha_1 \sqrt{q_o^2 - |z_1|^2} e^{i(\Psi_1 + \alpha_1^+)} \mathrm{sech}(S_1 - x_1^+), \tag{6.19a}$$

$$q_3(x,t) \sim q_+ e^{i\alpha_1} e^{2i\alpha_2}[\cos\alpha_1 - i\sin\alpha_1 \tanh(S_1 - x_1^+)], \tag{6.19b}$$

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where

$$\mathbf{p}_1^+ = \frac{1}{\chi}\left[\mathbf{p}_1^- + \frac{z_1(z_2-z_2^*)(q_o^2-|z_2|^2)}{z_2(z_2^*-z_1)(q_o^2-z_1z_2^*)}\left((\mathbf{p}_2^+)^\dagger\mathbf{p}_1^-\right)\mathbf{p}_2^+\right], \tag{6.20a}$$

$$\chi^2 = 1 - \left|(\mathbf{p}_1^-)^\dagger\mathbf{p}_2^+\right|^2 \frac{(z_1^*-z_1)(z_2-z_2^*)(q_o^2-|z_1|^2)(q_o^2-|z_2|^2)}{|z_1-z_2^*|^2|q_o^2-z_1z_2^*|^2}, \tag{6.20b}$$

$$e^{x_1^+} = \frac{\|\mathbf{c}_1\||z_1|}{2\nu_1\sqrt{q_o^2-|z_1|^2}}\frac{|z_1-z_2|}{|z_1-z_2^*|}\chi\,, \tag{6.20c}$$

$$\alpha_1^+ = \arg(z_1-z_2) - \arg(z_1-z_2^*)\,. \tag{6.20d}$$

Note one can easily verify that $\|\mathbf{p}_1^+\| = 1$.

- S_2 fixed and $t\to-\infty$ [corresponding to $e^{-S_1}\to\infty$],

$$\begin{pmatrix} q_1(x,t)\\ q_2(x,t)\end{pmatrix} \sim -i\mathbf{p}_2^-\,\sin\alpha_2\sqrt{q_o^2-|z_2|^2}e^{i(\Psi_2+\alpha_2^-)}\text{sech}(S_2-x_2^-), \tag{6.21a}$$

$$q_3(x,t) \sim q_+\,e^{i\alpha_2}e^{2i\alpha_1}[\cos\alpha_2 - i\sin\alpha_2\tanh(S_2-x_2^-)], \tag{6.21b}$$

where

$$\mathbf{p}_2^- = \frac{1}{\chi}\left[\mathbf{p}_2^+ + \frac{z_2(z_1-z_1^*)(q_o^2-|z_1|^2)}{z_1(z_1^*-z_2)(q_o^2-z_1^*z_2)}\left((\mathbf{p}_1^-)^\dagger\mathbf{p}_2^+\right)\mathbf{p}_1^-\right], \tag{6.22a}$$

$$e^{x_2^-} = \frac{\|\mathbf{c}_2\||z_2|}{|z_2-z_2^*|\sqrt{q_o^2-|z_2|^2}}\frac{|z_1-z_2|}{|z_1-z_2^*|}\chi\,, \tag{6.22b}$$

$$\alpha_2^- = \arg(z_1-z_2) - \arg(z_1^*-z_2)\,. \tag{6.22c}$$

Note one can easily verify that $\|\mathbf{p}_2^-\| = 1$.

Previous formulas (6.20a) and (6.22a) show that soliton interactions result in non-trivial polarization shifts, i.e., energy exchanges between the bright components of the interacting solitons. In a sense, these formulas are the analogue of the well-known Manakov's formulas for the polarization shift of two bright solitons in a 2-component focusing NLS system. To the best of our knowledge, this is the first time that non-trivial soliton polarization interactions have been reported in a defocusing system.

It is convenient to rewrite the previous formulas for the polarization shifts so as to express the ones as $t\to+\infty$ in terms of the ones as $t\to-\infty$. This can be done as follows. Introduce for simplicity notations: $\mathbf{p}_1\equiv\mathbf{p}_1^-$, $\mathbf{p}_2\equiv\mathbf{p}_2^-$, then $\hat{\mathbf{p}}_1=\mathbf{p}_1^+$ and $\hat{\mathbf{p}}_2=\mathbf{p}_2^+$. The formulas we have derived above then read

$$\mathbf{p}_2 = \frac{1}{\chi}\left[\hat{\mathbf{p}}_2 + A(\mathbf{p}_1^\dagger\hat{\mathbf{p}}_2)\mathbf{p}_1\right]\,,\qquad \hat{\mathbf{p}}_1 = \frac{1}{\chi}\left[\mathbf{p}_1 + B(\hat{\mathbf{p}}_2^\dagger\mathbf{p}_1)\hat{\mathbf{p}}_2\right]\,,$$

with

$$\chi^2 = 1 - C\left|\mathbf{p}_1^\dagger\hat{\mathbf{p}}_2\right|^2,$$

and where we have denoted for brevity,

$$C = \frac{(z_1^*-z_1)(z_2-z_2^*)(q_o^2-|z_1|^2)(q_o^2-|z_2|^2)}{|z_1-z_2^*|^2|q_o^2-z_1z_2^*|^2}\,,$$

$$A = \frac{z_2(z_1-z_1^*)(q_o^2-|z_1|^2)}{z_1(z_1^*-z_2)(q_o^2-z_1^*z_2)}\,,\qquad B = \frac{z_1(z_2-z_2^*)(q_o^2-|z_2|^2)}{z_2(z_2^*-z_1)(q_o^2-z_1z_2^*)}\,.$$

Then we also have

$$\mathbf{p}_2^\dagger = \frac{1}{\chi}\left[\hat{\mathbf{p}}_2^\dagger + A^*(\hat{\mathbf{p}}_2^\dagger\mathbf{p}_1)\mathbf{p}_1^\dagger\right],$$

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and if we multiply from the right by $\mathbf{p}_1$ and use that $\|\mathbf{p}_1\| = 1$, we then obtain

$$\mathbf{p}_2^\dagger \mathbf{p}_1 = \frac{1}{\chi}(1 + A^*)(\hat{\mathbf{p}}_2^\dagger \mathbf{p}_1)\,.$$

i.e.,

$$\frac{1}{\chi}\hat{\mathbf{p}}_2^\dagger \mathbf{p}_1 = \frac{1}{1 + A^*}\mathbf{p}_2^\dagger \mathbf{p}_1\,.$$

As a consequence, we also have

$$\left|\mathbf{p}_1^\dagger \hat{\mathbf{p}}_2\right|^2 = \chi^2 \frac{1}{(1 + A)(1 + A^*)}|\mathbf{p}_1^\dagger \mathbf{p}_2|^2\,.$$

The latter allows us to express χ in terms of $|\mathbf{p}_1^\dagger \mathbf{p}_2|^2$, since substituting the LHS into the expression for χ^2 yields

$$\chi^2 = \left[1 + \frac{C}{(1 + A)(1 + A^*)}|\mathbf{p}_1^\dagger \mathbf{p}_2|^2\right]^{-1}\,.$$

Substituting into the first equation and solving for $\hat{\mathbf{p}}_2$ gives

$$\hat{\mathbf{p}}_2 = \chi\left[\mathbf{p}_2 - \frac{A}{1 + A}(\mathbf{p}_1^\dagger \mathbf{p}_2)\mathbf{p}_1\right]\,, \tag{6.23a}$$

which gives the polarization of the second soliton after the interaction as a function of the "initial" polarizations of the two. Then we can substitute it into the second equation for $\hat{\mathbf{p}}_1$ and obtain

$$\hat{\mathbf{p}}_1 = \chi\left[\mathbf{p}_1 + \frac{B}{1 + A^*}(\mathbf{p}_2^\dagger \mathbf{p}_1)\mathbf{p}_2\right]\,, \tag{6.23b}$$

where the explicit expression of the coefficients can be easily computed in terms of A, B, C above

$$\frac{1}{1 + A} = \frac{z_1(z_1^* - z_2)(q_o^2 - z_1^* z_2)}{z_1^*(z_1 - z_2)(q_o^2 - z_1 z_2)}\,,$$

$$\frac{A}{1 + A} = \frac{z_2(z_1 - z_1^*)(q_o^2 - |z_1|^2)}{z_1^*(z_1 - z_2)(q_o^2 - z_1 z_2)}\,, \qquad \frac{B}{1 + A^*} = \frac{z_1^*(z_2^* - z_2)(q_o^2 - |z_2|^2)}{z_2(z_1^* - z_2^*)(q_o^2 - z_1^* z_2^*)}\,,$$

$$\frac{C}{(1 + A)(1 + A^*)} = \frac{(z_1^* - z_1)(z_2 - z_2^*)(q_o^2 - |z_1|^2)(q_o^2 - |z_2|^2)}{|z_1 - z_2|^2 |q_o^2 - z_1 z_2|^2}\,.$$

One can easily check that the vectors $\hat{\mathbf{p}}_j$ for $j = 1, 2$ are indeed norm 1 vectors.

VII. CONCLUSIONS

We have presented novel exact dark-bright soliton solutions for the three-component defocusing VNLS equation with nonzero boundary conditions, and we have discussed their interactions. The solutions have been obtained within the framework of the recently developed inverse scattering transform for the multi-component VNLS equation.[14,23]

Importantly, unlike the dark-bright soliton solutions in the two-component system in the same dispersion regime, the solutions discussed in this work exhibit nontrivial polarization interactions between the two bright components. Recall that the interaction-induced polarization shifts between bright solitons in focusing media are a well known effect,[2,15] which have been experimentally observed.[3] To the best of our knowledge, however, the interaction-induced polarization shift between dark-bright solitons in defocusing media is a novel physical effect that had not previously been reported in the literature.

We emphasize that, in spite of the interaction-induced redistribution of energy between the bright components along q_1 and q_2, the total energy of each soliton and that of its bright part are both conserved, and a polarization shift is still consistent with elastic interactions, just like a position shift.

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We should also note that formulae (6.23) for the polarization shift in the defocusing 3-component VNLS equation with NZBC are similar to those for the equivalent effect in the focusing 2-component VNLS equation with ZBC.[15] Also, the defocusing 2-component VNLS equation with NZBC admits solutions arising from double zeros of the analytic scattering coefficients, leading to logarithmic interactions between dark-bright solitons.[6] Such solutions are not allowed in the scalar defocusing NLS equation,[29] and their behavior is similar to that of double-pole solutions of the scalar focusing NLS equation with ZBC.[28] Thus, in both instances, the defocusing case of the VNLS equation with NZBC allows similar degrees of freedom as the focusing case with ZBC and one less component. It is an interesting question whether this analogy persists as the number of components increases further.

Many other interesting questions also remain to be addressed on a more theoretical setting. For example, an open problem is the derivation of so-called trace formulae that allow one to recover the analytic minors of the scattering matrix in terms of the reflection coefficients and discrete eigenvalues. In Ref. 14, a partial answer to this question was given, but only in the case when the extra minors are identically zero. It was also shown there that, once all the analytic minors are known, one can recover the whole scattering matrix in terms of them and the reflection coefficients. A generalization of those results to the case when the extra minors are non-zero is still an open problem, however. Such trace formulae are usually derived by formulating appropriate scalar RHPs.[1,14] Note however that the extra analytic minors are not on the same footing as the principal minors, since the latter tend to one as $z \to \infty$, while the former tend to zero. This suggests that, even if appropriate RHPs can be formulated to recover all the minors, the principal minors and the extra minors will need to be treated differently.

A related open problem is that of identifying a minimal set of scattering data that are in one-to-one correspondence with the solution of the VNLS equation. Note that, in general, this is a hard problem in the framework of IST for multicomponent integrable systems. Typically, the minimal set of spectral data consists of a certain number of reflection coefficients, a certain number of discrete eigenvalues and associated norming constants. Even in the simpler case of the pure soliton (reflectionless) solutions presented in this work, the answer to the question is not obvious, since we have seen that different combinations of zeros among the principal and non-principal analytic minors can lead to the same solution. Establishing a classification of the solutions in terms of the combination of zeros of the analytic minors therefore appears to be a non-trivial issue, which it is expected to require some appropriate combinatorial analysis.

A final related issue is that, as we have seen in Section V, when the extra minors are non-zero, their value at the points of the discrete spectrum enters the solution explicitly in the form of a norming constant. This is in contrast to the principal minors, for which their specific value at any points of discrete spectrum where they are non-zero does not explicitly enter the solution. This is yet another indication that the extra minors play a different role than the principal minors in the problem.

Of course all of the above questions also arise for the VNLS equation with a higher number of components, where their complexity is expected to increase further. It is hoped that these and many other interesting questions related to these systems can be addressed in the future.

ACKNOWLEDGMENTS

We thank M. J. Ablowitz for many insightful discussions related to this work. G.B. and B.P. also acknowledge financial support from NSF under Nos. DMS-1311847 and DMS-1311883, respectively.

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