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Dark Energy and Modified Gravity in Degenerate Higher-Order Scalar-Tensor (DHOST) theories: a review

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This article reviews scalar-tensor theories characterized by a Lagrangian that, despite the presence of second order derivatives, contain a single scalar degree of freedom. These theories, known as Degenerate Higher-Order Scalar-Tensor (DHOST) theories, include Horndeski and Beyond Horndeski theories. They propagate a single scalar mode as a consequence of the degeneracy of their Lagrangian and, therefore, are not plagued by an Ostrodradsky instability. They have been fully classified up to cubic order in second-order derivatives. The study of their phenomenological consequences restricts the subclass of DHOST theories that are compatible with observations. In cosmology, these theories can be described in the language of the unified effective approach to dark energy and modified gravity. Compact objects in the context of DHOST theories are also discussed.

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I. INTRODUCTION

General Relativity (GR), which is now more than centennial, has so far been confirmed by observations, at least provided a cosmological constant is added to the original theory in order to account for the present cosmological acceleration. The first direct observations of gravitational waves are also in agreement with the main predictions of GR, paving the way for refined tests of gravity in the strong field regime in the near future.

Despite these observational successes, there have been numerous attempts to modify or extend general relativity, spurred by various motivations. One motivation is to replace invisible ingredients that appear to be required by observations, such as dark energy or dark matter, by a modification of gravity on galactic or cosmological scales. Another motivation is to take into account, at least from an effective point of view, the corrections to general relativity that could resolve the singularities associated with black holes or with the early universe. A more down-to-earth motivation is to test general relativity quantitatively by constructing a parametrized space of theories around GR, which could be constrained by observations.

Because of their simplicity, scalar-tensor theories have played a prominent role in these attempts to go beyond GR. Many of the traditional scalar-tensor extensions of general relativity are described by a total action of the form

$$S = \frac{1}{16\pi\bar{G}} \int d^4x\sqrt{-g}\,[F(\phi)R - Z(\phi)g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 2U(\phi)] + S_m\,[\psi_m, g_{\mu\nu}]\ , \tag{1.1}$$

which can be seen as generalizations of the Brans-Dicke theories. The metric $g_{\mu\nu}$ in the above action corresponds to the physical (or Jordan) frame metric to which matter is minimally coupled, as indicated in the matter action S_m which does not depend on ϕ. Note that the general form (1.1) includes some of the most studied models in the cosmological context, such as quintessence and $f(R)$ theories which can be recast as scalar-tensor theories. K-essence is an extension of the traditional form (1.1), where the Lagrangian is allowed to be a non-linear function of the kinetic term $X \equiv \nabla^\mu\phi\,\nabla_\mu\phi$.

More recently, special attention has been devoted to scalar-tensor theories whose Lagrangians contain second-order derivatives of a scalar field, i.e. of the form

$$\mathcal{L}(\nabla_\mu\nabla_\nu\phi, \nabla_\mu\phi, \phi\,; g_{\mu\nu})\,. \tag{1.2}$$

Lagrangians of this type, which contain second-order time derivative $\ddot{\phi}$, are generically plagued by an instability due to the presence, in addition to the usual scalar mode and tensor modes, of an extra scalar degree of freedom (unless the higher order terms are restricted to be perturbative terms in the sense of low energy effective theories). This instability, known as Ostrogradsky instability, is related to a linear dependence of the Hamiltonian on one of the conjugate momenta [1, 2].

Until a few years ago, it was believed that only theories that yield second-order Euler-Lagrange equations, for both the scalar field and the metric, were free of this dangerous extra degree of freedom. Recently, it has been realized that this criterion is too restrictive and that there actually exists a much larger class of theories that satisfy this property, including theories that lead to higher-order Euler-Lagrange equations. What characterizes these theories is not the order of their equation of motion but the degeneracy of their Lagrangian, which guarantees that only one scalar degree of freedom is present. For this reason, these theories have been dubbed Degenerate Higher-Order Scalar-Tensor (DHOST) theories. The goal of this review is to present these theories and some of their phenomenological consequences.

The outline of this article is the following. In the next section, we briefly present the main scalar-tensor theories with higher-order derivatives that have been investigated in the last few years, from Horndeski to DHOST theories. Section III is devoted to the notion of degeneracy which is crucial to understand how the presence of an extra scalar degree of freedom can be avoided. The idea of degeneracy is then used in section IV where quadratic DHOST theories are explicitly constructed. The consequences of DHOST theories in cosmology and for astrophysical objects are then discussed in sections V and VI, respectively. Section VII concentrates on the status of DHOST theories after the observation of a binary neutron star merger via both gravitational waves and gamma-rays. Some other aspects and perspectives are discussed in the conclusion.

II. SCALAR-TENSOR THEORIES WITH SECOND ORDER DERIVATIVES

In addition to the traditional scalar-tensor theories, whose Lagrangian depends at most on first order derivatives of the scalar field, the space of theories up to second order derivatives in their Lagrangian that contain a single scalar degree of freedom has been reappraised in the last few years. As a result, the "island" of safe theories has undergone an expansion from Horndeski theories up to DHOST theories (see Fig. 1).

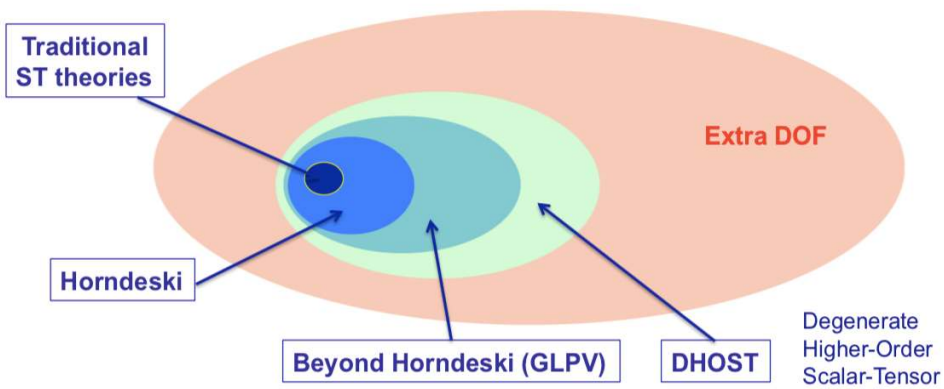

Figure 1. Landscape of scalar-tensor theories.

A. Horndeski theories

Scalar-tensor theories with a Lagrangian that contains second-order derivatives of a scalar field were constructed by Horndeski in 1974, with the specific requirement that the Euler-Lagrange equations are at most second order [3]. In the modern presentation, the general action of Horndeski theories are usually written in terms of five elementary Lagrangians:

$$L_2^{\rm H} \equiv G_2(\phi, X)\,, \qquad L_3^{\rm H} \equiv G_3(\phi, X)\,\Box\phi\,, \tag{2.1}$$

$$L_4^{\rm H} \equiv G_4(\phi, X)\,{}^{(4)}R - 2G_{4,X}(\phi, X)(\Box\phi^2 - \phi^{\mu\nu}\phi_{\mu\nu})\,, \tag{2.2}$$

$$L_5^{\rm H} \equiv G_5(\phi, X)\,{}^{(4)}G_{\mu\nu}\phi^{\mu\nu} + \frac{1}{3}G_{5,X}(\phi, X)(\Box\phi^3 - 3\,\Box\phi\,\phi_{\mu\nu}\phi^{\mu\nu} + 2\,\phi_{\mu\nu}\phi^{\mu\sigma}\phi^{\nu}{}_{\sigma})\,, \tag{2.3}$$

with the notation

$$\phi_\mu \equiv \nabla_\mu\phi\,, \qquad \phi_{\mu\nu} \equiv \nabla_\nu\nabla_\mu\phi\,, \qquad X \equiv \phi^\mu\phi_\mu\,. \tag{2.4}$$

Each of these Lagrangians depends on an arbitrary function of ϕ and X, denoted $G_A(\phi, X)$. Note that if one considers only $G_2(\phi, X)$ and $G_4(\phi)$, the Lagrangian does not contain any second order derivative and one recovers the general case of K-essence, mentioned in the introduction. Horndeski's work was mostly forgotten for many years, until it was resurrected in [4]. At that time, Horndeski theories had just been rediscovered as a generalization of the galileon models [5–7].

B. Beyond Horndeski (GLPV) theories

For a long time, it was believed that second-order Euler-Lagrange equations, as required by Horndeski, were necessary to have a single scalar degree of freedom. A first breach in this argument was provided in [8], where it was noticed that (invertible) field redefinitions of the metric of the form [9]

$$\tilde{g}_{\mu\nu} = C(X,\phi) g_{\mu\nu} + D(X,\phi)\, \phi_\mu\, \phi_\nu \tag{2.5}$$

in the Einstein-Hilbert action yields a new Lagrangian leading to third-order equations of motion, the number of degrees of freedom being preserved in an invertible field redefinition (see e.g. [10]).

Soon after, following a different line of reasoning, it was proposed in [11, 12] to extend the four elementary Horndeski Lagrangians (2.1-2.3) with the following two Lagrangians

$$L_4^{\rm bH} \equiv F_4(\phi,X)\epsilon^{\mu\nu\rho}{}_\sigma\, \epsilon^{\mu'\nu'\rho'\sigma}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\ , \tag{2.6}$$

$$L_5^{\rm bH} \equiv F_5(\phi,X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\phi_\mu\phi_{\mu'}\phi_{\nu\nu'}\phi_{\rho\rho'}\phi_{\sigma\sigma'}\ , \tag{2.7}$$

which depend on the two extra functions $F_4(\phi,X)$ and $F_5(\phi,X)$.

Despite leading to third order equations of motion, there exist theories that propagate a single scalar degree of freedom for appropriate combinations of the six Lagrangians (2.1-2.3) and (2.6-2.7). It should be stressed that not all combinations of these six Lagrangians are allowed, but only those for which the total Lagrangian is degenerate (in the sense that will be defined precisely below), which means a special tuning of the functions F_4 or F_5 (see [13–15] for details). This extended class of theories, which include Horndeski theories as a particular case, is usually denoted Beyond Horndeski or GLPV in the literature.

C. DHOST theories

Beyond Horndeski theories turned out to be an intermediate step on the way to a much more general class of scalar-tensor theories, based on the idea of degeneracy ensuring the absence of any extra scalar degree of freedom. We present these scalar-tensor theories, introduced in [13] and now known as Degenerate Higher-Order Scalar-Tensor (DHOST) theories[1].

DHOST theories were constructed and fully classified up to quadratic order (in second-order derivatives) in [13] (see also [16–18]) and later up to cubic order in [15]. The corresponding Lagrangians can be written in the form

$$S[g,\phi] = \int d^4x\,\sqrt{-g}\,\Bigg[F_{(2)}(X,\phi)\,{}^{(4)}R + P(X,\phi) + Q(X,\phi)\Box\phi + \sum_{I=1}^{5} A_I(X,\phi)L_a^{(2)} + F_{(3)}(X,\phi)\,G_{\mu\nu}\phi^{\mu\nu} + \sum_{I=1}^{10} B_I(X,\phi)L_a^{(3)}\Bigg]\ , \tag{2.8}$$

where we have introduced five elementary Lagrangians that are quadratic in $\phi_{\mu\nu}$, namely

$$L_1^{(2)} = \phi_{\mu\nu}\phi^{\mu\nu}\,, \qquad L_2^{(2)} = (\Box\phi)^2\,, \qquad L_3^{(2)} = (\Box\phi)\phi^\mu\phi_{\mu\nu}\phi^\nu\,, \tag{2.9}$$

$$L_4^{(2)} = \phi^\mu\phi_{\mu\rho}\phi^{\rho\nu}\phi_\nu\,, \qquad L_5^{(2)} = (\phi^\mu\phi_{\mu\nu}\phi^\nu)^2\,. \tag{2.10}$$

as well as ten cubic Lagrangians

$$L_1^{(3)} = (\Box\phi)^3\,, \quad L_2^{(3)} = (\Box\phi)\,\phi_{\mu\nu}\phi^{\mu\nu}\,, \quad L_3^{(3)} = \phi_{\mu\nu}\phi^{\nu\rho}\phi^\mu_\rho\,, \tag{2.11}$$

$$L_4^{(3)} = (\Box\phi)^2\,\phi_\mu\phi^{\mu\nu}\phi_\nu\,, \quad L_5^{(3)} = \Box\phi\,\phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,, \quad L_6^{(3)} = \phi_{\mu\nu}\phi^{\mu\nu}\phi_\rho\phi^{\rho\sigma}\phi_\sigma\,, \tag{2.12}$$

$$L_7^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^{\rho\sigma}\phi_\sigma\,, \quad L_8^{(3)} = \phi_\mu\phi^{\mu\nu}\phi_{\nu\rho}\phi^\rho\,\phi_\sigma\phi^{\sigma\lambda}\phi_\lambda\,, \tag{2.13}$$

$$L_9^{(3)} = \Box\phi\,(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,, \quad L_{10}^{(3)} = (\phi_\mu\phi^{\mu\nu}\phi_\nu)^3\,. \tag{2.14}$$

The above quadratic and cubic Lagrangians represent all the possible contractions of the second-order derivatives $\phi_{\mu\nu}$ with the metric $g_{\mu\nu}$ and the scalar field gradient ϕ_μ.

---

[1] This name was introduced in [16]. The same theories were also dubbed "Extended Scalar-Tensor" theories in [17].

The action (2.8) contains 19 functions of X and ϕ but these functions cannot be chosen arbitrarily to get a DHOST theory. Except the functions P and Q, which are arbitrary, the other functions must satisfy degeneracy conditions so that the corresponding theory contains a single scalar mode. Taking into account all these degeneracy conditions[2], which restrict the allowed functions $F_{(2)}$, A_I, $F_{(3)}$ and all B_a, one can then classify all DHOST theories of the form (2.8), whose functions must satisfy specific constraints [15]:

- Purely quadratic theories (with no cubic terms, i.e. such that $F_{(3)}$ and all B_I vanish) : 7 subclasses (4 subclasses with $F_{(2)} \neq 0$, 3 with $F_{(2)} = 0$).
- Purely cubic theories (with no quadratic term, i.e. such that $F_{(2)}$ and all A_I vanish): 9 subclasses (2 subclasses with $F_{(3)} \neq 0$, 7 with $F_{(3)} = 0$).
- Mixed quadratic and cubic theories: 25 subclasses (out of $7 \times 9 = 63$) are degenerate. It must be stressed that the sum of two degenerate Lagragians is not necessarily degenerate.

Some details about how DHOST theories were constructed and classified will be given in section IV, after the notion of degeneracy is explained in detail in the next section.

Particular cases

DHOST theories (4.1) include as a particular case the Horndeski theories, with the identifications

$$F_{(2)} = G_4\,, \qquad A_1 = -A_2 = 2G_{4,X}\,, \qquad A_3 = A_4 = A_5 = 0\,. \tag{2.15}$$

for the quadratic terms, corresponding to $L_4^{\rm H}$ in (2.2), and

$$F_{(3)} = G_5\,, \qquad 3B_1 = -B_2 = \frac{3}{2}B_3 = G_{5,X}\,, \qquad B_I = 0\ (I = 4, \ldots, 10) \tag{2.16}$$

for the cubic terms, corresponding to $L_5^{\rm H}$ in (2.3).

DHOST theories also include the larger class of Beyond Horndeski (GLPV) theories, with the identifications

$$F_{(2)} = G_4\,, \qquad A_1 = -A_2 = 2G_{4,X} + XF_4\,, \qquad A_3 = -A_4 = 2F_4\,, \qquad A_5 = 0\,. \tag{2.17}$$

for the quadratic terms, corresponding to $L_4^{\rm H} + L_4^{\rm bH}$, and

$$\begin{aligned} &F_{(3)} = G_5\,, \quad 3B_1 = -B_2 = \frac{3}{2}B_3 = G_{5,X} + 3XF_5\,, \quad -2B_4 = B_5 = 2B_6 = -B_7 = 6F_5\,, \\ &B_8 = B_9 = B_{10} = 0\,, \end{aligned} \tag{2.18}$$

for the cubic terms, corresponding to $L_5^{\rm H} + L_5^{\rm bH}$.

Interestingly, Beyond Horndeski theories, and therefore Horndeski theories, belong to the same subclass of DHOST theories, dubbed class Ia in [16], although some theories in class Ia are not Horndeski or Beyond Horndeski theories. Moreover, when matter is ignored, all the Lagrangians in class Ia can be mapped, via field redefinition of the metric (see discussion in subsection IV E), into a Horndeski Lagrangian.

III. DEGENERATE THEORIES

Despite the presence of higher order time derivatives in their (Euler-Lagrange) equations of motion, *degenerate* scalar-tensor theories contain a single degree of freedom. This can be more easily understood by considering a simple toy model in the context of classical mechanics. We will see in the next section how the notion of degeneracy operates for covariant scalar-tensor theories. The presentation of this section mainly follows the discussion given in [13].

---

[2] The degeneracy conditions at quadratic order will be given explicitly and the cubic ones can be found in [15].

A. Simple toy model

Let us consider the Lagrangian

$$L = \frac{1}{2} a\, \ddot{\phi}^2 + b\, \ddot{\phi}\, \dot{q} + \frac{1}{2} c\, \dot{q}^2 + \frac{1}{2} \dot{\phi}^2 - V(\phi, q)\,, \tag{3.1}$$

where the variable ϕ appears with second order time derivatives, while q is a standard variable, appearing in the Lagrangian with at most first order derivatives. The coefficients a, b and c are assumed here to be constant for simplicity.

The two equations of motion derived from (3.1) are given by

$$a\, \ddddot{\phi} - \ddot{\phi} + b\, \dddot{q} - V_\phi = 0\,, \tag{3.2}$$

$$c\ddot{q} + b\, \dddot{\phi} + V_q = 0\,, \tag{3.3}$$

where $V_q \equiv \partial V/\partial q$ and $V_\phi \equiv \partial V/\partial \phi$. As expected, these equations contain higher order time derivatives if a and b are nonzero: the first equation is fourth order if $a \neq 0$ and both equations are third order if $a = 0$ and $b \neq 0$. In general, these higher order terms can be associated to the existence of an extra degree of freedom. However, there are special cases where the extra degree of freedom can be avoided even if the equations of motion feature higher order time derivatives.

To compute the number of degrees of freedom (either in the Lagrangian or Hamiltonian frameworks), it is convenient to reformulate the theory in a way that eliminates explicit higher order time derivative in the Lagrangian. For that purpose, we simply replace $\dot{\phi}$ by a new variable Q in (3.1) and add a "constraint" that imposes $Q = \dot{\phi}$. Thus, we introduce the new Lagrangian

$$L = \frac{1}{2} a\, \dot{Q}^2 + b\, \dot{Q}\dot{q} + \frac{1}{2} c\, \dot{q}^2 + \frac{1}{2} Q^2 - V(\phi, q) - \lambda(Q - \dot{\phi})\,, \tag{3.4}$$

where Q and λ are two new variables.

The equations of motion derived from this new Lagrangian are

$$a\ddot{Q} + b\ddot{q} = Q - \lambda\,, \tag{3.5}$$

$$b\ddot{Q} + c\ddot{q} = -V_q \tag{3.6}$$

$$\dot{\phi} \;=\; Q \quad \text{and} \quad \dot{\lambda} = -V_\phi\,. \tag{3.7}$$

It is easy to check that this system of equations is equivalent to the original one (3.2 - 3.3).

We now introduce the kinetic matrix, or Hessian matrix, defined by

$$M = \left(\frac{\partial^2 L}{\partial v^A \partial v^B} \right) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}\,. \tag{3.8}$$

If M is invertible, which corresponds to the generic case, the equations (3.5-3.6) enable us to express the second derivatives $\ddot{Q}$ and $\ddot{q}$ in terms of quantities with at most first order derivatives. Together with the equations (3.7), the differential system thus requires 6 initial conditions, corresponding to the initial values for Q, $\dot{Q}$, q, $\dot{q}$, λ and ϕ. We thus conclude that the system contains 3 degrees of freedom, including an extra degree of freedom due to the presence of second derivatives in the Lagrangian. Note that this remains true even if $a = 0$, provided $b \neq 0$.

B. Degeneracy: eliminating the extra degree of freedom

If the kinetic matrix M is degenerate, i.e. if its determinant vanishes,

$$ac - b^2 = 0\,, \tag{3.9}$$

then the system does not contain any extra degree of freedom, as we show below.

An obvious way to make M degenerate is to choose $a = 0$ and $b = 0$. In this trivial case, all the higher order derivatives disappear in the original Lagrangian and the system describes 2 degrees of freedom as usual.

Let us now turn to more interesting situations where the degeneracy is nontrivial. Although the associated equations of motion involve higher order time derivatives (up to fourth order if $a \neq 0$, third order otherwise), one can show that the degeneracy guarantees that there is no extra degree of freedom.

In the following, we are going to assume that $c \neq 0$. Combining (3.5 - 3.6), one gets

$$Q + \frac{b}{c} V_q = \lambda \tag{3.10}$$

and the second equation can be rewritten as

$$c\,\ddot{x} + V_q = 0, \tag{3.11}$$

where we have introduced

$$x \equiv q + \frac{b}{c}\dot{\phi}\,. \tag{3.12}$$

Note that q can be replaced everywhere by $x - \frac{b}{c}\dot{\phi}$.

Taking the time derivative of (3.10) and using the second equation in (3.7), one gets the second-order equation of motion

$$\left(1 - \frac{b^2}{c^2} V_{qq}\right)\ddot{\phi} + \frac{b}{c} V_{qq}\,\dot{x} + V_\phi = 0\,. \tag{3.13}$$

We have thus obtained a second order system for the variables x and ϕ (the potential V and its derivatives depend on ϕ, $\dot{\phi}$ and x via the substitution $q = x - (b/c)\dot{\phi}$), which means that the system only requires 4 initial conditions, i.e. that there are only two degrees of freedom. Note that the system (3.11-3.13) involves the variables ϕ and x, but not q. However, once the system is solved for $\phi(t)$ and the $x(t)$, one can recover the original variables via the relations $q(t) = x(t) - b\dot{\phi}(t)/c$.

It is also instructive to count the number of degrees of freedom via a Hamiltonian analysis of the system, which can be found in the Appendix. In summary, the degeneracy of the Lagrangian implies the presence of a primary constraint. The time evolution of this primary constraint then shows the presence of a secondary constraint. These two constraints, which are second-class, eliminate one degree of freedom, which corresponds to the extra degree of freedom due to the higher derivatives. The linear dependence of the Hamiltonian on one of the momenta, which is a signature of the Ostrogradski instability, is also eliminated.

To conclude, the analysis of the Lagrangian (3.1) shows that the crucial ingredient to avoid the presence of an extra degree of freedom is the degeneracy of the kinetic matrix (3.8). The degeneracy of scalar-tensor theories can be investigated in a similar way, as will be illustrated in the next section. The scalar field, with second derivatives, will be analogous to our variable $\phi(t)$, while the metric field will play a rôle similar to that of the "regular" degrees of freedom $q(t)$. Of course, the mathematical structure will be more complicated but the essential features concerning the degeneracy of the kinetic matrix will turn out to be quite similar.

C. Generalization to multi-variable Lagrangians

In [19] and [20], the previous analysis was extended to more general Lagrangians containing several variables with accelerations, namely of the form

$$L(\ddot{\phi}^\alpha, \dot{\phi}^\alpha, \phi^\alpha; \dot{q}^i, q^i) \qquad (\alpha = 1, \cdots, n;\ i = 1, \cdots, m)\,, \tag{3.14}$$

with $\alpha = 1, \cdots, n$ and $i = 1, \cdots, m$. A priori, one generically expects $2n + m$ degrees of freedom, i.e. n extra DOFs, for this type of Lagrangians.

As shown in [19], one can eliminate these n extra DOFs by requiring, first, a degeneracy of order n of the kinetic matrix, or Hessian matrix:

$$K = \begin{pmatrix} L_{\dot{Q}^\alpha \dot{Q}^\beta} & L_{\dot{Q}^\alpha \dot{q}^j} \\ L_{\dot{Q}^\beta \dot{q}^i} & L_{\dot{q}^i \dot{q}^j} \end{pmatrix}\,, \tag{3.15}$$

The degeneracy of order n is expressed by the condition,

$$L_{\alpha\beta} - L_{\alpha i} L^{ij} L_{j\beta} = 0\,, \tag{3.16}$$

which is associated with the existence of n primary constraints in the Hamiltonian formulation.

If $n=1$, this primary condition is enough to eliminate the extra DOF, as we saw in the previous subsection. If $n>1$, by contrast, a secondary condition is required in order to ensure that there exist n secondary constraints which, together with the n primary constraints, can eliminate the n extra DOFs. This secondary condition reads

$$\begin{aligned}&L_{\dot{Q}^\alpha\dot{\phi}^\beta}-L_{\dot{Q}^\beta\dot{\phi}^\alpha}+L_{\dot{\phi}^\alpha\dot{q}^i}L^{ij}L_{\dot{q}^j\dot{Q}^\beta}-L_{\dot{Q}^\alpha\dot{q}^i}L^{ij}L_{\dot{q}^j\dot{\phi}^\beta}\\&+L_{\dot{Q}^\alpha\dot{q}^i}L^{ij}L_{q^j\dot{Q}^\beta}-L_{\dot{Q}^\alpha q^i}L^{ij}L_{\dot{q}^j\dot{Q}^\beta}\\&+L_{\dot{Q}^\alpha\dot{q}^i}L^{ij}\left(L_{\dot{q}^jq^k}-L_{q^j\dot{q}^k}\right)L^{kl}L_{\dot{q}^l\dot{Q}^\beta}=0\,,\end{aligned}\tag{3.17}$$

If $n=1$, one can check that this condition is automatically satisfied, as expected. If $m=0$, it reduces to

$$L_{\dot{Q}^\alpha\dot{\phi}^\beta}-L_{\dot{Q}^\beta\dot{\phi}^\alpha}=0\,.\tag{3.18}$$

Finally, it is also possible to consider generalized point particle Lagrangians with third or higher order time derivatives and look for theories without Ostrogradsky ghosts, as explored in [21, 22]. Analogous conditions for field theories were given in [23] (in particular, the property that the primary constraint is sufficiently when $n=1$ still holds for Lorentz-invariant theories).

IV. CONSTRUCTING DHOST THEORIES

The goal of this section is to show how the Lagrangians of DHOST theories can be constructed in a systematic way, illustrated by the case of quadratic DHOST theories. The construction up to third order is more involved and is presented in detail in [15]. This rather technical section can be skipped by the reader who is more interested by the phenomenological aspects of DHOST theories, discussed in the subsequent sections. The content of this section is mainly based on [13] and [16].

A. Quadratic higher-order Lagrangians

We consider scalar-tensor Lagrangians with a quadratic dependence on second derivatives of the scalar field, of the form

$$S[g,\phi]\equiv\int\sqrt{-g}\,[F(X,\phi)\,R+C^{\mu\nu\rho\sigma}\,\nabla_\mu\nabla_\nu\phi\,\nabla_\rho\nabla_\sigma\phi]\tag{4.1}$$

where the tensor $C^{\mu\nu\rho\sigma}$ depends only on ϕ, ϕ_μ and the metric $g_{\mu\nu}$. Given the way it is contracted in the action, one can impose, without loss of generality, the following symmetries:

$$C^{\mu\nu\rho\sigma}=C^{\nu\mu\rho\sigma}=C^{\mu\nu\sigma\rho}=C^{\rho\sigma\mu\nu}\,.\tag{4.2}$$

Therefore, this tensor can always be written in the form

$$\begin{aligned}C^{\mu\nu,\rho\sigma}=&\frac{1}{2}A_1\left(g^{\mu\rho}g^{\nu\sigma}+g^{\mu\sigma}g^{\nu\rho}\right)+A_2\,g^{\mu\nu}g^{\rho\sigma}+\frac{1}{2}A_3\left(\phi^\mu\phi^\nu g^{\rho\sigma}+\phi^\rho\phi^\sigma g^{\mu\nu}\right)\\&+\frac{1}{4}A_4(\phi^\mu\phi^\rho g^{\nu\sigma}+\phi^\nu\phi^\rho g^{\mu\sigma}+\phi^\mu\phi^\sigma g^{\nu\rho}+\phi^\nu\phi^\sigma g^{\mu\rho})+A_5\,\phi^\mu\phi^\nu\phi^\rho\phi^\sigma\,,\end{aligned}\tag{4.3}$$

where the A_I are arbitrary functions of ϕ and X.

As a consequence, the scalar part of the Lagrangian that depends on second-order derivatives of ϕ can be written as

$$L_\phi\equiv C^{\mu\nu\rho\sigma}\,\nabla_\mu\nabla_\nu\phi\,\nabla_\rho\nabla_\sigma\phi=\sum_I A_I L_I^{(2)}\,,\tag{4.4}$$

expressed in terms of the five elementary Lagrangians quadratic in second derivatives introduced in (2.9-2.10).

Instead of working directly with second order derivatives in the Lagrangian, it is more convenient to introduce the new variable

$$Q_\mu\equiv\nabla_\mu\phi\tag{4.5}$$

via a constraint in the Lagrangian. Our new Lagrangian is thus given by

$$S[g,\phi;Q_\mu,\lambda^\mu]=\int\sqrt{-g}\,\{F\,R+C^{\mu\nu\rho\sigma}\nabla_\mu Q_\nu\,\nabla_\rho Q_\sigma+\lambda^\mu(\nabla_\mu\phi-Q_\mu)\}\,,\tag{4.6}$$

where the tensor $C^{\mu\nu\rho\sigma}$ is now expressed in terms of Q_μ and ϕ. It is straightforward to verify that (4.6) and (4.1) are equivalent.

B. Kinetic matrix

To write the degeneracy conditions, one must identify the kinetic matrix, i.e. the analog of (3.8). This requires a 3+1 decomposition of spacetime in order to separate time derivatives from spatial ones.

We thus assume the existence of a slicing of spacetime with 3-dimensional spacelike hypersurfaces. We introduce their normal unit vector n^a, which is time-like, and satisfies the normalization condition $n_a n^a = -1$. This induces a three-dimensional metric, corresponding to the projection tensor on the spatial hypersurfaces, defined by

$$h_{ab} \equiv g_{ab} + n_a n_b \,. \tag{4.7}$$

Let us also introduce a time direction vector $t^a = \partial/\partial t$ associated with a time coordinate t that labels the spacelike hypersurfaces. One can always decompose t^a as

$$t^a = N n^a + N^a, \tag{4.8}$$

thus defining the lapse function N and the shift vector N^a orthogonal to n^a.

It is then convenient to decompose Q_a introduced in (4.5) into its normal and spatial projections, defined by

$$Q_* \equiv Q_a n^a \,, \qquad \hat{Q}_a \equiv h_a^b \, Q_b \,. \tag{4.9}$$

We find that the only terms in $\nabla_{(a} Q_{b)}$ that contain time derivatives and are thus relevant for the kinetic part of the Lagrangian reduce to

$$(\nabla_{(a} Q_{b)})_{\rm kin} = \lambda_{ab} \, \dot{Q}_* + \Lambda_{ab}{}^{cd} \, K_{cd} \,, \tag{4.10}$$

where K_{ab} is the extrinsic curvature tensor, defined by[3]

$$K_{ab} \equiv \frac{1}{2N} \left(\dot{h}_{ab} - D_a N_b - D_b N_a \right) \,, \tag{4.11}$$

and we have introduced the tensors

$$\lambda_{ab} \equiv \frac{1}{N} n_a n_b \,, \qquad \Lambda_{ab}{}^{cd} \equiv -Q_* \, h^c_{(a} h^d_{b)} + 2 \, n_{(a} h^{(c}_{b)} \hat{Q}^{d)} \,. \tag{4.12}$$

Strictly speaking, only the time derivative $\dot{h}_{ab}$ in (4.11) is relevant for the kinetic part of the action but we keep K_{ab} for convenience.

Substituting (4.10) into the action (4.6), one obtains the kinetic part of the Lagrangian, which can be written in the form

$$L_{\rm kin} = \mathcal{A} \, \dot{Q}_*^{\,2} + 2 \mathcal{B}^{ab} \, \dot{Q}_* K_{ab} + \mathcal{C}^{ab,cd} K_{ab} K_{cd} \,. \tag{4.13}$$

This is the analog of the Lagrangian (3.4), with Q_* in place of Q and K_{ab} (or $\dot{h}_{ab}$) in the rôle of $\dot{q}$. The coefficients, analogous to a, b and c in (3.4), are given by

$$\mathcal{A} \equiv C^{ef,gh} \lambda_{ef} \lambda_{gh} \,, \tag{4.14}$$

$$\mathcal{B}^{ab} \equiv 2 F_X \frac{Q_*}{N} \, h^{ab} + C^{ef,gh} \Lambda_{ef}{}^{ab} \lambda_{gh} \tag{4.15}$$

and

$$\begin{aligned} \mathcal{C}^{ab,cd} \equiv \, & \frac{1}{2} F \left(h^{ac} h^{bd} + h^{ad} h^{bc} - 2 h^{ab} h^{cd} \right) + 2 F_X \left(\hat{Q}^a \hat{Q}^b h^{cd} + \hat{Q}^c \hat{Q}^d h^{ab} \right) \\ & + C^{ef,gh} \Lambda_{ef}{}^{ab} \Lambda_{gh}{}^{cd} \,. \end{aligned} \tag{4.16}$$

It is worth stressing that the scalar curvature term $F \, {}^{(3)}\!R$ gives contributions to the coefficients $\mathcal{B}^{ab}$ and $\mathcal{C}^{ab,cd}$. For this reason, the scalar curvature term can be considered as part of the terms at quadratic order in $\phi_{\mu\nu}$.

Interestingly, in the case of the Horndeski Lagrangian $L_4^{\rm H}$, there is a cancellation between the Ricci term contribution and that from the terms quadratic in $\phi_{\mu\nu}$ so that the total coefficient $\mathcal{B}^{ab}$ vanishes. This is not surprising since Horndeski theories are, by construction, restricted to give second order equations of motion. By contrast, when $\mathcal{B}^{ab} \neq 0$, the equations of motion become higher order, as in the case of beyond Horndeski or other DHOST theories.

---

[3] The time derivative of any tensor corresponds to the Lie derivative of this tensor with respect to t^a.

C. Degeneracy conditions

Following the simple example of the previous section, we can now derive the degeneracy conditions by considering the kinetic matrix

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}^{cd} \\ \mathcal{B}^{ab} & \mathcal{C}^{ab,cd} \end{pmatrix} . \tag{4.17}$$

This matrix is degenerate if there exists an eigenvector with zero eigenvalue, i.e. if one can find v_0 and $\mathcal{V}_{cd}$ such that

$$v_0\,\mathcal{A} + \mathcal{B}^{cd}\mathcal{V}_{cd} = 0\,, \qquad v_0\,\mathcal{B}^{ab} + \mathcal{C}^{ab,cd}\,\mathcal{V}_{cd} = 0\,, \tag{4.18}$$

where $\mathcal{V}_{cd}$ is a symmetric tensor of order 2. Note that the latter can only be of the form

$$\mathcal{V}_{cd} = v_1\,h_{cd} + v_2\,\hat{Q}_c\,\hat{Q}_d\,, \tag{4.19}$$

and the contraction of $\mathcal{C}^{ab,cd}$ with $\mathcal{V}_{cd}$ can be similarly decomposed along h_{cd} and $\hat{Q}_c\hat{Q}_d$. In this way, the kinetic matrix can be seen as a 3×3 matrix. Requiring the determinant of the kinetic matrix to vanish yields (using $\hat{Q}^a\hat{Q}_a = X + Q_*^2$) an expression of the form

$$D_0(X) + D_1(X)\,Q_*^2 + D_2(X)\,Q_*^4 = 0\,, \tag{4.20}$$

with

$$D_0(X) \equiv -4(A_1+A_2)\left[XF(2A_2 + XA_4 + 4F_X) - 2F^2 - 8X^2F_X^2\right]\,, \tag{4.21}$$

$$\begin{aligned} D_1(X) \equiv\; & 4\left[X^2A_2(3A_1+A_2) - 2F^2 - 4XFA_1\right]A_4 + 4X^2F(A_1+A_2)A_5 \\ & +8XA_2^3 - 4(F + 4XF_X - 6XA_1)A_2^2 - 16(F+5XF_X)A_1A_2 \\ & +4X(3F - 4XF_X)A_2A_3 - X^2FA_3^2 + 32F_X(F+2XF_X)A_1 \\ & -16FF_XA_2 - 8F(F - XF_X)A_3 + 48FF_X^2\,, \end{aligned} \tag{4.22}$$

$$\begin{aligned} D_2(X) \equiv\; & 4\left[2F^2 + 4XFA_1 - X^2A_2(3A_1+A_2)\right]A_5 + 4(2A_1 - XA_3 - 4F_X)A_2^2 \\ & +4A_2^3 + 3X^2A_2A_3^2 - 4XFA_3^2 + 8(F+XF_X)A_2A_3 - 32F_XA_1A_2 \\ & +16F_X^2A_2 + 32F_X^2A_1 - 16FF_XA_3\,. \end{aligned} \tag{4.23}$$

Since the determinant must vanish for arbitrary values of Q_*, we deduce that degenerate theories are characterized by the three conditions

$$D_0(X) = 0, \qquad D_1(X) = 0, \qquad D_2(X) = 0\,. \tag{4.24}$$

Note that, if one ignores the dynamics of gravity, the system depends only the scalar field variables and is degenerate when $\mathcal{A} = 0$. Using the explicit expression for $\mathcal{A}$,

$$\mathcal{A} = \frac{1}{N^2}\left[A_1 + A_2 - (A_3+A_4)\,Q_*^2 + A_5\,Q_*^4\right]\,, \tag{4.25}$$

which must vanish for any value of Q_*, one finds the three conditions

$$A_1 + A_2 = 0\,, \qquad A_3 + A_4 = 0\,, \qquad A_5 = 0\,. \tag{4.26}$$

One can note that both quartic Horndeski and beyond Horndeski Lagrangians satisfy these three conditions.

The property that the degeneracy of the Lagrangian implies the absence of any extra scalar degree of freedom has been shown explicity in the general case of quadratic DHOST theories by resorting to a Hamiltonian formulation that uses the variable Q_* and does not assume any particular gauge [24]. Earlier Hamiltonian analyses were proposed for quadratic Beyond Horndeski theories described in the unitary gauge in [11, 12, 25] and for a particular Beyond Horndeski model in an arbitrary gauge in [26]. Note that there exists a special class of theories that appear degenerate in the unitary gauge but not in an arbitrary gauge (and therefore are not DHOST theories) and, in this case, the extra mode can be tamed via appropriate boundary conditions [27]. Another possibility to obtain healthy scalar-tensor theories with higher derivatives is to give up general covariance and work directly with spatially covariant theories [28, 29].

D. Classification of quadratic DHOST theories

In this subsection, we summarize the classification of quadratic DHOST theories, which can be found in detail in [13, 16, 17]. This classification has been extended to cubic order in [15].

For quadratic DHOST theories, the condition $D_0(X) = 0$ is the simplest of all three and allows to distinguish several classes of theories. Indeed, D_0 can vanish either if $A_1 + A_2 = 0$, which defines our first class of solutions, or if the term between brackets in (4.21) vanishes, which defines our second class, as well as our third class corresponding to the special case where $F = 0$. Below, we present explicitly only the first subclass of quadratic DHOST theories, which contains Horndeski and Beyond Horndeski theories, as this is the only class which turns out to be viable for phenomenology as we will explain later.

1. *Class I*

This class is characterized by $A_2 = -A_1$.

- Subclass Ia (or N-I): $F \neq XA_1$

  One can use the conditions $D_1(X) = 0$ and $D_2(X) = 0$ to express, respectively, A_4 and A_5 in terms of A_2 and A_3, provided $F + XA_2 \neq 0$. This defines the subclass Ia, characterized by

$$\begin{aligned} A_4 = \frac{1}{8(F - XA_1)^2} \big[& -16XA_1^3 + 4(3F + 16XF_X)A_1^2 - X^2FA_3^2 \\ & -(16X^2F_X - 12XF)A_3A_1 - 16F_X(3F + 4XF_X)A_1 \\ & +8F(XF_X - F)A_3 + 48FF_X^2 \big] \end{aligned} \tag{4.27}$$

  and

$$A_5 = \frac{(4F_X - 2A_1 + XA_3)\left(-2A_1^2 - 3XA_1A_3 + 4F_XA_1 + 4FA_3\right)}{8(F - XA_1)^2} \,. \tag{4.28}$$

  Degenerate theories in class Ia thus depend on three arbitrary functions A_1, A_3 and F.

- Subclass Ib (or N-II): $F = XA_1$

  In this subclass, $A_3 = 2\left(F - 2XF_X\right)/X^2$, and where F, A_4 and A_5 are arbitrary functions. The metric sector is degenerate[4].

2. *Class II*

This class is characterized by $F \neq 0$ and $A_2 \neq -A_1$.

- Subclass IIa (or N-IIIi): $F \neq XA_1$

  Three arbitrary functions: F, A_1 and A_2

- Subclass IIb (or N-IIIii): $F = XA_1$

  Three arbitrary functions: A_1 (or F), A_2 and A_3. Like class Ib, the metric sector is degenerate.

3. *Class III*

This class is characterized by $F = 0$.

---

[4] In addition to the degeneracy of the kinetic matrix for the scalar modes, there could also be a degeneracy in the tensor part of the kinetic matrix, in which case one would lose one or more of the usual tensors modes. This possibility is due to the presence of extra primary constraints in the system. This point is discussed in [16].

- Subclass IIIa (or M-I): $A_1 + 3A_2 \neq 0$

  Three arbitrary functions: A_1, A_2 and A_3. Note that this subclass has a non empty intersection with class Ia, which is parametrized by i two arbitrary functions, A_1 and A_3. This intersection includes the Lagrangian $L_4^{\rm bh}$ (for which $A_1/X = A_3/2 = F_4$).

- Subclass IIIb (or M-II): $A_1 + 3A_2 = 0$

  Three arbitrary functions: A_3, A_4 and A_5, and, in general, a degenerate metric sector. Another special case corresponds to the class

$$F = 0\,, \qquad A_1 = 0\,, \qquad \text{(class IIIc)} \tag{4.29}$$

  which depends on four arbitrary functions. Since $F - A_1 X = 0$, this class is also degenerate in the metric sector.

- Subclass IIIc (or M-III): $A_1 = 0$

  Four arbitrary functions. The metric sector is also degenerate since $F = X A_1 = 0$.

E. Disformal transformations

The higher-order Lagrangians that we have considered can always be modified by a field redefinition, in particular via disformal transformations of the metric [9]

$$\tilde{g}_{\mu\nu} = C(X,\phi) g_{\mu\nu} + D(X,\phi)\,\phi_\mu\,\phi_\nu\,. \tag{4.30}$$

Via this transformation, any action $\tilde{S}$ written as a functional of $\tilde{g}_{\mu\nu}$ and ϕ gives a new action S for $g_{\mu\nu}$ and ϕ, when one substitutes the above expression for $\tilde{g}_{\mu\nu}$ into $\tilde{S}$:

$$S[\phi, g_{\mu\nu}] \equiv \tilde{S}\left[\phi, \tilde{g}_{\mu\nu} = C\, g_{\mu\nu} + D\,\phi_\mu\phi_\nu\right]\,. \tag{4.31}$$

In this case, the actions S and $\tilde{S}$ are said to be related by the disformal transformation (4.30).

If matter is ignored, the actions S and $\tilde{S}$ describe the same physics, provided the disformal transformation is invertible (see e.g. [10] for a discussion on this point). However, if matter is included, such that the total actions read

$$S_{\rm tot} = S + S_m[\psi_m, g_{\mu\nu}]\,, \qquad \tilde{S}_{\rm tot} = \tilde{S} + S_m[\psi_m, \tilde{g}_{\mu\nu}] \tag{4.32}$$

correspond to different physical systems, since matter is minimally coupled to the metric $g_{\mu\nu}$ in the first case and to $\tilde{g}_{\mu\nu}$ in the latter case[5].

Interestingly, particular subsets of DHOST theories have some special properties under restricted disformal transformations:

- The family of Horndeski theories is stable under X-independent disformal transformations, i.e. with $C = C(\phi)$ and $D = D(\phi)$, as shown in [30].

- The family of Beyond Horndeski theories is stable under disformal transformations with $C = C(\phi)$ and $D = D(\phi, X)$, as demonstrated in [12].

- The family of DHOST theories is stable under the most general disformal transformations [16].

In particular, it has been checked explicitly that the structure of quadratic DHOST is preserved and that all seven subclasses are separately stable. Writing

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}}\left[\tilde{F}\,\tilde{R} + \sum_I \tilde{A}_I \tilde{L}_I^{(2)}\right]\,, \tag{4.33}$$

[5] By contrast, $\tilde{S} + S_m[\psi_m, \tilde{g}_{\mu\nu}]$ and $S + S_m[\psi_m, C\, g_{\mu\nu} + D\,\phi_\mu\phi_\nu]$ describe the same physics, the matter being non minimally coupled to $g_{\mu\nu}$.

the relations between the new and old Lagrangians have been obtained explicitly in [16], with in particular

$$\tilde{L}_I^{(2)} = \sum_J \mathcal{T}_{IJ} L_J^{(2)} + (\dots)\,, \tag{4.34}$$

where $\mathcal{T}_{ab}$ is an upper right triangular matrix and the dots indicate terms that are at most linear in $\phi_{\mu\nu}$ and therefore not relevant for the degeneracy.

Finally, let us note that mimetic theories [31] and their extensions (see e.g. [32] for a review), which are constructed via a non-invertible conformal or disformal transformation, also belong to the family of DHOST theories (even beyond the quadratic and cubic Lagrangians mentioned earlier), as discussed in [33, 34].

V. EFFECTIVE THEORY OF DARK ENERGY AND MODIFIED GRAVITY IN COSMOLOGY

In this section, we study the phenomenology of DHOST theories in a cosmological context. To do so, it is very useful to resort to the unified formalism providing an effective description of Dark Energy and Modified Gravity, which has been developed in a series of papers, especially [35–38] (see also [39, 40] and [41, 42] for related or alternative approaches). This formalism is based on a $3+1$ ADM decomposition of spacetime,

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\,, \tag{5.1}$$

in which the spatial slices coincide with uniform scalar field hypersurfaces (implicitly assuming that the scalar field gradient is time-like, which is natural in the cosmological context). In this specific gauge, often called unitary gauge, the action of DHOST theories can be written in the form [36, 38]

$$S = \int d^3x\, dt\, N\sqrt{h}\, L[N, K_{ij}, {}^{(3)}R_{ij}; t]\,, \tag{5.2}$$

where N is the lapse function which appears in the metric (5.1), K_{ij} is the extrinsic curvature tensor,

$$K_{ij} \equiv \frac{1}{2N}\left(\dot{h}_{ij} - D_i N_j - D_j N_i\right)\,, \tag{5.3}$$

and ${}^{(3)}R_{ij}$ the Ricci tensor associated with the spatial metric h_{ij}.

A. Homogeneous equations

The Friedmann equations associated with a spatially flat Friedmann-Lemaître-Robertson-Walker (FLRW) geometry,

$$ds^2 = -\bar{N}^2(t)dt^2 + a^2(t)\delta_{ij}dx^i dx^j\,, \tag{5.4}$$

are then derived from the (minisuperspace) homogeneous action

$$\bar{S} = \int dt\, \bar{N} a^3 \bar{L}\,, \tag{5.5}$$

which is simply obtained from the general action (5.2) via the substitution

$$\bar{L} \equiv L[N = \bar{N}(t), K^i_j = \frac{\dot{a}}{\bar{N}a}\delta^i_j, {}^{(3)}R_{ij} = 0; t]\,. \tag{5.6}$$

One must also take into account the action describing matter, which is assumed to be minimally coupled to the metric. The variation of the homogeneous matter action with respect to the lapse and the scale factor define the energy density $\rho_{\rm m}$ and the pressure $p_{\rm m}$, respectively, according to the expression

$$\delta\bar{S}_{\rm m} = \int d^4x \bar{N} a^3 \left(-\rho_{\rm m}\frac{\delta\bar{N}}{\bar{N}} + 3p_{\rm m}\frac{\delta a}{a}\right)\,. \tag{5.7}$$

Finally, the variation of the total homogeneous action $\bar{S}_{\rm total} = \bar{S} + \bar{S}_{\rm m}$ with respect to N and a yields the generalized Friedmann equations. Note that, whereas there is no explicit dependence on the derivative of N for Horndeski and Beyond Horndeski theories, this is no longer true for DHOST theories in general.

As an illustration, let us concentrate on the subclass Ia of DHOST theories (which will turn out to be the only viable subclass as shown later), following the recent analysis given in [43]. Considering only terms up to quadratic order in second derivatives, one finds that the homogeneous action can be written, taking into account the degeneracy conditions, in the form

$$\bar{S} \;=\; \int dt\, N a^3 \left\{ -6(F_{(2)} - XA_1)\left[\frac{\dot{a}}{Na} - \mathcal{V}\frac{\dot{\phi}}{N^2}\frac{d}{dt}\left(\frac{\dot{\phi}}{N}\right)\right]^2 \right.$$
$$\left. -3(Q + 2F_{(2)\phi})\frac{\dot{a}\,\dot{\phi}}{N^2 a} - Q\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{\phi}}{N}\right) + P \right\}, \tag{5.8}$$

where $\mathcal{V}$ is given by

$$\mathcal{V} \;\equiv\; \frac{4F_{(2)X} + XA_3 - 2A_1}{4(F_{(2)} - XA_1)}\,. \tag{5.9}$$

This action must be supplemented by the matter action $S_m[\psi_m, a, N]$. By varying the total action with respect to N, a and ϕ, and using (5.7), one obtains, respectively, the two Friedmann equations and the scalar equation of motion.

Note that these equations contain higher order time derivatives in general. However, due to the degeneracy of the Lagrangian, the equations of motion can be rewritten in a system which is not higher order. The degeneracy is manifest in the property that the terms quadratic in $\dot{a}$ and $\ddot{\phi}$ combine into a square term in the first line of (5.8). This suggests to introduce a new scale factor that absorbs the $\ddot{\phi}$ dependence in the square term, i.e. such that

$$\frac{\dot{b}}{b} = \frac{\dot{a}}{a} - \mathcal{V}\frac{\dot{\phi}}{N}\frac{d}{dt}\left(\frac{\dot{\phi}}{N}\right) + \dots\,. \tag{5.10}$$

This can be achieved by defining

$$a = e^{\lambda(X,\phi)}\, b \qquad \Rightarrow \qquad \frac{\dot{a}}{a} = \frac{\dot{b}}{b} + \lambda_X \dot{X} + \lambda_\phi \dot{\phi}\,, \tag{5.11}$$

such that

$$\lambda_X = -\frac{1}{2}\mathcal{V} = -\frac{4F_{(2)X} + XA_3 - 2A_1}{8(F_{(2)} - XA_1)}\,. \tag{5.12}$$

In terms of this new scale factor b, the cosmological action (5.8) becomes

$$\bar{S} = \int dt\, N b^3 \left[\hat{F}_{(2)}\frac{\dot{b}^2}{N^2 b^2} + \hat{Q}\frac{\dot{b}\,\dot{\phi}}{N^2 b} + \hat{P} + \hat{G}_{(1)}\frac{1}{N}\frac{d}{dt}\left(\frac{\dot{\phi}}{N}\right)\right], \tag{5.13}$$

with

$$\begin{aligned}
&\hat{F}_{(2)}(X,\phi) \equiv -6e^{3\lambda}(F_{(2)} - XA_1)\,,\\
&\hat{Q}(X,\phi) \equiv -3e^{3\lambda}\left[Q + 2F_{(2)\phi} + 4(F_{(2)} - XA_1)\lambda_\phi\right]\,, \qquad (5.14)\\
&\hat{G}_{(1)}(X,\phi) \equiv -e^{3\lambda}\left[Q + 6X(Q + 2F_{(2)\phi})\lambda_X\right]\,,\\
&\hat{P}(X,\phi) \equiv e^{3\lambda}\left[P + 3X(Q + 2F_{(2)\phi})\lambda_\phi + 6X(F_{(2)} - XA_1)\lambda_\phi^2\right]\,. \qquad (5.15)
\end{aligned}$$

In contrast with the previous action (5.8), the variation of the total action now yields directly a second order system. Note that, since

$$\frac{\delta a}{a} = -2X\lambda_X\frac{\delta N}{N} + \frac{\delta b}{b}\,, \tag{5.16}$$

as follows from (5.11), one finds that the variation of the matter action with respect to the lapse and scale factor, given in (5.7), is replaced by

$$\delta\bar{S}_m = b^3\Lambda^3\left[-\left(\rho_m + 6X\lambda_X p_m\right)\delta N + 3p_m\frac{\delta b}{b}\right]\,. \tag{5.17}$$

The properties of the homogeneous cosmological evolution, in particular the asymptotic de Sitter regime and the matter dominated era, are discussed in [43].

B. Quadratic action

The dynamics of linear perturbations is described by the action at quadratic order in perturbations. These perturbations are associated with the three basic ingredients of the action:

$$\delta N \equiv N - \bar{N}\,, \qquad \delta K^i_j = K^i_j - H\delta^i_j\,, \qquad \delta^{(3)}R^i_j = {}^{(3)}R^i_j\,, \tag{5.18}$$

where $H = \dot{a}/(\bar{N}a)$ is the Hubble parameter, and ${}^{(3)}R^i_j$ is already a perturbation since it vanishes in the background. The Lagrangian at quadratic order is then obtained via a Taylor expansion, which is formally written as

$$L(q_A) = \bar{L} + \frac{\partial L}{\partial q_A}\delta q^A + \frac{1}{2}\frac{\partial^2 L}{\partial q_A \partial q_B}\delta q^A \delta q^B + \dots\,. \tag{5.19}$$

where $q^A = \{N, K^i_j, {}^{(3)}R^i_j\}$.

All (quadratic and cubic) DHOST theories lead to a Lagrangian quadratic in linear perturbations of the form

$$\begin{aligned} S_{\rm quad} = \int d^3x\, dt\, a^3 \frac{M^2}{2}\Big\{ & \delta K_{ij}\delta K^{ij} - \left(1+\frac{2}{3}\alpha_{\rm L}\right)\delta K^2 + (1+\alpha_{\rm T})\left(\delta^{(3)}R\,\frac{\delta\sqrt{h}}{a^3} + \delta_2{}^{(3)}R\right) \\ & + H^2\alpha_{\rm K}\delta N^2 + 4H\alpha_{\rm B}\delta K\delta N + (1+\alpha_{\rm H})\,{}^{(3)}R\,\delta N \\ & + 4\beta_1\,\delta K\delta\dot{N} + \beta_2\,\delta\dot{N}^2 + \frac{\beta_3}{a^2}(\partial_i\delta N)^2\Big\}\,, \end{aligned} \tag{5.20}$$

where $\delta_2{}^{(3)}R$ denotes the second order term in the perturbative expansion of ${}^{(3)}R$, where the parameters M, $\alpha_{\rm L}$, $\alpha_{\rm T}$, $\alpha_{\rm K}$, $\alpha_{\rm B}$, $\alpha_{\rm H}$, β_1, β_2 and β_3 are time-dependent functions.

Only the four coefficients M, $\alpha_{\rm T}$, $\alpha_{\rm K}$, $\alpha_{\rm B}$ are necessary for Horndeski theories. Beyond Horndeski theories require in addition the parameter $\alpha_{\rm H}$. Finally, for general DHOST theories, one must also include the new coefficients $\alpha_{\rm L}$ and the β_I, although only one is independent since the four parameters are related by three constraints induced by the degeneracy conditions. Let us discuss briefly these nine parameters (the first four have been introduced in [44], $\alpha_{\rm H}$ in [37] and the last four in [38]):

- $\alpha_{\rm K}$: kineticity (associated with the kinetic term of the scalar field). This is the only coefficient that appears for quintessence, k-essence.

- $\alpha_{\rm B}$: so-called braiding parameter, corresponding to a mixing between the kinetic terms of the metric and of the scalar field[6].

- $\alpha_{\rm M}$: variation rate of the effective Planck mass squared (defined as the coefficient of the kinetic terms of the tensor modes)

$$\alpha_{\rm M} = \frac{1}{H}\frac{d}{dt}\ln M^2\,. \tag{5.21}$$

- $\alpha_{\rm T}$: associated to the tensor propagation velocity, according to

$$c_{\rm T}^2 = 1 + \alpha_{\rm T} \tag{5.22}$$

- $\alpha_{\rm H}$: this coefficient vanishes for Horndeski theories and becomes non zero for beyond Horndeski theories or DHOST theories.

- $\alpha_{\rm L}$: coefficient characterizing the detuning between the $\delta K_{ij}\delta K^{ij}$ and K^2 terms

- β_1, β_2 and β_3: these new coefficients appear for general DHOST theories, because their Lagrangian involves derivatives of the lapse function N, even if no extra scalar degree of freedom is involved. But they are not independent as discussed now.

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[6] Note that the definitions of $\alpha_{\rm B}$ in [44] and [37] differ by a factor -2.

For (quadratic and cubic) DHOST theories, the degeneracy conditions imply that the above parameters satisfy either one of the following sets of conditions [38]:

$$\mathcal{C}_{\rm I}: \quad \alpha_{\rm L}=0\,, \qquad \beta_2=-6\beta_1^2\,, \qquad \beta_3=-2\beta_1\left[2(1+\alpha_{\rm H})+\beta_1(1+\alpha_{\rm T})\right]\,, \tag{5.23}$$

or

$$\mathcal{C}_{\rm II}: \quad \beta_1=-(1+\alpha_{\rm L})\frac{1+\alpha_{\rm H}}{1+\alpha_{\rm T}}\,, \quad \beta_2=-6(1+\alpha_{\rm L})\frac{(1+\alpha_{\rm H})^2}{(1+\alpha_{\rm T})^2}\,, \quad \beta_3=2\frac{(1+\alpha_{\rm H})^2}{1+\alpha_{\rm T}}\,. \tag{5.24}$$

The category $\mathcal{C}_{\rm I}$ contains the subclass of Horndeski theories and DHOST theories related to Horndeski via disformal transformations, i.e. the subclass Ia (see Fig. 2). The category $\mathcal{C}_{\rm II}$ contains all the other subclasses.

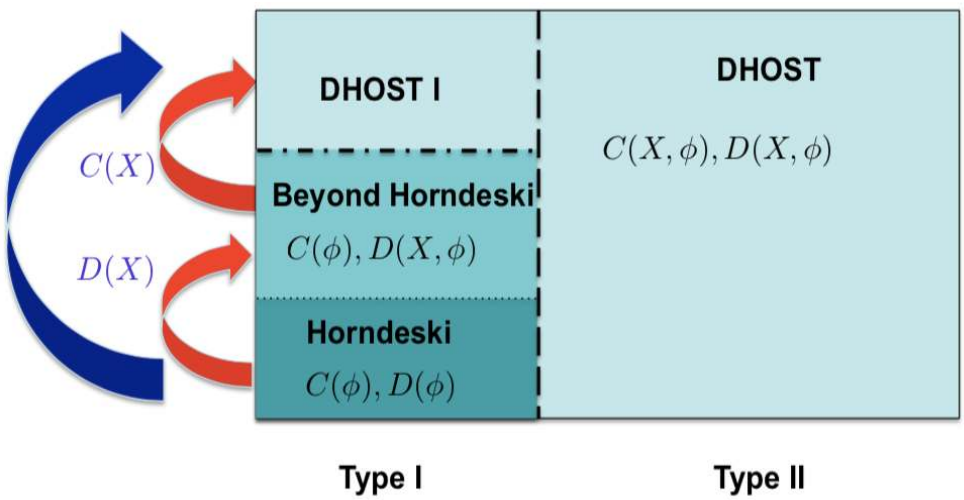

Figure 2. Type I theories (corresponding to Ia or N-I for quadratic theories, and ^{3}N-I for cubic ones, according to the terminology used in [13, 16, 17]) are related to Horndeski and Beyond Horndeski theories via disformal transformations (4.30). By contrast, type II theories cannot be related to Horndeski and Beyond Horndeski theories via disformal transformations.

C. Physical degrees of freedom

From the action (5.20), one can extract the physical degrees of freedom, which consist of one scalar mode and two tensor modes for DHOST theories.

1. *Tensor modes*

At linear order, the tensor modes correspond to the perturbations of the spatial metric

$$h_{ij}=a^2(t)\left(\delta_{ij}+\gamma_{ij}\right)\,, \tag{5.25}$$

with γ_{ij} traceless and divergence-free, i.e. $\gamma_{ii} = 0 = \partial_i \gamma_{ij}$. Substituting (5.25) into (5.20), one finds that the tensor quadratic action is given by

$$S_{\rm quad,tensor} = \int d^3x\, dt\, a^3 \left\{ \frac{M^2}{8} \left[\dot\gamma_{ij}^2 - \frac{1+\alpha_{\rm T}}{a^2} (\partial_k \gamma_{ij})^2 \right] \right\}, \tag{5.26}$$

which shows that the propagation speed of gravitational waves $c_{\rm T}$ is directly related to the coefficient $\alpha_{\rm T}$, as announced in (5.22). The corresponding equation of motion is given, in Fourier space, by

$$\ddot\gamma_{ij} + \left(3\frac{\dot a}{a} + 2\frac{\dot M}{M} \right) \dot\gamma_{ij} + (1+\alpha_{\rm T}) \frac{k^2}{a^2} \gamma_{ij} = 0\,, \tag{5.27}$$

or

$$\ddot\gamma_{ij} + (3+\alpha_{\rm M}) H \dot\gamma_{ij} + (1+\alpha_{\rm T}) \frac{k^2}{a^2} \gamma_{ij} = 0\,. \tag{5.28}$$

2. *Scalar modes*

The scalar modes can be described by the metric perturbations

$$N = 1 + \delta N, \quad N^i = \delta^{ij} \partial_j \psi, \quad h_{ij} = a^2(t) e^{2\zeta} \delta_{ij}\,. \tag{5.29}$$

After substitution into (5.20), one obtains an action that depends on δN, ψ and ζ. By solving the constraints associated with δN and ψ, respectively the linearized Hamiltonian and momentum constraints, one ends up with a quadratic action for the physical degree of freedom.

DHOST with $\alpha_{\rm L} = 0$ (category $\mathcal{C}_{\rm I}$)

Due to the degeneracy, all the time derivatives of δN can be combined with those of ζ through the variable

$$\tilde\zeta \equiv \zeta - \beta_1 \delta N\,. \tag{5.30}$$

Varying the action with respect to ψ yields the scalar component of the momentum constraint. In terms of the new variable, it reads

$$\delta N = \frac{\dot{\tilde\zeta}}{H(1+\alpha_{\rm B}) - \dot\beta_1}\,. \tag{5.31}$$

Substituting this expression into the action and performing some integration by parts, we finally obtain the quadratic action for the propagating degree of freedom $\tilde\zeta$ [38]

$$S^{\rm quad} = \int d^3x\, dt\, a^3 \frac{M^2}{2} \left[A_{\tilde\zeta} \dot{\tilde\zeta}^2 - B_{\tilde\zeta} \frac{(\partial_i \tilde\zeta)^2}{a^2} \right]. \tag{5.32}$$

Here $A_{\tilde\zeta}$ and $B_{\tilde\zeta}$ are background-dependent functions whose explicit expressions are

$$A_{\tilde\zeta} = \frac{1}{(1+\alpha_{\rm B} - \dot\beta_1/H)^2} \left[\alpha_{\rm K} + 6\alpha_{\rm B}^2 - \frac{6}{a^3 H^2 M^2} \frac{d}{dt} \left(a^3 H M^2 \alpha_{\rm B} \beta_1 \right) \right], \tag{5.33}$$

$$B_{\tilde\zeta} = -2(1+\alpha_{\rm T}) + \frac{2}{aM^2} \frac{d}{dt} \left[\frac{aM^2 \big(1 + \alpha_{\rm H} + \beta_1 (1+\alpha_{\rm T})\big)}{H(1+\alpha_{\rm B}) - \dot\beta_1} \right]. \tag{5.34}$$

No instability (ghost or gradient instability) requires

$$A_{\tilde\zeta} > 0\,, \qquad B_{\tilde\zeta} > 0\,. \tag{5.35}$$

When $\beta_1 = 0$, one recovers the results obtained for Horndeski and Beyond Horndeski theories (see [37]).

DHOST models in category $\mathcal{C}_{\rm II}$

For models in the category $\mathcal{C}_{\rm II}$, one finds that the coefficient of the gradient term in the quadratic action for the scalar mode reduces to [38, 45]

$$B = -2(1+\alpha_{\rm T}) \qquad (\text{category } \mathcal{C}_{\rm II})\,. \tag{5.36}$$

Comparing with the tensor action (5.26), one sees that the coefficients of the gradient terms for the scalar and tensor modes have opposite signs and therefore, these modes cannot be stable simultaneously. This signals a problematic instability for all theories satisfying $\mathcal{C}_{\rm II}$, which therefore cannot be considered as viable theories.

D. Cosmological constraints on DHOST models

One finds in the literature many papers where Horndeski models are confronted with cosmological data (see e.g. [46–59] and further references in the review [60]). Most of these papers use either the phenomenological parameters $\mu \equiv G_{\rm matter}/G$ and $\Sigma \equiv G_{\rm light}/G$, or the functions $\alpha_I(t)$ introduced in [44]. In the latter case, one of the difficulties is to write these time-dependent functions, which in principle depends on the background evolution, in a parametrized form. Ad hoc parametrizations are mostly adopted (see discussions in e.g [61] and [62]), with the exception of [63] where the background equations are explicitly solved.

In the papers mentioned above, all matter species are minimally coupled to the metric. It is nevertheless possible to relax this assumption by considering that only ordinary matter is minimally coupled to the Horndeski metric while dark matter is non-minimally coupled. This leads to scenarios of interacting dark energy, such as those studied in [64, 65] or [66].

Beyond Horndeski cosmology has also been considered in [67–70], while the cosmology of DHOST models has been explored in [71] and [43]. Finally, let us mention that for some models, cosmological perturbations have been studied not only at the linear level but also at the nonlinear level. In particular, the matter bispectrum has been studied for Horndeski theories [47, 72] and Beyond Horndeski theories [73].

VI. ASTROPHYSICAL BODIES IN DHOST THEORIES

This section summarizes the gravitational properties of astrophysical bodies in the context of DHOST theories. For higher-order scalar-tensor theories, scalar interactions can be suppressed, and therefore standard gravity recovered, via the so-called Vainshtein screening [74], introduced originally in the context of massive gravity. The Vainshtein mechanism has been investigated in detail in Horndeski theories [75–77]. For Beyond Horndeski theories, [78] found the surprising result that the Vainshtein mechanism works only partially in the sense that the usual gravitational law is modified *inside matter*. This specific deviation from standard gravity was studied for various astrophysical objects in order to obtain constraints on the parameters of the Beyond Horndeski models (see e.g. [79–86]), as recently summarized in [87]. More recently, the Vainshtein mechanism has been studied for DHOST theories [45, 88–90].

A. Vainshtein mechanism

Let us consider a nonrelativistic object, characterized by a spherically-symmetric mass density $\rho(r)$. The presence of this object induces a (small) deformation of spacetime, described by the perturbed metric

$$ds^2 = -\left[1 + 2\Phi(r)\right] dt^2 + \left[1 - 2\Psi(r)\right] \delta_{ij}\, dx^i dx^j \,, \tag{6.1}$$

where we have introduced the two gravitational potentials $\Phi(r)$ and $\Psi(r)$. Our goal is to determine the gravitational field, i.e. Φ and Ψ, generated by the mass distribution $\rho(r)$. The scalar field ϕ also acquires a perturbation due to the presence of the object and can be written as

$$\phi = \phi_c(t) + \chi(r) \,, \tag{6.2}$$

where $\phi_c(t)$ denotes the cosmological value of the scalar field far from the object. Here, in contrast to the metric, one needs to include the cosmological dependence of the scalar field because it can be relevant even for a local observer. However, the time dependence of χ can be neglected if one is interested in time and spatial scales much smaller than the cosmological ones, i.e. $Hr \ll 1$.

The relations between the two metric perturbations, the scalar perturbation and the mass distribution can be obtained by writing down the equations of motion for the scalar field and for the metric derived from the action (2.8). These equations are treated perturbatively, since χ and the metric perturbations are small for a nonrelativistic object. However, it is important to keep the nonlinear terms in higher-order derivatives of χ, which become important in the Vainshtein mechanism.

Let us give the results of these calculations in the particular case of viable quadratic DHOST theories verifying

$$A_1 = A_2 = 0 \,, \tag{6.3}$$

i.e. such that $c_{\rm T} = 1$ (as discussed in the next section), following the derivation presented in [45]. In this case, keeping only the most nonlinear terms in higher-order derivatives of χ, which are relevant in regions inside or near the central

object, one obtains the following three equations, corresponding, respectively, to the scalar field equation of motion and the time and radial components of the metric equations of motion:

$$\begin{aligned}
&(8A_3+6A_4)x^3+2(A_3+A_4)rx\left(6xx'+rx'^2+rxx''\right)\\
&\quad+\left(12F_X-(A_3-6A_4)v^2\right)x\,y+\left(4F_X+(A_3+2A_4)v^2\right)rx\,y'\\
&\quad-24F_X\,x\,z-8F_X rx\,z'=0\,,\\
&\left(4F_X+(3A_3+2A_4)v^2\right)x^2+\left(4F_X+(A_3+2A_4)v^2\right)rxx'\\
&\quad+2v^2(4F_X+A_4v^2)y-4(F+2F_Xv^2)z+2\mathcal{A}=0\,,\\
&2F_X(x^2+rxx')+(F+2F_Xv^2)y-Fz=0\,,
\end{aligned}\tag{6.4}$$

where we have used the variables $v\equiv\dot{\phi}_c$, $x\equiv\chi'/r$, $y\equiv\Phi'/r$, $z\equiv\Psi'/r$, and

$$\mathcal{A}(r)\equiv\frac{\mathcal{M}(r)}{8\pi r^3}\,,\qquad \mathcal{M}(r)\equiv 4\pi\int_0^r\bar{r}^2\rho(\bar{r})\mathrm{d}\bar{r}\,.\tag{6.5}$$

The terms A_I, F and F_X appearing in (6.4) are all evaluated on the background.

One can then use the last two equation of (6.4) to express y and z in terms of x and $\mathcal{A}$. Substituting these expressions into the first one gives an equation involving only the functions x and $\mathcal{A}$. This equation a priori involves derivatives of x but, remarkably, substituting the expression (4.27) for A_4 imposed by the degeneracy conditions yields an equation with no derivative of x, which reads

$$\begin{aligned}
x\Big\{&\left[3(4F_X+A_3v^2)^2v^2-4F(12F_X+7A_3v^2)\right]\mathcal{A}\\
&-(4F_X+A_3v^2)(4F+4F_Xv^2+A_3v^4)r\mathcal{A}'\\
&+4FA_3(4F+4F_Xv^2-3A_3v^4)x^2\Big\}=0\,.
\end{aligned}\tag{6.6}$$

Concentrating on the nontrivial solution, the above expression yields x^2 in terms of the matter function $\mathcal{A}$ and its derivative $\mathcal{A}'$. Substituting back into the earlier expressions for y and z, it is then easy to write y and z in terms of $\mathcal{A}$ and its derivatives. This finally gives the modified gravitational laws [45] (see also [88])

$$\begin{aligned}
\frac{d\Phi}{dr}&=\frac{G_\mathrm{N}\mathcal{M}(r)}{r^2}+\Xi_1G_\mathrm{N}\mathcal{M}''(r)\,,\\
\frac{d\Psi}{dr}&=\frac{G_\mathrm{N}\mathcal{M}(r)}{r^2}+\Xi_2\frac{G_\mathrm{N}\mathcal{M}'(r)}{r}+\Xi_3\,G_\mathrm{N}\mathcal{M}''(r)\,,
\end{aligned}\tag{6.7}$$

with the effective Newton's constant

$$(8\pi G_\mathrm{N})^{-1}=2F+2F_Xv^2-\frac{3}{2}A_3v^4\equiv 2F(1+\Xi_0)\tag{6.8}$$

and the dimensionless coefficients

$$\Xi_1=-\frac{(4F_X+A_3v^2)^2}{16FA_3}\,,\qquad \Xi_2=\frac{2F_Xv^2}{F}\,,\qquad \Xi_3=\frac{16{F_X}^2-A_3^2v^4}{16A_3F}\,.\tag{6.9}$$

Outside the matter source, $\mathcal{M}$ is constant and one recovers the usual gravitational behaviour. But, inside the matter distribution, the above equations differ from those of standard gravity. In the Beyond Horndeski case, we have $4F_X+XA_3=0$, and therefore, $\Xi_3=0$ and $\Xi_2=-\Xi_0=-2\,\Xi_1$ in this particular case.

Interestingly, the three coefficients in (6.9) are not independent but satisfy the consistency relation

$$\Xi_3^2-\Xi_1^2=\frac{1}{2}\Xi_1\,\Xi_2\,.\tag{6.10}$$

It is also worth noting that the coefficients Ξ_i can be related to two of the cosmological effective parameters, namely α_H and β_1. One finds the very simple relations

$$\Xi_1=-\frac{(\alpha_\mathrm{H}+\beta_1)^2}{2(\alpha_\mathrm{H}+2\beta_1)}\,,\qquad \Xi_2=\alpha_\mathrm{H}\,,\qquad \Xi_3=-\frac{\beta_1(\alpha_\mathrm{H}+\beta_1)}{2(\alpha_\mathrm{H}+2\beta_1)}\,.\tag{6.11}$$

The above expressions are valid when $\alpha_\mathrm{T}=0$, corresponding to $A_1=A_2=0$. In the more general situation where α_T is left unconstrained, viable quadratic DHOST theories still lead to gravitational laws of the form (6.7) with more general expressions for the coefficients Ξ_I, which can be found in [89].

B. Newtonian stars

For Newtonian stars, only the first equation of (6.7) is relevant, i.e.

$$\frac{d\Phi}{dr} = \frac{G_{\rm N}\mathcal{M}(r)}{r^2} + \Xi_1\, G_{\rm N}\, \mathcal{M}''(r)\,. \tag{6.12}$$

In the literature, the coefficient $\Upsilon \equiv 4\,\Xi_1$ is often used.

A theoretical upper bound is [80]

$$\Xi_1 > -\frac{1}{6}\,, \tag{6.13}$$

just to ensure that the mass density ρ decreases when going outwards from the center. Indeed, near the center, we have

$$\mathcal{M} \approx \frac{4\pi}{3}\rho_c r^3\,, \qquad \mathcal{M}'' \approx 6\frac{\mathcal{M}}{r^2}\,, \tag{6.14}$$

where ρ_c is the central mass density, and therefore

$$\frac{d\Phi}{dr} \approx (1+6\,\Xi_1)\frac{G_{\rm N}\mathcal{M}}{r^2}\,, \tag{6.15}$$

which gives the above constraint in order to have $\Phi' > 0$, i.e. an attractive gravitational force.

Various astrophysical constraints on Ξ_1 have been given in the literature. An upper bound was obtained by considering the lowest mass hydrogen burning stars [81]. Indeed, for $\Xi_1 > 0$, the onset of hydrogen burning in low mass stars occurs at higher masses than in standard gravity. The lowest mass observed ($\simeq 0.1 M_\odot$) thus gives the upper bound

$$\Xi_1 < 0.4\,. \tag{6.16}$$

A more recent upper bound,

$$\Xi_1 \leq 3.5\times 10^{-2} \quad (2\sigma)\,, \tag{6.17}$$

has been obtained [91] from the observation of white dwarfs (radius, mass and temperature): one of the difficulties in using white dwarf is to take into account the finite temperature effects and the existence of an envelope. This result improves a previous bound also obtained from observations of white dwarfs in [83]. The same reference found a lower bound $\Xi_1 \geq -4.5\times 10^{-2}$ (1σ), based on the Chandrasekhar mass limit of white dwarfs[7]. However, taking into account relativistic effects, one finds simply $\Xi_1 \geq -0.11$ [85]. Finally, analysis of weak lensing and X-ray profiles of 58 galaxy clusters have given the constraint $\Upsilon = 4\Xi_1 = -0.11^{+0.93}_{-0.67}$ [84].

C. Neutron stars

Neutron stars in the context of DHOST theories have been explored in several papers, not only within Horndeski theories (see e.g. [92–94] and the review [95]) but also for Beyond Horndeski and DHOST theories, as we discuss below.

Neutron stars in the context of Beyond Horndeski gravity have been studied numerically in [85] and [86] with an action of the form

$$S = \int d^4\sqrt{-g}\left[M_P^2\left(\frac{R}{2} - k_0\Lambda\right) - k_2 X - \frac{\zeta}{2}X^2 + f_4 L_4^{\rm bH}\right]\,, \tag{6.18}$$

where k_0, k_2, ζ and f_4 are assumed to be constant. The action is expressed in the physical (Jordan) frame where matter is minimally coupled to the metric. Reference [85] studied the case $\zeta = 0$, with a polytropic equation of state

$$\rho = \left(\frac{P}{K}\right)^{1/2} + P\,, \qquad (K = 123 M_\odot^2)\,, \tag{6.19}$$

[7] Taking $\Xi_1 < 0$ decreases the Chandrasekhar mass of white dwarfs. Requiring that it must remain higher than the mass of the most massive white dwarf ($1.37 M_\odot$) gives a lower bound on Ξ_1.

as well as two realistic equations of state, called SLy4 and BSK20. The case $\zeta \neq 0$ was considered in [86] with a broad range of realistic equations of state.

Since local gravity depends on the cosmological velocity of the scalar field, it is instructive to consider a star embedded in a cosmological spacetime. The simplest cosmological solution is de Sitter, which can be written, in Schwarzschild-like coordinates, as

$$ds^2 = -(1 - H^2 r^2)dt^2 + \frac{dr^2}{1 - H^2 r^2} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{6.20}$$

while the scalar field profile $\phi = vt_c$ (where t_c is the usual cosmological time) becomes

$$\phi(r, t) = vt + \frac{v}{2H} \ln(1 - H^2 r^2) . \tag{6.21}$$

One can then insert a spherical symmetric object within this cosmological solution by trying to solve the Einstein equations for a static and spherically symmetric metric of the form

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) , \tag{6.22}$$

going asymptotically to (6.20).

In practice, the energy-momentum tensor is the sum of the scalar field contribution and that of a perfect fluid assumed to model the neutron star's matter. Remarkably, for theories of the form (6.18), one finds neutron star solutions where the metric outside the star is an exact Schwarzschild-de Sitter. In fact, they belong to a subset of theories among Beyond Horndeski theories leading to Schwarzschild-de Sitter, characterized by four specific conditions verified by the six functions in the Lagrangian, as discussed in [96].

Inside the star, the equations of motion can be solved numerically, assuming some equation of state, in order to determine the matter density profile and the internal geometry. When $\Xi_1 < 0$, one finds that stars with fixed mass have a larger radius than their GR counterparts. Moreover, for the same equation of state, the maximum mass can increase significantly with respect to GR [85]. This illustrates how modified gravity could provide a solution to the hyperon puzzle.

Slowly-rotating neutron stars were also studied in [86]. Interestingly, one can derive a relation between the dimensionless moment of inertia Ic^2/G^2M^3 and the compactness GM/Rc^2, which is robust in the sense that it weakly depends on the equation of state and which can discriminate between modified gravity and GR.

The action (6.18) gives models where the speed of gravitational waves differs from that of light. Such models are in contradiction with the recent observation of a neutron star merger via gravitational waves and gamma rays, as we will discuss in the next section. The most recent studies have taken into account this constraint and investigated models for which $c_g = c$. For example, a Beyond Horndeski model characterized by $F_4 = -2G_{4X}/X$, so that $A_1 = A_2 = 0$ according to (2.17), was studied in [97].

Finally, a DHOST model satisfying $c_g = c$ was considered in [98], with an action of the form

$$S = \int d^4\sqrt{-g} \left[FR + A_3 L_3 + A_4 L_4 + A_5 L_5\right] , \tag{6.23}$$

where

$$F = \frac{M_p^2}{2} + \alpha X^2 , \qquad A_3 = -8\alpha - \beta , \tag{6.24}$$

and the functions A_4 and A_5 are fixed in terms of F and A_3 by the degeneracy conditions. Note that the limit $\beta = 0$ corresponds to a Beyond Horndeski model. Using the polytropic equation of state (6.19), neutron stars that differ significantly from GR ones were obtained, even for small values of

$$\bar{\alpha} = \alpha \frac{v^4}{M_p^2} , \qquad \bar{\beta} = \beta \frac{v^4}{M_p^2} . \tag{6.25}$$

Moreover, no qualitative difference was noticed between the cases $\beta = 0$ and $\beta \neq 0$, i.e. between Beyond Horndeski and other DHOST models.

Black holes have been studied in DHOST theories, in particular in Horndeski and Beyond Horndeski theories (see e.g. [97, 99, 100] and the review [95]).

VII. DHOST THEORIES AFTER GW170817

The propagation of gravitational waves in a cosmological background can differ from that of light, as illustrated by (5.22). The simultaneous detection of a gravitational wave signal and its optical counterpart has thus provided very interesting constraints on dark energy models (see e.g. [101] for a review and references therein). This was dramatically illustrated by the recent observation of a neutron star merger, associated to the gravitational wave event GW170817 seen by LIGO/Virgo [102] and the gamma-ray burst GRB170817A [103], showing in particular that the speed of gravitational waves and that of light coincide to a very high precision, namely

$$-3\times 10^{-15} \leq \frac{c_g}{c} - 1 \leq 7\times 10^{-16}\,. \tag{7.1}$$

For simplicity here, we will ignore the possibility of some (extreme) fine-tuning and assume that this observation is a strong indication that c_g and c *strictly* coincide. The implications of $c_g = c$ for higher-order scalar-tensor theories, summarized in several papers (see e.g. [45, 87, 88, 104–106]), are given below, considering first the general case of DHOST theories, and then specializing to Beyond Horndeski and Horndeski theories.

Note that these implications of GW170817 for DHOST theories considered as dark energy candidates assume that the regime of validity of these effective theories remains valid up to an energy scale corresponding to the frequency of gravitational waves observed by LIGO/Virgo, $f \sim 100$ Hz. However, this is very close to the typical strong coupling scale associated with these dark energy models, $\Lambda \sim (M_P H_0^2)^{1/3} \sim 260$ Hz. As pointed in [107], one could envisage that the effective description of modified gravity near this scale is such that c_g is very close to 1, even if c_g can significantly deviate from 1 on much lower energy scales. The restrictions on DHOST theories presented below should thus be taken with a grain of salt.

A. Constraints on DHOST theories

Since the speed of gravitational waves (with respect to a cosmological background) has been computed for all DHOST theories, it is a straightforward exercise to identify the DHOST theories that "survive" after GW170817. In fact, it is more useful to consider directly the full ADM decomposition of the DHOST Lagrangian (in a gauge where the scalar field is spatially uniform), because it provides nonlinear information and thus makes the background-dependent effects manifest. Concentrating on the terms in the Lagrangian that contribute to the kinetic and spatial gradient terms of the tensor modes, one finds [38]

$$L_{\rm ADM} = (F - XA_1)\, K_{ij}K^{ij} + F\,{}^{(3)}\!R + \ldots \tag{7.2}$$

Note that the cubic terms also contribute to the gravitational speed but in an explicitly background-dependent way. The requirement $c_g = c$ for any background thus imposes the very simple condition

$$A_1 = 0 \tag{7.3}$$

for the quadratic terms, while all cubic terms must vanish.

Going back to the quadratic DHOST theories identified in [13], one sees that the condition $c_g = c$ dramatically restricts the set of viable theories, which must now depend on only two arbitrary functions of ϕ and X, namely F and A_3. The other functions A_I are given, according to (7.3) and (4.27-4.28), by

$$\begin{aligned} A_1 &= A_2 = 0\,,\\ A_4 &= \frac{1}{8F}\left[48{F_X}^2 - 8(F - XF_X)A_3 - X^2A_3^2\right]\,,\\ A_5 &= \frac{1}{2F}\left(4F_X + XA_3\right)A_3\,, \end{aligned} \tag{7.4}$$

which corresponds to the Lagrangian

$$\begin{aligned} L^{\rm DHOST}_{c_g=1} &= P + Q\,\Box\phi + F\,{}^{(4)}\!R + A_3\phi^\mu\phi^\nu\phi_{\mu\nu}\Box\phi\\ &+ \frac{1}{8F}\left(48{F_X}^2 - 8(F - XF_X)A_3 - X^2A_3^2\right)\phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}\\ &+ \frac{1}{2F}\left(4F_X + XA_3\right)A_3(\phi_\mu\phi^{\mu\nu}\phi_\nu)^2\,. \end{aligned} \tag{7.5}$$

B. Constraints on Horndeski and Beyond Horndeski theories

One can then specialize the above results to Beyond Horndeski theories, characterized by

$$F = G_4\,, \quad A_1 = -A_2 = 2G_{4X} + XF_4\,, \quad A_3 = -A_4 = 2F_4\,, \quad A_5 = 0\,. \tag{7.6}$$

Combining this with the condition $A_1 = 0$ combined implies

$$F_4 = -\frac{2}{X}G_{4X} = -\frac{2}{X}F_X\,. \tag{7.7}$$

The corresponding Lagrangian is thus obtained by substituting $A_3 = -4F_X/X$ in (7.5), which gives

$$L^{\rm bH}_{c_g=1} = P + Q\,\Box\phi + F\,{}^{(4)}R - \frac{4}{X}F_X\left(\phi^\mu\phi_{\mu\nu}\phi^\nu\Box\phi - \phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}\right)\,. \tag{7.8}$$

Finally, one can further specialize to the case of Horndeski theories, characterized by $F_4 = 0$, which implies $F_X = 0$. The corresponding Lagrangian thus reduces to

$$L^{\rm bH}_{c_g=1} = P(X,\phi) + Q(X,\phi)\,\Box\phi + F(\phi)\,{}^{(4)}R\,. \tag{7.9}$$

VIII. CONCLUSIONS

In this review, we have presented the family of higher-order scalar-tensor theories, dubbed DHOST theories, that do not contain any problematic extra degree of freedom because of the degeneracy of their Lagrangian. It should be pointed out that similar analyses have been performed for vector-tensor theories (see in particular [108, 109] and other references in the review [110]) and for exploring theories beyond Lovelock [111].

As we have seen, from a very large family of theories, explicitly constructed up to cubic order in second derivatives $\phi_{\mu\nu}$, emerges a subset of viable theories, safe from gradient instabilities (in both scalar and tensor sectors), which can be disformally related to Horndeski theories (but matter cannot always be minimally coupled to the metric in the Horndeski formulation). These theories are in principle potential models of dark energy and can be confronted to cosmological and astrophysical observations. Remarkably, the recent observation of a binary neutron star merger, showing that the speed of gravitational waves coincides with that of light, has put severe constraints on DHOST theories[8], assuming that they are still applicable at energy scales probed by LIGO/Virgo.

Although we have focused our attention on scenarios of dark energy, it should be mentioned that higher-order scalar-tensor theories have also been considered in the context of early universe cosmology. In particular, Horndeski and Beyond Horndeski theories have been used to construct scenarios such as cosmological bounces or Genesis, which require a violation of the Null Energy Condition (see e.g. [113–119]).

Another issue concerning DHOST theories is the robustness of their Lagrangian with respect to quantum corrections. Galileon theories are protected by the so-called galileon symmetry $\phi \to \phi + b_\mu x^\mu$. When gravity is taken into account, this galilean symmetry is no longer exact but it is nevertheless possible to identify theories with weakly broken galileon invariance, which are still stable with respect to quantum corrections [120]. This analysis has been recently extended to quadratic DHOST theories [121], showing that theories with these properties form a subset of Beyond Horndeski theories[9].

A more mathematical issue is whether the initial value problem, at least locally, is well posed. A necessary condition for local well-posedness is the strong hyperbolicity of the equations of motion in some gauge, such as Einstein's equations in the harmonic gauge. It was shown in [122, 123] that one can find a generalized harmonic gauge in which the equations of motion, for a generic weak-field background, are strongly hyperbolic only for a very limited subclass of Horndeski theories (satisfying $G_3 = G_{4X} = G_5 = 0$). These results hold for the generalized harmonic gauge and there could exist, in principle, a different gauge where a larger subset of theories could be shown to be strongly hyperbolic.

As this review has tried to illustrate, scalar-tensor theories with higher order derivatives have revealed fascinating uncharted territories, generating a series of new issues at both fundamental and phenomenological levels.

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[8] It has been recently argued in [112] that more theories are in fact ruled out as dark energy candidates by the direct detection of gravitational waves, because they predict a rapid decay of gravitational waves into scalar field fluctuations. This would leave as acceptable theories only the DHOST theories (7.5) with the additional constraint $A_3 = 0$, i.e. with $L = P + Q\,\Box\phi + F\,{}^{(4)}R + 6({F_X}^2/F)\phi^\mu\phi_{\mu\nu}\phi_\lambda\phi^{\lambda\nu}$. Note that this effect assumes again that the effective description on cosmological scales remains valid on the much higher energy scale of LIGO/Virgo gravitational waves.

[9] Two conditions on quadratic DHOST theories have been obtained in [121] by computing the loop corrections. Interestingly, the first condition turns out to be equivalent to $\mathcal{A} = 0$, meaning that the theory is non degenerate when gravity is ignored, while the second condition is the same as that obtained from the degeneracy conditions in the case $\mathcal{A} = 0$, as mentioned in [13].

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Appendix A: Degeneracy and Hamiltonian formulation

The Hamiltonian point of view provides the most systematic approach to count the numbers of physical degrees of freedom and to study the stability of a system. As a complement to the Lagrangian formulation described in section III, we present here the Hamiltonian analysis, following [13]. The extension of this analysis to the case of quadratic DHOST theories can be found in [24].

In the Hamiltonian framework, the configuration variables and their respective conjugate momenta satisfy the Poisson brackets:

$$\{P, Q\} = 1 \ , \ \{p, q\} = 1 \ , \ \{\pi_\phi, \phi\} = 1 \,, \tag{A1}$$

while all other Poisson brackets vanish. For simplicity, λ is not considered as another variable, but simply as the momentum π_ϕ since, according to (3.4), $\pi_\phi \equiv \partial L/\partial\dot\phi = \lambda$. One also finds that the momenta P and p_i are related to $\dot Q$ and $\dot q$ by

$$\begin{pmatrix} P \\ p \end{pmatrix} = \begin{pmatrix} \frac{\partial L}{\partial \dot Q} \\ \frac{\partial L}{\partial \dot q} \end{pmatrix} = \begin{pmatrix} a\dot Q + b\dot q \\ b\dot Q + c\dot q \end{pmatrix} = M \begin{pmatrix} \dot Q \\ \dot q \end{pmatrix} , \tag{A2}$$

where M is the kinetic matrix (3.8).

In the nondegenerate case, one can invert the system (A2) in order to express the velocities $\dot Q$ and $\dot q$ in terms of the momenta. The Hamiltonian is thus given by

$$H = P\dot Q + p\dot q + \pi_\phi\dot\phi - L \tag{A3}$$

$$= \frac{1}{2}\,(\,P,\ p\,)\,M^{-1}\begin{pmatrix} P \\ p \end{pmatrix} + V(\phi, q) - \frac{1}{2}Q^2 + \pi_\phi Q\,. \tag{A4}$$

The Hamiltonian H is a function of the 6 canonical variables (Q, q, ϕ) and (P, p, π_ϕ), corresponding to 3 degrees of freedom, in agreement with the results of the Lagrangian analysis. Moreover, one observes that the Lagrangian is linear in π_ϕ, which makes the Hamiltonian unbounded from below. This is the characteristic signature of Ostrogradski's instability [2].

In the degenerate case, one cannot express the velocities $\dot Q$ and $\dot q$ in terms of the momenta. Instead, one finds a primary constraint relating the canonical momenta, given by

$$\Omega = \frac{b}{c}p - P \approx 0. \tag{A5}$$

As usual, we use the notation $\approx$ to denote weak equality in phase space. Taking into account this constraint, the total Hamiltonian is given by

$$H_{\rm tot} = P\dot Q + p\dot q + \pi_\phi\dot\phi - L + \mu\,\Omega \tag{A6}$$

where μ is a Lagrange multiplier enforcing the primary constraint.

After some straightforward manipulations to eliminate the velocities, one finds that the expression of $H_{\rm tot}$ in terms of the canonical variables reduces to

$$H_{\rm tot} = \frac{1}{2c}p^2 - \frac{1}{2}Q^2 + V(\phi, q) + \pi_\phi Q + (\mu + \dot Q)\Omega\,. \tag{A7}$$

One can then redefine the Lagrange multiplier μ so that $\mu + \dot Q$ becomes μ.

The invariance under time evolution of the constraint Ω leads to the secondary constraint

$$\Psi = \dot{\Omega} = \{\Omega, H_{\rm tot}\} = Q + \frac{b}{c} V_q - \pi_\phi \;\approx\; 0\,. \tag{A8}$$

To see whether the time evolution of Ψ leads to a tertiary constraint, it is sufficient to compute the Poisson bracket between the primary and secondary constraints

$$\{\Omega, \Psi\} = 1 - \frac{b^2}{c^2} V_{qq} \equiv \Delta\,, \tag{A9}$$

Leaving aside the special case $\Delta = 0$ (which would generate new constraints and further reduce the physical number of degrees of freedom), the analysis stops in the generic case where $\Delta \neq 0$, because imposing $\dot{\Psi} = 0$ simply fixes the Lagrange multiplier μ without generating any new constraint.

In summary, we have obtained a Hamiltonian system in a 6-dimensional phase space, restricted by two second-class constraints Ω and Ψ. This implies that the number of physical degrees of freedom is only 2. This confirms that there is no extra degree of freedom in the degenerate case.

Moreover, one can construct the physical phase space spanned by the variables $(q, p; Q, \phi)$, with a Poisson algebra defined from the Dirac bracket

$$\{F, G\}_D = \{F, G\} - \frac{1}{\Delta}\left(\{F, \Psi\}\{\Omega, G\} - \{F, \Omega\}\{\Psi, G\}\right)\,. \tag{A10}$$

The associated Hamiltonian $H_{\rm phys}$ is obtained from the total Hamiltonian after elimination of P and π_λ via the second class constraints:

$$H_{\rm phys} = \frac{1}{2c} p^2 + \frac{1}{2} Q^2 + \frac{b}{c} V_q Q + V(\phi, q)\,. \tag{A11}$$

The linear dependence on a canonical momentum has disappeared in the above Hamiltonian. This confirms that the Ostrogradski ghost has disappeared as a consequence of the degeneracy of the kinetic matrix M.

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