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Data Compression for Storing and Transmitting ECG's/VCG's

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Abstract—A data compression technique is given which yields a compression ratio slightly better than 12 to 1 for cardiograms and is implementable with either a mini- or microcomputer. The technique involves two applications of the discrete Karhunen-Loève expansion, intrinsic components, principal factors, or principal components, all synonyms. The first application reduces the effects of respiration and the various orientations of different patients' hearts, and requires the solution of a 3 × 3 matrix eigenvalue, eigenvector problem for each beat. The second application involves expressing the transformed cardiogram in a Karhunen-Loève series, and requires the solution of the eigenvalue, eigenvector problem for a large matrix. However, the solution, which must be obtained only once for all time, can be performed off line. (The same eigenvectors are used for all patients.) Comparisons are given of the cardiograms reconstructed from the compressed data with the original cardiograms.

I. Introduction

MANY PAPERS are available in the engineering literature which review the mathematical, physiological, and engineering aspects of cardiography. McFee and Baule [1] and Cox *et al.* [2] are excellent examples and both articles have extensive bibliographies. This paper will deal with specific problems: Transmitting cardiograms from one place to another and storing large quantities of cardiograms in machine readable form. However, the techniques presented are applicable to any stochastic signals.

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The United States Air Force School of Aerospace Medicine (USAFSAM) presently has stored in its Central Electrocardiographic Library more than 800 000 ECG's, and these files are growing at a rate of over 100 per day. In order to ease the workload on Air Force physicians and to make this valuable source of data more accessible to medical researchers, techniques have been investigated for 1) automating the measurements a cardiologist extracts from the cardiograms and 2) storing the cardiograms in a machine readable form. The first problem encountered is how to get cardiograms stored on microfilm and paper into a digital computer. Two approaches to this problem are being tried. An optical scanner is being developed for digitizing ECG's/VCG's stored on paper and microfilm. At the same time, systems are being investigated whereby the cardiograms are transmitted via telephone to USAFSAM, digitized and stored [3]-[5]. This is routine procedure at many hospitals; however, organizations that offer this service have reported noise problems associated with transmitting ECG's/VCG's in the form of FM modulated signals over the telephone lines [5]-[7]. One obvious solution is to digitize the signal at the site it is taken and to transmit digitized data—0's and 1's are much less susceptible to noise than is an analog signal. In order to meet the American Heart Association recommendations, one must use 9 bits and sample at a minimum of 500 samples/second [8] which for a 12 lead ECG yields a grand total of 54 000 bits/s (BAUD). The conclusion reached is that in order to either transmit digitized ECG's/VCG's or store large quantities of them, some data compression technique is required. Furthermore, whatever the data compression technique, the cardiogram must be able to be faithfully reconstructed from the compressed data. Faithfully reconstructed, a very imprecise term, is used to mean that there is no discernible difference between the original and the reconstructed cardiogram; at the present time it is impossible to define what is clinically significant beyond the cardiologist not being able to detect any significant difference between the two.

Several data compression techniques exist which yield from 3 to 1 up to 10 to 1 reductions in the amount of data required. These include techniques whereby a beat is stored and transmitted; after which, only the differences in successive beats are transmitted [9], techniques whereby only changes from sample-to-sample are transmitted [7], [10] and techniques whereby combinations of changes and slopes are transmitted [11], [12]. When one knows precisely how the cardiograms reconstructed from the compressed data are to be used and, if the reconstructions adequately satisfy that use, one is not concerned with discernible differences. While the cardiograms reconstructed from the compressed data of [7], [9]-[12] certainly perform their desired use adequately, a comparison of the reconstructions with the original waveforms show that there are discernible differences which for some uses would lead to difficulties. Other researchers have reported various compression ratios using the technique whereby the cardiograms are represented as some linear combination of known functions. Fourier series were used in [13], orthonormal exponentials were used in [14], and Karhunen-Loève series were used on portions of the waveform in [15]-[17]. Suboptimal versions of the Karhunen-Loève series were used in [18], [19]. Haar transforms and the discrete cosine transform were also used in [19]. Observe that if one uses orthogonal functions to represent cardiograms, one is faced with the problem of identifying a heartbeat so that it can be aligned with the functions (the epoch problem [14]). Once the epoch problem is determined, any of the series could be implemented with analog equipment. In [20] and [21], techniques are given for solving both the epoch and series representation problems with analog, orthonormal exponential filters.

Because of the digital computer power available today (micro, mini, mega) and its low cost, the approach to data compression reported in this paper has been implemented on a digital computer. It is a combination of sending and storing deviations from a known reference signal, and representing those deviations as linear combinations of orthogonal functions. As a first step, the VCG's are filtered to attenuate high-frequency noise with a fourth order Butterworth filter set for a 100-Hz cutoff frequency, straightens and digitized at a rate of 500 samples/second and passed through a preprocessor which shifts the baseline to zero, and calibrates the signals [22]. Next, a beat is identified by locating a window of fixed length N on the point on the x lead corresponding to the maximum negative slope (fiducial point) so that 45 percent of the beat precedes the fiducial point and 55 percent follows it [22]. At this point the data compression, which is the subject of this paper, begins.

II. First Expansion

Since it is well known that the discrete Karhunen-Loève expansion [23], intrinsic components [24], principal factors [25], [26], or principal components [27], all synonyms, are optimal with respect to mean-square error and entropy, they will be used twice. First, in order to remove the effects of respiration and the various orientations of different patients' hearts [28], a special case of the expansion as applied in [16], [17], [24]-[26], [28] will be used. That is, the digitized VCG signals are used to compute a 3 X 3, symmetric matrix

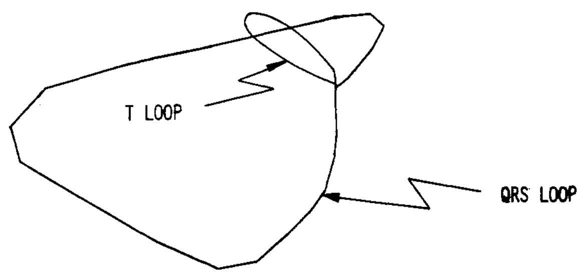

Fig. 1. *QRS* and *T* loops in principal plane.

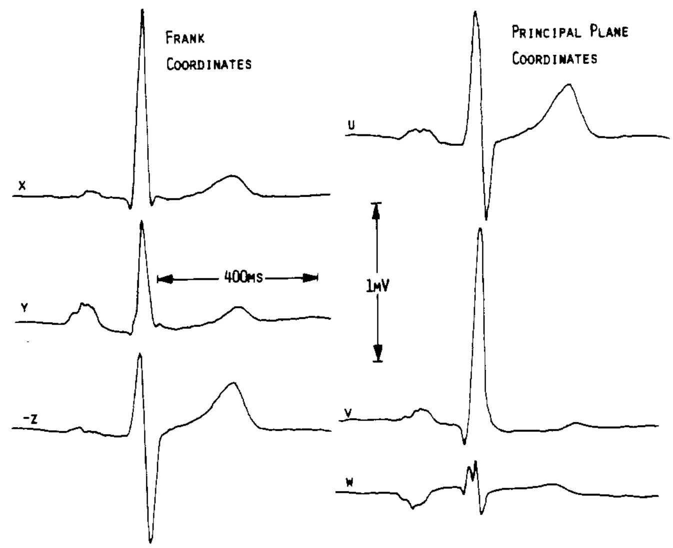

Fig. 2. A VCG coordinatized on the left in the Frank system and on the right in the principal plane coordinate system.

$$S = \frac{1}{N} \sum_{i=1}^{N} \begin{bmatrix} x(i) \\ y(i) \\ z(i) \end{bmatrix} [x(i)\, y(i)\, z(i)] \tag{1}$$

where $x(i)$, $y(i)$, and $z(i)$ are the x, y, and z components of the VCG coordinatized in the Frank coordinate system [1], [2], [29], and N is the number of samples in the beat. The eigenvalues and eigenvectors of this matrix are determined and the VCG transformed from the Frank coordinate system to the eigenvector coordinate system. As was previously reported [1], [24]-[26], this transformation causes over 90 percent of the energy in the VCG to be contained in the plane defined by the eigenvectors corresponding to the two largest eigenvalues (principal plane). Observe that eigenvectors are not unique (if $+u$ is an eigenvector, so is $-u$) and that when the two largest eigenvalues are almost the same, small perturbations can cause large changes in the eigencoordinate system. Therefore, a second rotation is performed on the data. An examination of Fig. 1 shows that the angular spread of the T loop is much smaller than the angular spread of the QRS loop; therefore, the first axis of the final coordinate system (u) is determined from the T loop in the principal plane. It is aligned with the longest vector in the T loop. The v axis also lies in the principal plane, is perpendicular to the u axis and is oriented such that the QRS loop rotates counterclockwise in the principal plane. The w axis, which is perpendicular to the principal plane, is chosen so that (u, v, w) is a right-handed coordinate system.

An example of the transformation is shown in Fig. 2, where a single beat of a VCG is shown coordinatized in the Frank (x, y, z) system and in the (u, v, w) coordinate system. Observe that, in the latter system, the major portion of the *QRS* and *T* loops are contained in the u and y leads, while the *P* loop is split equally among the three leads. This was expected—the eigenvectors corresponding to the two largest eigenvalues define the plane which contains the most energy and most of the energy (amplitude squared) is contained in the *QRS* and *T* loops.

The planar nature of the *QRS* portion of the VCG has long been known by cardiologists and several approaches have been made to display the VCG in that plane for examination [30], [31]. However, these techniques involved human beings entering rotation angles by turning knobs, and the results were not as consistent as the results obtained mathematically.

III. Second Expansion

Prior to the next expansion, a previously determined "population mean heartbeat" is subtracted from the transformed VCG. The mean beat was determined by averaging 900 patients' rotated VCG's. The purpose of this subtraction was twofold. First, it was anticipated that the range of the differences would be smaller than the range of the VCG's, which would reduce the number of bits required. Secondly, the statistical expected value of the difference is zero. Unfortunately, when a large ensemble of patients was used, the range of the deviations from the "mean heartbeat" was as large as the range of the original VCG. However, because of the simplifications resulting from using variables with a mean value of zero, the differences were used.

The expansion as applied in [15]-[17] was applied to the differences, where the basis functions were previously determined and stored in the computer.

The formula used for calculating the coefficients is

$$\alpha_i = \boldsymbol{\varphi}_i^T \boldsymbol{\rho}, \qquad i = 1, 2, \cdots, M \tag{2}$$

where the α_is are the coefficients in the expansion, the $\boldsymbol{\varphi}_i$s are the basis functions, $\boldsymbol{\rho}$ is the patient vector and M is the number of terms in the expansion. The components of the patient vector are defined to be the differences in the heartbeat and the mean beat.

$$\boldsymbol{\rho} = \begin{bmatrix} u(1) - u_m(1) \\ u(2) - u_m(2) \\ \vdots \\ u(N) - u_m(N) \\ v(1) - v_m(1) \\ v(2) - v_m(2) \\ \vdots \\ v(N) - v_m(N) \\ w(1) - w_m(1) \\ w(2) - w_m(2) \\ \vdots \\ w(N) - w_m(N) \end{bmatrix}$$

where $u_m(i)$, $v_m(i)$, and $w_m(i)$ are the values of the mean VCG in the transformed coordinate system.

The basis functions, which are determined once and for all time off-line, are the eigenvectors corresponding to the M largest eigenvalues of the matrix

$$\Lambda = \frac{1}{N} \sum_{i=1}^{N} \boldsymbol{\rho}_i \boldsymbol{\rho}_i^T \tag{3}$$

where N is the number of patients used in determining the functions and $\boldsymbol{\rho}_i$ is the patient vector of the ith patient—the eigenvectors are calculated once off-line and then stored. Functions are obtained from the eigenvectors via the same rule that was used to convert the difference between the heartbeat and the mean beat to a vector.

$$\boldsymbol{\varphi}_i^T = [\varphi_i(1)\, \varphi_i(2) \cdots \varphi_i(N)].$$

That is, the jth component of the ith eigenvector is the value of the ith basis function at the jth sample.

The VCG's are reconstructed by adding to the mean VCG

$$\sum_{i=1}^{N} \alpha_i \boldsymbol{\varphi}_i \tag{4}$$

and transforming the results from the principal plane coordinates (u, v, w) to the Frank coordinate system.

IV. Eigenvector Calculation

The expansions of both Sections II and III are based on determining the eigenvalues and eigenvectors of nonnegative definite and symmetric matrices. The first expansion requires the determination of the eigenvalues and eigenvectors for each patient; however, the matrices are only 3×3 [see (1)]. In order to perform the second expansion, a one-time determination of the M largest eigenvalues and corresponding eigenvectors of a $3N \times 3N$ matrix is required [see (3)]—a much more formidable task.

The algorithm used to solve the second eigenvalue eigenvector problem is from Wilkinson and Reinsch [32]. The algorithm which was originally coded in Algol has been translated into Fortran [16] and has been implemented on an IBM 360/75 computer. Typical examples of the core and times required by the program are

	10 Eigenvectors of 150 × 150 Matrix	20 Eigenvectors 300 × 300 Matrix
Time	3.62 min	9.61 min
Core	224K	352K

V. Results

Over 1000 VCG's have now been compressed, reproduced and the reproduction compared to the original waveforms. In order to ease the eigenvalue, eigenvector calculation, the sampling rate was reduced to 250 samples/second and a 200 sample window ($N = 200$) was used. The mean VCG and the first four eigenvectors are shown in Figs. 3 and 4. Observe that the w component (normal to the principal plane) had only negligible strength in the first four eigenvectors. In fact, w had only negligible effects on the first 11 eigenvectors. This is as expected—the major portion of the energy is contained by the u, v leads; therefore, they will be represented first. Compari-

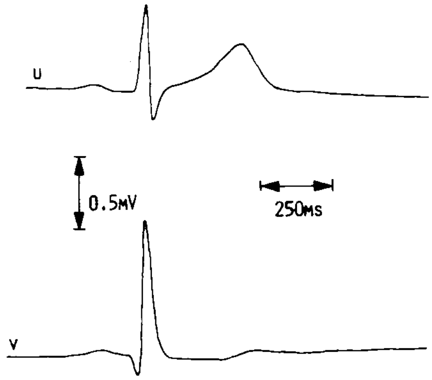

Fig. 3. The mean VCG in the principal plane.

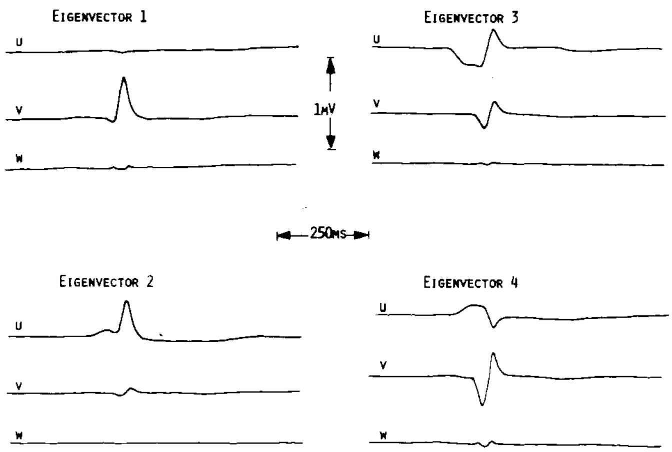

Fig. 4. The first four eigenvectors.

sons of reconstructions from 40 and 60 coefficients ($M = 40$ and $M = 60$) with the original are given in Fig. 5 for a typical VCG. Finally, the expansions were performed on each of the leads separately. Comparisons of reconstructions from 20 coefficients per lead, a total of 60 coefficients, with the original are given in Fig. 6.

In order to demonstrate the advantage that using an optimal expansion offers over using the discrete Fourier transform, reconstructions of the same VCG were obtained by applying the fast Fourier transform to each of the leads separately. The sampling rate used was 500 samples/second and 512 samples were used. This made the fundamental frequency approximately 1 hertz. Observe that, since the coefficients are now complex numbers, two real numbers are required for each frequency. Comparisons of reconstructions from the first 10 Fourier coefficients (20 numbers/lead), which corresponds to 10 Hz, with the original VCG are shown in Fig. 6. Comparing the Karhunen-Loève results to the Fourier results confirms the theory and clearly shows that, if one wishes to either transmit or store 60 numbers and reconstruct the 3 leads of a VCG, then Karhunen-Loève expansion is far superior to the Fourier Transform. This is a result of the highly nonstationary characteristics of the VCG—although the VCG's of different patients are not identical, there is a definite observable similarity. The theory predicts that the Karhunen-Loève transform is superior to the Fourier transform for nonstationary random signals; however, it has been shown that, for periodic stationary signals, the Karhunen-Loève transform becomes the Fourier transform [16].

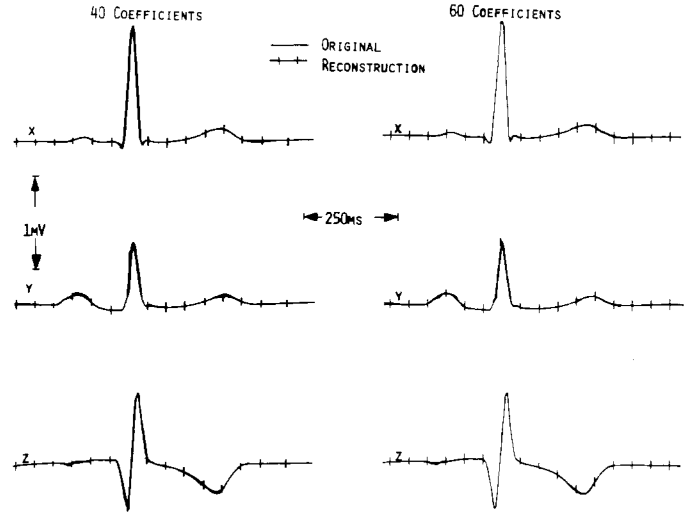

Fig. 5. Reconstructions obtained with 40 and 60 KL coefficients.

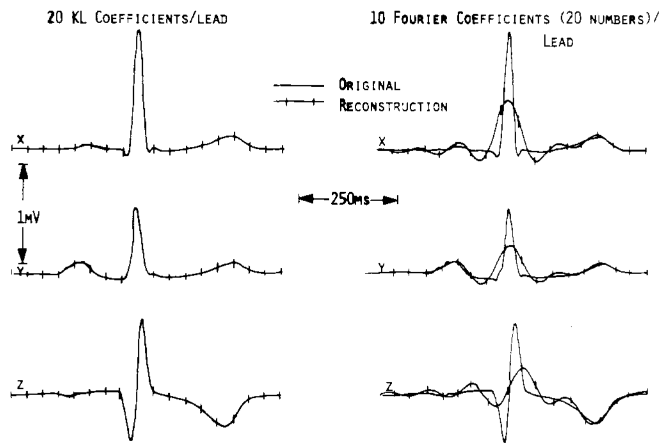

Fig. 6. Reconstructions obtained with 20 KL coefficients/lead and 20 Fourier coefficients/lead.

VI. CONCLUSIONS

A data compression technique for VCG's based on two applications of "principal components" (discrete Karhunen-Loève expansions) has been derived. It has been observed that both expansions, or transformations, require the determination of the eigenvectors of real, nonnegative definite and symmetric matrices. The first transformation, which is used to remove diagnostically insignificant patient-to-patient variations from the VCG's, requires the eigenvectors of a 3×3 matrix for each patient—a relatively easy task. The second transformation, which is the key ingredient to the data compression, requires that the eigenvectors of a representative matrix be determined *once* and for all time (all patients use the same eigenvectors); however, the matrix is rather large. A technique for determining the largest eigenvectors of large matrices was developed by Wilkinson and Reinsch and programmed in Algol. These programs have been translated to Fortran IV and the technique is operational on an IBM 360/75 computer.

Results obtained with over 1000 VCG's sampled at a rate of 250 samples/second have been given. An examination of those results shows that the reconstructions using 60 coefficients (20 per lead) faithfully reproduce all but the P wave portion of the VCG's. The reduction from 250 sample/second (approximately one beat for a resting VCG) to 20 coefficients per lead is a little better than 12 to 1. This compression ratio has been obtained by others; however, these reconstructions are far superior to those obtained with other techniques. Finally, others have observed that the discrete Karhunen-Loève functions are the most efficient to use; however, they have stated that the computations were too complex and proceeded to use suboptimal functions [14], [18]-[21]. We have now shown that the calculations are *not* too complex; furthermore, since the large-matrix-eigenvalue, eigenvector problem is solved, once and for all time off-line, this data compression scheme can be implemented with either a microcomputer or minicomputer.

Although only VCG's have been considered to date, applying the second expansion to the 12 leads of an ECG, one lead at a time, requires no modification. If the 12 leads of the ECG are partitioned into appropriate groups of threes and if each group of 3 leads is recorded simultaneously, then each group can be processed as a VCG. With either approach, the compression ratios will be the same as the VCG's (12 to 1).

As was pointed out in the Introduction, the American Heart Association recommends that cardiograms be digitized at a rate of 500 samples/second, and, in this paper a rate of 250 samples/second was used. The programs developed are being extended to handle the recommended rate. Also, techniques for better representing the *P* waves are being developed. These include performing separate expansions on the *P* wave and the *QRST* portion of the signals, and weighting the *P* wave more when calculating the eigenvectors. The results obtained will be reported.

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