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DATA VARIATION IN FRACTAL INTERPOLATION

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ABSTRACT. **A fractal interpolant of the proper interpolation data is fully determined by the functional equation of Read-Bejraktarevic type and a free vertical scaling vector v such that $\|v\| < 1$. In this note, it is shown how insertion of the data impacts the related Read-Bejraktarevic equation and the interpolant. Some examples support the theory.**

1. INTRODUCTION

The recent investigations of fractal functions (see [2, 3, 4]) reveal that some properties of the classical interpolation methods are preserved by the fractal interpolating methods. In this note, the problem of node and knot insertion will be studied.

Let $x_0 < x_1 < \cdots < x_n$ be an interpolating mesh in $\mathbf{R}$, and let $B[x_0, x_n]$ denote all bounded real functions on $[x_0, x_n]$. For $i = 1, ..., n$, let the set of homeomorphisms $u_i : [x_0, x_n] \to \mathbf{R}$ and the set of $(\mathrm{Lip}, \mathrm{Lip}^{(<1)})$-mappings $v_i : \mathbf{R}^2 \to \mathbf{R}$ be given. Then the Read-Bajraktarević functional equation

$$\phi(x) = v_i(u_i^{-1}(x), \phi(u_i^{-1}(x))) , \quad x \in [x_{i-1}, x_i] , \ i = 1, \dots, n , \tag{1}$$

uniquely defines the function $\varphi \in B[x_0, x_n]$. Consequently, its graph Γ_φ is the fixed point of the operator $W(\cdot) = \cup_i w_i(\cdot)$, where

$$w_i(x, y) = (u_i(x), v_i(x, y)). \tag{2}$$

In general, its Hausdorff dimension satisfies the inequality $1 \le D_H(\Gamma_\varphi) < 2$. Let an interpolating data set $Y = \{(x_i, y_i)\}_{i=0}^n$ $(n \ge 2)$ be given such that $\Delta x_i = x_{i+1} - x_i > 0, \ i = 0, 1, ..., n-1$. If the homeomorphisms u_i and v_i satisfy

$$u_i(x_0) = x_{i-1}, \ u_i(x_n) = x_i; \ v_i(x_0, y_0) = y_{i-1}, \ v_i(x_n, y_n) = y_i, \tag{3}$$

then φ interpolates the data set Y, i.e. $\varphi(x_i) = y_i, \quad i = 1, \dots, n$.

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If $\mathbf{C}^*[Y] = \{f \in \mathbf{C}[x_0, x_n] \mid f(x_i) = y_i, i = 1, \ldots, n\}$, it is easy to show [5] that the operator $\Phi : B[x_0, x_n] \to \mathbf{C}^*[Y]$, given by

$$\Phi(f) = v_i(u_i^{-1}(\cdot), f(u_i^{-1}(\cdot))) \tag{4}$$

for all i, is contractive on $\mathbf{C}^*[Y]$ w.r.t. the appropriate metric, and φ is a fixed point of Φ, namely $\Phi(\varphi) = \varphi$. Since $D_H(\Gamma_\varphi)$ in general is a fractional number, φ is usually referred to as *fractal interpolating function.*

There are two problems arising in interpolation theory that are relevant for applications:

1.1. **Node insertion problem.** Let us call the pair (x_i, y_i) an *interpolation node*. The problem of *node insertion* refers to introducing a new node say, $(\hat{x}, \hat{y})$, $x_0 < \hat{x} < x_n$, such that the new interpolating data set will be $\hat{Y} = Y \cup \{(\hat{x}, \hat{y})\}$, and to find the new Read-Bajraktarević operator $\hat{\Phi}$ and its fixed point $\hat{\varphi}$. The problem generalizes to multiple node insertion, i.e. insertion of many nodes at the same time. The reason for node insertion is increasing the degrees of freedom of an interpolant.

1.2. **Knot Insertion problem.** It is usual to call *knot* an interpolating abscissa point, x_i. To *insert a new knot* means to add a knot $\hat{x}$, $x_0 < \hat{x} < x_n$, to interpolating abscissas $\{x_1, \ldots, x_n\}$, such that $\hat{y} = \varphi(\hat{x})$. In this sense, knot insertion is a special case of a node insertion, the case when the interpolant remains unchanged after getting a new node. The reason is the same, to increase degrees of freedom, i.e.the dimensionality of interpolating points, but now without changing the function φ.

2. Node Insertion

It has already been mentioned that the pair $w_i(x, y) = (u_i(x), v_i(x, y))$ defines a contraction in the metric space $(\mathbf{R}^2, d)$. Insertion of a node $(\hat{x}, \hat{y})$ updates the set of data Y to $\hat{Y}$. Since each subinterval $[x_{i-1}, x_i]$ is responsible for one contraction, provided $x_{i-1} < \hat{x} < x_i$, two more subintervals arise: $[x_{i-1}, \hat{x}]$ and $[\hat{x}, x_i]$, and therefore two new mappings say left and right ones, w^L and w^R.

Lemma 2.1. *Let the Read-Bajraktarević functional equation* (1) *be given so that* (3) *is satisfied. Then the introduction of the new node* $(\hat{x}, \hat{y})$, $x_{i-1} < \hat{x} < x_i$ *will result in the new equation*

$$\phi(x) = \hat{v}_i(\hat{u}_i^{-1}(x), \phi(\hat{u}_i^{-1}(x))), \quad x \in [\hat{x}_{i-1}, \hat{x}_i], \ i = 1, \ldots, n, \tag{5}$$

where

$$\hat{x}_j = \begin{cases} x_j, & j = 1, \dots i-1, \\ \hat{x}, & j = i, \\ x_{j-1}, & j = i+2, \dots, n+1, \end{cases}$$

and

$$\hat{w}_j = \begin{cases} w_j, & j = 1, \dots i-1, \\ w^L, & j = i, \\ w^R, & j = i+1, \\ w_{j-2}, & j = i+2, \dots, n+1, \end{cases}$$

where $w^L = (u^L, v^L)$ *and* $w^R = (u^R, v^R)$ *are defined by*

$$u^L(x_0) = x_{i-1},\ u^L(x_n) = \hat{x};\ v^L(x_0, y_0) = y_{i-1},\ v^L(x_n, y_n) = \hat{y},$$

and

$$u^R(x_0) = \hat{x},\ u^R(x_n) = x_i;\ v^R(x_0, y_0) = \hat{y},\ v^R(x_n, y_n) = y_i.$$

If w^L *and* w^R *are* $(Lip, Lip^{(<1)})$*-mappings, then the unique solution* $\hat{\varphi}$ *interpolates* $\hat{Y}$*, i.e.* $\hat{\varphi}(\hat{x}_i) = \hat{y}_i$*,* $i = 0, 1, \dots, n+1$*.*

There is an important case of u_i and v_i being affine functions, i.e.

$$w_i : (x, y) \mapsto (a_i x + e_i,\ c_i x + d_i y + f_i). \tag{6}$$

Then, if $|d_i| < 1$, $i = 1, \dots, n$, and the coefficients a_i, c_i, e_i and f_i are given by

$$\begin{aligned} a_i &= \tfrac{\Delta x_{i-1}}{x_n - x_0}\,, \quad c_i = \tfrac{\Delta y_{i-1}}{x_n - x_0} - d_i \tfrac{y_n - y_0}{x_n - x_0}, \\ e_i &= x_i - a_i x_n\,, \quad f_i = y_i - c_i x_n - d_i y_n\,, \end{aligned} \tag{7}$$

the solution of (1) is a fractal interpolating function [1].

Theorem 2.2. *Let the mappings in* (1) *be given by* (6)*, and let the node* $(\hat{x}, \hat{y})$*,* $x_{i-1} < \hat{x} < x_i$*, be inserted. Then the new Read-Bajraktarević functional equation will be given as in Lemma 2.1, and the coefficients of* w^L *and* w^R *are given by*

$$a_i^L = \lambda a_i,\ c_i^L = \mu c_i + (\mu d_i - d_i^L)(y_n - y_0)/(x_n - x_0),\ e_i^L = \hat{x} + \lambda(e_i - x_i),$$

$$f_i^L = f_i + [x_0(y_i - \hat{y}) + (x_0 y_n - x_n y_0)(d_i^L - d_i)]/(x_n - x_0),$$

$$a_i^R = (1-\lambda) a_i,\ c_i^R = (1-\mu) c_i + ((1-\mu) d_i - d_i^R)(y_n - y_0)/(x_n - x_0),$$

$$e_i^R = \lambda x_i + (1-\lambda) e_i,$$

$$f_i^R = f_i + [x_n(\hat{y} - y_{i-1}) + (x_0 y_n - x_n y_0)(d_i^R - d_i)]/(x_n - x_0),$$

where $\lambda = (\hat{x} - x_{i-1})/(x_i - x_{i-1})$ *and* $\mu = (\hat{y} - y_{i-1})/(y_i - y_{i-1})$*.*

Proof. Applying (6) and Lemma 2.1 on the new set of nodes $\hat{Y} = Y \cup \{(\hat{x}, \hat{y})\}$ one gets

$$a_i^L = \frac{\hat{x} - x_{i-1}}{x_n - x_0} = \frac{\hat{x} - x_{i-1}}{x_i - x_{i-1}} \frac{x_i - x_{i-1}}{x_n - x_0} = \lambda a_i,$$

and similarly $a_i^R = (1 - \lambda) a_i$. By setting $D = (y_n - y_0)/(x_n - x_0)$, one gets

$$c_i^L = \frac{\hat{y} - y_{i-1}}{x_n - x_0} - d_i^L D = \mu \frac{y_i - y_{i-1}}{x_n - x_0} - \mu d_i D + \mu d_i D - d_i^L D = \mu c_i + (\mu d_i - d_i^L) D,$$

$$\begin{aligned} c_i^R = \frac{y_i - \hat{y}}{x_n - x_0} - d_i^R D &= (1 - \mu) \frac{y_i - y_{i-1}}{x_n - x_0} - (1 - \mu) d_i D + (1 - \mu) d_i D - d_i^R D \\ &= (1 - \mu) c_i + ((1 - \mu) d_i - d_i^R) D. \end{aligned}$$

The expressions for e_i^L and e_i^R follow from

$$e_i^L = \hat{x} - a_i^L x_n, \quad e_i^R = x_i - a_i^R x_n,$$

and from

$$f_i^L = \hat{y} - c_i^L x_n - d_i^L y_n, \quad f_i^R = y_i - c_i^R x_n - d_i^R y_n,$$

the expressions for f_i^L and f_i^R can be derived. □

3. Knot Insertion

Let $\Sigma_k = \{1, \dots, n\}^{\{1, \dots, k\}}$ be the finite tree of length k with n branches [5]. The elements of Σ_k, $\mathbf{i(k)} = (\mathbf{i_1}, \mathbf{i_2}, \dots \mathbf{i_k})$ are finite codes of length k. For the set of n mappings $\{f_i\}_{i=0}^n$, let $f_{\mathbf{i}(k)}$ denote the composition $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}$. Let $Y = Y^0 \{(x_i, y_i)\}_{i=0}^n$ $(n \geq 2)$ be the set of interpolating points, and $\varphi \in \mathbf{C}^*(Y)$. Then, it is easy to see that besides Y^0, φ also interpolates the set of points

$$Y^{k+1} = \cup_{\mathbf{i(k)} \in \Sigma_\mathbf{k}} w_{\mathbf{i}(k)}(Y^0) \quad (k = 1, 2, ...)\,, \tag{8}$$

w_i being given by (2). Note that

$$\Gamma_\varphi = \lim_{k \to \infty} Y^k. \tag{9}$$

Lemma 3.1. *We have $Y^{k-1} \subset Y^k$ for $k = 1, 2, \dots,$.*

Proof. The mapping $w_j \circ w_{\mathbf{i}(k-1)}$ is of the type $w_{\mathbf{i}(k)}$. Now $Y^k = \cup_{\Sigma_k}(Y) = \cup_{\substack{j=1,\dots,n \\ \mathbf{i}(k-1) \in \Sigma_{k-1}}} (Y) = \cup_j \big(\cup_{\Sigma_{k-1}} w_{\mathbf{i}(k-1)}(Y)\big) = \cup_j w_j(Y^{k-1})$, which implies that Y^{k-1} is a subset of Y^k. □

Lemma 3.2. *The iterated function systems $\{\mathbf{R^2}, \{w_i\}_{i=1}^n\}$ and $\{\mathbf{R^2}, \{w_{\mathbf{i}(k)}\}_{i=1}^{n^k}\}$, $k \geq 2$, have the same attractor, the set Γ_φ.*

Proof. Consider the operator $W_k = \cup_{\Sigma_k} w_{\mathbf{i}(k)}$. Then $W_1 = \cup_i w_i$ and its $(k-1)$ autocomposition is $W_1^{k-1} = W_k$. This means that

$$W_1^{k-1}(Y) = W_k(Y) = Y^k. \tag{10}$$

From the theory of iterated systems, it is known that $\Gamma_\varphi = \lim_{k\to\infty} W_1^{k-1}(Y)$, which, in combination with (9) and (10), proves the assertion. □

Theorem 3.3. *Let the knot $(\hat{x},\hat{y}) \notin Y$ but $(\hat{x},\hat{y}) \in Y^k$. Let φ be the fractal interpolating function given by* (1) *and* (3) *and let $w_{\mathbf{i}(k)} = (u_{\mathbf{i}(k)}, v_{\mathbf{i}(k)})$. Then the same function is the solution of the Read-Bajraktarević equation*

(11)
$$\phi(x) = v_{\mathbf{i}(k)}\big(u^{-1}_{\mathbf{i}(k)}(x), \phi(u^{-1}_{\mathbf{i}(k)}(x))\big)\,, \quad x \in [u_{\mathbf{i}(k)}(x_0), u_{\mathbf{i}(k)}(x_n)]\,,\ \mathbf{i}(k) \in \Sigma_k\,.$$

Proof. Since u_i and v_i are contractions for all $i = 1,\dots,n$, their autocompositions $u_{\mathbf{i}(k)}$ and $v_{\mathbf{i}(k)}$ $(k = 1,2,\dots)$ are contractions too. So, the operator Φ_k, given by $\Phi_k(f) = v_{\mathbf{i}(k)}\big(u^{-1}_{\mathbf{i}(k)}(\cdot), f(u^{-1}_{\mathbf{i}(k)}(\cdot))\big)$, is contractive on $C^*[Y^k]$ with respect to some functional metric. By virtue of Lemma 3.1, $Y^{k-1} \subset Y^k$ which implies $C^*[Y^{k-1}] \subset C^*[Y^k]$, so Φ_k is also contractive on $C^*[Y^{k-1}]$.This means that the fixed point of Φ_k, say φ_k, interpolates $Y^0 = Y$ as well.

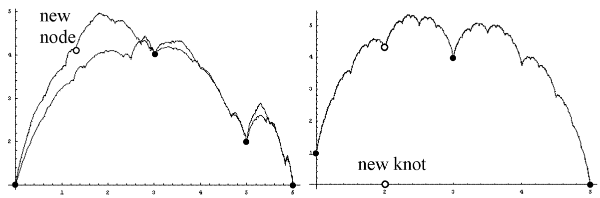

Figure 1. Insertion of a new node (left) and a new knot (right).

The following assertion can be proved by induction: The solution of the Read-Bajraktarević functional equation $\Phi_\nu(\phi) = \phi$ also satisfies the equation $\Phi_{\nu+1}(\phi) = \phi$. For $\nu = 1$, the first equation becomes

$$\phi(x) = v_i\big(u_i^{-1}(x), \phi(u_i^{-1}(x))\big),\ x \in [u_i(x_0), u_i(x_n)],\ i \in \Sigma_1\,, \tag{12}$$

and the second becomes

(13)
$$\phi(x) = v_{ij}\big(u_{ij}^{-1}(x), \phi(u_{ij}^{-1}(x))\big)\ x \in [u_{ij}(x_0), u_{ij}(x_n)],\ (i,j) \in \Sigma_2\,.$$

Let φ and f be the solutions of (12) and (13), respectively. From (13),

$$\phi(x) = v_{ij}\Big(u_i^{-1}(u_j^{-1}(x)), \phi(u_i^{-1}(u_j^{-1}(x)))\Big),$$

and from the obvious relation $v_j(u_i(x), v_i(x,y)) = v_{ij}(x,y)$, one gets

$$\phi(x) = v_j\Big(u_i(u_i^{-1}(u_j^{-1}(x))), v_i(u_i^{-1}(u_j^{-1}(x)), \phi(u_i^{-1}(u_j^{-1}(x))))\Big),$$

which simplifies by the substitution $x = u_j(t)$ to

$$\phi(u_j(t)) = v_j(t, v_i(u_i^{-1}(t), \phi(u_i^{-1}(t))).$$

It transforms to $\phi(u_j(t)) = v_j(t, \phi(t))$, or having x variable back, $\phi(x) = v_j(u_j^{-1}(x), \phi(u_j^{-1}(x)))$, whence, regarding (12), $\phi(x) = \varphi(x)$, $x \in [x_0, x_n]$. But since $\phi(x) = f(x)$, $x \in [x_0, x_n]$ is also a solution of (13), it follows that $\varphi(x) = f(x)$, $x \in [x_0, x_n]$. By a similar procedure, the case $\nu = k$ is to be handled. Namely, the solution of $\Phi_k(\phi) = \phi$, i.e. (11) is the solution of $\Phi_{k+1}(\phi) = \phi$, which can be derived from (11) by replacing the indices $\mathbf{i}(k)$ by $\mathbf{i}(k+1) = \mathbf{i}(k) \cdot i_{k+1} = \{(i_1, \ldots, i_k\} \times \{i_{k+1}\}$.

So the solution of (11) φ_k also satisfies (1), provided (3) is satisfied. Since φ_k interpolates Y, it is equivalent with the fractal interpolant φ, defined by (1). By Lemma 3.2, the graphs of the functions φ and φ_k are the same. □

4. Conclusion

This note presents some results of investigations of gaining more flexibility to fractal interpolating functions, by inserting a new node or knot. In the first case, the function changes to another one, with one extra node that can be used in modelling purposes. In the other case, the function does not change, which is even more suitable in modelling fractal functions. A generalization is possible by repeating the insertion process many times. Further investigations should also be focused on the node and knot deletion processes.

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