DE GIORGI'S COUNTEREXAMPLE IN ELASTICITY*

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Abstract. The framework of linear elastostatics is used to interpret De Giorgi's example concerning the non-regularity of extremals of a certain Dirichlet integral with measurable and bounded coefficients.

1. Introduction. Some years ago De Giorgi invented a counterexample [1] which rendered obsolete many optimistic conjectures on the regularity of weak solutions of variational systems of elliptic type. Precisely, De Giorgi was able to exhibit an unbounded and discontinuous extremal of the finite Dirichlet integral of a certain differential operator with measurable and bounded coefficients.

Thus De Giorgi's counterexample also proves to be vacuous the widespread belief that most of the classical problems of mathematical physics can be given the aspect of variational problems with finite energy; and that, conversely, variational problems with finite energy should always describe physically meaningful situations.

In this note we use the framework of linear elastostatics to construct an interpretation of De Giorgi's result which clarifies the reasons for the failure of this latter conjecture. We consider a suitable displacement problem for a monoparametric family of spherical shells, of unit outer radius and comprised of a certain aelotropic and inhomogeneous elastic material, taking as a parameter the radius ϵ of the cavity. We then show that this problem has a C^∞ solution which converges to De Giorgi's extremal when $\epsilon \to 0$. However, if the cavity shrinks under a certain limit value ϵ_0, the solution loses its physical meaning. Accordingly, De Giorgi's counterexample is given a precise status as the *mathematical* problem which results from a family of physically meaningful problems via a well-defined limiting process.

2. Notation. Let V be the vector space associated with the n-dimensional Euclidean space E, with inner product $\mathbf{v}\cdot\mathbf{w}$. If $\mathbf{v} \in V$ we denote by $v = (\mathbf{v}\cdot\mathbf{v})^{1/2}$ the magnitude of $\mathbf{v}$.

We write $\mathrm{Lin}(V)$ for the space of all linear transformations on V, with inner product $\mathbf{V}\cdot\mathbf{W}^T\mathrm{j} = \mathrm{tr}\,(\mathbf{VW}^T)$ (here $\mathbf{W}^T$ is the transpose of $\mathbf{W}$, $\mathbf{VW}^T$ is the composition of $\mathbf{W}^T$ and $\mathbf{V}$, and tr is the trace functional). We denote by $V = (\mathbf{V}\cdot\mathbf{V})^{1/2}$ the magnitude of $\mathbf{V}$, and by $\mathbf{1}$ the identity of $\mathrm{Lin}(V)$. If $\mathbf{a} \in V$ and $a = 1$, the orthogonal projection of V onto the span of $\mathbf{a}$ is the element $\mathbf{a} \otimes \mathbf{a}$ of $\mathrm{Lin}(V)$ defined by

$$\mathbf{a} \otimes \mathbf{a}(\mathbf{v}) = (\mathbf{a}\cdot\mathbf{v})\mathbf{a}, \qquad \forall\ \mathbf{v} \in V.$$

We also write Lin(Lin) for the space of all linear transformation on $\mathrm{Lin}(V)$. In partic-

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ular, we consider the following members of Lin(Lin):

(i) $\mathbf{1}$, the identity mapping.

(ii) $\mathbf{V} \otimes \mathbf{W}$, the mapping defined by

$$\mathbf{V} \otimes \mathbf{W}(\mathbf{Z}) = (\mathbf{W} \cdot \mathbf{Z})\mathbf{V}, \qquad \forall\, \mathbf{Z} \in \mathrm{Lin}(V).$$

If $\mathbf{A} \in \mathrm{Lin}(V)$ and $A = 1$, $\mathbf{A} \otimes \mathbf{A}$ is the orthogonal projection of $\mathrm{Lin}(V)$ onto the span of $\mathbf{A}$.

(iii) sym (skw), the mapping which associates to any element $\mathbf{V} \in \mathrm{Lin}(V)$ its symmetric (skew-symmetric) part:

$$\mathrm{sym}\,(\mathbf{V}) = \tfrac{1}{2}(\mathbf{V} + \mathbf{V}^T) \qquad (\mathrm{skw}\,(\mathbf{V}) = \tfrac{1}{2}(\mathbf{V} - \mathbf{V}^T)).$$

The images of $\mathrm{Lin}(V)$ under sym and skw are the (orthogonal) subspaces $\mathrm{Sym}(V)$ and $\mathrm{Skw}(V)$ of $\mathrm{Lin}(V)$, respectively.

Let Ω be a smooth, bounded neighborhood of the origin $\mathbf{0}$ of E, with boundary $\partial\Omega$. We call

$$V \ni \mathbf{p}(\mathbf{x}) = \mathbf{x} - \mathbf{0}$$

the position vector of the point $\mathbf{x} \in E$, and

$$\mathrm{Sym}\,(V) \ni \mathbf{P}(\mathbf{x}) = p^{-2}\mathbf{p} \otimes \mathbf{p}$$

the radial projection of V onto the line spanned by $\mathbf{p}$.

For conciseness, we write H^1, in place of $(H^1(\Omega))^n$ [1], for the Hilbert space of square-integrable (in the sense of Lebesgue) vector fields on Ω with square-integrable first gradient. If $\mathbf{u} \in H^1$ we denote the H^1 norm of $\mathbf{u}$ by

$$\|\mathbf{u}\|_1 = \left(\int_\Omega (\mathbf{u} \cdot \mathbf{u} + \mathrm{grad}\,\mathbf{u} \cdot \mathrm{grad}\,\mathbf{u})\right)^{1/2} = (\|u\|_0^2 + |u|_1^2)^{1/2};$$

here $\|\mathbf{u}\|_0$ is the L^2 norm and $|\mathbf{u}|_1$ the seminorm of the first partial derivatives of $\mathbf{u}$. By C_0^∞ we denote, as usual, the space of vector fields vanishing on $\partial\Omega$ with their gradients of any order. We write H_0^1 for the completion of C_0^∞ with respect to the norm of H^1. We also make use of another seminorm of $\mathbf{u}$ in H^1, namely

$$|||\mathbf{u}||| = \left(\int_\Omega \mathbf{E}(\mathbf{u}) \cdot \mathbf{E}(\mathbf{u})\right)^{1/2},$$

where

$$\mathbf{E}(\mathbf{u}) = \mathrm{sym}\,(\mathrm{grad}\,\mathbf{u}).$$

It follows from this last definition and the definition of the inner product of $\mathrm{Lin}(V)$ that

$$\mathrm{div}\,\mathbf{u} = \mathbf{1} \cdot \mathrm{grad}\,\mathbf{u} = \mathbf{1} \cdot \mathbf{E}(\mathbf{u}).$$

3. De Giorgi's integral. Let $\mathbf{B}(\mathbf{x}) \in \mathrm{Lin}(V)$, and let

$$\mathrm{Lin(Lin)} \ni \mathbf{C}(\mathbf{x}) = \mathbf{1} + (n-2)^2\mathbf{B} \otimes \mathbf{B} \quad \text{for any} \quad \mathbf{x} \in E. \tag{3.1}$$

De Giorgi [1] considers the bilinear form

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{C}(\mathrm{grad}\,\mathbf{u}) \cdot \mathrm{grad}\,\mathbf{v} \tag{3.2}$$

[1] We use similar abbreviations in analogous cases.

and calls $\mathbf{u}$ an *extremal* of the integral

$$I(\mathbf{u}) = a(\mathbf{u}, \mathbf{u}) \tag{3.3}$$

if $\mathbf{u} \in H^1$ and

$$a(\mathbf{u}, \mathbf{v}) = 0, \qquad \forall\, \mathbf{v} \in C_0^\infty. \tag{3.4}$$

He further selects

$$\mathbf{B} = \mathbf{1} + \frac{n}{n-2}\mathbf{P}, \qquad n \geq 3, \tag{3.5}$$

so that the integro-differential form (3.1) has measurable and bounded coefficients in $\bar{\Omega}$. Finally he proves that

$$\hat{\mathbf{u}} = p^\alpha \mathbf{p} \tag{3.6}$$

is an extremal of the integral (3.3) if

$$\alpha = -\frac{n}{2}\left(1 - \frac{1}{(1 + 4(n-1)^2)^{1/2}}\right). \tag{3.7}$$

It is easily seen that $\hat{\mathbf{u}}$ is neither continuous nor bounded at $\mathbf{0}$. As condition $(3.5)_2$ is essential to the proof, De Giorgi's example shows that the regularity of the extremals of integrals as (3.3) may depend, among other things, on the *dimension* n of the underlying space.

De Giorgi's result can be easily cast into a result concerning the non-regularity of weak solutions of linear systems of elliptic type (see also [2] and [3]).

For future convenience, we first replace the single mapping C defined in (3.1) by the class

$$\mathsf{C} = \mathsf{1} + c\mathbf{B} \otimes \mathbf{B}, \tag{3.8}$$

where

$$\mathrm{Sym}(V) \ni \mathbf{B} = \mathbf{1} + b\mathbf{P}, \tag{3.9}$$

and $c > 0$, b are constants which may depend on n. By merely repeating the steps of the procedure outlined in [1], with C as above, it can be shown that $\hat{\mathbf{u}}$ as given by (3.6) is still an extremal of the family of integrals

$$I(\mathbf{u}; b, c) = \int_\Omega (|u|_1^2 + c(\operatorname{div}\mathbf{u} + b\mathbf{P}\cdot\operatorname{grad}\mathbf{u})^2) \tag{3.10}$$

provided n, b and α obey certain restrictions to be derived later on.

In view of (3.8) and (3.9) we now write (3.2) as

$$a(\mathbf{u}, \mathbf{v}) = \int_\Omega (\operatorname{grad}\mathbf{u}\cdot\operatorname{grad}\mathbf{v} + c(\operatorname{div}\mathbf{u} + b\mathbf{P}\cdot\operatorname{grad}\mathbf{u})(\operatorname{div}\mathbf{v} + b\mathbf{P}\cdot\operatorname{grad}\mathbf{v})). \tag{3.11}$$

$a(\mathbf{u}, \mathbf{v})$ is a *symmetric, continuous* and *coercive* bilinear form on H^1.[2]

[2] Symmetry and continuity of $a(\mathbf{u}, \mathbf{v})$ follow from straightforward verifications. On the other hand, for any b, positiveness of c implies the following property of C:

$$\text{(strong ellipticity)} \quad \exists\ \nu > 0 : \mathbf{v} \otimes \mathbf{w} \cdot \mathsf{C}(\mathbf{v} \otimes \mathbf{w}) \geqq \nu\, v^2 w^2, \quad \forall\ \mathbf{v}, \mathbf{w} \in V - \{\mathbf{0}\}. \tag{*}$$

Coerciveness of $a(\mathbf{u}, \mathbf{v})$ is then proved by exploiting the strong ellipticity condition and Korn's inequality.

Let us henceforth identify Ω with the open ball $B = B(\mathbf{0}; 1) \subset E$ of center $\mathbf{0}$ and unit radius. As the form $a(\mathbf{u}, \mathbf{v})$ in continuous and coercive, a classical result of Lax and Milgram insures that there exists a unique solution to the problem of finding a field $\mathbf{u} \in H^1$ such that $(\mathbf{u} - \mathbf{p}) \in H_0^{\ 1}$ and

$$a(\mathbf{u}, \mathbf{v}) = 0, \qquad \forall\, \mathbf{v} \in H_0^{\ 1}. \tag{3.12}$$

As it is well known (see, e.g., [4]), it follows further from the symmetry of $a(\mathbf{u}, \mathbf{v})$ that the former boundary-value problem is completely equivalent to the minimum problem (in $K = \mathbf{p} + H_0^{\ 1}$) for the quadratic functional (3.10).

We may regard the Euler equation (3.12) of this functional as a weak formulation of the following class of elliptic second-order systems:

$$\operatorname{div} \mathbf{C}\,(\operatorname{grad} \mathbf{u}) = \mathbf{0} \quad \text{in} \quad B, \qquad \mathbf{u} - \mathbf{p} = \mathbf{0} \quad \text{in} \quad \partial B. \tag{3.13}$$

By routine calculations it can be shown that $\hat{\mathbf{u}}$ given by (3.6) solves problem (3.13) in $B - \{\mathbf{0}\}$ if

$$\alpha = -\frac{n}{2}\left(1 \mp \left(1 - \frac{4bc(n-1)(n+b)}{n^2(1 + c(1+b)^2)}\right)^{1/2}\right). \tag{3.14}$$

Moreover, $\hat{\mathbf{u}} \in H^1$ (and $a(\mathbf{u}, \mathbf{u}) < +\infty$) if

$$2\alpha + n > 0. \tag{3.15}$$

Finally, if

$$\alpha + 1 \leq 0, \tag{3.16}$$

$\hat{\mathbf{u}}$ is discontinuous at the origin (in addition, $\hat{\mathbf{u}}$ is unbounded at $\mathbf{0}$ when the last inequality holds strictly). Conditions (3.14), (3.15), (3.16) imply the following restrictions on the choice of n, b and α:

$$n \geq 3; \qquad b \geq \frac{1 + c}{c(n-2)}\,; \qquad \alpha = -\frac{n}{2}\left(1 - \left(1 - \frac{4bc(n-1)(n+b)}{n^2(1 + c(1+b)^2)}\right)^{1/2}\right). \tag{3.17}$$

Thus (cf. [1]), for any choice of the parameters compatible with (3.17), $\hat{\mathbf{u}}$ is the non-regular weak (variational) solution of a problem of class (3.13). (We will comment further on De Giorgi's example in the Appendix.)

4. Application to elasticity. In order to interpret De Giorgi's example in the context of elasticity, we first notice that if one replaces $\mathbf{C}$ in (3.8) with

$$\mathbf{C} = \operatorname{sym} + c\mathbf{B} \otimes \mathbf{B}, \tag{4.1}$$

and, consequently, I in (3.10) with

$$I(\mathbf{u}; b, c) = \int_\Omega (|||\mathbf{u}|||^2 + c(\operatorname{div} \mathbf{u} + b\mathbf{P}\cdot\mathbf{E}(\mathbf{u}))^2), \tag{4.2}$$

all the developments of the previous section go through unaltered.[3] In particular, under the restrictions (3.17) $\hat{\mathbf{u}}$ is a variational solution of the problem

$$\operatorname{div} \mathbf{C}(\mathbf{E}(\mathbf{u})) = \mathbf{0} \quad \text{in} \quad B, \qquad \mathbf{u} - \mathbf{p} = \mathbf{0} \quad \text{in} \quad \partial B. \tag{4.3}$$

[3] In fact, replacing $\mathbf{1}$ with sym has no effect as $\operatorname{grad} \mathbf{u} = p^\alpha(\mathbf{1} + \alpha\mathbf{P}) \in \operatorname{Sym}(V)$.

We remark that $\mathbf{C}$ as given by (4.1) is endowed with the canonic symmetries of a response function in linear elasticity,[4] namely (see, e.g., Gurtin [5], Sec. 20)

(i) (minor symmetries) $\mathbf{C}(\mathbf{V}) = \mathbf{C}\,(\text{sym}(\mathbf{V})) \in \text{Sym}(V)$, $\forall\, \mathbf{V} \in \text{Lin}(V)$, so that $\mathbf{V}\cdot\mathbf{C}(\mathbf{W}) = \text{sym}\,(\mathbf{V})\cdot\mathbf{C}(\text{sym}\,(\mathbf{W}))$, $\forall\, \mathbf{V}, \mathbf{W} \in \text{Lin}(V)$.

(ii) (major symmetry) $\mathbf{V}\cdot\mathbf{C}(\mathbf{W}) = \mathbf{W}\cdot\mathbf{C}(\mathbf{V})$, $\forall\, \mathbf{V}, \mathbf{W} \in \text{Lin}(V)$.[5]

We now identify Ω with a reference configuration of a continuous body, and let $\mathbf{u}(\mathbf{x})$ denote the *displacement* of a point $\mathbf{x} \in \Omega$ from its place in the reference configuration to its place in some strained configuration; accordingly, $\mathbf{E}(\mathbf{u}(\mathbf{x}))$ denotes the usual "infinitesimal" measure of the *strain* at $\mathbf{x}$. We then let the *stress* $\mathbf{T}$ (modulo an inessential dimensional coefficient, bearing the dimensions of pressure) be delivered by the response function

$$\mathbf{T}(\mathbf{x}, \mathbf{u}) = \mathbf{C}(\mathbf{E}), \tag{4.4}$$[6]

and investigate the symmetry properties (at a fixed point $\mathbf{x} \neq \mathbf{0}$) of the special *linearly elastic material* defined by (4.4).

By definition (see, e.g., [5], Sec. 21), the set of symmetry mappings for the present material is the following subgroup $G_{\mathbf{x}}$ of the full orthogonal group Orth (V):

$$G_{\mathbf{x}} = \{\mathbf{Q} \in \text{Orth}\,(V) : \mathbf{Q}\mathbf{C}(\mathbf{E})\mathbf{Q}^T = \mathbf{C}(\mathbf{Q}\mathbf{E}\mathbf{Q}^T),\ \forall\, \mathbf{E} \in \text{Sym}\,(V)\}.$$

It is easy to show that a necessary and sufficient condition for $\mathbf{Q} \in G_{\mathbf{x}}$ is

$$\mathbf{QP} = \mathbf{PQ}. \tag{4.5}$$

But $\mathbf{Q}$ commutes with $\mathbf{P}$ if and only if $\mathbf{Q}$ leaves each of the characteristic spaces of $\mathbf{P}$ invariant (cf. Gurtin [5], Thm. (3.3)), and these spaces are the line spanned by $\mathbf{p}$ and the plane perpendicular to $\mathbf{p}$. Therefore, for any $\mathbf{x} \in \Omega - \{\mathbf{0}\}$,

$$G_{\mathbf{x}} = \{\mathbf{Q} \in \text{Orth}(V) : \mathbf{Qp} = \pm\mathbf{p}\}.$$

As $G_{\mathbf{x}}$ is a proper subgroup of Orth(V), the material is an *aelotropic solid.* Moreover, this material has *transverse isotropy* (with respect to the direction of $\mathbf{p}$), and is characterized by the *elastic moduli* b and c.

Remark 1. Only the modulus b is responsible for aelotropy. In fact, if one puts $b = 0$ in (4.1) C reduces to

$$\mathbf{C} = \text{sym} + c\mathbf{1} \otimes \mathbf{1},$$

the familiar response function of *isotropic* linearly elastic materials. Not surprisingly, the above choice of b is excluded under De Giorgi's hypotheses, namely, $c > 0$ and $(3.17)_{1,2}$.

We pass on now to formulate a suitable boundary-value problem for a monoparametric family of elastic bodies comprised of the material just described.

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[4] It is vacuous to pursue an interpretation in the domain of *finite* elasticity because the candidate response function $\mathbf{T}_R = \mathbf{C}$ (grad $\mathbf{u}$), where $\mathbf{T}_R$ is the Piola-Kirchhoff stress tensor, does not obey the principle of material frame-indifference.

[5] It is perhaps of some interest to record the aspect of properties (*i*) and (*ii*) when a Cartesian coordinate system is used. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be an orthonormal basis. Then the components of $\mathbf{C}$ with respect to this basis are $C_{ijkl} = \mathbf{e}_i \otimes \mathbf{e}_j \cdot \mathbf{C}(\mathbf{e}_k \otimes \mathbf{e}_l)$, and (*i*), (*ii*) read, respectively, $C_{ijkl} = C_{ijlk} = C_{jikl}$, $C_{ijkl} = C_{klij}$.

[6] Recall that in (4.4) an *explicit* dependence of $\mathbf{C}$ on $\mathbf{x}$ is understood. Notice also that $\mathbf{C}$ is *not* defined at the point $\mathbf{0}$. It follows from (4.4) that the reference configuration is stress-free.

Let $B_\epsilon = B(\mathbf{0}; \epsilon)$, and let $S_\epsilon = B - B_\epsilon$ denote the generic representative of an ϵ-family of *shells* which are elastic in the sense of (4.4). Further, let body forces vanish in S_ϵ, and let the displacement field be prescribed on ∂S_ϵ as follows:

$$\begin{aligned} \mathbf{u} &= \mathbf{0} \quad \text{on} \quad \partial B_\epsilon, \\ &= \mathbf{p} \quad \text{on} \quad \partial B. \end{aligned} \tag{4.6}$$

Then,

$$\operatorname{div} \mathbf{C}(\mathbf{E}) = \mathbf{0} \quad \text{in} \quad S_\epsilon \tag{4.7}$$

is the equation of equilibrium, and (4.6), (4.7) (cf. (4.3)) is an *equilibrium problem* (in terms of displacements) for the shell S_ϵ.

If one seeks centrally symmetric solutions of (4.7), i.e. solutions of the form

$$\mathbf{u}(\mathbf{x}) = f(p)\mathbf{p}, \tag{4.8}$$

it is a simple matter to show that f in (4.8) is to be the general solution of the well-known ordinary differential equation of Euler

$$f_{,pp} + (n+1)p^{-1}f_{,p} + \frac{bc(n-1)(n+b)}{1+c(1+b)^2}\,p^{-2}f = 0,$$

so that

$$f(p) = ap^{\alpha} + \bar{a}p^{\bar{\alpha}}, \tag{4.9}$$

where a, $\bar{a}$ are constants and α, $\bar{\alpha}$ are specified by (3.14).[7]

In view of (4.8) and (4.9), if we chose a, $\bar{a}$ to satisfy (4.6), the *elastic state* (cf. Gurtin [5], Sec. 28) on $\bar{S}_\epsilon$ at equilibrium is the triplet $\{\mathbf{u}_\epsilon, \mathbf{E}_\epsilon, \mathbf{T}_\epsilon\}$ where

$$\begin{aligned} \mathbf{u}_\epsilon &= \mathbf{u}(\epsilon; \mathbf{x}) = \frac{p^{\bar{\alpha}}}{1-\epsilon^{\alpha-\bar{\alpha}}}\,(p^{\alpha-\bar{\alpha}} - \epsilon^{\alpha-\bar{\alpha}})\mathbf{p}, \\ \mathbf{E}_\epsilon &= \mathbf{E}(\mathbf{u}_\epsilon) = \frac{p^{\bar{\alpha}}}{1-\epsilon^{\alpha-\bar{\alpha}}}\,((p^{\alpha-\bar{\alpha}} - \epsilon^{\alpha-\bar{\alpha}})\mathbf{1} + (\alpha p^{\alpha-\bar{\alpha}} - \bar{\alpha}\epsilon^{\alpha-\bar{\alpha}})\mathbf{P}), \\ \mathbf{T}_\epsilon &= \mathbf{T}(\mathbf{u}_\epsilon) = \frac{p^{\bar{\alpha}}}{1-\epsilon^{\alpha-\bar{\alpha}}}\,((1+c(n+b))(p^{\alpha-\bar{\alpha}} - \epsilon^{\alpha-\bar{\alpha}}) + c(1+b)(\alpha p^{\alpha-\bar{\alpha}} - \bar{\alpha}\epsilon^{\alpha-\bar{\alpha}}))\mathbf{1} \\ &\quad + \frac{p^{\bar{\alpha}}}{1-\epsilon^{\alpha-\bar{\alpha}}}\,(bc(n+b)(p^{\alpha-\bar{\alpha}} - \epsilon^{\alpha-\bar{\alpha}}) + (1+bc(1+b))(\alpha p^{\alpha-\bar{\alpha}} - \bar{\alpha}\epsilon^{\alpha-\bar{\alpha}}))\mathbf{P}. \end{aligned} \tag{4.10}$$

As to the former result, some comments are in order. Let first ϵ be fixed. We see that

(i) $\mathbf{u}(\epsilon, \cdot)$ is a C^∞ mapping on S_ϵ which satisfies the boundary conditions (4.6) on ∂S_ϵ. We may write this property as

$$\left((\mathbf{u}(\epsilon, \mathbf{p}) - \frac{p-\epsilon}{p(1-\epsilon)}\,\mathbf{p}\right) \in C^\infty(\epsilon, 1) \cap C_0(\epsilon, 1). \tag{4.11}$$

Up to this point of the present section we have had no need of restricting the choice of n, b and α. We now assume

$$n = 3, \qquad \alpha + 1 < 0^8$$

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[7] Notice that α, i.e. De Giorgi's exponent $(3.17)_3$, is greater than $\bar{\alpha}$ if c is positive.

[8] Cf. (3.16). Using the strict inequality conforms to the original prescription of De Giorgi [1].

(and, consequently, $b > (1 + c)/c$). We also denote by $\mathbf{F}_\epsilon$ the gradient of the mapping $\mathbf{x} \to \mathbf{x} + \mathbf{u}_\epsilon$, so that

$$\mathbf{F}_\epsilon = \mathbf{1} + \operatorname{grad} \mathbf{u}_\epsilon = \mathbf{1} + \mathbf{E}_\epsilon ,$$

and

$$\det \mathbf{F}_\epsilon = \left(1 + \frac{p^{\bar\alpha}}{1 - \epsilon^{\alpha - \bar\alpha}} ((1 + \alpha)p^{\alpha - \bar\alpha} - (1 + \bar\alpha)\epsilon^{\alpha - \bar\alpha})\right)\left(1 + \frac{p^{\bar\alpha}}{1 - \epsilon^{\alpha - \bar\alpha}} (p^{\alpha - \bar\alpha} - \epsilon^{\alpha - \bar\alpha})\right)^2.$$

It is not difficult to see that

$$\det \mathbf{F}_\epsilon > 0 \quad \text{in} \quad S_\epsilon ,$$

provided that $\epsilon > \epsilon_0$, where $\epsilon_0 \in (0, 1)$ and solves the equation

$$\frac{(\alpha - \bar\alpha)(1 + \bar\alpha)}{\alpha} \left(\frac{\bar\alpha(1 + \bar\alpha)}{\alpha(1 + \alpha)}\right)^{\bar\alpha/\alpha - \bar\alpha} \epsilon^\alpha + \epsilon^{\alpha - \bar\alpha} = 1.$$

Therefore, under the condition $\epsilon > \epsilon_0$, we also have the following consequences of (4.10):

(ii) The strained configuration is a diffeomorph of S_ϵ .

(iii) The elastic state has finite magnitude.

(iv) The strain energy

$$a_\epsilon(\mathbf{u}_\epsilon , \mathbf{u}_\epsilon) = \int_{S_\epsilon} (E_\epsilon^2 + c(\operatorname{div} \mathbf{u}_\epsilon + b\mathbf{P}\cdot\mathbf{E}_\epsilon)^2)$$

is finite. Moreover,

$$a_\epsilon(\mathbf{u}_\epsilon , \mathbf{u}_\epsilon) = a(\hat{\mathbf{u}}, \hat{\mathbf{u}}) + o(\epsilon). \tag{4.12}$$

Thus, we have solved a physically meaningful displacement problem in linear elastostatics for a family of spherical shells, inhomogeneous and with radially bundled structure.

If we further let $\mathbf{p} \neq \mathbf{0}$ be fixed, it can be read off at once from (4.10) that

(v) $$\lim_{\epsilon \to 0} \mathbf{u}(\epsilon, \mathbf{p}) = \hat{\mathbf{u}}(\mathbf{p}).$$

We now extend $\mathbf{u}_\epsilon$ to $\bar{B}$ as follows:

$$\begin{aligned} \mathbf{u}_\epsilon &= \mathbf{0} && \text{for} \quad p < \epsilon \\ &= \mathbf{u}(\epsilon, \mathbf{p}) && \text{for} \quad \epsilon \le p \le 1. \end{aligned}$$

Clearly, $\mathbf{u}_\epsilon \in H^1$. From (4.12) we see that

(vi) the sequence $\{\mathbf{u}_\epsilon\}$ converges to $\hat{\mathbf{u}}$ in energy, or, equivalently, in H^1. Alternatively, we may write this as a statement of convergence concerning the sequence of triplets:

$$\{\mathbf{u}_\epsilon , \mathbf{E}_\epsilon , \mathbf{T}_\epsilon\} \to \{\hat{\mathbf{u}}, \hat{\mathbf{E}}, \hat{\mathbf{T}}\} \quad \text{in} \quad L^2.$$

(Here, of course, $\hat{\mathbf{E}} = \mathbf{E}(\hat{\mathbf{u}})$, $\hat{\mathbf{T}} = \mathbf{T}(\hat{\mathbf{u}})$.)

Thus, we conclude that (4.3) is the limit problem of the sequence (4.6), (4.7). However, we are not allowed to term $\{\hat{\mathbf{u}}_\epsilon , \hat{\mathbf{E}}_\epsilon , \hat{\mathbf{T}}_\epsilon\}$ an elastic state, whenever $\epsilon \le \epsilon_0$. In particular, by direct inspection of $\{\hat{\mathbf{u}}, \hat{\mathbf{E}}, \hat{\mathbf{T}}\}$ it becomes clear that this is *not* a plausible elastic state at equilibrium on a ball B comprised of the elastic material (4.4), or, in other words, that (4.3) *cannot* be interpreted as a displacement problem for B.

In fact, not only $\{\hat{\mathbf{u}}, \hat{\mathbf{E}}, \hat{\mathbf{T}}\}$ is discontinuous and unbounded at the origin, but also

$$\det \hat{\mathbf{F}} = 0 \quad \text{at} \quad p_* = \left(\frac{-1}{1+\alpha}\right)^{1/\alpha}.$$

Therefore, B is mapped into the *infinite* domain exterior to the sphere of radius $(p_* + \hat{u}(p_*))$, the inverse mapping is not single-valued in the annular region defined by $p \in (p_* + \hat{u}(p_*), 2)$, and the physical *axiom of impenetrability of matter* is violated.[9]

We close the paper with two further remarks illustrating the role of the modulus of aelotropy b.

Remark 2. For any $n \leq 3$, if $b = 0$ (so that $\alpha = 0, \bar{\alpha} = -n$), the sequence of elastic states on $\bar{S}_\epsilon$ converges to $\{\mathbf{p}, \mathbf{1}, (1 + nc)\mathbf{1}\}$, i.e. the elastic state on $\bar{B}$ at equilibrium (cf. Remark 1).

Remark 3. If $b = (1 + c)/c$ (so that $\alpha = -1, \bar{\alpha} = -2$), the sequence of the elastic states on the spherical shells S_ϵ converges to $\{p^{-1}\mathbf{p}, p^{-1}(\mathbf{1} - \mathbf{P}), (1 + 2c)p^{-1}(\mathbf{1} + \mathbf{P})\}$. Again this triplet is unacceptable as an elastic state on the ball B at equilibrium, although $\hat{\mathbf{u}}$ is now bounded and $\det \hat{\mathbf{F}} > 0$ on $\bar{B}$, due e.g. to the unbounded increase of $\hat{\mathbf{E}}$ and $\hat{\mathbf{T}}$ near the origin.

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Appendix. Here we collect two remarks which enlarge the scope of De Giorgi's counterexample and bring to light its curious adaptability. It seems to us that this latter feature was exploited, but scantly acknowledged, in the past (see, e.g. [2, 3]).

Remark 1. The developments of Secs. 3 and 4 show that

$$\hat{\mathbf{u}} = p^\alpha \mathbf{p} \tag{A.1}$$

is an extremal of integrals of both class (3.10) and (4.2), associated with *linear variational systems* of elliptic type. Let us introduce a scalar function $g(p, u)$ defined on $\mathbf{R}^+ \times \mathbf{R}^+$, and such that

$$g(p, \hat{u}) = \hat{u}^{-2}. \tag{A.2}$$

If we replace $\mathbf{P}(\mathbf{x})$ in (3.9) with

$$\mathbf{Q}(\mathbf{x}, \mathbf{u}) = g(p, u)\mathbf{u} \otimes \mathbf{u}, \tag{A.3}$$

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[9] We realize that this axiom is often violated in the *linear* theory and is, in fact, not a requirement of that theory.

we see that, *ceteris paribus*, $\hat{\mathbf{u}}$ solves the resulting class of *quasilinear* problems. In fact, (A.1) implies

$$\hat{u}^{-1}\hat{\mathbf{u}} = p^{-1}\mathbf{p},$$

so that, by (A.2) and (A.3),

$$\mathbf{Q}(\mathbf{x}, \hat{\mathbf{u}}) = \mathbf{P}(\mathbf{x}).$$

The simplest choice of g is of course

$$g(p, u) = u^{-2}.$$

As (A.1) implies further

$$\hat{u} = p^{\alpha+1},$$

another possible choice is

$$g(p, u) = \frac{2p^{-2(\alpha+1)}}{1 + p^{-2(\alpha+1)}u^2},$$

leading to a non-linear system considered by Nečas and Stará [3] when one takes De Giorgi's parameters $b = n/(n-2)$, $c = (n-2)^2$, and to the case minutely analyzed by Giusti and Miranda [2] when one takes, as they do, $b = 2/(n-2)$, $c = 1$ (so that, in particular, $\alpha = -1$).

Remark 2. As was already noticed by De Giorgi [1],

$$\hat{\mathbf{u}} = \frac{1}{\alpha + 2}\,\mathrm{grad}\,\hat{h},$$

where $\hat{h}(\mathbf{x}) = p^{\alpha+2}$. It follows that $\hat{h}$ (for $n \geq 5$) is a non-regular weak solution of the *linear fourth-order equation* associated with the integral

$$I(h; b, c) = \int_\Omega \mathbf{C}(\mathrm{grad}\,\mathrm{grad}\,h)\cdot\mathrm{grad}\,\mathrm{grad}\,h.$$

Quite naturally, this result may be combined with the observation leading to (A.3) in the previous Remark to show that $\hat{h}$ solves the corresponding class of *quasilinear* equations (cf. [2]).