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De-pinning of disordered bosonic chains

N Vogt[1,2], J H Cole[2] and A Shnirman[1,3]

[1] Institut für Theorie der Kondensierten Materie, Karlsruhe Institute of Technology, D-76128 Karlsruhe, Germany
[2] Chemical and Quantum Physics, School of Applied Sciences, RMIT University, Melbourne, 3001, Australia
[3] L. D. Landau Institute for Theoretical Physics RAS, Kosygina street 2, 119334 Moscow, Russia

E-mail: Nicolas.Vogt@rmit.edu.au

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Abstract

We consider onset of transport (de-pinning) in one-dimensional bosonic chains with a repulsive boson–boson interaction that decays exponentially on large length-scales. Our study is relevant for (i) de-pinning of Cooper-pairs in Josephson junction arrays; (ii) de-pinning of magnetic flux quanta in quantum-phase-slip ladders, i.e. arrays of superconducting wires in a ladder-configuration that allow for the coherent tunneling of flux quanta. In the low-frequency, long wave-length regime these chains can be mapped onto an effective model of a one-dimensional elastic field in a disordered potential. The standard de-pinning theories address infinitely long systems in two limiting cases: (a) of uncorrelated disorder (zero correlation length); (b) of long range power-law correlated disorder (infinite correlation length). In this paper we study numerically chains of finite length in the intermediate case of long but finite disorder correlation length. This regime is of relevance for, e.g., the experimental systems mentioned above. We study the interplay of three length scales: the system length, the interaction range, the correlation length of disorder. In particular, we observe the crossover between the solitonic onset of transport in arrays shorter than the disorder correlation length to onset of transport by de-pinning for longer arrays.

1. Introduction

Depinning theory describes the onset of propagation in many different physical systems. Examples include electrical transport in charge density waves (CDW) [1, 2], the critical current of magnetic flux lattices [3, 4] in type II superconductors, the propagation of magnetic domain boundaries [5] and crack formation in strained materials [6]. It was recently shown that the onset of electrical transport in one-dimensional arrays of Josephson junctions is also determined by the de-pinning of the charge-configuration along the array [7].

In this paper we consider a more general model, a discrete chain macroscopically occupied by bosons with a repulsive interaction that decays exponentially on long length-scales. In such a model the interaction between neighboring islands can be expressed by introducing a continuous variable, quasi-charge/flux, whose value is determined by the distribution of bosons along the chain. Assuming that the continuous variable changes adiabatically, an effective model can be derived with the help of the Born–Oppenheimer approximation [7, 8]. In the case that disorder is present in the system, de-pinning theory can be applied to find the critical driving force that leads to a steady boson transport through the system. The Josephson junction arrays studied in [7] represent a particular realization of this model. Alternatively, our results describe the dual system of quantum phase slip (QPS) ladders. In the latter case (QPS ladders) the bosons are magnetic flux quanta that tunnel through QPS elements [9–11] that separate the loops in a ladder.

In voltage biased Josephson junction arrays, the de-pinning theory describes the transition from an insulating regime at low voltages to a transport regime at higher voltages. The critical voltage of the transition is referred to as the switching voltage. The insulating regime of the arrays is governed by an effective sine-Gordon-like quasi-charge model.

In the study of the onset of transport in Josephson junction arrays, [7], the connection to standard de-pinning theory was established under the assumption of a very particular capacitance matrix consisting to the junction capacitances C and the capacitances to the ground C_0. Here we generalize that connection to the case of a general matrix of capacitances provided it is characterized by a finite screening length. Moreover, [7] addressed the regime of strongly disordered background charges (also referred to as the maximal disorder model). Under this assumption the disorder-term in the effective model is spatially uncorrelated, allowing one to apply the standard de-pinning theory. Additionally the mapping to the standard de-pinning theory assumes the array length N to be much larger than all other length-scales of the problem.

In this paper we study the de-pinning behaviour of more general chain models. In the case of short chains we find a saturation regime, in which the Larkin length controlled by the range of repulsive interaction exceeds the length of the system. In this regime the array behaves essentially as a lumped zero-dimensional element. Another case is that of weak charge (flux) disorder, which corresponds to a long disorder correlation length in the effective sine-Gordon-like description. In this case we observe a crossover between a regime, in which the onset of transport is due to injection of ballistic solitons and a regime of pinning weakened by the correlations. The de-pinning process in systems with infinite range (power law) disorder correlations has been studied in [12].

2. Methods

2.1. The model

We start by introducing the generalized model that contains the recently studied model for Josephson junction arrays [7] as a special case. We consider a chain of islands

$$H = \sum_{i,j} \frac{1}{2} \tilde{n}_i M_{i,j} \tilde{n}_j - \sum_i t\,(b_{i+1}^\dagger b_i \ + \text{h.c.}) \tag{1}$$

$$\tilde{n}_i = N_i - N_0 - f_i, \tag{2}$$

where N_i are the discrete bosonic occupation numbers of the islands, N_0 is the average occupation number at equilibrium (chemical potential, positive charge background), f_i are the random gate charges, b_i, $b_i^\dagger$ are the bosonic annihilation and creation operators ($N_i = b_i^\dagger b_i$) and the bosonic tunnelling amplitude is given by t. In the limit $N_0 \gg 1$ we can introduce the phase variables φ_i and replace $b_i \approx \sqrt{N_0}\, e^{-i\varphi_i}$, $b_i^\dagger \approx \sqrt{N_0}\, e^{i\varphi_i}$, which leads to

$$H = \sum_{i,j} \frac{1}{2} \tilde{n}_i M_{i,j} \tilde{n}_j - \sum_i E_t \cos(\varphi_{i-1} - \varphi_i), \tag{3}$$

where $E_t \equiv N_0 t/2$. For convenience we also introduce $n_i \equiv N_i - N_0$, so that $\tilde{n}_i = n_i - f_i$ and n_i and φ_i are conjugate variables.

We assume that the long-range behaviour of the interaction matrix $M_{i,j} = M_{|i-j|}$ is determined by an exponential decay on a length-scale Λ

$$M_{i,j} \propto e^{-\frac{|i-j|}{\Lambda}} \qquad \text{for} \quad |i-j| \geqslant \Lambda \tag{4}$$

This general class of models contains Josephson junction arrays, chains of superconducting island coupled via Josephson junctions and with self-capacitances C_0 (capacitances to the ground) and capacitances C between the neighboring islands (junction capacitances). In this case $E_t = E_J$ is the Josephson energy whereas the coupling matrix is given by

$$M_{i,j} = (2e)^2 [\hat{C}^{-1}]_{i,j}, \tag{5}$$

where

$$\hat{C} = \begin{pmatrix} -C & & & & \\ -C & C_0 + 2C & -C & & \\ & -C & C_0 + 2C & -C & \\ & & \vdots & \vdots & \end{pmatrix}. \tag{6}$$

In the large Λ-regime the interaction matrix is well approximated by the following expression:

$$M_{|i-j|} \approx \Lambda E_C e^{-\frac{|i-j|}{\Lambda}}, \tag{7}$$

where the junction charging energy is defined by $E_C \equiv (2e)^2/2C$. In particular the energy cost of a single charge in such an array is of the order $M_{j,j} \approx \Lambda E_C$. Activated behaviour with activation energy of order ΛE_C was observed in [13].

It has been realized long ago [14, 15] that for $\Lambda \gg 1$ a description based on the number of Bosons that have tunnelled between neighbouring chain sites is more appropriate than a description based on the occupation

number of the islands of the chain. We introduce, thus

$$m_i = \sum_{j=1}^{i-1} n_j, \qquad F_i = \sum_{j=1}^{i-1} f_j. \tag{8}$$

The resulting Hamiltonian has the form,

$$H = \frac{1}{2}\sum_{i,j}(m_i - F_i)\,U_{i,j}\,(m_j - F_j) - \sum_i E_t \cos(\theta_i), \tag{9}$$

with the modified coupling matrix

$$U_{i,j} = U_{|i-j|} = 2M_{|i-j|} - M_{|i-j|-1} - M_{|i-j|+1}. \tag{10}$$

Here $\theta_i \equiv \varphi_{i-1} - \varphi_i$. One can easily check that m_i and θ_i are conjugate variables.

A qualitative picture in the low energy, long wave length regime can be obtained from the Fourier transform of the coupling matrices

$$U(k) = \mathcal{FT}(U_{i-j}) = 2(1 - \cos(k))M(k) \tag{11}$$

$$M(k) \equiv \mathcal{FT}(M_{i-j}), \tag{12}$$

where $k \in [-\pi, \pi]$. The $k \to 0$ behaviour of $M_{|i-j|}$ is dominated by the exponential decay and for small $k \ll \Lambda^{-1}$ the Fourier transform of the interaction matrix $M_{|i-j|}$ is approximately constant, $M(k) \approx M(0)$, which leads to

$$U(k) \approx M(0)k^2. \tag{13}$$

In the case of a Josephson junction array the exact expressions are

$$M(k) = \Lambda E_C \frac{\frac{2}{\Lambda}}{\frac{1}{\Lambda^2} + 2(1 - \cos(k))}, \tag{14}$$

$$M(0) = 2\Lambda^2 E_C. \tag{15}$$

2.2. Standard villain transformation

The model (9) can be treated by a standard technique involving the Villain approximation [15]. We omit all the details and only mention the fact that the spin wave part of the resulting model in the limit $k \ll \Lambda^{-1}$, where $U(k) \approx M(0)k^2$ is quadratic, corresponds to a Luttinger liquid [16]

$$H = \frac{v}{2}\int \mathrm{d}x \left[K\,[\theta(x)]^2 + \frac{1}{K}[\partial_x m(x)]^2\right], \tag{16}$$

$$[\theta(x_1),\, m(x_2)] = \mathrm{i}\delta(x_1 - x_2), \tag{17}$$

where the Luttinger liquid velocity v and the Luttinger liquid parameter K scale with the original model parameters as

$$v = \sqrt{M(0)E_t}, \tag{18}$$

$$K = \sqrt{\frac{E_t}{M(0)}}. \tag{19}$$

The corrections to (17) due to vortices (phase slips) in this type of theories (see, e.g., [17]) are of the form $\propto\cos(2\pi p\,[m(x) + F(x)])$ with $p \in \mathbb{Z}$. The amplitude in front of this term (fugacity of vortices) is predicted to be small in the limit $\Lambda \gg 1$ [14, 15] so that without disorder the critical value of K is close to $2/\pi$ (it may be renormalized by disorder [16]).

We assume that $M(0) \gg E_t$ and therefore $K \ll 2/\pi$. In this case the system is firmly in the CDW-regime [16] and dominated by classical charge dynamics. The disorder $F(x)$ enters the relevant phase slip terms $\propto\cos(2\pi\,[m(x) + F(x)])$ and can pin the charge density profile.

2.3. Alternative derivation

Here we generalize the derivation given in [8] for the case of a Josephson array described by the capacitance matrix (6) to the case of a more general matrix $M_{i,j}$ characterized by a screening length Λ. We start by rewriting the Hamiltonian (9) as

$$H = \frac{1}{2}\sum_j U_0 (m_j - F_j)^2 - \sum_j E_t \cos(\theta_j) + \frac{1}{2}\sum_{i,j}(m_i - F_i)\delta U_{i,j}(m_j - F_j), \tag{20}$$

where $U_0 = U_{j,j}$ and $\delta U_{i,j} = U_{i,j} - U_0\delta_{i,j}$. Next we transform the third term with the help of the Hubbard–Stratonovich transformation, which introduces a new degree of freedom Q_i, often referred to as the quasi-charge. This gives

$$H\{Q\} = \frac{1}{2}\sum_j U_0(m_j - F_j)^2 - \sum_j E_t\cos(\theta_j) - U_0\sum_j Q_j(m_j - F_j) - \frac{U_0^2}{2}\sum_{i,j} Q_i[\delta U^{-1}]_{ij}Q_j, \tag{21}$$

such that $H = \min_Q[H\{Q\}]$. Next

$$H\{Q\} = \frac{1}{2}\sum_j U_0(m_j - F_j - Q_j)^2 - \sum_j E_t\cos(\theta_j) + \frac{1}{2}\sum_{i,j} Q_i B_{i,j}Q_j, \tag{22}$$

where $B_{i,j} = -U_0^2[\delta U^{-1}]_{ij} - U_0\delta_{i,j}$. The Fourier transform reads

$$B(k) = -U_0 - \frac{U_0^2}{U(k) - U_0}. \tag{23}$$

For small wave vectors $k \ll \Lambda^{-1}$ we obtain $U(k) \ll U_0$ and $B(k) \approx U(k) \approx M(0)k^2$. Thus, assuming Q_i changes slowly enough as a function of the coordinate i, i.e., changes on length scales longer than Λ, we can approximate

$$H\{Q\} \approx \frac{1}{2}\sum_j U_0(m_j - F_j - Q_j)^2 - \sum_j E_t\cos(\theta_j) + \frac{M(0)}{2}\sum_i (Q_i - Q_{i+1})^2. \tag{24}$$

For the Josephson arrays with the capacitance matrix (6) we obtain $U_0 = 2E_C$ and $M(0) = 2\Lambda^2 E_C = 2E_{C0}$, where $E_{C0} \equiv (2e^2)/2C = \Lambda^2 E_C$. In this case the form of the third term of (24) is exact [8].

The adiabatic dynamics of the model (24) without disorder was analyzed in [8]. The inclusion of disorder is straightforward. The aim is to integrate out the degrees of freedom (m_i, θ_i). For a given (adiabatic) trajectory $Q_i(t)$ the dynamics factorizes to independent dynamics of single junctions governed by the Hamiltonians

$$H_i(Q_i) = \frac{1}{2}U_0(m_i - F_i - Q_i)^2 - E_t\cos(\theta_i). \tag{25}$$

The Born–Oppenheimer periodic potential is given by the ground state of the well known Hamiltonian (25), $E_Q(Q_i + F_i)$, where $Q_i + F_i$ serves here as the total quasi-charge. The function $E_Q(Q)$ is periodic with period 1 as can be seen from (25). In the limit $E_t \gg U_0$ ($E_J \gg E_C$) it is given by $E_Q(Q) = E_S\cos(Q)$, where E_S is the quantum phase slip amplitude [18].

Thus we obtain the effective potential energy of the whole array of the form

$$\begin{aligned} U_C &= \frac{1}{2}\sum_{i,j} Q_i B_{i,j}Q_j + \sum_j E_Q(Q_i + F_i) \\ &\approx \frac{1}{2}\sum_i M(0)(Q_i - Q_{i+1})^2 + \sum_i E_Q(Q_i + F_i) \\ &= \sum_i E_{C0}(Q_i - Q_{i+1})^2 + \sum_i E_Q(Q_i + F_i). \end{aligned} \tag{26}$$

This potential is supplemented by the kinetic energy. In the limit $E_t \gg U_0$ ($E_J \gg E_C$) it reads $T = (1/2)\sum_i L\dot{Q}_i^2$, where the L is the Josephson inductance $L \approx L_J = 1/E_t = 1/E_J$. The quadratic part of the Lagrangian $T - U_C$ gives again the Luttinger liquid with the parameters (18). Since we assume $K \ll 2/\pi$, the periodic potential $E_Q(Q_i + F_i)$ is relevant and pins the density profile. In what follows we investigate the charge pinning in this setup.

2.4. Edge bias

After generalizing the Josephson junction array model to generic bosonic chains with an exponentially decaying interaction we now turn to the effects of finite size and finite correlation length of disorder. While the effects we study from here on should appear in a similar way in the generic bosonic chains we limit ourselves to the specific form of interaction M_{ij} that is found in Josephson junction arrays and QPS ladders, i.e. the last line of equation (26). Consequently we employ the terminology of Josephson junction arrays and put $2e = 1$. As we are primarily interested in the transport properties of the chains, we introduce a bias V at the edge:

$$U_C = \sum_i \frac{(Q_i - Q_{i+1})^2}{2C_0} + E_Q(Q_i + F_i) - \frac{V}{C}Q_1. \tag{27}$$

Biasing at the edge is somewhat unusual in the traditional de-pinning models [3], it is however the physical reality in many artificially fabricated mesoscopic systems. It is for example not experimentally feasible to apply a homogeneous external electric field to Josephson junction arrays. Instead these arrays always have to be voltage biased at the edges in real experiments.

To simplify the treatment in terms of the de-pinning theory we transform the system from a boundary biased situation to a spatially homogeneous driving by introducing a parabolic shift in Q and F

$$\tilde{Q}_i = Q_i - \frac{C_0}{C} V \frac{(N+1-i)(N-i)}{2N}, \tag{28}$$

$$\tilde{F}_i = F_i + \frac{C_0}{C} V \frac{(N+1-i)(N-i)}{2N} \tag{29}$$

and the corresponding potential part of the Hamiltonian with a driving force V/N

$$U_C = \sum_i \frac{(\tilde{Q}_i - \tilde{Q}_{i+1})^2}{2C_0} + E_Q(\tilde{Q}_i + \tilde{F}_i) - \frac{V}{N}\frac{\tilde{Q}_i}{C}. \tag{30}$$

Here we dropped a constant shift in the potential energy that does not depend on any of the variables Q_i. This energy shift has no effect on the equations of motion of the system and can therefore be safely ignored at this point.

In this formulation the problem corresponds to the discrete version of the well known de-pinning problem in one-dimension [2]. The elastic energy of the field Q_i is determined by the elastic constant C_0. The elastic field is pinned by the random pinning potential $E_Q(\tilde{Q}_i + \tilde{F}_i)$ and driven by the homogeneous driving force V/N. In the pinned regime the applied force V/N is not strong enough to overcome the potential barrier imposed on the elastic object Q_i by the random pinning potential.

In the case that no driving force is applied, $V = 0$, the form of the elastic object is determined by a competition between the elastic term $(Q_i - Q_{i+1})^2$ and the pinning term $E_Q(Q_i + F_i)$ in U_C. On small length-scales, where the elastic energy term dominates, Q_i is approximately constant. The field Q_i changes on large length-scales where the pinning potential dominates. The crossover between the two regimes happens at the length scale L_p, which was first determined by Larkin for a flux line lattice in type II superconductors [4]. The length L_p goes by many names depending on the physical systems that are pinned. In type II superconductors it is called Larkin length, in ferromagnets with domain boundaries Imry–Ma length [5] and for CDW it is called Fukuyama–Lee length [1]. In this work we use the term Larkin length.

Once the driving force V/N exceeds a critical force V_{cr}/N, the pinning potential is overcome and the elastic object starts to move through the disordered medium. An intuitive argument to find the value of the critical driving force can be found by comparing the driving force to the pinning force at the Larkin length [4]. The distribution of Q is rigid on length-scales up to the Larkin length. The elastic object can only start to move when the driving force exceeds the collective pinning force on a segment with length $L = L_p$.

2.5. Numerics

Besides the analytical considerations pertaining to the onset of transport we also numerically simulate the time evolution of the systems we are interested in. We use the numerical simulations to confirm the analytical results of the paper. We focus on numerically determining the point of the de-pinning transition and continue to use a model with a Josephson array type interaction matrix M_{ij}.

We obtain the critical driving force V_{cr} by numerically solving the equations of motion of the field Q_i that can be obtained from the Hamiltonian (30)

$$\mathcal{M}\ddot{Q}_i + \frac{2Q_i - Q_{i-1} - Q_{i+1}}{C_0} + \alpha_R \dot{Q}_i + V_Q(Q_i + F_i) = 0, \tag{31}$$

$$\mathcal{M}\ddot{Q}_1 + \frac{Q_1 - Q_2}{C_0} + \alpha_R \dot{Q}_2 + V_Q(Q_1) = \frac{V}{C}, \tag{32}$$

$$\mathcal{M}\ddot{Q}_N + \frac{Q_N - Q_{N-1}}{C_0} + \alpha_R \dot{Q}_N + V_Q(Q_N + F_N) = 0. \tag{33}$$

The function V_Q is the force exerted by the pinning potential given by

$$V_Q(Q) \equiv \partial_Q E_Q(Q). \tag{34}$$

To guarantee numerical convergence we have introduced a mass $\mathcal{M}$ and a linear dissipative term with a dissipation constant α_R. Similar numerically simulations of the switching voltage in arrays of normal tunnel contacts have been conducted in [19].

The critical driving force V_{cr} is determined by adiabatically applying the boundary force V and determining whether a stable solution for the field Q_i can be found. Although V is increased slowly, the switch-on time of the driving force V in the numerical simulation is finite. The phenomenological dissipative term has to be included to compensate the small transport velocity $\dot{Q}_i$ introduced by the switch-on of V. The introduction of a phenomenological term is also a standard tool in the derivation of the de-pinning force V_{cr} in renormalization-group-treatments of pinned systems [2].

The mass $\mathcal{M}$ and the dissipation parameter α_R both affect the dynamical properties of the system, however they have no influence on the breakdown of the static state. In the example of a Josephson junction array, the mass $\mathcal{M}$ corresponds to an inductance and α_Q corresponds to an Ohmic resistance. In a quantum phase slip ladder $\mathcal{M}$ corresponds to a capacitance. In all simulations we choose the tunnelling amplitude and the coupling energy to be equal, $E_J = E_C$, so that the potential E_Q is close to a cosine potential. The length of the chain N and the parameter Λ are varied.

3. Results

3.1. Strong disorder

We first consider the strongly disordered model for which the results of the standard de-pinning theory [1, 2] are directly applicable. To make the connection to these results the difference between the original disorder (f_i) and the effective quasi-disorder before (F_i) and after ($\tilde{F}_i$) the parabolic shift in the quasicharge is important. In terms of the original disorder the strongly disordered model is defined by

$$f_i \in [-1/2,\, 1/2], \tag{35}$$

$$p(f_i) = \Theta_H\left(\frac{1}{2} - |f_i|\right), \tag{36}$$

where $p(f_i)$ is the probability distribution of the disorder f_i and Θ_H is the Heaviside Θ-function. This model corresponds to the strongest possible disorder in the considered chain model. A frustration f_i with an absolute value larger than $1/2$ is compensated by placing an additional (anti)-boson on the ith island of the chain. The disorder is bounded by $\pm 1/2$ and a box-distribution of disorder-charges inside these boundaries is the maximal disorder. While f_i itself is not spatially correlated, in the effective quasi-charge model, the quasi-disorder $\tilde{F}_i$ is correlated between different islands i and j

$$\langle \tilde{F}_i \tilde{F}_j \rangle_{dis} \neq 0 \quad \text{for} \quad i \neq j. \tag{37}$$

At first this seems to deviate from the normal situation in de-pinning theory where the disorder in the system is not spatially correlated [2]. However, in the de-pinning theory, only correlations in the pinning potential are relevant to the behaviour of the system. The potential E_Q is a function of the quasi-charge with a periodicity of 1. Since the background charges f_i are homogeneously distributed in an interval that corresponds to the periodicity of the potential, the offset $\tilde{F}_i$ can be absorbed into another uncorrelated and flatly distributed disorder variable f_i^b

$$\tilde{F}_i = F_{i-1} + \frac{C_0}{C} V \frac{(N+1-i)(N-i)}{2N} + f_i \to f_i^b, \tag{38}$$

$$E_Q(\tilde{Q}_i + \tilde{F}_i) \to E_Q(\tilde{Q}_i + f_i^b), \tag{39}$$

$$f_i^b \in \left[-\frac{1}{2}, \frac{1}{2}\right], \tag{40}$$

$$p(f_i^b) = \Theta_H\left(\frac{1}{2} - |\tilde{f}_i|\right). \tag{41}$$

From the point of view of the potential E_Q, the quasi-disorder $\tilde{F}_i$ is equivalent to a spatially uncorrelated disorder field f_i^b in the maximally disordered model.

Another way to determine whether spatial correlations in $\tilde{F}_i$ affect the quasi-charge model, is to calculate the disorder-averaged correlation function of the pinning potential:

$$\langle E_Q(\tilde{Q}_1 + \tilde{F}_i) E_Q(\tilde{Q}_2 + \tilde{F}_j) \rangle_{dis} = R(Q_2 - Q_1)\delta_{i,j}, \tag{42}$$

where the correlation function $R(Q)$ is given by

$$R(Q) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{dF}\; E_Q(Q+F) E_Q(F). \tag{43}$$

Since the correlator of the pinning potential is proportional to a Kronecker delta, the pinning potential is not spatially correlated.

We have now seen that for the maximal disorder model we arrive at an effective model that conforms with the standard assumptions of de-pinning theory. In this case the Larkin length and the critical driving force are well known (see for example [1]).

The approximate value of the Larkin length in one-dimensional systems is given by [1]

$$L_{\mathrm{p}} = 3^{-\frac{2}{3}}\Lambda^{\frac{4}{3}}\left[\tilde{R}\left(\frac{E_{\mathrm{J}}}{E_{\mathrm{C}}}\right)\right]^{-\frac{2}{3}}. \tag{44}$$

The relevant parameters of the chain are the energy E_{C}, the tunnelling amplitude E_{J}, the chain length N and Λ. To express the Larkin length in terms of these parameters we have defined the function $\tilde{R}$

$$\tilde{R}\left(\frac{E_{\mathrm{J}}}{E_{\mathrm{C}}}\right) = \frac{(E_Q^{\max})^2}{16E_{\mathrm{C}}^2}, \tag{45}$$

where $E_Q^{\max}$ is the amplitude of the random pinning potential $E_Q(\tilde{Q}_i + \tilde{F}_i)$. The function $\tilde{R}$ is a function of the dimensionless ratio of the tunnelling matrix element and E_{C}. The function needs to be determined numerically only once for all possible values of chain length and C_0.

Similarly the de-pinning force can be expressed in terms of $\tilde{R}$ and is given by [2]

$$V_{\mathrm{cr}} \approx N\frac{1}{C_0}l\frac{1}{L_{\mathrm{p}}^2} \tag{46}$$

$$= N3^{\frac{4}{3}}\,\Lambda^{-\frac{2}{3}}\left\{\tilde{R}\left(\frac{E_{\mathrm{J}}}{E_{\mathrm{C}}}\right)\right\}^{\frac{2}{3}}. \tag{47}$$

Further corrections to this intuitive approach can be obtained from renormalization-group-approaches [2, 20, 21]. We use the approximation equation (46) in this work.

In most systems, where de-pinning theory is applicable, the system size is much larger than the Larkin length and it is a good approximation to assume infinite system size. We now turn our attention specifically to short finite chains. From equation (47) we see that the critical driving force decreases with increasing Λ. At the same time the Larkin length increases

$$L_{\mathrm{p}} \propto \Lambda^{\frac{4}{3}}. \tag{48}$$

In finite chains the Larkin length becomes equal to the system size N when Λ reaches the value,

$$\Lambda_N = N^{\frac{3}{4}}3\left\{\tilde{R}\left(\frac{E_{\mathrm{J}}}{E_{\mathrm{C}}}\right)\right\}^{\frac{1}{4}}. \tag{49}$$

Increasing Λ further while keeping E_{C} constant only increases the elastic energy E_{C0} coupling neighbouring islands. The Larkin length, the length-scale on which Q is approximately constant, should increase. However in this limit the Larkin length is equal to the system size and therefore the field Q is approximately constant along the whole array.

For $\Lambda \gg \Lambda_N$ the critical driving force is independent of the interaction length as long as E_{C} is kept constant. A lower boundary for V_{cr} is approximately given by

$$V_{\mathrm{cr}} \approx \sqrt{N}3^{-\frac{1}{2}}\left\{\tilde{R}\left(\frac{E_{\mathrm{J}}}{E_{\mathrm{C}}}\right)\right\}^{\frac{1}{2}}, \tag{50}$$

which is the critical driving V_{cr} one finds for $\Lambda = \Lambda_N$. In reality V_{cr} saturates for smaller Λ, when N is of the same order of magnitude as L_{p} (for comparison see the numerical simulations in section 3.2). This leaves the principal behaviour of equation (50) unchanged and contributes a prefactor of order one in the expression for the critical driving force.

3.2. Weak disorder

In the weak disorder case the distribution of the bare disorder f_i is not flat in the interval $[-1/2,\ 1/2]$, instead the probability distribution either decreases for larger absolute values of f_i or is cut-off at an absolute value smaller than $1/2$.

We consider two models of weak disorder: (i) the weak box disorder

$$f_i \in \left[-\frac{\gamma}{2}, \frac{\gamma}{2}\right], \tag{51}$$

$$p(f_i) = \frac{1}{\gamma}\Theta_{\mathrm{H}}\left(\frac{\gamma}{2} - |f_i|\right), \tag{52}$$

with the disorder strength $\gamma < 1$; (ii) Gaussian disorder

$$p(f_i) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{f_i^2}{\sigma^2}}, \tag{53}$$

with a standard deviation $\sigma < 1/2$.

Weak disorder is particularly relevant for a ladder of QPS junctions [10, 11]. In such a system, superconducting wires are arranged in a ladder configuration, such that a one-dimensional chain of superconducting loops is formed. The superconducting wires that are shared by neighbouring loops contain a very thin section that forms the QPS-junction. Magnetic flux quanta in the loops assume the role of the bosonic particles. The QPS-junctions between the superconducting loops provide the hopping matrix element and the coupling matrix $M_{i,j}$ is the inverse inductance matrix of the system. In a ladder configuration of superconducting wires the inductance matrix has the exactly the previously mentioned tridiagonal form equation (6), as long as the kinetic inductance dominates over the geometric inductance. Due to the lack of large magnetic dipoles in the vicinity of such a system, a weak disorder limit is more likely to be realized than in Josephson junction arrays.

In the weak disorder models the spatial correlation in the quasi-disorder $\tilde{F}_{i+1}$ can not be neglected and does lead to a finite spatial correlation in the pinning potential. The maximal value of the disorder f_i is smaller than the periodicity of the potential E_Q and the long range correlation in the quasi-disorder $\tilde{F}_i$ can not be absorbed in the potential. The correlation function of the pinning potential therefore acquires a long range correlation component. We decompose the correlation function into short and long-range components

$$\langle E_Q(Q + \tilde{F}_i) E_Q(\tilde{F}_j)\rangle_{\text{dis}} = R(Q)\delta_{i,j} + R_2(Q, i, j), \tag{54}$$

with the δ-correlated component $R(Q)$ and the long range correlation function $R_2(Q, i, j)$. Due to the long range correlations the intuitive picture of the de-pinning-transition is not valid anymore. For a long range correlation function

$$R_2(Q, i, j) \propto |i - j|^{-a}, \tag{55}$$

that decays with a power law, the problem has been approached with the functional renormalization group method in [12, 22].

It has been shown [6, 23] that these long-range correlations lead to the emergence of a new length-scale in the pinned system, the typical correlation length L_{corr}. The roughness function $w(x)$ of a pinned system shows a different behaviour, namely a variation in the roughness exponent ζ_{rough}, depending on whether the system is probed at length-scales smaller or larger than the correlation length [6]. We derive typical correlation lengths for the two weak disorder models under the assumption that E_Q can be approximated as a cosine-potential

$$E_J \sim E_C, \tag{56}$$

$$E_Q(Q) \approx E_Q^{\max}\,[1 - \cos(2\pi Q)]. \tag{57}$$

To calculate the correlation function of the pinning-potential of two different chain sites j and k we set, without loss of generality, $j < k$. The correlation function is given by an integral over the disorder

$$R_2(Q, j, k) = (E_Q^{\max})^2 \int_{-\infty}^{\infty} \mathrm{d}F_j \tilde{p}(F_j) \left(\frac{1}{\gamma}\right)^{k-j} \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} \mathrm{d}f_j \ldots \int_{-\frac{\gamma}{2}}^{\frac{\gamma}{2}} \mathrm{d}f_{k-1} \cos(Y_1)\cos(Y_2), \tag{58}$$

with

$$Y_1 = Q + F_j + V\frac{(N + 1 - j)(N - j)}{2N},$$
$$Y_2 = Q + F_j + \sum_{l=j}^{k-1} f_l + V\frac{(N + 1 - k)(N - k)}{2N}, \tag{59}$$

where $\tilde{p}(F_j)$ is the probability distribution of the quasi-disorder F_j. Expressing the cosine in terms of exponentials one finds that the absolute value of the correlation function R_2 is bounded by an envelope function, R_E for the case of a weak box-disorder and R_G for weak Gaussian disorder. In the weak box-disorder case we find

$$|R_2(Q, j, k)| \leqslant R_E(Q, k - j) = 2\,(E_Q^{\max})^2 \left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)^{k-j}. \tag{60}$$

The long-range correlation function decays exponentially with the distance $k - j$ and the correlation of the pinning-potential decays on the length-scale

$$L_{\mathrm{corr}} = -\frac{1}{\ln\left(\frac{\sin(\pi\gamma)}{\pi\gamma}\right)}. \tag{61}$$

As expected the correlation length goes to zero in the limit of the maximal disorder and diverges in the clean limit without disorder

$$\gamma \to 1 \qquad \Rightarrow \qquad L_{\mathrm{corr}} \to 0, \tag{62}$$

$$\gamma \to 0 \qquad \Rightarrow \qquad L_{\mathrm{corr}} \to \infty. \tag{63}$$

For a Gaussian distribution of the bare disorder f_i, the envelope function R_{G} is:

$$|R_2(Q, j, k)| \leqslant R_{\mathrm{G}}(Q, k-j) = 2\,(E_Q^{\max})^2\,(\mathrm{e}^{-2\pi^2\sigma^2})^{k-j}. \tag{64}$$

The correlation length is determined by the standard deviation σ of the bare disorder

$$L_{\mathrm{corr}} = \frac{1}{2\pi^2\sigma^2}. \tag{65}$$

We can again test the limits of infinitely broad and non-disordered distributions

$$\sigma \to \infty \qquad \Rightarrow \qquad L_{\mathrm{corr}} \to 0, \tag{66}$$

$$\sigma \to 0 \qquad \Rightarrow \qquad L_{\mathrm{corr}} \to \infty. \tag{67}$$

In the broad limit the Gaussian disorder shows the same asymptotic behaviour as the box-disorder distribution when approaching the maximal disorder limit. The maximal disorder limit is consistent with a very broad bare Gaussian distribution. In the opposite limit the Gaussian distribution corresponds to a homogeneous shift in the definition of the quasi-charge and the correlation length diverges.

The correlation length L_{corr} marks the crossover between an almost disorder free and a strongly disordered array. On length-scales smaller than the correlation length the value of the disorder F_i is approximately constant and constitutes a mere shift in the field Q. If the weakly disordered system is probed on these length-scales it behaves like a clean chain. On larger length-scales the value of the disorder changes significantly and the system behaves like a disordered chain. This transition is shown in the next section with the example of the dependence of the threshold voltage on the length of the chains.

3.3. The clean chain

We now turn to numerical simulations of the system under consideration to compare to the analytic results of the previous section.

We first simulate clean chains with zero disorder. This model has been used as the default model in a number of experimental papers on Josephson junction arrays [24, 27, 28] and its analytical known properties provide us with a benchmark for the numerical simulation method.

Also, while this model does not account for the relevant charge disorder in Josephson arrays [29], it might be more relevant for QPS-arrays than Josephson junction arrays as the former lack the strong charge disorder that can be found in the latter.

In the clean case, a simple argument to determine the critical driving force V_{cr} can be found in [24]. In the continuum limit ($\Lambda \gg 1$) for long chains ($\Lambda \ll N$) the effective model of the clean chain is equal to the sine-Gordon model with a modified potential. The solutions of the standard sine-Gordon equation of motion are the well known solitons [8, 26]

$$Q(x) = \frac{2}{\pi}\arctan\left(\mathrm{e}^{\gamma_{\mathrm{sol}}\frac{x-vt}{\Lambda}}\right), \tag{68}$$

$$\gamma_{\mathrm{sol}} = \frac{1}{\sqrt{1-\frac{v^2}{LC_0}}}, \tag{69}$$

with the soliton velocity v. The spatial derivative of a static soliton $v = 0$ has a maximal value of

$$\partial_x Q(x)|_{v=0} \leqslant \frac{1}{\pi}\frac{1}{\Lambda}. \tag{70}$$

The boundary driving force V takes the form of a boundary condition on the spatial derivative at $x = 0$

$$\partial_x Q(x)|_{x=0} = \frac{C_0}{C}V. \tag{71}$$

This can be used to estimate the maximal boundary force V for which a static soliton can exist at the array ends

$$V_{\mathrm{cr}} = \frac{4}{\sqrt{\pi}}\sqrt{\frac{C}{2C_0}V_Q^{\max}} \propto \Lambda, \tag{72}$$

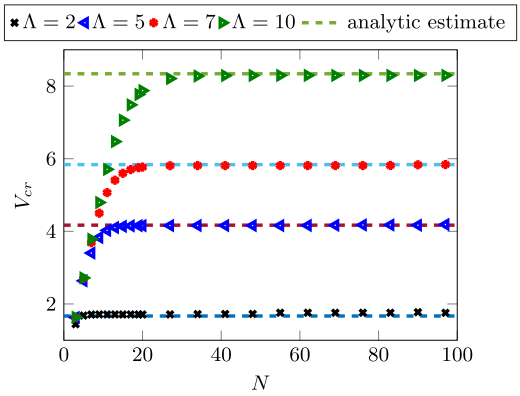

Figure 1. The critical driving force $V_{\rm cr}$ of the clean chain is plotted as a function of the chain-length for several values of Λ. As long as the chain is more than twice as long as Λ, $V_{\rm cr}$ is independent of the length N and proportional to Λ. The critical driving force has the value predicted by an analytic estimate by Haviland and Delsing [24]. In the region where the chain is shorter than Λ the system is in the zero-dimensional limit. The critical driving force is proportional to N and does not depend on Λ.

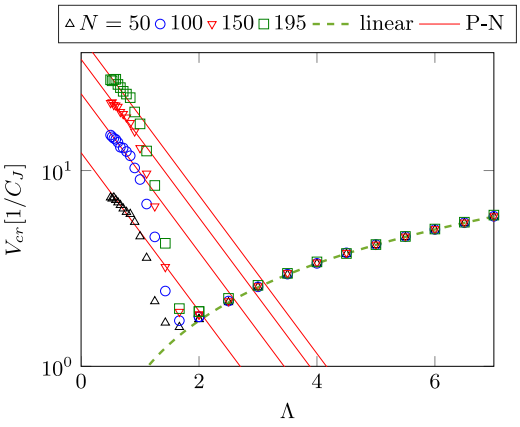

Figure 2. The critical driving force $V_{\rm cr}$ as a function of Λ for different chain-lengths N in the clean chain. As long as Λ is larger than 2, the analytic estimate equation (72) is reproduced and $V_{\rm cr} \propto \Lambda$, as it is shown by the linear fit (dashed line) in the plot. For smaller Λ the de-pinning force has been fitted (red lines) to an exponential function $V_{\rm sol} = \beta {\rm e}^{-\gamma\Lambda}$ as it arises from the Peierls–Nabarro-potential [25, 26].

$$V_Q^{\max} = \max_Q \left(\partial_Q E_Q(Q)\right). \tag{73}$$

In the Josephson junction arrays this force corresponds to the switching voltage at which the array switches from insulating to transport behaviour.

The critical driving force does not depend on the array length and is proportional to the interaction length Λ. Both features are confirmed by the numerical simulations in figures 1 and 2.

In the limit $\Lambda > N$ the spatial dependent field Q_i takes the same value on all islands of the chain, $Q_i \to Q$ and the coupled equations of motion simplify to a single equation of motion

$$\mathcal{M}\ddot{Q} + \alpha_R \dot{Q} + V_Q(Q) = \frac{V}{C}. \tag{74}$$

The one-dimensional clean chain model reduces to a zero-dimensional model. In this limit, the critical driving force increases linearly with array size and is independent of Λ (figure 1).

When the interaction length Λ is comparable to the inter-site distance $\Lambda < 2$ we are no longer in the continuum limit and the analytic approximation equation (72) is not valid. The switching-voltage is proportional to the length N and the Λ-dependence can be fitted to an exponential behaviour

$$V_{\rm cr} = N\beta {\rm e}^{-\gamma\Lambda}, \tag{75}$$

as seen in figure 2. Only one set of fitting parameters β, γ is used for all four simulated chain-lengths.

The Λ-dependence in equation (75) corresponds to the de-pinning of a single charge excitation in a disordered discrete lattice with large lattice spacing. This transition is determined by the Peierls–Nabarro-

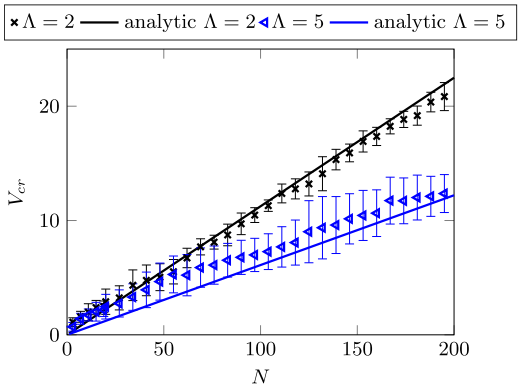

Figure 3. The critical driving force V_{cr} is plotted as a function of the length N of the disordered chain. For $\Lambda = 2$ (black crosses) the critical force grows linearly with the chain-length N as expected from the analytic estimate (black line). For $\Lambda = 5$ (blue triangles) V_{cr} is proportional to N at larger chain-lengths when $N \approx 100 \approx 2.5 L_p$ and fits the analytic estimate (blue solid line) from equation (47). Due to the strong dependence on the random disorder-configuration the linear dependence is only realized on average over 20 disorder configurations. The error-bars give the standard-deviation of the critical driving force in the sample of disorder-configurations.

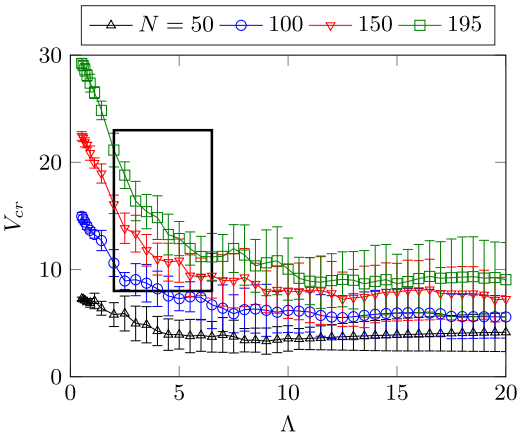

Figure 4. The critical driving force is plotted as a function of Λ in disordered arrays for a wide range of Λ. For $\Lambda < 2$ the de-pinning-theory for the continuum limit is not applicable. For large Λ the Larkin length L_p is comparable to the chain length and V_{cr} is independent of Λ, see also equation (50). For intermediate Λ the behaviour (black rectangle) is shown in figure 5.

potential [25, 26]. In the context of Josephson-junction-arrays this was discussed by Fedorov *et al* for the de-pinning of a single $2e$-charge-excitation [26].

3.4. The maximally disordered array

Here we present the critical driving force obtained from numerical simulations of the maximally disordered model. In figure 3 we compare the dependence of V_{cr} on the parameter N with analytic estimate in equation (47). At large N, where the array is longer than the Larkin length $N > L_p$, we find that the numerical simulations fit to the expected linear dependence on the system length. For small system lengths the switching voltage does not increase linearly with N, as expected in the saturation regime where the Larkin length is comparable to the system size (equation (50)).

The numerically determined dependence of V_{cr} on Λ is shown in figures 4 and 5. For small Λ the inter-site distance is comparable to Λ and the continuum limit of the standard de-pinning-picture does not apply. For large Λ the Larkin-length is comparable to the chain-length N and we observe a saturation of V_{cr} with Λ. The saturation sets in for

$$N \approx \alpha_{sat} L_p, \tag{76}$$

where α_{sat} is of order of one. Comparing the analytic estimate equation (50) with the saturation points we expect α_{sat} in the range $2.5 \leqslant \alpha_{sat} \leqslant 3.5$.

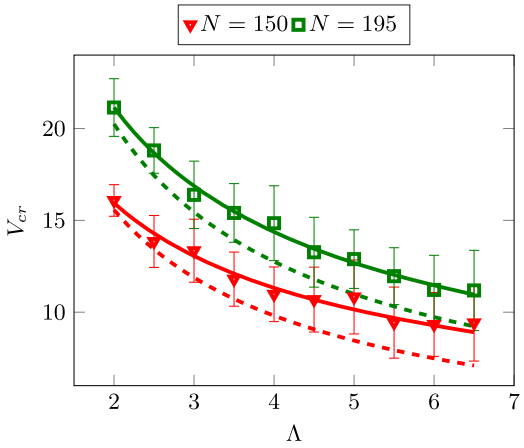

Figure 5. A comparison of V_{cr} in the intermediate Λ regime of figure 4 (black rectangle) with a fitted power-law decay (solid lines) and the analytic estimate equation (47) (dashed lines). From the fit we obtain an exponent of -0.49 ($N = 150$) and -0.56 ($N = 195$) while de-pinning theory predicts an exponent of $-\frac{2}{3}$.

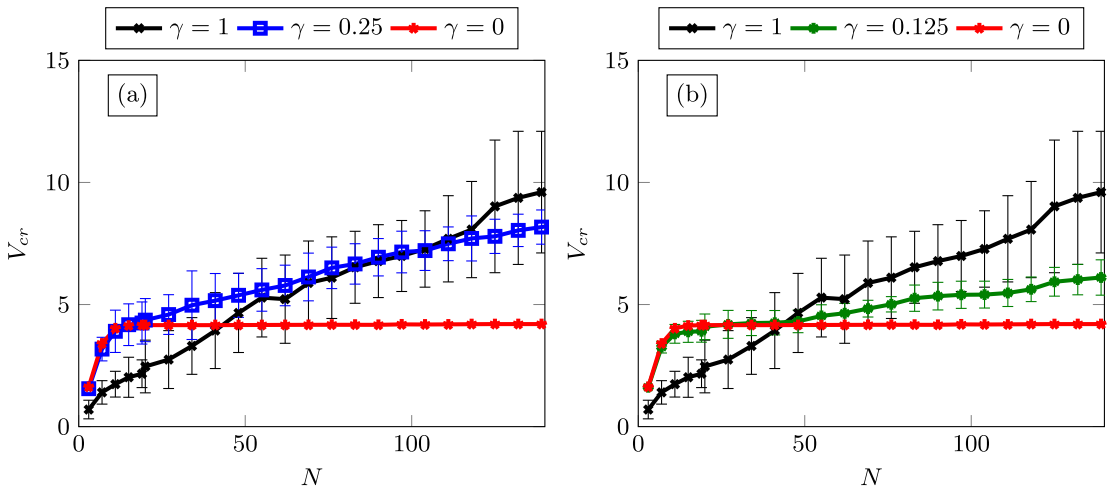

Figure 6. The critical driving force V_{cr} is plotted as a function of the chain-length in a weak box-disorder model. To enhance visibility we show two subplots for different disorder strengths: $\gamma = 0.25$, blue markers, subplot (a) and $\gamma = 0.125$ green markers subplot (b). For comparison the V_{cr} of the clean case ($\gamma = 0$, red asterisk) and the maximally disordered model ($\gamma = 1$, black crosses) are included in the plots. The behaviour of V_{cr} changes when the chain length is equal to the correlation length $N = L_{corr}$. Below $N = L_{corr}$, the critical driving force has approximately the same value as in the clean case. Above $N = L_{corr}$ it increases linearly with N as in the maximal disorder model, $L_{corr}(\gamma = 0.25) \approx 10$ and $L_{corr}(\gamma = 0.125) \approx 40$.

In this intermediate regime we expect a power-law behaviour with an exponent of $-\frac{2}{3}$ (equation (47)). Fitting the numerical data to a power-law we obtain the exponents -0.49 ± 0.05 ($N = 150$) and -0.56 ± 0.03 ($N = 195$). However this is limited by the numerically accessible chain-lengths and we can not obtain a robust confirmation of the value of the exponent of Λ from the numerical simulations.

3.5. Weak disorder and emergent correlation length

To validate our analytic model of the introduction of a new length-scale by weak disorder, we also simulate the de-pinning-transition of chain with weak box-disorder. We choose the disorder strengths $\gamma = 0.25$, $L_{corr}(\gamma = 0.25) \approx 10$ and $\gamma = 0.125$, $L_{corr}(\gamma = 0.125) \approx 40$.

In figure 6 it is shown that the system undergoes a transition when the array-length is equal to the correlation length, $N = L_{corr}$. Below $N < L_{corr}$ the chain is described by the clean chain model ($\gamma = 0$). Above the transition the critical driving force increases linearly with N. The N-dependence of V_{cr} matches the maximally disordered model $\gamma = 1$.

When the correlation length is significantly larger than the array size we can approximate all correlated disorder terms F_i by a single value $F_i \approx F$. The perfectly correlated disorder term F can be absorbed into the definition of the quasi-charge and the system is equivalent to the clean array without disorder $F_i = 0$.

When the length of the chain exceeds the correlation length one has to distinguish between two cases. The case when the correlation length is shorter than Λ and the case where it is longer. The first case requires a careful treatment to map the weakly disordered case to an effective strongly disordered model. Here we limit ourselves to the simpler second case. In this case one can understand the behaviour of the critical driving force with the following simple argument. The typical length of a soliton Λ is smaller than the correlation length and static solitonic solutions of the field Q can exist in the chain. Since the chain is longer than L_{corr} it can be subdivided into N/L_{corr} domains of length L_{corr}. To switch into the conduction regime, the applied driving force needs to overcome the transport threshold in each domain, where the transport threshold is proportional to the critical driving force in the clean chain equation (72). This gives

$$V_{\mathrm{cr}} \propto \sqrt{\frac{C}{2C_0} V_Q^{\max}} \frac{N}{L_{\mathrm{corr}}}. \tag{77}$$

This mechanism explains the linear increase in V_{cr} seen in figure 6 for $N > L_{\mathrm{corr}}$.

4. Discussion and conclusions

In this paper we have studied the de-pinning behaviour of discrete bosonic chains that can be described by an effective disordered sine-Gordon model. The most experimentally relevant realization of this model are linear arrays of Josephson junctions, however another possible realization is a ladder configuration of superconducting wires with QPS elements separating neighboring superconducting loops.

We used analytical considerations and numerical simulations to determine the critical driving force required to overcome the pinning of bosons in the chain. In the parameter regime that corresponds to experimentally studied arrays we reproduce the recently observed behaviour [7].

The classic depinning works [1–5] are mostly concerned with the infinitely long systems in which depinning can be analyzed as a critical phenomenon. In particular the investigated systems are much longer than the interaction range and the correlation length of disorder. Another extreme is the power-law correlated disorder with an infinite correlation length [12]. In this work we analyze numerically the depinning transition in the experimentally relevant regime of all three length (system length, disorder correlation length, interaction range) being of the same order.

We see a saturation regime in short chains where the Larkin length exceeds the system length and the critical driving force is independent of the range Λ of the repulsive interaction. In the weak disorder regime we observe the emergence of a new correlation length-scale L_{corr}. Both effects show good agreement between the analytic results and the numerical simulations. The obtained results could be important for future experiments on arificial chains with a low level of intrinsic disorder such as QPS ladders.

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