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De Sitter Superalgebras and Supergravity*

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Abstract. A general analysis of all possible super-extensions of anti-de Sitter and de Sitter algebras $O(3,2)$ and $O(4,1)$ is presented. It is shown that actions with de Sitter local supersymmetry exist, but contain vector-ghosts.

1. Introduction

Classical solutions of supergravity models with cosmological constants have been constructed from anti-de Sitter metrics with space-time symmetry $O(3,2)$, but not from de Sitter metrics with $O(4,1)$. A number of arguments are usually put forward for the non-existence of supergravity models with a positive cosmological constant. Such arguments are often based on the *non-existence of Majorana spinors* for $O(4,1)$. Indeed, one can use the "Noether coupling" approach to supergravity to directly show that Majorana gravitini are incompatible with a positive cosmological constant: the cosmological term in the Lagrangian, $a_1\sqrt{g}$, is accompanied by a gravitino mass term $a_2\sqrt{g}\,\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu$ and by a term $\delta\psi_\mu = D_\mu\varepsilon + a_3\gamma_\mu\varepsilon + \ldots$ in the gravitino transformation law. Demanding invariance of the Lagrangian, one finds relationships $a_1 = a_2 a_3$ and $a_2 = a_3$. The Majorana property of ψ_μ fixes a_3 to be real, with the result that $a_1 \geqq 0$. The details of this are dependent on notation and conventions, but the result is not.

There clearly is a way out of this type of no-go situation, as there is no need to insist on the existence of Majorana spinors. We may simply accept that for every spinor its charge-conjugate is also present and independent. The usual rules for counting spinors in supergravity then mean that in the de Sitter case we must have extended supergravity with *even* N. Once we have included all charge-conjugates in the basic set of N spinors, there will be a symplecitic Majorana condition

$$(Q_{\alpha i})^* = E_i{}^j D_\alpha{}^\beta Q_{\beta j} \tag{1}$$

for the de Sitter case, with $E^T = -E$, rather than a straightforward one with $E = \mathbf{1}$.

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This particular solution to the Majorana dilemma was recently highlighted by the construction of super-de Sitter algebras as quarternionic matrix algebras $UU_\alpha\left(1,1;\frac{N}{2};\mathbb{H}\right)$ by Lukierski and Nowicki [1] who indeed arrived at Majorana conditions of this type. These authors then went on to construct an action for the $N=2$ case, using the methods for the "gauging" of superalgebras which were developed by MacDowell and Mansouri [2] and Townsend and van Nieuwenhuizen [3]. Thus, it was claimed, de Sitter supergravity exists after all.

There is, however, a second and more serious objection to de Sitter supergravity. The very structure of the superalgebra itself indicates a serious problem. The basic anticommutator

$$\{Q_{\alpha i}, Q_{\beta j}\} = \omega_{ij}(\Gamma^{ab}C^{-1})_{\alpha\beta}M_{ab} + C^{-1}{}_{\alpha\beta}T_{ij}, \tag{2}$$

is the same for both the anti-de Sitter and the de Sitter case, but the properties of the Γ-matrices are different, and if we use the reality condition (1) we can show that

$$\sum_{\alpha=1}^{4}\{Q_{\alpha i}, (Q_{\alpha j})^*\} = \begin{cases} 8\,\omega_{ij}M_{45} & \text{for} \quad O(3,2) \\ 0 & \text{for} \quad O(4,1)\,. \end{cases} \tag{3}$$

If we assume that our operators act on a positive definite Hilbert space, the left-hand side of this is positive definite for $i=j$. In the anti-de Sitter case of $O(3,2)$, and if the hermitian matrix $i\omega$ with matrix elements $i\omega_{ij}$ is positive (or negative) definite, we get a positivity theorem for the eigenvalues of the hermitian operator $-iM_{45}$ (or iM_{45}), i.e. for the energy. In all other cases we find that *there is no non-trivial representation of the algebra on a positive definite Hilbert space.*

This situation leads us to expect ghosts in the corresponding supergravity theories. However, it is not always true that every local gauge algebra realized on fields has a corresponding global algebra realized on states which is obtained as the linearized limit for constant parameters. Hence the absence of an acceptable super-de Sitter symmetry on states cannot be used as conclusive proof that no action with de Sitter supersymmetry exists. We have therefore explicitly constructed the transformation rules and the action for the multiplet $e_\mu{}^m$, $\psi_{\mu\alpha i}$, and A_μ of $N=2$ de Sitter supergravity and find that the *action exists* but that it contains terms

$$L = \sqrt{g}\left(-\tfrac{1}{2}R + \ldots + \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \ldots\right) \tag{4}$$

with the *wrong relative sign* between the kinetic terms for graviton and photon, making one or the other a ghost. Tracing back in [1], we find that the invariant tensors which multiply the squares of curvatures, and which are determined by the superalgebra, were incorrectly chosen to be the same as in the anti-de Sitter case.

Our conclusions thus are: if one insists on positive energy at the perturbative level (something which nowadays is not always wanted [4]) then de Sitter supergravity does not exist.

A general treatment, on the other hand, of those (anti-)de Sitter superalgebras which cannot be represented on a positive definite space does not seem to exist. We begin the body of this article by filling this gap. Our analysis will be purely algebraic and does not require the assumptions of the analysis of Haag et al. [5] for the super-Poincaré algebra, such as non-degenerate vacuum, non-trivial S-matrix, positive definite Hilbert space. The reason that we can still get conclusive answers for the (anti-)de Sitter case under less assumptions is that the (anti-)de Sitter algebras, being simple, lead to more restrictions on their super-extensions than the Poincaré algebra. Our conclusion will be that if the matrix ω is non-singular then the internal symmetry group – whose structure is the only open question – is $O(p,q)$ with $p+q=N$ for the anti-de Sitter case (it is $O(N)$ for the "good" case which allows a positive definite Hilbert space). For the de Sitter case, the internal symmetry group is $O^*(N)=O\left(\frac{N}{2};\mathbb{H}\right)$, i.e.,

$$\begin{aligned} &O^*(2)=O(2) && \text{for} \quad N=2,\\ &O^*(4)=\mathrm{SU}(2)\otimes\mathrm{SU}(1,1) && \text{for} \quad N=4,\\ &O^*(6)=\mathrm{SU}(3,1) && \text{for} \quad N=6,\\ &O^*(8)=O(6,2) && \text{for} \quad N=8. \end{aligned} \tag{5}$$

The superalgebras are simple for non-singular ω and are $\mathrm{OSp}(p,q;4;\mathbb{R})$ for the anti-de Sitter case and $UU_\alpha\left(1,1;\frac{N}{2};\mathbb{H}\right)$ for the de Sitter case. This agrees with the results of [1]. If ω is singular, we get group contractions of $O(p,q)$ or $O^*(N)$ and the superalgebras are not simple and not contained in Kac's list [6]. In fact, these contracted superalgebras contain internal central charges and the structure of the internal group is a semidirect product of a semisimple group with a nilpotent (rather than merely a solvable) group.

The article is organized as follows. In Sect. 2 we derive the most general de Sitter superalgebras. The structure of the internal symmetry group is analyzed in Sect. 3. Finally, in Sect. 4, the action for $N=2$ supergravity with local de Sitter supergravity is constructed, and the presence of ghosts is exhibited.

2. The de Sitter Superalgebras

The anti-de Sitter and de Sitter superalgebras in four-dimensional space-time have the following generators:

(i) $M_{ab}=-M_{ba}$; $a,b=1,\ldots,5$. These generate the groups $O(3,2)$ or $O(4,1)$.

(ii) T_I; $I=1,\ldots,n$. These generate an internal symmetry group and commute with the M_{ab}.

(iii) $Q_{\alpha i}$; $\alpha=1,\ldots,4$; $i=1,\ldots,N$. These are fermionic generators which transform as spinors under the (anti-)de Sitter group.

We assume that the algebra is closed under an antilinear involution *. This can always be achieved by extending the algebra to a bigger one which together with every generator G also contains its conjugate G^* (if the generators are operators on a Hilbert space, G^* could be the hermitian conjugate of G). We take the generators M_{ab} to be anti-self-conjugate (i.e. antihermitian),

$$(M_{ab})^*=-M_{ab}. \tag{6}$$

The conjugates of the remaining generators can be expressed as

$$(Q_{\alpha i})^*=A_{\alpha i}{}^{\beta j}Q_{\beta j}, \qquad (T_I)^*=\mathrm{B}_I{}^J T_J, \tag{7}$$

since * does not mix fermionic with bosonic generators, nor internal with space-time ones.

The involution property $(*)^2=1$ gives consistency conditions

$$A_{\alpha i}{}^{\beta j} A^{*}_{\beta j}{}^{\gamma k} = \delta_\alpha{}^\gamma \delta_i{}^k, \qquad B_I{}^J B^{*}_J{}^K = \delta_I{}^K. \tag{8}$$

Further constraints on the constants A and B will be derived from the closure of the superalgebra under the involution.

In order to derive super-extensions of the anti-de Sitter and de Sitter algebras we shall assume the most general (anti-)commutation relations with arbitrary structure constants and subsequently look for the most general solution of the Jacobi identities. Under the assumptions (i)–(iii), the most general ansatz for the algebra is:

$$[M_{ab}, M_{cd}] = \eta_{bc} M_{ad} + \eta_{ad} M_{bc} - \eta_{ac} M_{bd} - \eta_{bd} M_{ac}, \tag{9}$$

$$[T_I, T_J] = c_{IJ}{}^K T_K, \tag{10}$$

$$[M_{ab}, T_I] = 0, \tag{11}$$

$$[Q_{\alpha i}, M_{ab}] = \tfrac{1}{2}(\Gamma_{ab})_\alpha{}^\beta Q_{\beta i}, \tag{12}$$

$$[Q_{\alpha i}, T_I] = b_{I\alpha i}{}^{\beta j} Q_{\beta j}, \tag{13}$$

$$\{Q_{\alpha i}, Q_{\beta j}\} = X_{\alpha i\beta j}{}^{ab} M_{ab} + W_{\alpha i \beta j}{}^I T_I. \tag{14}$$

Here η_{ab} is the metric $(+++--)$ or $(++++-)$, and (9) is just the algebra of the (anti-)de Sitter group. The spinorial character of $Q_{\alpha i}$ implies (12) with

$$\Gamma_{ab} = \tfrac{1}{2}(\Gamma_a\Gamma_b - \Gamma_b\Gamma_a); \qquad \Gamma_a\Gamma_b + \Gamma_b\Gamma_a = 2\eta_{ab}\mathbf{1}. \tag{15}$$

The matrices $\frac{1}{2}\Gamma_{ab}$ are a representation of the algebra (9), and the Γ_α are the 4×4 Dirac matrices for five space-time dimensions. The symmetry of the anti-commutator (14) implies that

$$X_{\alpha i \beta j}{}^{ab} = X_{\beta j \alpha i}{}^{ab}; \qquad W_{\alpha i\beta j}{}^I = W_{\beta j \alpha i}{}^I. \tag{16}$$

It is straightforward to check that the (M, M, M), (M, M, Q), (M, M, T), and (M, T, T) Jacobi identitities are satisfied. Next we consider the (M, Q, Q) identity. It is satisfied provided that for each ij the structure constants X and W are numerically invariant tensors of $O(3, 2)$ or $O(4, 1)$, with the indices α and β in the spinorial and the index pair ab in the antisymmetric tensor (adjoint) representations. Such numerical invariants are unique up to a constant (the direct product of two spinors contains the tensor only once) and we get

$$X_{\alpha i\beta j}{}^{ab} = \omega_{ij}(\Gamma^{ab} C^{-1})_{\alpha\beta}, \tag{17}$$

$$W_{\alpha i \beta j}{}^I = w_{ij}{}^I (C^{-1})_{\alpha\beta}, \tag{18}$$

where C is the charge conjugation matrix in *five* dimensions which has the following properties:

$$C^T = -C; \qquad (\Gamma_a C^{-1})^T = -\Gamma_a C^{-1}; \qquad (\Gamma_{ab} C^{-1})^T = \Gamma_{ab} C^{-1}. \tag{19}$$

We see that of this complete set of 4×4 matrices ten are symmetric and six are antisymmetric. These symmetry properties imply, together with Eqs. (16) that ω is symmetric and w antisymmetric,

$$\omega_{ij}=\omega_{ji}; \qquad w_{ij}{}^{I}=-w_{ji}{}^{I}. \tag{20}$$

We observe that in the $\{Q,Q\}$ anticommutator the internal symmetry generators appear in the form

$$T_{ij}=w_{ij}{}^{I}T_I, \qquad T_{ij}=-T_{ji}, \tag{21}$$

and we will use T_{ij} rather than T_I to determine the structure of the algebra. For this we multiply the generators T_I in (10), (11), and (13) with $w_{ij}{}^{I}$.

Next we analyze the (Q,Q,Q) Jacobi identity. Using Eq. (12), this identity reads

$$O=\tfrac{1}{2}\omega_{ij}(\Gamma^{ab}C^{-1})_{\alpha\beta}(\Gamma_{ab}Q)_{\gamma k}+C^{-1}{}_{\alpha\beta}[Q_{\gamma k},T_{ij}]+(\text{cyclic } \alpha i\to\beta j\to\gamma k),$$

and uniquely determines the $[Q,T]$ commutator to be

$$[Q_{\alpha i},T_{jk}]=-2(\omega_{ij}Q_{\alpha k}-\omega_{ik}Q_{\alpha j}). \tag{22}$$

(Note that the homogeneous equation for $[Q,T]$ has no non-trivial solution, as one may show by taking an explicit representation for C^{-1}.)

Having obtained the $[Q,T]$ commutator, we deduce the $[T,T]$ commutator from the (Q,Q,T) Jacobi identity and finally check that the (Q,T,T), (T,T,T), and (Q,M,T) Jacobi identities are satisfied automatically. This concludes the analysis of the Jacobi identities and gives the full superalgebra in the form

$$[M_{ab},M_{cd}]=\eta_{bc}M_{ad}+\eta_{ad}M_{bc}-\eta_{ac}M_{bd}-\eta_{bd}M_{ac}, \tag{23}$$

$$[T_{ij},T_{kl}]=-2(\omega_{jk}T_{il}+\omega_{il}T_{jk}-\omega_{ik}T_{jl}-\omega_{jl}T_{ik}), \tag{24}$$

$$[M_{ab},T_{ij}]=0, \tag{25}$$

$$[Q_{\alpha i},M_{ab}]=\tfrac{1}{2}(\Gamma_{ab})_{\alpha}{}^{\beta}Q_{\beta i}, \tag{26}$$

$$[Q_{\alpha i},T_{jk}]=-2(\omega_{ij}Q_{\alpha k}-\omega_{ik}Q_{\alpha j}), \tag{27}$$

$$\{Q_{\alpha i},Q_{\beta j}\}=\omega_{ij}(\Gamma^{ab}C^{-1})_{\alpha\beta}M_{ab}+C^{-1}{}_{\alpha\beta}T_{ij}. \tag{28}$$

The only unknown quantity in this is the symmetric matrix ω whose properties we shall now proceed to determine.

We first turn to the conditions which follow from the existence of the involution *. Taking the conjugate of (26) we find, using (7),

$$A_{\alpha i}{}^{\beta j}(\Gamma_{ab})_{\beta}{}^{\gamma}=((\Gamma_{ab})_{\alpha}{}^{\beta})^{*}A_{\beta i}{}^{\gamma j}. \tag{29}$$

This means that for each fixed i and j, A is an interpolating matrix between the irreducible representation matrices Γ_{ab} and $(\Gamma_{ab})^{*}$ and thus unique up to a constant (Schur's lemma). Therefore A factorizes,

$$A_{\alpha i}{}^{\beta j}=E_i{}^{j}D_{\alpha}{}^{\beta}, \quad \text{i.e.} \quad (Q_{\alpha i})^{*}=E_i{}^{j}D_{\alpha}{}^{\beta}Q_{\beta j}, \tag{30}$$

and the first of the conditions (8) becomes

$$EE^{*}DD^{*}=\mathbf{1}. \tag{31}$$

We now construct a representation of the Dirac matrices Γ_a in five dimensions from the γ_m and γ_5 in four dimensions which we take in the Majorana representation, and multiply with factors of i such that

$$\begin{array}{lll} O(3,2): & \Gamma_k (k=1,2,3): & \text{real and symmetric, } (\Gamma_k)^2=+\mathbf{1} \\ & \Gamma_4 \text{ and } \Gamma_5: & \text{real and antisymmetric, } (\Gamma_4)^2=(\Gamma_5)^2=-\mathbf{1}. \\ O(4,1): & \Gamma_k (k=1,2,3): & \text{real and symmetric, } (\Gamma_k)^2=+\mathbf{1} \\ & \Gamma_4: & \text{imaginary and antisymmetric, } (\Gamma_4)^2=+\mathbf{1} \\ & \Gamma_5: & \text{real and antisymmetric, } (\Gamma_5)^2=-\mathbf{1}. \end{array} \tag{32a}$$

In this representation one has for $O(3,2)$ that $(\Gamma_{ab})^* = \Gamma_{ab}$, while for $O(4,1)$ the same is true except that $(\Gamma_{a4})^* = -\Gamma_{a4}$. We can now easily determine the matrices C and D. One finds

$$\begin{array}{lll} O(3,2): & D=\mathbf{1}; & C=\Gamma_4\Gamma_5 \text{ (antisymmetric, real)} \\ O(4,1): & D=\Gamma_4; & C=\Gamma_4\Gamma_5 \text{ (antisymmetric, imaginary)}. \end{array} \tag{32b}$$

From $DD^* = \mathbf{1}$ for $O(3,2)$ and $DD^* = -\mathbf{1}$ for $O(4,1)$ we see that the consistency condition (31) implies that

$$EE^* = \begin{cases} +\mathbf{1} & \text{for } O(3,2) \\ -\mathbf{1} & \text{for } O(4,1). \end{cases} \tag{33}$$

We now take the conjugate of the $\{Q,Q\}$ anticommutator

$$\{Q^*, Q^*\} = -\omega^* \Gamma_{ab}{}^* C^{-1*} M^{ab} + C^{-1*} T^*$$

and convert it back into $\{Q,Q\}$, using the reality condition (30). The result should be Eq. (28) again. This gives us conditions

$$(E\omega E^T)_{ij} = \mp (\omega_{ij})^*, \qquad (ETE^T)_{ij} = \pm (T_{ij})^* \equiv \pm (T^\dagger)_{ji}, \tag{34}$$

where the upper sign refers to $O(3,2)$ and the lower to $O(4,1)$. From $EE^* = \pm\mathbf{1}$ we get $(E^T)^{-1} = \pm E^\dagger$ and find the following reality properties

$$(E\omega)^\dagger = -E\omega, \qquad (ET)^\dagger = -ET \text{ (in both cases)}. \tag{35}$$

The reality conditions for the remaining $[Q, T]$ and $[T, T]$ commutators should now be satisfied, since these commutators followed from the Jacobi identities. We checked this and found agreement.

3. Structure of Internal Symmetry Group

Before going on, we note that one can choose a basis for the $Q_{\alpha i}$ such that E is the unit matrix for the anti-de Sitter case and a symplectic metric in the de Sitter case:

$$\begin{array}{ll} E=\mathbf{1} & \text{for } O(3,2) \\ E=\begin{pmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{pmatrix} & \text{for } O(4,1). \end{array} \tag{36}$$

The proof is given in the appendix.

In the *anti-de Sitter* case, ω and T are both antihermitian and since ω is symmetric and T antisymmetric, $i\omega$ is a real symmetric matrix, while the operators T_{ij} are hermitian, $(T_{ij})^* = T_{ij}$. Redefining

$$Q'_{\alpha i} = M_i{}^j Q_{\alpha j}, \tag{37}$$

one finds the same algebra but with ω_{ij} replaced by

$$\omega'_{ij} = M_i^{\ k} M_j^{\ l} \omega_{kl}\,. \tag{38}$$

Hence, with M being an appropriate real orthogonal matrix one can always cast $i\omega$ into diagonal form with p entries equal to $+1$, q entries equal to -1 and r entries equal to zero:

$$\omega_{ij} = \mathrm{diag}(1, \ldots, 1, -1, \ldots, -1, 0, \ldots, 0)\,. \tag{39}$$

For $p+q=N$ one clearly obtains the OSp$(p,q;4)$ superalgebra with $O(p,q)$ as the internal symmetry group, see Eq. (24).

For $p+q<N$, the $T_{\tilde{i}\tilde{j}}$ (with $\tilde{i}$ and $\tilde{j}$ running from $p+q+1$ to N) commute with all the generators in the superalgebra but still can be obtained by the $\{Q, Q\}$ anticommutator. Hence these $T_{\tilde{i}\tilde{j}}$ are central charges. The mixed $T_{i\tilde{j}}$ (with $i=1, \ldots, p+q$), on the other hand, rotate a Q_i into $Q_{\tilde{i}}$ and vice-versa, while $[T_{i\tilde{j}}, T_{k\tilde{l}}]$ produces only a $T_{\tilde{j}\tilde{l}}$. Thus the system $(T_{ij}, T_{i\tilde{j}}, T_{\tilde{i}\tilde{j}})$ forms a (bosonic) graded algebra with central charges:

$$\begin{aligned} &[T_{ij}, T_{kl}] \sim T_{mn}; \quad &&[T_{ij}, T_{k\tilde{l}}] \sim T_{m\tilde{n}}; \quad &&[T_{ij}, T_{\tilde{k}\tilde{l}}] = 0 \\ &[T_{i\tilde{j}}, T_{k\tilde{l}}] \sim T_{\tilde{m}\tilde{n}}; \quad &&[T_{i\tilde{j}}, T_{\tilde{k}\tilde{l}}] = 0; \quad &&[T_{\tilde{i}\tilde{j}}, T_{\tilde{k}\tilde{l}}] = 0\,. \end{aligned} \tag{40}$$

The internal symmetry group is thus a semidirect product of the simple group $O(p,q)$ with the nilpotent group generated by the $T_{i\tilde{j}}$ and $T_{\tilde{i}\tilde{j}}$. We shall call this non-semisimple superalgebra the inhomogenous IOSp$(p,q,r;4)$.

Turning to the *de Sitter* case, we begin by noting that the general solution of the reality conditions for ω and T reads

$$\begin{aligned} \omega_{ij} &= \begin{pmatrix} S & iH \\ -iH^T & S^* \end{pmatrix} \quad && S = \text{complex symmetric} \\ & && H = \text{hermitian} \\ T_{ij} &= \begin{pmatrix} a & ih \\ -ih^T & a^\dagger \end{pmatrix} \quad && a = \text{complex antisymmetric} \\ & && h = \text{hermitian}, \ (h_{ij})^\dagger = h_{ji}\,. \end{aligned} \tag{41}$$

As a special case which suggests the general treatment, we consider first $S=0$ and $H=\mathbf{1}$. In that case the generators T_{ij} act on $Q_{\alpha i}$ $\left(i=1, \ldots, n;\ n=\frac{N}{2}\right)$ and $\tilde{Q}_{\alpha i} = Q_{\alpha i+n}$ as follows

$$\begin{aligned} [Q_{\alpha i}, a_{jk}] &= 0; \quad [Q_{\alpha i}, h_{jk}] = 2\delta_{ik} Q_{\alpha j}\,, \\ [\tilde{Q}_{\alpha i}, (a^\dagger)_{jk}] &= 0; \quad [\tilde{Q}_{\alpha i}, h_{jk}] = -2\delta_{ij} \tilde{Q}_{\alpha k}\,, \\ [Q_{\alpha i}, (a^\dagger)_{jk}] &= -2i\delta_{ij}\tilde{Q}_{\alpha k} + 2i\delta_{ik}\tilde{Q}_j\,, \\ [\tilde{Q}_{\alpha i}, a_{jk}] &= -2i\delta_{ij} Q_{\alpha k} + 2i\delta_{ik} Q_{\alpha j}\,. \end{aligned} \tag{42}$$

Let us expand the generators h_{ij} on a complete basis of hermitian $n \times n$ matrices λ^r_{ij}:

$$h_{ij} = \lambda^r_{ij} R_r; \qquad R_r \ \text{hermitian}\,. \tag{43}$$

For example, if $N=2$, we have $\lambda=1$, while for $N=4$ one may take

$$\lambda^r_{ij} R_r = (\vec{\tau} \cdot \vec{R} + \mathbf{1} R_0)_{ij}\,,$$

and for $N=6$ and 8:

$$\lambda^r_{ij} R_r = 2i\delta^{kl}_{[ij]} R_{[kl]} + 2\delta^{kl}_{(ij)} R_{(kl)} + \left(\frac{2}{n}\right)^{1/2} \delta_{ij} R_0$$

with $k>1$. If the λ^r_{ij} satisfy the orthogonality relation

$$\lambda^r_{ij}\lambda^s_{ji}=2\delta^{rs}, \tag{44}$$

one obtains

$$[Q_{\alpha i}, R^r]=\lambda^r_{ij}Q_{\alpha j}; \quad [\tilde{Q}_{\alpha i}, R^r]=-\lambda^r_{ji}\tilde{Q}_{\alpha j}. \tag{45}$$

The generators R^r must have the same algebra as the matrices λ^r. This follows from the (Q, R, R) Jacobi identity, and we conclude that the R^r generate the group $U\left(\frac{N}{2}\right)$.

Next we expand the a_{ij} on a basis of antisymmetric and real matrices O^I_{ij}:

$$a_{ij}=O^I_{ij}(P^I+iW^I); \quad P^I \text{ and } W^I \text{ hermitian}. \tag{46}$$

For example, for $N=2$, a_{ij} vanishes, while for $N=4$ one may take $O_{ij}=\varepsilon_{ij}$; for $N=6$ one may take $O^I_{ij}=\varepsilon^k{}_{ij}$, and for $N=8$ one may take $O^I_{ij}=\varepsilon^{kl}{}_{ij}$ with $k>l$. Normalizing these O^I_{ij} to

$$O^I_{ij}O^J_{ji}=-2\delta^{IJ}, \tag{47}$$

one obtains

$$\begin{aligned}&[Q_{\alpha i}, P^I]=-iO^I_{ji}\tilde{Q}_{\alpha j}; \quad [Q_{\alpha i}, W^I]=O^I_{ji}\tilde{Q}_{\alpha j}\\ &[\tilde{Q}_{\alpha i}, P^I]=iO^I_{ji}Q_{\alpha j}; \quad [\tilde{Q}_{\alpha i}, W^I]=O^I_{ji}Q_{\alpha j}.\end{aligned} \tag{48}$$

In the space $Q_{\alpha i}\oplus\tilde{Q}_{\alpha i}$, the operators P^I and W^I act as

$$[Q_{\alpha i}, P]=(P)_{ij}Q_{\alpha j},$$

with the matrix representation given by

$$(P^I)_{ij}=-\sigma_2\otimes O^I_{ij}, \quad (W^I)_{ij}=-\sigma_1\otimes O^I_{ij}. \tag{49}$$

These representation matrices are antihermitian although the P^I and W^I are hermitian: we conclude that these generators are non-compact.

In the same way one may deduce the explicit matrix representation of the R^r generators on $Q_{\alpha i}\otimes\tilde{Q}_{\alpha i}$ from (46). Splitting the λ^r_{ij} into a traceless part λ'^r_{ij} and a trace part proportional to δ_{ij}, one finds

$$\begin{aligned}(R'^r)_{mn}&=\tfrac{1}{2}(\mathbf{1}+\sigma_3)\otimes\lambda'^r_{mn}-\tfrac{1}{2}(\mathbf{1}-\sigma_3)\otimes\lambda'^r_{nm},\\ (R^0)_{mn}&=\sigma_3\otimes\delta_{mn}.\end{aligned} \tag{50}$$

From this explicit matrix representation of the generators T_{ij} we can now deduce the group which the T_{ij} generate. Note that the reality properties of T_{ij} were fixed; one can, of course, change the reality properties of P^I, W^I, R'^r, and R^0 by redefining $P^I=i\tilde{P}^I$, etc., but this does not affect the reality properties of the T_{ij} themselves. Clearly, the λ matrices generate the same group on Q as do the $-\lambda^T$ on $\tilde{Q}$. The matrices satisfy the additional commutation relations

$$\begin{aligned}&[P^I, P^J]=[W^I, W^J]=\mathbf{1}\otimes[O^I, O^J],\\ &[W^I, P^J]=i\sigma_3\otimes\{O^I, O^J\},\\ &[P^I, R_r]=\frac{1}{2}\sigma_2\otimes[O^I, \lambda^T_r-\lambda_r]-\frac{i}{2}\sigma_1\otimes\{O^I, \lambda^T_r+\lambda_r\},\\ &[W^I, R_r]=\frac{1}{2}\sigma_1\otimes[O^I, \lambda^T_r-\lambda_r]+\frac{i}{2}\sigma_2\otimes\{O^I, \lambda^T_r+\lambda_r\}.\end{aligned} \tag{51}$$

Since $\{O^I, O^J\}$ and $i[O^I, O^J]$ are hermitian and $i[O^I, \lambda^T - \lambda]$ and $\{O^I, \lambda^T + \lambda\}$ are antisymmetric and real, the algebra closes.

This whole structure suggests that the internal symmetry group is $O^*(N)$, which is that complex form of $O(N)$ whose maximal compact subgroup is $U\left(\frac{N}{2}\right)$. Rather than work these details out any further, we now present a general proof that the internal symmetry group is indeed $O^*(N)$ for any non-singular ω_{ij}. Let us begin by stressing that one must look at those linear combinations of T_{ij} which are antihermitian. Without such a restriction, statements like "the internal symmetry group is $O(p, q)$" are empty, since one can always go to another complex form where the internal symmetry group is different, e.g., $O(p+q)$. The first problem to solve is thus: for which $\lambda^{ij} = -\lambda^{ji}$ are the $\lambda^{ij} T_{ij}$ antihermitian. The solution is given by the following Lemma.

Lemma I. *$\lambda^{ij} T_{ij}$ is antihermitian if and only if λ^{ij} are matrices of $O^*(N)$.*

Proof. $O^*(N)$ is defined as that subgroup of $O(N;\mathbb{C})$ which leaves the sesquilinear antisymmetric form given by E invariant [7]. Since λ^{ij} are complex antisymmetric matrices, they are generators of $O(N;\mathbb{C})$. Using that $(T_{ij})^* = -(ETE^T)_{ij}$, see Eq. (34), we have

$$(\lambda^{ij} T_{ij})^* = -\lambda^{*ij} E_i{}^k E_j{}^l T_{kl} = -\lambda^{kl} T_{kl} \,. \tag{52}$$

It follows from Eq. (27) that for non-singular ω the T_{kl} are linearly independent. We can therefore drop them and multiply by E_{lm} to get

$$\lambda^\dagger E + E\lambda = 0 \,, \tag{53}$$

the condition for λ to be generators of $O^*(N)$.

Let us write the set of generators as $\lambda_I{}^{ij} = -\lambda_I{}^{ji}$, $I = 1, \ldots, \frac{N}{2}(N-1)$. Our task is to determine which group the $T_I \equiv \lambda_I{}^{ij} T^i{}_j$ generate. It will turn out to be more convenient to use the matrices

$$\tilde{\lambda}_{Iij} \equiv (\sqrt{\omega})^{-1}{}_{ik} \lambda_I{}^{kl} (\sqrt{\omega})^{-1}{}_{lj} \,, \tag{54}$$

which also form a basis of $O^*(N)$ according to the next lemma.

Lemma II: *$\tilde{\lambda}_{Iij}$ defined in* (54) *form a basis for the algebra of* $O^*(N)$.

Proof. First note that $(\sqrt{\omega})^{-1}$ can be defined by a power series in ω (provided that ω is non-singular, as we have assumed), and that it is a symmetric matrix. This implies that the $\tilde{\lambda}_I$ are antisymmetric. Next, we must show that (53) holds with λ_I replaced by $\tilde{\lambda}_I$, i.e.,

$$(\sqrt{\omega})^{*-1} \lambda_I^\dagger (\sqrt{\omega})^{*-1} E + E(\sqrt{\omega})^{-1} \lambda_I (\sqrt{\omega})^{-1} = 0 \,. \tag{55}$$

Using that $\omega^* E = E\omega$ according to (35) we move the first E to the left through the power series in ω^* which is $\sqrt{\omega^*}^{-1}$ and the second E to the right through $(\sqrt{\omega})^{-1}$. Then multiplying by $\sqrt{\omega^*}$ from the left and by $\sqrt{\omega}$ from the right one obtains Eq. (53) which is satisfied as we have assumed that the λ_I generate $O^*(N)$. Thus the $\sqrt{\omega}^{-1} \lambda_I \sqrt{\omega}^{-1}$ also generate $O^*(N)$.

We can now prove the main theorem, namely that the antihermitian operators

$$T_I = \tilde{\lambda}_I{}^{ij} T_{ij} \tag{56}$$

also generate the group $O^*(N)$.

Proof. Using (24) one finds that the commutator $[T_I, T_J]$ reads

$$\begin{aligned}[T_I, T_J] &= 8\,\mathrm{Tr}\{(\sqrt{\omega})^{-1}\lambda_I(\sqrt{\omega})^{-1}\omega(\sqrt{\omega})^{-1}\lambda_J(\sqrt{\omega})^{-1}T\} \\ &= 8\,\mathrm{Tr}\{(\sqrt{\omega})^{-1}\lambda_I\lambda_J(\sqrt{\omega})^{-1}T\} \\ &= 4\,\mathrm{Tr}\{(\sqrt{\omega})^{-1}(\lambda_I\lambda_J - \lambda_J\lambda_I)(\sqrt{\omega})^{-1}T\} \\ &= 4c_{IJ}{}^K T_K,\end{aligned} \tag{57}$$

where $c_{IJ}{}^K$ are the structure constants of $O^*(N)$, defined by

$$[\lambda_I, \lambda_J] = c_{IJ}{}^K \lambda_K. \tag{58}$$

Hence, the internal symmetry group is indeed $O^*(N)$.

4. The $N=2$ Action with de Sitter Supersymmetry

In the previous section we have obtained superextensions of the de Sitter algebra $O(4,1)$ for any even N. We now turn to the question whether there exists local Lagrangian field theories with a related set of local symmetries. To keep the discussion simple, we only consider the case of $N=2$. The $N=2$ extended supergravity model, as in the Poincaré and anti-de Sitter cases [8], contains only gauge fields, namely the vielbein $e_\mu{}^m$, two gravitini $\psi_\mu{}^i$ $(i=1,2)$ and a photon A_μ. These fields satisfy reality conditions, namely the vielbein and the photon fields are real, while the gravitino, being the gauge field associated with the odd charges $Q_{\alpha i}$, satisfies the symplectic Majorana condition

$$\bar{\psi}_\mu{}^i = \psi_{\mu j}^T C^{-1} \varepsilon^{ji} \tag{59}$$

with $C^{-1} = C^{-1}{}_5$ the charge conjugation matrix in five dimensions. [Since the internal symmetry group is $O(2)$ there is no significance in the position of indices i,j.]

The appearance of the symplectic metric ε^{ij} suggests that an acceptable action may exist. If in the gravitino mass term and its supersymmetry transformation law the symplectic metric is present, then the variation of the cosmological term $\delta\sqrt{g}$ must cancel against terms involving the square of the symplectic metric (instead of the square of the unit matrix as in the anti-de Sitter case), thus containing an extra minus sign. However, one can immediately rule out this possibility because $\bar{\psi}_\mu{}^i\gamma^{\mu\nu}\psi_\nu{}^j\varepsilon_{ij}$ vanishes identically, independent of whether the $\psi^i{}_\mu$ are Majorana or symplectic Majorana spinors (note that C_4 and C_5 differ by a factor of γ_5 and that $\gamma^{\mu\nu}\gamma_5$ is a linear combination of $\gamma^{\mu\nu}$).

The leading terms in the transformation laws of the fields can be determined, according to general rules, from the structure of the superalgebra, using the fact that each gauge field is associated with an appropriate charge. This will be enough to decide, using the Noether coupling construction [9], whether the complete nonlinear theory exists.

The supersymmetry transformation law of the gravitini follows from the $[Q,M]$ and $[Q,T]$ commutators (26) and (27) by identifying M_{5m} with P_m, and taking the vielbein as the gauge field associated with P_m. It reads

$$\delta\psi_\mu{}^i = D_\mu\zeta^i + bA_\mu\varepsilon^{ij}\zeta^j + a_3\gamma_\mu\zeta^i\,, \tag{60}$$

where we have identified the gauge field A_μ^{ij} associated with the internal symmetry generator $T_{ij} = -T_{ji}$ with the photon as $A_\mu^{ij} = \varepsilon^{ij}A_\mu$. The constants b and a_3, which in principle could be fixed from the algebra, will be more easily determined later by the Noether construction.

The transformation law of the vielbein is found from the $\{Q,Q\}$ anti-commutator. It reads

$$\delta e_\mu{}^m = \bar{\psi}_\mu{}^{i\alpha}\bar{\zeta}^{j\beta}\omega_{ij}(\Gamma^m{}_5 C^{-1})_{\alpha\beta}\,.$$

The bars denote here the symplectic Majorana conjugate (50). Taking $\omega_{ij} = \delta_{ij}$, which is allowed by (41), and evaluating (60) with $C^{-1} = \gamma_5\gamma_4$, one gets

$$\delta e_\mu{}^m \sim \bar{\zeta}^{Ti}\gamma_4\gamma^m\psi_\mu^i\,.$$

Note that the *symplectic metrics have cancelled* ($\varepsilon^2 = -\mathbf{1}$). To facilitate the calculations we can thus use the ordinary Majorana conjugation. Then the vielbein transformation law (with the conventional normalization) reads

$$\delta e_\mu{}^m = \tfrac{1}{2}\bar{\zeta}^i\gamma^m\psi_\mu{}^i\,. \tag{61}$$

In a similar way one arrives at the photon transformation rule

$$\delta A_\mu = \frac{q}{\sqrt{2}}\bar{\zeta}^i\gamma_5\varepsilon^{ij}\psi_\mu^j\,, \tag{62}$$

where q is a constant to be determined.

The Noether procedure gives the following most general form of the action

$$\begin{aligned}\frac{1}{\sqrt{g}}L = &-\frac{1}{2}R(e,\omega) - \frac{1}{2}\bar{\psi}_\mu^i\gamma^{\mu\varrho\sigma}(D_\varrho(\omega)\psi_\sigma{}^i + b\varepsilon^{ij}A_\varrho\psi_\sigma^j)\\ &-\frac{1}{4}F_{\mu\nu}^2 + \frac{r}{4\sqrt{2}}\bar{\psi}_\mu^i(2F^{\mu\nu}\gamma_5 + \frac{1}{\sqrt{g}}\varepsilon^{\mu\nu\varrho\sigma}F_{\varrho\sigma})\psi_\nu^j\varepsilon_{ij}\\ &+a_1 + a_2\bar{\psi}_\mu^i\gamma^{\mu\nu}\gamma_5\psi_\nu^i + \psi^4\text{-terms}\,.\end{aligned} \tag{63}$$

The requirement of invariance of this action under transformations (60)–(62) fixes all the constants. From the variation of the cosmological term $\sqrt{g}\,a_1$ we get

$$a_1 = -12a_2a_3\,, \tag{64}$$

while the $\psi D\zeta$ variation gives

$$a_2 = a_3\,. \tag{65}$$

Thus if a_2 were real, we would have a de Sitter supergravity model. (The details of this are slightly different from the anti-de Sitter case sketched in the introduction.) However, we must still check the other variations.

The $\psi F\zeta$ variation tells us to add to the transformation law of the gravitino a term of the form

$$\delta\psi_\mu{}^i = \frac{p}{2\sqrt{2}} \varepsilon^{ij} \gamma^{\varrho\sigma} \gamma_\mu \gamma_5 \zeta_j F_{\varrho\sigma} \tag{66}$$

with an arbitrary constant p. To cancel the terms one must have

$$ra_3 + a_2 p = 0\,, \qquad ra_3 - a_2 p = -b\sqrt{2}\,. \tag{67}$$

From the $\bar{\zeta}\psi DF$, the $D\zeta\psi F$ and the $\bar{\zeta}\psi FF$ variations, one gets respectively

$$p+q=0\,, \qquad p+r=0\,, \qquad pq=1\,. \tag{68}$$

The final result is

$$p=-q=-r=i\,, \qquad a_2=a_3=-\frac{ib}{\sqrt{2}}\,, \qquad a_1=6b^2\,. \tag{69}$$

Since q is imaginary, we find that δA_μ is imaginary: we have chosen the wrong phase for A_μ and must redefine $A_\mu = iA'_\mu$, where A'_μ is taken to be real. Also, a_2 and a_3 must be real to ensure the reality of the gravitino mass term and the symplectic Majorana condition of $\delta\psi_\mu$. This fixes b to be imaginary and everything is consistent. However, the *Maxwell action changes sign* by this redefinition, and one ends up with a photon ghost.

We conclude that $N=2$ supergravity can be gauged such that one has a de Sitter rather than the usual anti-de Sitter model, but the price one has to pay if one wants to have a *real* action is violation of positive energy. Although certain efforts are being made nowadays to show that actions with perturbative ghosts may yield at the nonperturbative level unitary theories [4], these efforts are at this moment rather speculative, and we reject, for the time being, de Sitter supergravities.

Appendix

We show that one can choose a basis for the $Q_{\alpha i}$ in which the matrix E, defined by $E_i{}^j Q_{\alpha j} = E(Q_{\alpha i})$, with $E(Q)$ given by $E(Q_{\alpha i}) = D^{-1}{}_\alpha{}^\beta Q^*_{\beta i}$, becomes the unit matrix for $O(3,2)$ and the symplectic metric for $O(4,1)$. We shall treat these two cases separately. Note that if $E(Q)=\lambda Q$, then $\lambda \neq 0$ and $E(\alpha Q) = (\alpha^*/\alpha)\lambda(\alpha Q)$. Hence one can always make λ real and positive.

$O(3,2)$ *Case*. Pick a Q, and consider $E(Q)$. If $E(Q)=\lambda Q$ make λ positive. From $EE^* = \mathbf{1}$ it then follows that $\lambda=1$. Hence $E(Q)=Q$ in that case. By induction we complete the proof. Namely, if one has $Q_1, \ldots, Q_k$ satisfying $E(Q_i)=Q_i$, then an S which is linearly independent from these Q's satisfies either

(i) $E(S)$ is proportional to S. Then one can make $E(S)=S$.

(ii) $E(S)$ is linearly independent of $Q_1, \ldots, Q_k$ and S. In this case one can take $Q_{k+1} = S + E(S)$.

(iii) $E(S)$ is a linear combination of S and Q's, namely $E(S) = \lambda S + \alpha^i Q_i$ with $\lambda > 0$. Acting on it with E one finds $S = \lambda^* E(S) + \alpha^{i*} Q_i$. Hence $\lambda = 1$ and α^i are purely imaginary. One can take $Q_{k+1} = S + \frac{1}{2}\alpha^i Q_i$.

Thus, E becomes the unit matrix.

$O(4,1)$ *Case*. Since now $EE^* = -\mathbf{1}$, one cannot have $E(Q)=\lambda Q$ because iteration would yield $-Q = \lambda^*\lambda Q$. Take now an arbitrary Q and construct $Q^\pm = Q \pm E(Q)$.

Then $E(Q^+) = -Q^-$ and $E(Q^-) = Q^+$. Hence, E acts on (Q^+, Q^-) as a symplectic metric. Now suppose one has constructed $2k$ charges $Q_1, \ldots, Q_{2k}$ on which E acts as a symplectic metric $\Omega_k = -i\sigma_2 \otimes \mathbf{1}_k$. Then, if S is linearly independent from $Q_1, \ldots, Q_{2k}$, also $E(S)$ will be linearly independent from $Q_1, \ldots, Q_{2k}$ and S. [Suppose $E(S) = \lambda S + \alpha^i Q_i$. Acting with E once more one would find $+S = \lambda^* E(S) + \alpha^{i*} E(Q_i) = \lambda^* \lambda S + Q$-terms, which is impossible.] So one can take $Q_{2k+1} = S + E(S)$ and $Q_{2k+2} = S - E(S)$. This concludes the proof.

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