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Review Article

Debris flow modeling: A review

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A debris flow represents a mixture of sediment particles of various sizes and water flowing down a confined, channel-shaped region (*e.g.*, gully, ravine or valley) down to its end, at which point it becomes unconfined and spreads out into a fan-shaped mass. This review begins with a survey of the literature on the physical-mathematical modeling of debris flows. Next, we discuss the basic aspects of their phenomenology, such as dilatancy, internal friction, fluidization, and particle segregation. The basic characterization of a debris flow as a mixture motivates the application of the continuum thermodynamical theory of mixtures to formulate a model for a debris flow as a viscous fluid-granular solid mixture. A major advantage of such a formulation, which goes beyond the most general models in the literature, *e.g.*, Takahashi (1991), is that it can be used to expose and better understand the assumptions underlying existing models, as well as to derive new, more sophisticated models. Finally, we delve into the issue of how such models have been or can be implemented numerically, as well as general boundary conditions for debris flows.

1 Introduction

Broadly speaking, a debris flow represents the gravity-driven flow of a mixture of various sizes of sediment (from clay to boulders), water and air, down a steep slope, often initiated by heavy rainfall and/or landslides. The volume solids concentration in the front part of such a flow varies between about 30 and 65%, and generally decreases toward the rear. The flow depth is of the order of 1 to several meters, mean velocities may be as high as 15 m/s, and channel gradients vary from about 40° in the starting zones to about 3° in the deposition zone.

In Switzerland, such flows cause considerable damage in mountain torrents during flood events. They seem to have increased in occurrence in the last few years, possibly by the retreat of the glaciers and the permafrost areas to higher elevations owing to the global climatic change. As a result, their scientific study seems to become more and more pressing in order to gain a better understanding of the physical mechanisms that govern their *initiation* at steep slopes, their *motion* along these, and their relatively *abrupt settlement* in the deposition areas.

Logically, research on debris flows has been chiefly conducted in countries with areas that are prone to debris flows. The most comprehensive and detailed observations and mathematical-physical studies have been conducted in Japan and China; however, countries with some natural potential of rockfalls, landslides, sturzstroms and avalanches, such as the western US and Canada, as well as France and Switzerland, have

Dedicated to Professor Reint de Boer upon the occasion of his 60th birthday also been involved. In fact, the present day theoretical formulations of the dynamics of snow avalanches are quite similar to, though somewhat less complex than, the proposed models of debris flows.

A perusal of the existing literature on debris flow models leaves the impression that very few of these are formulated on the basis of continuum mechanical principles. This fact often obscures the physical assumptions such models are based on, and makes the comparison of these with each another difficult. In particular, practically all models in the literature are based on a single-constituent material even though a debris flow is clearly a mixture, as discussed above. On this basis, the role played by the water in such flows, for example, can at best be incorporated parametrically, and certainly not dynamically. It is well-known, however, that water is the main contributor to debris flow initiation, and as such must be incorporated as a distinct constituent in any realistic mathematical model which aims at describing the initiation of such flows.

Beyond this issue, debris flow models are generally of a *global nature*, *i.e.*, the physical laws of balance of mass and momentum are stated in *integrated form*, either as hydraulic equations (of the Boussinsq type) or vertically integrated. This procedure hides primarily the configurational restrictions the equations are based upon. Without the necessary precautions, such formulations are generally only valid if the flow region conforms with the "thin layer" assumption, in which curvature effects of the basal topography and the free surface are insignificant. It is unlikely that this is the case when the debris material is flowing through gullies with relatively fast changes from steep to shallow topographies or with transition zones from channel to fan geometries.

Global models generally are fraught with the additional difficulty of finding constitutive relations for their closure variables in a particular local formulation. More specifically, in a hydraulic model, the basal shear traction must be formally, *i.e.*, constitutively, related to the cross sectional area (or debris flow depth), the volume flux and the distribution of the mass (which is generally assumed to be uniform). Such functional relations ought to be derived from the local form of the constitutive relations plus velocity and particle concentration profiles, which are, however, not explicitly known in a global model. As a result, closure conditions in global models are often postulated without recourse to detailed knowledge of constitutive behaviour (by, say, extending Chezy-Manning-Stickler-type formulas).

These examples may suffice to justify an in-depth analysis of the principles involved with the modeling of debris flows. This review will go beyond a mere discussion of existing models; indeed, new, more general models will also be derived on a sound continuum-mechanical basis, allowing us to delimit the range of applicability of the existing models. In doing so, a hierarchy of the proposed and existing models naturally emerges. As a by-product, in each of the presented class of models, the general form of the governing equations to be solved, as well as the relative importance of their terms, is obtained, the latter on the basis of dimensional analysis. This procedure involves a *physical aspect* through the postulates of material behaviour, as well as a *geometric aspect* in the form of the configurations that are analysed. It delimits automatically in the various cases the dominant (physical) processes to which the emerging field equations are applicable.

1.1 Structure of this review

This review begins with a survey of the existing literature and work on debris flows (Sect. 2). Next, we delve into the phenomenology of such flows (Sect. 3), something upon which constitutive models for the material behaviour of these are based. Typical features observed in debris flows that should be incorporated in general into any model for these include (1), dilatancy, (2), internal friction, (3), cohesion, (4) fluidization, and (5), particle segregation.

Debris flows exhibit *dilatancy* because shearing in such flows at constant confining pressure leads to expansion of the interstitial (pore) space. On the other hand, shearing at constant volume induces normal stresses in such bodies, and so should result in normal stress effects. Beyond these, such bodies exhibit, under given circumstances, both solid- and fluid-like material behaviour. For example, quasi-statically, such flows can pile up, a process influenced by *internal friction* and *cohesion*, implying that they support quasi-static shear stresses – a solid-like behaviour. At large strain rates, however, debris flows behave more like non-linear, highly-viscous fluids. Stress constitutive relations for the debris flow must then account for these possibilities. There are a number of ways to go about formulating such relations, and we discuss one such approach in detail (Sect. 3).

Fluidization is the process by which internal friction and cohesion in the flow are reduced or eliminated via the fluctuation of the solid particles and motion of the interstitial fluid, leading to an increase in the interstitial or pore volume (fraction) and consequently a greater mobility of the bulk material. A proper theoretical treatment of this process should include a model for the solid particle fluctuations, something that has not yet been done. Instead, one simply assumes that the viscosities depend on solid volume fraction (see Sect. 4).

Debris flows consist of grains which differ in size, shape, composition, and so on; *particle segregation* in a debris flow represents the process by which particles of different sizes are redistributed during motion. In particular, over longer timescales, larger particles are observed to move toward the top, and smaller ones correspondingly toward the bottom, of a debris flow. A realistic model of a debris flow should incorporate a description for such a spatio-temporal evolution of the particle size distribution, and in particular the phenomena of *inverse grading*; presently, however, no such model exists.

Section 4 details a viscous fluid-granular solid mixture model for a debris flow as an application of a continuum thermodynamical theory for a mixture of granular materials. The viscous-fluid constituent accounts for the slurry (*i.e.*, water and dissolved fine sediment) constituent, and the granular solid for the coarse sediment constituent, of a debris flow. In this approach, the two constituents are treated as interpenetrating material bodies interacting with each other mechanically at the same temperature. Consequently, each of these is described dynamically by distinct mass and momentum balance relations, and common energy and entropy balances (*i.e.*, those of the mixture as a whole). The fluid and solid are assumed incompressible in the sense that their true mass densities (*i.e.*, mass densities per unit constituent volume) are assumed constant; such an assumption, however, does not imply that the mass density of the mixture is constant, since the constituent volume fractions can still vary. We also distinguish between *saturated* and *unsaturated* mixtures, the former being the case in which the fluid constituent fills the interstitial space in the granular solid constituent completely, and the latter when it does not. Debris flow models are generally restricted to the saturated case.

Mass and momentum balance relations for the granular solid and viscous fluid constituents embody the evolution equations of the model, in which the constituent partial Cauchy stress and mechanical interaction rate density represent constitutive quantities depending on the constituent volume fractions, their spatial gradients, the constituent spatial velocities, their spatial gradients, and the rate of change of the solid volume fraction. In particular, this latter dependence is a direct result of the assumed *granular* nature of the solid constituent. We apply the results of the full thermodynamic formulation (Svendsen & Hutter 1995), and give explicit relations for the static and dynamic parts of the granular solid and viscous fluid partial Cauchy stresses, as well as the solid-fluid mechanical interaction rate density. In particular, the granular solid static stress tensor can be incorporated into the Mohr-Coulomb criterion for yielding of the granular solid, something which relates the mixture specific inner free energy to the internal friction and cohesion properties of the granular solid. We stress that this formulation is more general and encompassing than any other model for debris flows of which we are aware. Such a formulation and model are in fact necessary as a basis for the rational and physically coherent derivation of simplified models which can be used to develope numerical simulations for such flows. One such simplified model for the channel-flow regime of a debris flow can be obtained with the help of the so-called thin-layer approximation. The resulting simplified model equations reduce, in the context of two further assumptions, to the model proposed by Takahashi (1991). This thin-layer approximation can easily be extended to deal with the fan-flow regime. In addition, it points out the direction research on the subject must take to obtain simplified models which can be numerically implemented.

The issue of numerical implementation is considered in Sect. 5. In so doing, the viewpoint taken is somewhat more general than usual in the sense that the analytical reduction of a given model to a simpler form is also considered to be part of the numerical implementation. To begin, the existing or potential models are discussed in a hierarchical framework (see Table 5.1). Accordingly, the physical complexities range from two-constituent mixture and diffusive models to single-constituent models, and each of these classes can be subdivided into subclasses according to whether an equation for the fluctuation of the solid particles and/or turbulence in the fluid is taken into account. Each class is characterized as well by its own set of constitutive relations required to close the system of balance and/or evolution relations. For example, two-constituent mixture models require constitutive relations for the solid and fluid stresses, as well as the solid-fluid interaction rate density, while two-constituent diffusive models and single-constituent models require but a single constitutive relation for the mixture stress. Further details particular to each of these model classes are summarized in Table 5.1.

The above classification of debris flow models is based upon the relative physical sophistication of each model. Such models, however, can also be classified on the basis of the complexity inherent in their numerical implementation. In particular, we define hydraulic models as global descriptions of debris flows that are obtained from the established moving initial boundary value problem by averaging the full field equations over depth or cross-section (depending on whether or not the flow is confined cross-sectionally, as in the channel-flow regime, or free, as in the fan-flow regime). This averaging is achieved by integration of the full field relations in the directions approximately perpendicular to the main flow direction(s). In the existing literature, only Cartesian coordinates are used to do this; as shown by the work of Hutter and associates (see, *e.g.*, Hutter 1996), however, curvilinear coordinates should be used, in which one of the coordinate surfaces follows more or less the topographic surface over which the debris flow moves. Besides this, hydraulic models involve simplifications which are connected with the averaging process and assumed profiles of the field variables. Such profiles should also be predicted by the averaged field relations. The thin-layer approximation offers the possibility to do this in a more systematic fashion than is done in the works to be found in the reference list.

These disadvantages are circumvented to some degree by the higher-order hydraulic models which refine the averaging procedure via a spectral expansion of the fields in terms of functions from a complete function set such as Tshebyshev polynomials. The higher-order models can be developed by using the method of weighted residuals in conjunction with the above spectral expansion. The so-averaged equations can then be solved by the finite-difference or finite-elements methods.

A general difficulty with the numerical implementation of debris flow models consists in the specification of initial and/or boundary conditions, in particular since debris flows possess free boundaries, such that the region occupied by the debris flow moves with time, and the boundary of the debris flow must be determined along with the independent fields. Use of the Eulerian description in numerical models of debris flow often prove to be numerically unstable, while the Lagrangian description, at least in the context of granular avalanches, appears to lead to more stable numerical schemes, implying that this latter description should be used in any finite-difference or finite-element implementation of the model.

Finally, we come to the issue of boundary conditions in Sect. 6. The complexity of a particular debris flow model is also determined in part by the boundary conditions involved. In general, two bounding surfaces of a debris flow can be identified, *i.e.*, the free surface, and base, of the flow, the latter of which is in contact with the surface over which the debris flow moves. The nature and complexity of the boundary conditions at each of these is dependent on the class of models (see Table 1) chosen to model the flow. There are, however, common aspects to all cases, *i.e.*, the *kinematic conditions* allowing the incorporation of processes such as the erosion/deposition of sediment, and the drainage/supply of water, at the base. The remaining boundary conditions are *dynamic* in nature, and derived from the mass and momentum jump balance relations at each boundary.

Tables 1 and 2 summarize the boundary conditions formulated at the free surface and base of the debris flow. It is in particular evident that deposition and erosion, as well as supply and drainage of water, at the base, can reasonably be described only by a full two-constituent mixture model. In addition, erosion should probably be modeled in conjunction with field relations for the solid particle fluctuations and fluid turbulence. The kinematic relation at the base accommodates in fact the incorporation of these processes. In a similar fashion, precipitation can be accounted for at the free surface.

The dynamic conditions at the free surface imply that it is stress-free for all constituents, and so the mixture. At the base, however, no-slip conditions, or a viscous sliding relation, must be incorporated. These are also to be found in Tables 1 and 2.

The boundary conditions discussed in Sect. 6 are of a classical nature, and unproblematic, only if the field relations and/or constitutive relations do not require that boundary conditions be specified for the constituent volume fraction gradients. For models in which this is not the case, *i.e.*, that formulated in Sect. 4, such additional boundary conditions are then of course necessary. Unfortunately, these are quite difficult to formulate in a physically-meaningful fashion. In any case, such boundary conditions should be found prior to the numerical implementation of any debris flow model in which they play a role. Similarly, if a balance relation for the fluctuation energy is included, boundary conditions for this energy at both the free surface and base must be found before any numerical implementation of the corresponding model is undertaken.

2 Review of previous work

We give here a brief review of the variety of the current work on the mechanics of granular materials, and in particular that on debris flows. Since dry granular flows, avalanches, and debris flows are in principle related phenomena, the following survey is not exclusively restricted to the debris flow literature.

Perhaps the most up-to-date literature source available at the current time on the mechanics and modeling of debris flows is Takahashi's (1991) IAHR monograph[1], which gives a fairly critical account on the mechanisms of debris flows from their onset to deposition. It summarizes Takahashi's own extensive research work, and presents a detailed understanding of the mechanics of the flow of a layer of a particle-fluid mixture under simple gravity driven shear for Bagnold's (1954) grain inertia and macroviscous regimes. The model equations of the two-constituent model are eventually simplified to essentially a one-constituent model, and this view is maintained throughout. Time dependent processes, *i.e.*, development of a debris flow hydrograph and its deformation as well as snout behaviour are also discussed as are inverse grading and the transportation of large boulders on the free surface of a debris flow and the processes of deposition of sediments in the run-out zone. Considerations are all based on two-dimensional plane flow.

In a similar spirit is the work of Cheng-Lung Chen (1987). For simple plane shear flows under gravity (in which a shear stress and a normal stress are the only materially dependent stress variables that are introduced), Chen presents a detailled analysis of rheological models and deduces with these velocity profiles for steady gravity driven flow of a strictly parallel sided slab. We shall discuss these equations later on (see Table 1). In the debris flow literature, there appears to be no other work[2] that goes beyond Takahashi (1991, and previous work referred to there) and Chen (1987), except perhaps the in-depth, though descriptive, account of Iverson & Denlinger (1987). These authors delineate the range of applicability of the formulations and, in particular, point out the severe limitation "that steady uniform flows can exist only when the debris travels down a slope with a specific inclination. Chen (1987) discusses this phenomenon in detail, but does not seem to be bothered by this. The reason stated by Iverson & Denlinger seems to be that the variation of the grain concentration across the debris flow depth is ignored. The problem is that four equations for three unknowns exist in this case; they mandate a consistency condition which seems to be the reason for the mentioned peculiarity.

Somewhat hidden in existing formulations of the rheological behaviour of debris flows is the fact that *these relations cannot uniquely be extended to a three-dimensional form of the constitutive relations. In other words, two sets of general constitutive relations can in plane simple shear be indistinguishable*. When attempting to describe a dispersion of a channelized debris flow into the fanned deposition area this might be of some importance. Furthermore, debris flow specialists also generally abstain from introducing a variable and associated field equation for the internal structure, say the fluctuations of the velocity and particle concentration fields due to grain collisions and/or possible turbulence in the interstitial fluid flow. In the granular flow literature this field is generally of scalar nature: the *collisional fluctuation energy* or so-called *granular temperature*. From this point of view, the granular literature should also be consulted, e.g., Scheiwiller & Hutter (1982), or Hutter & Rajagopal (1994). Both works address the formulation of the constitutive relations for granular materials under rapid shearing. Both contain extensive literature reviews on constitutive modelling, but they do not present formulations of flow models deduced from a set of constitutive relations. Hutter & Rajagopal (1994) also do not address the models suggested by molecular dynamics, in which a large number (several thousands) of rigid particles are followed in time under free motion and colliding with each other. Interaction rules for collisions are formulated, the equations of motion of all the particles integrated, and followed through time, taking into account the free flow and collisions. Campbell (1990) reviews these methods,

[1] This book contains a large number of references to Takahashi's work and that of others, but it is probably fair to say that it gives a coherent account of the theoretical formulation of debris flow as Takahashi sees it.

[2] Jan & Shen (1993), Hungr (1994), Coussot (1994), O'Brien et al. (1993), Laigle & Coussot (1993) and others show particular aspects of the rheological behaviour, which, however, all are more restrictive than Chen & Takahashi.

and Straub (1995) demonstrates in a voluminous dissertation its use in pyroclastic flows. Its application to fluid-grain-grain interaction has not been attempted so far.

Consider next the problem of the derivation of *evolution equations*. In particular, hydraulic or Boussinesq-type theories have been obtained by, for instance, MacArthur & Schamber (1986), Coussot (1994), Laigh & Coussot (1993), O'Brien et al. (1993), and Montefusio (1994), and exclusively consist in establishing vertically, or cross sectionally, integrated, balance laws of mass and momentum in a Cartesian reference frame, in one occasion restricted to the *kinematic wave approximation*. In this approximation, one restricts considerations to a global mass balance relation for the mixture as a whole,

$$h_{,t} + Q_{,x} = 0 \quad , \tag{2.1}$$

in which h is flow depth, and Q the volume flux, $h_{,t} = \partial h/\partial t$, $Q_{,x} = \partial Q/\partial x$, and writes a constitutive equation for Q, usually by considering steady state momentum balance to connect Q with basal and turbulent friction, etc., see, *e.g.*, Hutter (1983). Only in a single case were these balance laws complemented by a balance of mass for the solids, thus allowing particle segregation mechanisms and deposition or erosion along the debris flow path to be accounted for (Takahashi et al. 1992). In a single paper by Jenkins & Askasi (1994), a hydraulic theory for a debris flow is presented in which the particle fluctuation energy affects the evolution of the flow.

The drawbacks of these formulations have been pointed out before – use of a Cartesian formulation requires that the topography is flat, expressions for the basal drag cannot clearly be related to constitutive postulates, nonlinear advective terms in the momentum equation cannot be properly estimated. Very similar concepts, however, have been developed in the theory of snow and granular avalanches[3]. A fairly up-to date summary on this subject is contained in Hutter (1996). Through comparison of theory and laboratory experiments it is shown that the curvature of the topography affects the solution non-negligibly and thus should not be ignored. Hutter's (1996) review also contains an extensive treatment of powder snow avalanches, which are two-phase mixtures with balance laws of mass and momentum for both constituents. The works[4] discussed there indicate, how (i) density variations and thus particle segregation including deposition and erosion can be dealt with, (ii) how microstructural effects could be incorporated (*e.g.*, turbulence) and (iii) how hydraulic models can be constructed that amend the above mentioned drawbacks.

From another viewpoint, the existing literature may be characterized according to whether a model for debris or granular flows is formulated, or applied in the context of a physically-relevant initial-boundary value problem. In the former category, one finds such works as Chen (1987), Takahashi (1991), Hutter & Rajagopal (1994), and Hutter (1995); in these, the constitutive behaviour of a granular material that may exhibit debris flow characteristics is discussed. Such constitutive models can be formulated on a sound continuum thermodynamical basis, as shown by, *e.g.*, Goodman & Cowin (1972), Passman et al. (1984), or more recently in an extended context by Svendsen & Hutter (1995). In the latter category belongs the work of, *e.g.*, O'Brien et al. (1993), who present a depth-averaged hydraulic model for the fan-flow regime of a debris flow. Focusing on the computer implementation of their debris flow model, they do not, unfortunately, invest time in discussing or appreciating its theoretical limitations. Such limitations are discussed, *e.g.*, in the works of Hutter and his associates (see, *e.g.*, Hutter (1996)), and are outlined in Sect. 5.

Finally, it is also perhaps worth mentioning that no model appears sufficiently general to deal with processes such as erosion and/or deposition of sediment. Such processes are governed predominantly by turbulence in the fluid and agitation of the solid particles at the base of the flow. Consequently, these processes cannot be left out of any model hoping to address erosion/deposition. Ideas on how these processes can be modeled are to be found in the literature on turbidity currents and powder-snow avalanches, and are briefly reviewed in Hutter (1996).

[3] see, *e.g.*, Hutter et al. (1988a,b), Hutter (1989), Savage & Hutter (1989, 1991), Hutter & Nohguchi (1990), Hutter (1991), Hutter & Koch (1991), Greve & Hutter (1993), Hutter & Greve (1993), Greve, Hutter & Koch (1994), Koch, Greve & Hutter (1994), Hutter et al. (1993a,b), Koch (1994), Wieland (1995).

[4] In particular, we draw attention to Parker (1982), Parker et al. (1986), Tesche (1986, 1987), Scheiwiller, Hutter & Hermann (1987). Of interest, but not mentioned, is Gauer (1994).

3 Aspects of the phenomenology of debris flows

The subject of this review is the theory of continuous structured mixtures of solid grains of various sizes and an interstitial fluid which may itself be a slurry, *i.e.*, a mixture of water with very small suspended particles. Granular materials are of this kind, and they exhibit certain features that are common with debris flows: *dilatancy, internal friction, cohesion, fluidization* and *particle segregation*, but only the first three are incorporated in the theoretical models that were so far proposed. In fact, none of the concepts presented in this review exhibits enough complexity to incorporate particle segregation.

3.1 Dilatancy

If an array of identical spherical particles at closest packing is subject to a load so as to cause a shear deformation, then from pure geometrical consideration that particles must ride over one another it follows that an increase in volume of the bulk material will occur. This property was termed *dilatancy* by Reynolds (1889). It also occurs in rapidly sheared granular systems but is there due to the particle collisions and the *dispersive pressure* that is generated by these particle encounters (Bagnold 1954, 1966). Accompanied with this dilatancy is the phenomenon of *normal stress effects* that granular materials encounter.

Quite naturally, dispersive pressure can only develop when particles in a grain-fluid mixture flow feel their neighbours, *i.e.*, if neighbouring particles do interact with each other. If they don't, or if they do only in a limited fashion, they modify the viscous properties of the interstitial fluid (Einstein 1906, 1911; Mooney 1951; Moni and Ototake 1956; Batchelor and Green 1972). This flow regime is called the *macro-viscous regime* and the constitutive response is essentially of Newtonian type. In the other extreme, only the particle collisions are significant and the role played by the interstitial fluid can be ignored. This regime was coined *grain-inertia regime* by Bagnold (1954). Rapid shearing is one mechanism that gives rise to dispersive pressure; this is the manifestation of dilatancy mentioned above.

This description suggest that the two limiting cases are likely describable by a single-constituent continuum model, wereas such a simplification is less obviously suggested when both the grains and the interstitial fluid interact dynamically.

3.2 Internal friction and cohesion

Debris flows exhibit both solid and fluid behaviour. At rest they can be piled up in a heap and the free surface can be inclined at some maximum corresponding to the material's angle of repose. This behaviour is the result of friction and possibly cohesion between grains and is described generally by the *Mohr-Coulomb yield criterion.* It states that yielding will occur in the material on a surface if the resolved shear stress τ on this surface reaches the critical value

$$|\tau| = c + \sigma \tan\phi \tag{3.1}$$

where σ is the resolved normal stress, ϕ the internal angle of friction, and c the cohesion (see Sect. 4). The quasi-static form of ϕ generally takes values between 20° and 45°, while its dynamic counterpart is about 3-4° smaller under rapid shear (but slow enough that rubbing-type friction between the particles occurs). The Mohr-Coloumb criteria (3.1) only describes the conditions for yield; a corresponding (non-associated) flow rule for the ongoing motion beyond plastic yield must, if needed, be separately formulated. Most descriptions restrict themselves to simple gravity driven shear, see, *e.g.*, Kupper (1967); a full account is given by Ehlers (1993).

This is perhaps the place where the flow behaviour beyond yield should be more closely addressed on this more general level. In general terms, the constitutive behaviour in this range of stress and strain rate exhibits *plastic* as well as *viscous* behaviour. Here the term "plastic" means that the stress is related to strain rate in a rate-independent fashion, while "viscous" means that this dependence is rate dependent. The notion of a yield surface need not necessarily be associated with this definition of plasticity. For instance, if $\mathbf{D}$ is strain rate and II_D its second invariant (see (4.32)), then $\mathbf{D}/\sqrt{|II_D|}$ is rate independent. A constitutive relation for the Cauchy stress tensor $\mathbf{T}$ of the form $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{D}/\sqrt{|II_D|})$ is then a *plastic* relation, $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{D})$ a *viscous* one, and $\mathbf{T} = \hat{\mathbf{T}}(\mathbf{D}, \mathbf{D}/\sqrt{|II_D|})$ *viscoplastic*. Other such constitutive relations can be formulated as well.

Experience indicates that soil under quasistatic loads exhibits plastic behaviour, that rapid shearing of a debris flow is predominantly viscous. Evidently, if the flow of a debris current in the channelized downhill area as well as in the fan-shaped settlement zone are of interest, both viscous and plastic features must be incorporated in the constitutive model, and indeed they are.

Examples of purely viscous behaviour are the Newtonian fluids, the Bagnold fluid and essentially all constitutive models derived from statistical mechanics (see Hutter and Rajagopal 1994). Purely plastic behaviour is exhibited by all Mohr-Coulomb models (see Ehlers 1993). Simple viscoplastic models are the Bingham and Herschel-Bulkley bodies (see Chen 1987; Coussot 1994). We shall not deal at all with these latter models here because they do not account properly for the static stresses in granular materials.

3.3 Fluidization

Large masses of rocks, soils and snow occasionally spread out into very thin layers, flow on surfaces that are much less inclined than the angle of repose of the material and travel distances that are correspondingly large. The reason is that during their motion, granular flows behave like non-Newtonian fluids, due mainly to the fact that the high shear deformation leads to high dispersive pressure and thus reduced effective viscosity. Of course, in water laden debris flow the dilation that is accompanied with the increased dispersive pressure, water is accumulated in these high mobility zones and may serve as an additional lubricant. In dry granular flow experience from laboratory experiments indicates that large shearing is restricted to a very thin zone in a basal near boundary layer. Thus the effect of fluidization can be absorbed in a basal boundary condition, *e.g.*, by introducing a Coulomb-type basal friction law with reduced bed friction angle δ. In water saturated debris flows large shearing occurs throughout the flow depth (Takahashi 1991), *i.e.*, the entire flow depth consists of the boundary layer. It follows that if global (hydraulic) models are used, where only the basal tractions enter the formulation, bulk and boundary effects cannot easily be separated.

3.4 Particle segregation

Debris flows consist of grains which differ in size, shape, *etc.* (but generally not density). When such a material is agitated or deformed in the presence of a gravitational field, *segregation* or *grading of the particles* can occur and particles having the same or similar properties tend to collect together. In a tear-drop like debris flow starting from a uniform distribution of particle size, shape etc. large particles tend with time to accumulate at the front and at the free surface. In the geological literature this phenomenon is called *reverse grading* or *inverse grading* (Middleton 1970; Middleton and Hampton 1976; Sallenger 1979; Naylor 1980). Early physical explanations on the basis of a granular fluid with indistinguishable grains (Bagnold 1954; Takahashi 1981, 1983) was found unsuitable (Iverson and Denlinger 1987). Adequate explanations require a granular fluid model with grains of different sizes.

The quantitative explanation by Savage and Lun (1988) is based on what they call the *random fluctuating sieve mechanism* and requires high particle concentration and slow shearing dry rubbing friction and diffusion due to particle collisions can be ignored. If we envisage a layer of particles parallel to its mean motion, then at any instant if the void space is large enough, then a particle from the layer above can fall into it as the adjacent layers move relative to one another. The probability of finding a hole that a small particle can fall into is larger than the probability of finding a hole that a large particle can fall into. This is the basic segregation mechanism that gives rise to the inverse grading.

Iverson and Denlinger (1987) lists two other possible mechanisms by which inverse grading can be achieved. In the first the dispersive stress model due to particle collisions is used: assume that the mean velocity profile is convex upward and shear as well as dispersive pressure are proportional to the velocity gradient squared. Under such a situation lower layers are subject to larger shearing than higher layers of a gravity current; thus collisions are more intense at depth than above and particles of finite diameter are more often and more intensely hit from below than from above suggesting an upward drift that must be larger for large than for small particles. Inverson points out that "this logic to explain inverse grading is necessarily circular and therefore unsatisfactory: it relies on uniform-concentration and uniform-size assumptions to explain nonuniform-concentrations of nonuniformly sized grains."

The other mechanism requires that the grain mixture is sheared, so *rotations* are imparted to particles during collision during frictional interactions. In a shear flow, large particles will tend to climb over small particles when they have prolonged contacts. This process eventually will lead to an increased concentration of large grains toward the flow surface.

The above mechanisms that explain particle segregation make use of the presence of particles of different sizes and their acquaintance of angular momentum. It follows that theoretical concepts not incorporating any one of these cannot describe particle segregation. To date there is no theory that would be general enough to do so. Its least complex structure must be a classical mixture of water and grains *that is complemented by an evolution equation for the paticle size distribution.* This probably does not suffice; inclusion of rotation would require a *polar theory* that accounts explicitly for the nontrivial grain rotations. Such models have been proposed by Kanatani (1979a,b, 1980a,b), but have only seen a limited exploration.

4 Fluid-solid mixture models for debris flows

4.1 Basic considerations and balance relations

Consider a model of a debris flow as a mixture of (1), a slurry (made up of fine-grained sediment and water) modeled as a viscous fluid, and (2), a coarse-grained sediment consisting of solid sediment "particles" and interstitial space modeled as a *granular* solid. Let the underscript "f" stand for the slurry (*i.e.*, water and fine sediment), and "s" for the granular solid (*i.e.*, coarse sediment and interstitial space), in what follows. The mass density ϱ of such a *two-constituent* mixture is given by

$$\varrho = \bar{\varrho}_f + \bar{\varrho}_s \quad , \tag{4.1}$$

where $\bar{\xi}_\alpha$ ($\alpha = s, f$) represents the constituent *partial* mass density, *i.e.*, its mass density per unit *mixture* (fluid plus granular solid) volume. An alternative form of (4.1) is obtained by dividing both sides by ϱ, yielding

$$1 = \bar{\xi}_s + \bar{\xi}_f \quad , \tag{4.2}$$

with

$$\bar{\xi}_\alpha = \bar{\varrho}_\alpha / \varrho \tag{4.3}$$

the constituent mass fraction in the mixture. Now, since the mass of a granular solid is determined solely by that of its *solid particles*, *i.e.*, the interstitial space is massless, $\bar{\varrho}_s$ must depend in some fashion on the mass density ϱ_s of the solid particles, *i.e.*, the solid mass per unit volume of *solid particles* in the mixture. Indeed, we have in general

$$\frac{\text{constituent mass}}{\text{unit mixture volume}} = \frac{\text{unit constituent volume}}{\text{unit mixture volume}} \, \frac{\text{constituent mass}}{\text{unit constituent volume}} \quad \Longrightarrow \quad \bar{\varrho}_\alpha = \nu_\alpha \varrho_\alpha \tag{4.4}$$

for $\alpha = s, f$, where ν_α represents the constituent *volume fraction* (density), *i.e.*, the unit constituent volume per unit mixture volume, taking values as such between 0 and 1. Analogous to $\bar{\varrho}_\alpha = \nu_\alpha \varrho_\alpha$, a superposed bar over any variable in what follows signifies that that variable is weighted with the corresponding volume fraction.

Now, if the fluid happens to fill the entire interstitial space in the granular solid, the mixture is referred to as *saturated*, and the sum of the fluid and *solid particle* volumes equals that of the mixture, *i.e.*,

$$1 = \nu_s + \nu_f \tag{4.5}$$

holds. On the other hand, when the fluid only partly fills this space, the mixture is said to be *unsaturated*; in this case, ν_f and ν_s vary independently.

In this work, we assume for simplicity that the fluid and solid constituents are *incompressible* in the sense that

$$\varrho_{\alpha} = \text{const.}\ , \qquad \alpha = \mathrm{s},\mathrm{f}\ , \tag{4.6}$$

holds, *i.e.*, the constituent true mass densities are constant. In this case, it's useful to write ϱ in the form

$$\varrho = \begin{cases} \varrho_{\mathrm{s}}[r\,\nu_{\mathrm{f}} + \nu_{\mathrm{s}}] & \text{unsaturated} \\ \varrho_{\mathrm{s}}[r + (1-r)\nu_{\mathrm{s}}] & \text{saturated} \end{cases} \tag{4.7}$$

via (4.4), where

$$r = \varrho_{\mathrm{f}} / \varrho_{\mathrm{s}} \tag{4.8}$$

represents the ratio of the fluid to granular solid true mass density, and is constant. Under the assumption (4.6), then, ϱ, and so ξ_{f} and ξ_{s} via (4.3), vary only with ν_{f} and ν_{s} in the unsaturated, and only with ν_{s} in the saturated, case.

Although it is possible that, in the course of the flow, collisions of solid particles could lead to the creation of fine sediments which would increase the mass fraction of the viscous slurry, we assume for simplicity that this process is negligible. In this case, the fluid and solid constituents may be assumed not to exchange mass. Under this assumption, the constituent mass and momentum balance relations are given by

$$\bar{\varrho}_{\alpha,t} + \operatorname{div}(\bar{\varrho}_{\alpha}\mathbf{v}_{\alpha}) = 0\ , \qquad \alpha = \mathrm{s},\mathrm{f}\ , \tag{4.9}$$

and

$$\bar{\varrho}_{\alpha}\grave{\mathbf{v}}_{\alpha} = \operatorname{div}\mathbf{T}_{\alpha} + \bar{\varrho}_{\alpha}\mathbf{g} + \mathbf{m}_{\alpha}\ , \qquad \alpha = \mathrm{s},\mathrm{f}\ , \tag{4.10}$$

respectively. In these relations, $f_{,t} = \partial f/\partial t$ and ∇f represent the partial time derivative and gradient, respectively, of any time-dependent field f, $\mathbf{v}_{\alpha}$ is the constituent spatial velocity, $\grave{f} = f_{,t} + (\nabla f)\,\mathbf{v}_{\alpha}$ the material time derivative of any time-dependent field f with respect to $\mathbf{v}_{\alpha}$, $\mathbf{T}_{\alpha}$ the constituent *partial* stress tensor, $\mathbf{g}$ the specific gravitational body force, and $\mathbf{m}_{\alpha}$ the constituent momentum exchange rate density, which satisfies

$$\mathbf{m}_{\mathrm{f}} = -\mathbf{m}_{\mathrm{s}} \tag{4.11}$$

by Newton's third law in the fluid-solid mixture under consideration here. Note that entrainment and other such processes are not taken into account in (4.9) and (4.10); indeed, we would have to add a mass supply rate density term to (4.9), and a momentum supply rate density term to (4.10), to account for such processes in the balance relations.

Note that the constituent mass balance relations (4.9) represent in general evolution relations for the *two* independent variables $\bar{\varrho}_{\mathrm{s}} = \nu_{\mathrm{s}}\,\varrho_{\mathrm{s}}$ and $\bar{\varrho}_{\mathrm{f}} = \nu_{\mathrm{f}}\,\varrho_{\mathrm{f}}$, respectively. On the basis of the constant true density assumption (4.6), however, these reduce to such relations for the ν_{s} and ν_{f}, and further to two evolution relations for *one* unknown ν_{s} in the context of the saturation constraint (4.5). Consequently, this latter constraint leads to the loss of an independent variable. As such, this latter constraint is formally analogous to the classical constant density ("incompressibility") constraint for a fluid, in which case the mass density is lost as such a variable, to be "replaced" by the pressure maintaining the constraint as a new unknown. By analogy, in the current mixture context, this new unknown pressure p replaces ν_{f} lost via (4.5) as an unknown in the model, and maintains (4.5). Hence, we refer to p as the saturation pressure. In this context, (4.9) and (4.10), which originally represented 8 scalar equations containing in general the 8 independent unknowns $\bar{\varrho}_{\mathrm{s}}$, $\bar{\varrho}_{\mathrm{f}}$, $\mathbf{v}_{\mathrm{s}}$ and $\mathbf{v}_{\mathrm{f}}$, as well as the 15 constitutive unknowns $\mathbf{T}_{\mathrm{s}}$, $\mathbf{T}_{\mathrm{f}}$ and $\mathbf{m}_{\mathrm{s}}$ (with $\mathbf{m}_{\mathrm{f}} = -\mathbf{m}_{\mathrm{s}}$), become 8 equations in the 8 unknowns p, ν_{s}, $\mathbf{v}_{\mathrm{s}}$ and $\mathbf{v}_{\mathrm{f}}$ in the saturated case, and those ν_{f}, ν_{s}, $\mathbf{v}_{\mathrm{s}}$ and $\mathbf{v}_{\mathrm{f}}$ in the unsaturated case. It remains then to determine the dependences of $\mathbf{T}_{\mathrm{s}}$, $\mathbf{T}_{\mathrm{f}}$ and $\mathbf{m}_{\mathrm{s}}$ on the independent variables, our next task.

4.2 General constitutive model

For a debris flow, we assume that the solid constituent, *i.e.*, the coarse sediment and interstitial space, can together be modeled as a *granular solid*. Further, the fluid constituent, *i.e.*, the water and fine sediment slurry occupying some and/or all of the interstitial space between the solid particles, behaves as a *viscous fluid.* For such a mixture, constitutive relations take the general form[5]

$$\mathscr{C} = \hat{\mathscr{C}}(\nu_s, \zeta, \mathbf{v}_s, \mathbf{v}_f, \mathbf{w}_s, \nabla \mathbf{v}_s, \nabla \mathbf{v}_f) \qquad (4.12)$$

by equipresence, with $\zeta = \nu_f$ in the unsaturated, and $\zeta = p$ in the saturated, case, as discussed in the last section, for $\mathscr{C} \in \{\mathbf{T}_s, \mathbf{T}_f, \mathbf{m}_s\}$, and

$$\mathbf{w}_s := \nabla \nu_s \qquad (4.13)$$

the solid constituent volume fraction gradient. Whereas $\nabla \nu_f$ does not appear as an independent variable in (4.12) since we assume the fluid constituent behaves as a viscous fluid, the dependence of $\hat{\mathscr{C}}$ on the solid volume fraction gradient $\nabla \nu_s$ reflects the *granular* nature[6] of the solid constituent, and represents in essence the effect of spatial *inhomogeneity* in the solid distribution on the material behaviour of the mixture, something that becomes especially important when the corresponding volume fraction becomes substantial. In the case of a debris flow, the solid volume fraction varies between 30 and 65% at the front of the flow, and decreases with distance away from the front. Consequently, longitudinal and cross-sectional solid volume fraction gradients in such flows are to be expected. Such inhomogeneity in the granular solid can lead to additional effects such as dilatant behaviour, *i.e.*, the expansion of the interstitial volume during deformation of a tightly-packed granular material.

Constitutive relations such as (4.12) are subject to general constitutive requirements such as observer independence, reducing it to the form

$$\mathscr{C} = \hat{\mathscr{C}}(\nu_s, \zeta, \mathbf{v}_{sf}, \mathbf{w}_s, \mathbf{D}_s, \mathbf{D}_f, \mathbf{W}_{sf}) \quad , \qquad (4.14)$$

with

$$\mathbf{v}_{sf} := \mathbf{v}_s - \mathbf{v}_f \qquad (4.15)$$

the difference or relative velocity, $\mathbf{D}_a$ the symmetric part of $\nabla \mathbf{v}_a$, $\mathbf{W}_a$ its skew-symmetric part, and

$$\mathbf{W}_{sf} := \mathbf{W}_s - \mathbf{W}_f \qquad (4.16)$$

the difference or relative spin. In addition, constitutive relations are required to be compatible with the second law of thermodynamics. Under the further assumption that the constituent material behaviour is isotropic (which holds in any case for the viscous fluid constituent), the corresponding restrictions on forms such as (4.14) have been obtained elsewhere (Svendsen and Hutter 1995). For example, on the basis of (4.14), $\mathbf{T}_a$ is reduced to the form

$$\mathbf{T}_a = \mathbf{S}_a + \boldsymbol{\Sigma}_a \quad , \qquad (4.17)$$

where

$$\mathbf{S}_a = \hat{\mathbf{T}}_a(\nu_s, \zeta, \mathbf{0}, \mathbf{w}_s, \mathbf{0}, \mathbf{0}, \mathbf{0}) \qquad (4.18)$$

is the static or equilibrium contribution, and

$$\boldsymbol{\Sigma}_a = \hat{\boldsymbol{\Sigma}}_a(\nu_s, \zeta, \mathbf{v}_{sf}, \mathbf{w}_s, \mathbf{D}_s, \mathbf{D}_f, \mathbf{W}_{sf}) \qquad (4.19)$$

the dynamic contribution, to $\mathbf{T}_a$, such that

[5] Since we assume the mixture is isothermal in this work, the temperature is left out of all constitutive functions.

[6] In more elaborate structured mixture approaches , *e.g.*, Passman et al. (1984), a dependence on the material time derivative of the solid volume fraction is also considered for a granular material. Since we have no sepearate balance relation for the "equilibrated inertia" to consider here, however, we leave this out.

$$\underset{a}{\hat{\boldsymbol{\Sigma}}}\,(\underset{s}{\nu},\zeta,\mathbf{0},\underset{s}{\mathbf{w}},\mathbf{0},\mathbf{0},\mathbf{0}) = \mathbf{0} \quad . \tag{4.20}$$

More specifically, the form

$$\underset{s}{\mathbf{S}} = -\underset{s}{\pi}\,\mathbf{I} + 2\,\underset{s}{\alpha}\,\underset{s}{\mathbf{M}} \tag{4.21}$$

is obtained for $\underset{s}{\mathbf{S}}$, where

$$\underset{s}{\pi} = \begin{cases} \underset{s}{\beta} & \text{unsaturated} \\ \underset{s}{\beta} + p & \text{saturated} \end{cases} \tag{4.22}$$

is the solid static pressure,

$$\underset{a}{\beta} := \varrho\,\psi_{,\underset{a}{\nu}} \tag{4.23}$$

the constituent configuration pressure (a = s, f),

$$\psi = \hat{\psi}(\underset{s}{\nu},\zeta,\underset{s}{\mathbf{w}}\cdot\underset{s}{\mathbf{w}}) \tag{4.24}$$

the mixture inner specific free energy,

$$\underset{s}{\alpha} = \varrho\,\psi_{,\underset{s}{\mathbf{w}}\cdot\underset{s}{\mathbf{w}}} \tag{4.25}$$

and

$$\underset{s}{\mathbf{M}} := \underset{s}{\mathbf{w}} \otimes \underset{s}{\mathbf{w}} \quad . \tag{4.26}$$

As pointed out by, *e.g.*, Savage (1979), the form (4.21) for $\underset{s}{\mathbf{S}}$ predicts non-zero shear stresses in the granular solid in equilibrium only if $\underset{s}{\nu}$ is inhomogeneous or non-uniform in at least two different directions. Turning next to the fluid static partial stress tensor $\underset{f}{\mathbf{S}}$, we have

$$\underset{f}{\mathbf{S}} = -\underset{f}{\pi}\,\mathbf{I} \tag{4.27}$$

with

$$\underset{f}{\pi} = \begin{cases} \underset{f}{\beta} & \text{unsaturated} \\ p & \text{saturated} \end{cases} \tag{4.28}$$

the corresponding static fluid pressure. The *result* (4.28) of the thermodynamic analysis lends credence to the standard *interpretation* of the saturation pressure p as the fluid static true pressure. Lastly, we obtain the form

$$\underset{s}{\hat{\mathbf{m}}}(\underset{s}{\nu},\zeta,\mathbf{0},\underset{s}{\mathbf{w}},\mathbf{0},\mathbf{0},\mathbf{0}) = \begin{cases} (1-\underset{s}{\bar{\xi}})\underset{s}{\beta}\underset{s}{\mathbf{w}} - \underset{s}{\bar{\xi}}\,\underset{f}{\beta}\,\underset{f}{\mathbf{w}} & \text{unsaturated}\ , \\ [p + (1-\underset{s}{\bar{\xi}})\underset{s}{\beta}]\underset{s}{\mathbf{w}} & \text{saturated}\ , \end{cases} \tag{4.29}$$

for the equilibrium or static part of the solid-fluid momentum exchange rate density $\underset{s}{\mathbf{m}}$.

The reduced forms (4.24)–(4.28) for the dependent constitutive quantities obtained via the second law are as yet too general to be of use in the formulation of initial-boundary value problems describing the debris flow. As such, we must introduce further assumptions into the formulation in order to obtain more specific forms for these constitutive quantities. A purely formal assumption in this direction, introduced mainly to simplify the formulation, is for example the "principle of phase separation" of Passman et al. (1984). In the current context, this represents the assumption that $\underset{s}{\hat{\boldsymbol{\Sigma}}}$ depends only on quantities associated with the solid, and $\underset{f}{\hat{\boldsymbol{\Sigma}}}$ only on those associated with the fluid, *i.e.*,

$$\begin{aligned} \underset{s}{\boldsymbol{\Sigma}} &= \underset{s}{\hat{\boldsymbol{\Sigma}}}\,(\underset{s}{\nu},\underset{s}{\mathbf{w}},\underset{s}{\mathbf{D}})\ , \\ \underset{f}{\boldsymbol{\Sigma}} &= \underset{f}{\hat{\boldsymbol{\Sigma}}}(\underset{f}{\nu},\underset{f}{\mathbf{D}})\ , \end{aligned} \tag{4.30}$$

with $\underset{f}{\nu} = 1 - \underset{s}{\nu}$ in the saturated case via (4.19). Note that $\underset{s}{\boldsymbol{\Sigma}}$ and $\underset{f}{\boldsymbol{\Sigma}}$ do not depend on p, $\underset{sf}{\mathbf{v}}$ and $\underset{sf}{\mathbf{W}}$ in this case because these quantities are not associated with any one constituent. This notion of phase separation does not apply to *mixture* properties such as ψ, or to constitutive quantities describing constituent interactions, *e.g.*, $\underset{s}{\mathbf{m}}$. The assumed isotropy of the fluid and solid reduces the forms (4.30) yet further to

$$\begin{aligned} \underset{s}{\Sigma} &= \underset{s}{\kappa}_1 \mathbf{I} + \underset{s}{\kappa}_2 \underset{s}{\mathbf{D}} + \underset{s}{\kappa}_3 \underset{s}{\mathbf{D}}^2 + \underset{s}{\kappa}_4 \underset{s}{\mathbf{M}} + \underset{s}{\kappa}_5 \langle \underset{s}{\mathbf{M}}, \underset{s}{\mathbf{D}} \rangle + \underset{s}{\kappa}_6 \langle \underset{s}{\mathbf{M}}, \underset{s}{\mathbf{D}}^2 \rangle , \\ \underset{f}{\Sigma} &= \underset{f}{\kappa}_1 \mathbf{I} + \underset{f}{\kappa}_2 \underset{f}{\mathbf{D}} + \underset{f}{\kappa}_3 \underset{f}{\mathbf{D}}^2 , \end{aligned} \tag{4.31}$$

where $\langle \mathbf{A}, \mathbf{B} \rangle = \mathbf{AB} + \mathbf{BA}$ for any tensors $\mathbf{A}$ and $\mathbf{B}$. The coefficients $\underset{s}{\kappa}_{1-6}$ are functions of $\underset{s}{\nu}$, $I_{\underset{s}{D}}$, $II_{\underset{s}{D}}$, $III_{\underset{s}{D}}$, $I_{\underset{s}{M}}$, $I_{\underset{s}{M}\underset{s}{D}}$, and $I_{\underset{s}{M}\underset{s}{D}^2}$, while $\underset{f}{\kappa}_{1-3}$ depend on $\underset{f}{\nu}$, $I_{\underset{f}{D}}$, $II_{\underset{f}{D}}$ and $III_{\underset{s}{D}}$, in general, where

$$\begin{aligned} I_C &= \mathrm{tr}(\mathbf{C}) , \\ II_C &= \tfrac{1}{2}(I_C^2 - I_{C^2}) , \\ III_C &= \tfrac{1}{6}(I_C^3 - 3 I_C I_{C^2} + 2 I_{C^3}) , \end{aligned} \tag{4.32}$$

for any tensor $\mathbf{C}$. Since the definition (4.26) of $\underset{s}{\mathbf{M}}$ implies $\underset{s}{\mathbf{M}}^2 = I_{\underset{s}{M}} \underset{s}{\mathbf{M}}$ with $I_{\underset{s}{M}} = \underset{s}{\mathbf{w}} \cdot \underset{s}{\mathbf{w}}$, note that $II_{\underset{s}{M}} = 0 = III_{\underset{s}{M}}$.

Besides (4.30), another very useful formal assumption is *quasi-linearity*, *i.e.*, that vector- and tensor-valued constituent quantities are assumed to depend *explicitly* and *linearly* on vector-valued and tensor-valued independent variables, respectively, via scalar-valued coefficients which themselves depend on these and the scalar-valued independent variables. A special case of this is *linearity*, which arises when the scalar-valued coefficients in the quasi-linear form are assumed to depend at most on the scalar-valued independent variables. Such a form is indeed the simplest, and when there are no observations, experiments or other physical reasons to believe that the constitutive processes involved are more complicated, it seems sensible to work with this linear form. Having no such information to the contrary, and for simplicity, we assume in this work that the constituent mechanical interaction rate density $\underset{s}{\mathbf{m}}$, as well as the fluid dynamic stress tensor $\underset{f}{\Sigma}$, can be adequately represented by their linear forms. For $\underset{s}{\mathbf{m}}$, then, we have

$$\underset{s}{\mathbf{m}} = m_D \underset{sf}{\mathbf{v}} + \begin{cases} (1-\underset{s}{\bar{\xi}})\underset{s}{\beta}\underset{s}{\mathbf{w}} - \underset{s}{\bar{\xi}}\,\underset{f}{\beta}\,\underset{f}{\mathbf{w}} & \text{unsaturated} \\ [p + (1-\underset{s}{\bar{\xi}})\underset{s}{\beta}]\underset{s}{\mathbf{w}} & \text{saturated} \end{cases} \tag{4.33}$$

in the context of (4.14) and (4.29), where

$$m_D = \hat{m}_D(\underset{s}{\nu}, \zeta) \tag{4.34}$$

is the drag coefficient between the solid and fluid. Recall that the result (4.33) for $\underset{s}{\mathbf{m}}$ also determines $\underset{f}{\mathbf{m}}$ via (4.11). As for $\underset{f}{\Sigma}$, its linear form is obtained from $(4.31)_2$ by setting $\underset{f}{\kappa}_1 = \underset{f}{\lambda}\, I_{\underset{f}{D}}$ with $\underset{f}{\lambda} = \hat{\underset{f}{\lambda}}(\underset{f}{\nu})$, $\underset{a}{\kappa}_2 = 2\,\underset{f}{\mu}$ with $\underset{f}{\mu} = \hat{\underset{f}{\mu}}(\underset{f}{\nu})$, and $\underset{f}{\kappa}_3$ equal to zero, yielding the simple Newtonian viscous fluid form

$$\underset{f}{\Sigma} = \underset{f}{\lambda}\, I_{\underset{f}{D}}\, \mathbf{I} + 2\,\underset{f}{\mu}\,\underset{f}{\mathbf{D}} \quad . \tag{4.35}$$

Here, $\underset{f}{\lambda} = \hat{\underset{f}{\lambda}}(\underset{f}{\nu})$ and $\underset{f}{\mu} = \hat{\underset{f}{\mu}}(\underset{f}{\nu})$ represent the volume and shear viscosities, respectively, of the fluid, such that $\underset{f}{\lambda} + \frac{2}{3}\underset{f}{\mu}$ is the bulk viscosity, usually negligible or assumed zero. Typical values for $\underset{f}{\mu}$ can be found in, *e.g.*, Mooney (1951), Moni and Ototake (1956), or Batchelor and Green (1972). Unlike in the case of $\underset{f}{\Sigma}$, more is known about the possible form that the granular solid dynamic stress tensor $\underset{s}{\Sigma}$ should take; in particular, the assumption of linearity would be inappropriate in general for $\underset{s}{\Sigma}$, as we discuss next.

4.3 Granular solid stress tensor

To specify the form of $\underset{s}{\mathbf{T}}$ further, the above formal assumptions such as (4.30) must be supplemented by those of an empirical nature, as based on, *e.g.*, laboratory observations and measurements. In particular, observations indicate (*e.g.*, Savage 1979) that granular solids can sustain shear stress in equilibrium, and that the critical stresses at which shearing begins depend on normal stress. In particular, because of this latter fact, the onset of flow in a solid granular material is generally modeled using the Mohr-Coulomb criterion for yielding or failure (*e.g.*, Savage 1979). According to this criterion, flow will begin in the granular material on a surface with normal $\mathbf{n}$ when the magnitude

$$|\underset{s\parallel}{\mathbf{t}}| = \sqrt{\underset{s\parallel}{\mathbf{t}} \cdot \underset{s\parallel}{\mathbf{t}}} \tag{4.36}$$

of the resolved solid static shear stress

$$\underset{s\parallel}{\mathbf{t}} = [\underset{s}{\mathbf{S}} - (\mathbf{n} \cdot \underset{s}{\mathbf{S}}\,\mathbf{n})\mathbf{I}\,]\mathbf{n} \tag{4.37}$$

on the surface reaches the critical value

$$|\underset{s\parallel}{\mathbf{t}}| = c + |\underset{s\perp}{\mathbf{t}}| \tan\phi \quad , \tag{4.38}$$

where

$$\underset{s\perp}{\mathbf{t}} = (\mathbf{n} \cdot \underset{s}{\mathbf{S}}\,\mathbf{n})\mathbf{n} \tag{4.39}$$

represents the resolved solid static normal stress on the surface, ϕ the angle of internal friction, and c the cohesion of the material. Although in wide use, the generality of (4.38) is not clear, in particular with respect to the angle of internal friction ϕ, which is usually assumed constant, but is known to depend on the state of deformation and $\underset{s}{\nu}$.

From (4.21) and (4.26), we obtain the expressions

$$\begin{aligned} \underset{s\perp}{\mathbf{t}} &= -\underset{s}{\bar{\pi}}\mathbf{n} + 2\,\underset{s}{\alpha}\,(\mathbf{n} \cdot \underset{s}{\mathbf{w}})^2\mathbf{n} \ , \\ \underset{s\parallel}{\mathbf{t}} &= 2\,\underset{s}{\alpha}\,(\mathbf{n} \cdot \underset{s}{\mathbf{w}})[\underset{s}{\mathbf{w}} - (\mathbf{n} \cdot \underset{s}{\mathbf{w}})\mathbf{n}] \ , \end{aligned} \tag{4.40}$$

for the normal and tangential stress vectors in the *saturated* case. The squared magnitudes

$$|\underset{s\perp}{\mathbf{t}}|^2 = \underset{s\perp}{\mathbf{t}} \cdot \underset{s\perp}{\mathbf{t}} = \underset{s}{\bar{\pi}}^2 - 4\,\underset{s}{\alpha}\,\underset{s}{\bar{\pi}}\,(\mathbf{n} \cdot \underset{s}{\mathbf{w}})^2 + 4\,\underset{s}{\alpha}^2\,(\mathbf{n} \cdot \underset{s}{\mathbf{w}})^4 \tag{4.41}$$

and

$$|\underset{s\parallel}{\mathbf{t}}|^2 = \underset{s\parallel}{\mathbf{t}} \cdot \underset{s\parallel}{\mathbf{t}} = 4\,\underset{s}{\alpha}^2\,(\mathbf{n} \cdot \underset{s}{\mathbf{w}})^2[(\underset{s}{\mathbf{w}} \cdot \underset{s}{\mathbf{w}}) - (\mathbf{n} \cdot \underset{s}{\mathbf{w}})^2] \tag{4.42}$$

of these two vectors satisfy the algebraic relation

$$(|\underset{s\perp}{\mathbf{t}}| - t)^2 + |\underset{s\parallel}{\mathbf{t}}|^2 = s^2 \quad , \tag{4.43}$$

representing a Mohr circle with radius

$$s = \underset{s}{\alpha}\,(\underset{s}{\mathbf{w}} \cdot \underset{s}{\mathbf{w}}) \tag{4.44}$$

and center $(t, 0)$ in the $(|\underset{s\perp}{\mathbf{t}}|, |\underset{s\parallel}{\mathbf{t}}|)$-plane, where

$$t = -\underset{s}{\bar{\pi}} - s \quad . \tag{4.45}$$

The geometry of the Mohr circle and yield criterion imply the relations

$$s = |\underset{s\parallel}{\mathbf{t}}| \ \sec\phi \tag{4.46}$$

with $\sec\phi = 1/\cos\phi$ and

$$t = |\underset{s\perp}{\mathbf{t}}| + s\ \sin\phi \tag{4.47}$$

for the radius and center of the circle, respectively, as functions of the critical shear and normal stress magnitudes and ϕ. Substituting the Mohr-Coulomb criterion (4.38), as well as (4.46) and (4.47), into (4.45), we obtain the relation

$$\underset{s}{\bar{\pi}} = c\ \cot\phi - (1 + \csc\phi)\,\underset{s}{\alpha}\,(\underset{s}{\mathbf{w}} \cdot \underset{s}{\mathbf{w}}) \tag{4.48}$$

between the cohesion c, the angle of internal friction ϕ, the mixture specific free energy ψ (see (4.23) and (4.25)), and, in the saturated case, the fluid static true pressure p (see (4.22)), with $\csc\phi = 1/\sin\phi$. In essence, (4.48) represents a compatibility relation between (4.21) and the Mohr-Coulomb criterion (4.38). Note that (4.48) implies the alternative form

$$\underset{s}{\mathbf{S}} = [(1 + \csc\phi)\,\underset{s}{\alpha}\,(\underset{s}{\mathbf{w}} \cdot \underset{s}{\mathbf{w}}) - c\ \cot\phi]\,\mathbf{I} - 2\,\underset{s}{\alpha}\,(\underset{s}{\mathbf{w}} \otimes \underset{s}{\mathbf{w}}) \tag{4.49}$$

for $\underset{s}{\mathbf{S}}$ in terms of ϕ, c and ψ via (4.25). A similar expression was obtained by, *e.g.*, Savage (1979).

Two aspects of (4.21) and (4.49) should perhaps be mentioned at this point. The first concerns the dependence of $\underset{s}{\mathbf{S}}$ on the volume fraction gradients $\underset{a}{\mathbf{w}}$, for which non-standard boundary conditions must be found when formulating initial-boundary value problems based on the constituent balance relations. Such boundary conditions are not well-understood physically. Because of this, one might be tempted to neglect $\underset{s}{\mathbf{S}}$ altogether; this, however, leads to equations which, for certain simple flow fields, have no solution (*e.g.*, Scheiwiller and Hutter 1982). Instead, one could assume that ψ is independent of the $\underset{s}{\mathbf{w}}$, which would alleviate the problem, at least as far as $\underset{s}{\mathbf{S}}$ is concerned. In this case, however, the Mohr-Coulomb criterion must be formulated with a dependence on, *e.g.*, $\underset{s}{\mathbf{D}}/\sqrt{|II_{\underset{s}{\mathbf{D}}}|}$, which, unlike $\underset{s}{\mathbf{D}}$, does not vanish in equilibrium. This has been done by, *e.g.*, Goddard (1985), and Norem et al. (1987), and is summarized by Hutter and Rajagopal (1994). Such a possibility has yet to be implemented in a mixture context, representing work in progress, and consequently is not dealt with here.

From the soil mechanics literature (*e.g.*, Means and Parcher 1963), it is known that ϕ decreases with the solid volume fraction $\underset{s}{\nu}$. In addition, Bagnold (1954) showed that there exists a solid volume fraction $\underset{s}{\nu_0}$ at which the granular solid exhibits zero resistance to shear, *i.e.*, "fluidization," for which $\phi = 0$, corresponding to values of $\underset{s}{\lambda}$ (see (4.53 below) of 12–14. On this basis, Savage (1979) assumed the simple linear relationship

$$\sin\phi = \begin{cases} k(\underset{s}{\nu} - \underset{s}{\nu_0}) & \underset{s}{\nu} \geq \underset{s}{\nu_0} \\ 0 & \underset{s}{\nu} < \underset{s}{\nu_0} \end{cases}\ , \tag{4.50}$$

where k is a constant determined by the critical solid volume fraction, *i.e.*, the solid volume fraction at which the volume of the granular solid does not change during shear. If the volume fraction is lower (higher) than the critical value, the volume decreases (increases) during shear. Indeed, k is chosen so that ϕ takes on a known value at this critical solid volume fraction (termed the critical void ratio by Goodman and Cowin, 1972). Note also that Savage (1979) assumed the simple form

$$\underset{s}{\alpha} = \begin{cases} \text{const.} & \underset{s}{\nu} \geq \underset{s}{\nu_0} \\ 0 & \underset{s}{\nu} < \underset{s}{\nu_0} \end{cases} \tag{4.51}$$

for $\underset{s}{\alpha}$, so that $\underset{s}{\mathbf{S}}$ vanishes in cohesionless granular solids for solid volume fractions less than that at which fluidization arises (see (4.49)). In view of (4.24) and (4.25), this assumption is consistent with $\hat{\psi}$ depending at most linearly on $\underset{s}{\mathbf{w}} \cdot \underset{s}{\mathbf{w}}$ in the unsaturated case.

It remains to determine a form for the granular solid dynamic partial stress tensor $\underset{s}{\boldsymbol{\Sigma}}$. Empirical considerations show that, while the individual particles in a granular solid themselves behave more or less elastically (*e.g.*, in collisions), the behaviour of a granular solid as a whole is very much like that of a non-Newtonian viscous fluid (*e.g.*, Savage 1979). This idea is to be found in the work of Bagnold (1954), who conducted experiments on neutrally bouyant, spherical particles suspended in Newtonian fluids undergoing shear in a coaxial rotating cylinder. For such mixtures, Bagnold (1954) observed two distinct types of behaviour. In his so-called "macro-viscous" regime, corresponding to low shear rates, the shear and normal stresses are linear functions of the velocity gradient, *i.e.*, Newtonian viscous-like, and the fluid viscosity dominates. On the other hand, in his so-called "grain-inertia" regime, the mixture flow is dominated by collisions between the solid particles, leading to a dependence of both the normal and shear stresses on the square of the velocity gradient. In this case, normal and shear stresses are proportional to each other, much like in the case of a cohesionless Mohr-Coulomb material deforming quasi-statically. Bagnold (1954) characterized the mixture quasi-static shear flow field $v_1(x_2)$ in terms of a dimensionless number

$$N = \sqrt{\underset{s}{\lambda}}\, \underset{s}{\varrho}\, d^2 v_{1,2} / \underset{f}{\mu} \quad , \tag{4.52}$$

i.e., the so-called Bagnold number, where $\underset{f}{\mu}$ is the shear viscosity of the fluid, d the solid particle diameter, $v_{1,2}$ the mixture velocity gradient, and $\underset{s}{\lambda}$ the linear particle concentration, defined by

$$\underset{s}{\lambda} = \frac{1}{(\underset{s}{\nu}_\infty / \underset{s}{\nu})^{1/3} - 1} \quad , \tag{4.53}$$

with $\underset{s}{\nu}_\infty$ the maximum possible solid volume fraction (*e.g.*, for spheres, $\underset{s}{\nu}_\infty \approx 0.74$) in the unsaturated mixture context. Note that large $\underset{s}{\lambda}$ corresponds to the particles being close together. Furthermore, the larger $\underset{s}{\lambda}$, the larger the stresses needed to shear the mixture at all. From his experiments, Bagnold (1954) established that the macro-viscous regime corresponds to $N < 40$, and the grain-inertia regime to $N > 450$, with the intermediate region being then in some sense a combination of these two end-regimes.

Beyond these effects, granular materials are observed to exhibit higher-order normal stress effects (*e.g.*, Rajagopal and Massoudi 1990; Hutter and Rajagopal 1994). All three of these types of behaviour are in principle included when we adopt the Reiner-Rivlin form

$$\underset{s}{\boldsymbol{\Sigma}} = \underset{s}{\lambda}\,\mathbf{I} + 2\underset{s}{\mu}\,\underset{s}{\mathbf{D}} + 4\underset{s}{\eta}\,\underset{s}{\mathbf{D}}^2 \tag{4.54}$$

for the granular solid dynamic partial stress tensor $\underset{s}{\boldsymbol{\Sigma}}$, where the viscosities $\underset{s}{\lambda}$, $\underset{s}{\mu}$ and $\underset{s}{\eta}$ are in general functions of $\underset{s}{\nu}$, and the invariants $I_{\underset{s}{\mathbf{D}}}$, $II_{\underset{s}{\mathbf{D}}}$, $III_{\underset{s}{\mathbf{D}}}$ of $\underset{s}{\mathbf{D}}$ (see (4.32)). The form (4.54) can be obtained from the general isotropic form (4.31) by setting $\underset{\mathfrak{a}}{\kappa}_1 = \underset{s}{\lambda}$, $\underset{\mathfrak{a}}{\kappa}_2 = 2\underset{s}{\mu}$, $\underset{\mathfrak{a}}{\kappa}_3 = 4\underset{s}{\eta}$, $\underset{\mathfrak{a}}{\kappa}_{4-6}$ equal to zero, and assuming $\underset{s}{\kappa}_{1-3}$ are independent of (1), $\underset{s}{\dot{\nu}}$, and (2), $\underset{s}{\mathbf{M}}$. Possible physical justification for (2) are discussed by Savage (1979). In particular, assuming the granular solid moves isochorically, *i.e.*, $I_{\underset{s}{\mathbf{D}}} = \operatorname{div} \underset{s}{\mathbf{v}} = 0$, in which case (4.9) for $\mathfrak{a} = \mathrm{s}$ implies $\underset{s}{\dot{\nu}} = 0$, Savage (1979) proposed a special case of (4.54) given by

$$\begin{aligned} \underset{s}{\lambda} &= 4c_1 \left(\frac{\underset{s}{\nu}_\infty - \underset{s}{\nu}_0}{\underset{s}{\nu}_\infty - \underset{s}{\nu}} \right)^8 II_{\underset{s}{\mathbf{D}}} \;, \\ \underset{s}{\mu} &= 2c_2 \left(\frac{\underset{s}{\nu}_\infty - \underset{s}{\nu}_0}{\underset{s}{\nu}_\infty - \underset{s}{\nu}} \right)^8 \sqrt{|II_{\underset{s}{\mathbf{D}}}|} \;, \qquad \text{for } \underset{s}{\nu}_0 < \underset{s}{\nu} < \underset{s}{\nu}_\infty \;, \\ \underset{s}{\eta} &= 0 \;, \end{aligned} \tag{4.55}$$

consistent with Bagnold's (1954) experimental results, where c_1 and c_2 are constants, and $\underset{s}{\nu}_0$ is the solid volume fraction at which fluidization arises.

In viscometric flows, only certain parts of the Reiner-Rivlin relation (4.54) are actually non-zero. When trying to fit the viscosities $\underset{s}{\lambda}$, $\underset{s}{\mu}$ and $\underset{s}{\eta}$ to experimental data, it turns out that there are many choices for these which yield the same material behaviour with respect to the non-zero components. As such, there exists no unique relationship for $\underset{s}{\boldsymbol{\Sigma}}$ as based on (4.54). Indeed, Jenkins and Cowin (1979) suggest a form of $\underset{s}{\boldsymbol{\Sigma}}$ which involves the second Rivlin-Eriksen tensor, yet can also be reduced in simple shear to exactly Savage's (1979) special case (4.55) of (4.54) as discussed above. This should be borne in mind, as different representations for the dynamic partial stress tensor will be observable in general flow situations. This point is dicussed in detail by Scheiwiller and Hutter (1982), as well as Hutter and Rajagopal (1994).

4.4 Two-dimensional thin-layer model for the channel-flow regime

After the onset of flow, debris masses are often constrained to move down gullies or narrow valleys in a channel-type flow whose thickness may be comparable to its width, and both are much smaller than its length. Upon reaching the mouth of the gully or valley, however, this constraint is removed, allowing the debris flow to spread out into a fan whose thickness is substantially smaller than its width or length. These geometric aspects of the flow can be used in a two-dimensional setting to obtain simplified models for the debris flow in these two regimes. In this section, we focus on the channel-flow case alone, primarily in order to address the model of Takahashi (1991) in the next section.

The channel-flow regime can be idealized by a two-dimensional mixture flowing downhill under the influence of gravity at an angle γ to the horizontal, with the x-axis aligned parallel to the flow, and the z-axis perpendicular to it. For this geometry, the constituent mass and momentum balances (4.9) and (4.10) take the forms

$$\begin{aligned} \underset{a}{\nu}_{,t} + (\underset{a}{\nu}\,\underset{a}{v}_x)_{,x} + (\underset{a}{\nu}\,\underset{a}{v}_z)_{,z} &= 0 \,, \\ \underset{a}{\bar{\varrho}}\,\underset{a}{\grave{v}}_x &= \underset{a}{T}_{xx,x} + \underset{a}{T}_{xz,z} + \underset{a}{\bar{\varrho}} g \sin\gamma + \underset{a}{m}_x \,, \\ \underset{a}{\bar{\varrho}}\,\underset{a}{\grave{v}}_z &= \underset{a}{T}_{xz,x} + \underset{a}{T}_{zz,z} - \underset{a}{\bar{\varrho}} g \cos\gamma + \underset{a}{m}_z \,, \end{aligned} \tag{4.56}$$

for (4.9) in two dimensions, with $\mathbf{e}_x \cdot \mathbf{g} = g \sin\gamma$ and $\mathbf{e}_z \cdot \mathbf{g} = -g \cos\gamma$. In the case of a debris flow, the bed angle γ varies from about 40° in the starting zones to about 3° in the runout zone.

Let $[f]$ now stand for the characteristic value of some quantity f (*e.g.*, x, y, t, v_x, *etc.*). In particular, let $[z]$ represent the characteristic thickness, $[x]$ the characteristic length, $[v_x]$ the characteristic velocity parallel to the flow, and $[v_z]$ that perpendicular to it, "vertically." Introducing then the characteristic time

$$[t] = \frac{[x]}{[v_x]} = \frac{[z]}{[v_z]} \quad , \tag{4.57}$$

characteristic stresses

$$[\underset{a}{T}_{ij}] = [\varrho]\, g\, [z] \quad , \tag{4.58}$$

as well as the aspect ratio

$$\epsilon = \frac{[z]}{[x]} = \frac{[v_z]}{[v_x]} \quad , \tag{4.59}$$

into (4.56), non-dimensionalization yields

$$\begin{aligned} \underset{a}{\nu}_{,t} + (\underset{a}{\nu}\,\underset{a}{v}_x)_{,x} + (\underset{a}{\nu}\,\underset{a}{v}_z)_{,z} &= 0 \,, \\ \frac{[v_x]}{g[t]} \frac{\underset{a}{\varrho}}{[\varrho]}\, \underset{a}{\nu}\,\underset{a}{\grave{v}}_x &= \epsilon\, \underset{a}{T}_{xx,x} + \underset{a}{T}_{xz,z} + \frac{1}{[\varrho]} \underset{a}{\varrho}\,\underset{a}{\nu} \sin\gamma + \underset{a}{m}_x \,, \\ \frac{[v_x]}{g[t]} \frac{\underset{a}{\varrho}}{[\varrho]}\, \epsilon\, \underset{a}{\nu}\,\underset{a}{\grave{v}}_z &= \epsilon\, \underset{a}{T}_{xz,x} + \underset{a}{T}_{zz,z} - \frac{1}{[\varrho]} \underset{a}{\varrho}\,\underset{a}{\nu} \cos\gamma + \underset{a}{m}_z \,, \end{aligned} \tag{4.60}$$

for the two-dimensional constituent mass and momentum balance relations, where

$$\begin{aligned} \underset{s}{m}_x &= \mathcal{M}_{\mathrm{D}}\, m_{\mathrm{D}} \underset{sf}{v}_x + \left\{ \underset{s}{\mathcal{P}} (1 - \underset{s}{\bar{\xi}}) \underset{s}{\beta} + p \right\} \epsilon\, \underset{s}{\nu}_{,x} \,, \\ \underset{s}{m}_z &= \mathcal{M}_{\mathrm{D}}\, \epsilon\, m_{\mathrm{D}} \underset{sf}{v}_z + \left\{ \underset{s}{\mathcal{P}} (1 - \underset{s}{\bar{\xi}}) \underset{s}{\beta} + p \right\} \underset{s}{\nu}_{,z} \,, \end{aligned} \tag{4.61}$$

from (4.22) and (4.33) in the saturated case with $[p] = [\varrho] g [z]$,

$$\underset{s}{\mathcal{P}} = \frac{1}{[p]} [\underset{s}{\beta}] \tag{4.62}$$

and

$$\mathscr{M}_{\mathrm{D}} = \frac{[m_{\mathrm{D}}][v_x]}{[\varrho]g} \quad , \tag{4.63}$$

where $m_{\mathrm{D}} = \hat{m}_{\mathrm{D}}(\underset{s}{\nu})$ is now a non-dimensionalized function of $\underset{s}{\nu}$. Note that all fields appearing in (4.60) and (4.61) are now non-dimensional.

Now, for debris flows, $[v_x]$ is on the order of 1 to 15 m/s (*i.e.*, for the channel- and fan-flow regimes); consequently, with g = 10 m/s^2, the ratio $[v_x]/g$ is on the order of 1 second. So, for timescales $[t]$ much greater than this, the acceleration terms appearing on the left hand side in $(4.60)_{2,3}$ will be negligible. As the channel and runout phases of debris flows generally last more than 10 seconds[7], it is in fact reasonable to neglect these terms here, such that $(4.60)_{2,3}$ reduce to

$$\begin{aligned}
\left\{\underset{s}{T}_{xx,x} + \left\{\underset{s}{\mathscr{P}}(1-\underset{s}{\bar{\xi}})\underset{s}{\beta} + p\right\}\underset{s}{\nu}_{,x}\right\}\epsilon + \underset{s}{T}_{xz,z} + \frac{1}{[\varrho]}\underset{s}{\varrho}\,\underset{s}{\nu}\sin\gamma + \mathscr{M}_{\mathrm{D}}\, m_{\mathrm{D}}\underset{sf}{v_x} &= 0 \,, \\
\left\{\underset{s}{T}_{xz,x} + \mathscr{M}_{\mathrm{D}}\, m_{\mathrm{D}}\underset{sf}{v_z}\right\}\epsilon + \underset{s}{T}_{zz,z} - \frac{1}{[\varrho]}\underset{s}{\varrho}\,\underset{s}{\nu}\cos\gamma + \left\{\underset{s}{\mathscr{P}}(1-\underset{s}{\bar{\xi}})\underset{s}{\beta} + p\right\}\underset{s}{\nu}_{,z} &= 0 \,,
\end{aligned} \tag{4.64}$$

for the solid constituent, and

$$\begin{aligned}
\left\{\underset{f}{T}_{xx,x} + \left\{\underset{s}{\mathscr{P}}(1-\underset{s}{\bar{\xi}})\underset{s}{\beta} + p\right\}\underset{s}{\nu}_{,x}\right\}\epsilon + \underset{f}{T}_{xz,z} + \frac{1}{[\varrho]}\underset{f}{\varrho}\,\underset{f}{\nu}\sin\gamma + \mathscr{M}_{\mathrm{D}}\, m_{\mathrm{D}}\underset{sf}{v_x} &= 0 \,, \\
\left\{\underset{f}{T}_{xz,x} - \mathscr{M}_{\mathrm{D}}\, m_{\mathrm{D}}\underset{sf}{v_z}\right\}\epsilon + \underset{s}{T}_{zz,z} - \frac{1}{[\varrho]}\underset{s}{\varrho}\,\underset{s}{\nu}\cos\gamma - \left\{\underset{s}{\mathscr{P}}(1-\underset{s}{\bar{\xi}})\underset{s}{\beta} + p\right\}\underset{s}{\nu}_{,z} &= 0 \,,
\end{aligned} \tag{4.65}$$

for the fluid one, via (4.11) and (4.61). As discussed in the introduction, the length of a debris flow is generally much larger than its width and thickness. Consequently, one can work with the *thin-layer (shallowness) approximation*

$$\epsilon \ll 1 \tag{4.66}$$

in the channel-flow regime. In the context of this approximation, one can formulate a hierarchy of constituent momentum balances (4.65) to various orders in ϵ via the constitutive relations for $\underset{s}{T}_{ij}$ and $\underset{f}{T}_{ij}$. In this work, however, we restrict ourselves to the derivation of the $O(1)$ forms for (4.65). To carry this out, we first need to determine the dependence of ψ and $\underset{a}{\mathbf{T}}$ on ϵ, and then retain only the $O(1)$ parts, *i.e.*, those parts independent of ϵ.

To begin, consider the mixture specific inner free energy ψ. From (4.24) for the saturated case, we see that

$$\hat{\psi}(\underset{s}{\nu}, \underset{s}{\mathbf{w}}\cdot\underset{s}{\mathbf{w}}) = \hat{\psi}_1(\underset{s}{\nu}, \underset{s}{\nu}^2_{,z}) + O(\epsilon) \quad , \tag{4.67}$$

so that to $O(1)$, ψ is a function of $\underset{s}{\nu}$ and $\underset{s}{\nu}^2_{,z}$, implying in turn via (4.23) and (4.25) that $\underset{s}{\beta}$ and $\underset{s}{\alpha}$, respectively, can be replaced by corresponding functions $\underset{s}{\beta}_1$ and $\underset{s}{\alpha}_1$, respectively, to $O(1)$, which are analogous to $\hat{\psi}_1$. On this basis, we obtain the form

$$\begin{aligned}
\underset{s}{S}_{xx} &= -\underset{s}{\nu}p - \underset{s}{\mathscr{P}}\,\underset{s}{\bar{\beta}}_1 + O(\epsilon^2) \,, \\
\underset{s}{S}_{xz} &= O(\epsilon) \,, \\
\underset{s}{S}_{zz} &= -\underset{s}{\nu}p - \left\{\underset{s}{\mathscr{P}}\,\underset{s}{\bar{\beta}}_1 + \underset{s}{\mathscr{A}}\,\underset{s}{\alpha}_1\,\underset{s}{\nu}_{,z}\,\underset{s}{\nu}_{,z}\right\} + O(\epsilon) \,,
\end{aligned} \tag{4.68}$$

for the non-dimensionalized constituents of the solid static partial stress tensor $\underset{s}{\mathbf{S}}$ from (4.21) and (4.22), with

$$\underset{s}{\mathscr{A}} = \frac{1}{[\varrho]g[z]^3}\,[\underset{s}{\alpha}] \quad . \tag{4.69}$$

Again, like in (4.64) and (4.65), all fields appearing in (4.68) are now non-dimensional. As for the components of the non-dimensionalized fluid static partial stress is given by

[7] In real situations, because of topographic variations, the acceleration terms may not in fact be negligible. They in fact will survive the non-dimensionalization when we assume, *e.g.*, $[v_x] = g[t]$ instead of $(4.57)_1$.

$$
\begin{aligned}
\underset{f}{S}_{xx} &= -\underset{f}{\nu} p \,, \\
\underset{f}{S}_{xz} &= 0 \,, \\
\underset{f}{S}_{zz} &= -\underset{f}{\nu} p \,,
\end{aligned}
\tag{4.70}
$$

via (4.27) and (4.28) in the saturated case.

To determine the non-dimensionalized forms of the constituent dynamic stresses $\underset{\alpha}{\Sigma}$ and their dependence on ϵ, we first need to obtain those for $\underset{\alpha}{\mathbf{D}}$, $\underset{\alpha}{\mathbf{D}}^2$ and $\underset{\alpha}{\mathbf{D}}^3$. From (4.57) and (4.59), we obtain

$$
\begin{aligned}
\underset{\alpha}{D}_{xx} &= O(\epsilon) \,, \\
\underset{\alpha}{D}_{xz} &= \tfrac{1}{2} \underset{\alpha}{v}_{x,z} + O(\epsilon^2) \,, \\
\underset{\alpha}{D}_{zz} &= O(\epsilon) \,,
\end{aligned}
\tag{4.71}
$$

for the components of the non-dimensionalized constituent deformation rate tensor $\underset{\alpha}{\mathbf{D}}$, those

$$
\begin{aligned}
(\underset{\alpha}{\mathbf{D}}^2)_{xx} &= \tfrac{1}{4} \underset{\alpha}{v}^2_{x,z} + O(\epsilon^2) \,, \\
(\underset{\alpha}{\mathbf{D}}^2)_{xz} &= O(\epsilon) \,, \\
(\underset{\alpha}{\mathbf{D}}^2)_{zz} &= \tfrac{1}{4} \underset{\alpha}{v}^2_{x,z} + O(\epsilon^2) \,,
\end{aligned}
\tag{4.72}
$$

for the components of $\underset{\alpha}{\mathbf{D}}^2$, and those

$$
\begin{aligned}
(\underset{\alpha}{\mathbf{D}}^3)_{xx} &= O(\epsilon) \,, \\
(\underset{\alpha}{\mathbf{D}}^3)_{xz} &= \tfrac{1}{8} \underset{\alpha}{v}^3_{x,z} + O(\epsilon^2) \,, \\
(\underset{\alpha}{\mathbf{D}}^3)_{zz} &= O(\epsilon) \,,
\end{aligned}
\tag{4.73}
$$

for the components of $\underset{\alpha}{\mathbf{D}}^3$. The results (4.71)–(4.73) determine in turn the non-dimensionalized forms

$$
\begin{aligned}
I_{\underset{\alpha}{D}} &= O(\epsilon) \,, \\
II_{\underset{\alpha}{D}} &= -\tfrac{1}{4} \underset{\alpha}{v}^2_{x,z} + O(\epsilon^2) \,, \\
III_{\underset{\alpha}{D}} &= O(\epsilon) \,,
\end{aligned}
\tag{4.74}
$$

for the invariants of $\underset{\alpha}{\mathbf{D}}$ from (4.32) and (4.71).

On the basis of (4.71) and (4.74), (4.35) yields

$$
\begin{aligned}
\underset{f}{\Sigma}_{xx} &= O(\epsilon) \,, \\
\underset{f}{\Sigma}_{xz} &= \underset{f}{\mu}\, \underset{f}{v}_{x,z} + O(\epsilon^2) \,, \\
\underset{f}{\Sigma}_{zz} &= O(\epsilon) \,,
\end{aligned}
\tag{4.75}
$$

for the components of the non-dimensionalized fluid dynamic stress tensor with $\underset{f}{\mu} = \underset{f}{\hat{\mu}}(\underset{s}{\nu})$ via (4.5). Similarly,

$$
\begin{aligned}
\underset{s}{\Sigma}_{xx} &= \underset{s}{\lambda} + \underset{s}{\eta}\, \underset{s}{v}^2_{x,z} + O(\epsilon) \,, \\
\underset{s}{\Sigma}_{xz} &= \underset{s}{\mu}\, \underset{s}{v}_{x,z} + O(\epsilon) \,, \\
\underset{s}{\Sigma}_{zz} &= \underset{s}{\lambda} + \underset{s}{\eta}\, \underset{s}{v}^2_{x,z} + O(\epsilon) \,,
\end{aligned}
\tag{4.76}
$$

follows for the components of the non-dimensionalized granular solid dynamic stress tensor from (4.54), (4.71) and (4.74). To obtain (4.76), we assumed that $\underset{s}{\lambda}$, $\underset{s}{\mu}$ and $\underset{s}{\eta}$ are at least $O(1)$. For example, in Savage's (1979) model, $\underset{s}{\lambda}$ and $\underset{s}{\mu}$ are both $O(1)$. Under this assumption, then, the solid viscosities $\underset{s}{\lambda}$, $\underset{s}{\mu}$ and $\underset{s}{\eta}$ are, to $O(1)$, functions of $\underset{s}{\nu}$ and $\underset{s}{II}_{\mathbf{D}}$ alone via $(4.74)_2$.

The corresponding components of the static (4.70) and dynamic (4.76) parts of $\underset{s}{\mathbf{T}}$ can be combined to obtain those

$$
\begin{aligned}
\underset{s}{T}_{xx} &= -\underset{s}{\nu} p - \underset{s}{\mathscr{P}}\, \underset{s}{\bar{\beta}}_1 + \underset{s}{\lambda} + \underset{s}{\eta}\, \underset{s}{v}_{x,z}^2 + O(\epsilon) , \\
\underset{s}{T}_{xz} &= \underset{s}{\mu}\, \underset{s}{v}_{x,z} + O(\epsilon) , \\
\underset{s}{T}_{zz} &= -\underset{s}{\nu} p - \left\{ \underset{s}{\mathscr{P}}\, \underset{s}{\bar{\beta}}_1 + \underset{s}{\mathscr{A}}\, \underset{s}{\alpha}_1\, \underset{s}{\nu}_{,z}\, \underset{s}{\nu}_{,z} \right\} + \underset{s}{\lambda} + \underset{s}{\eta}\, \underset{s}{v}_{x,z}^2 + O(\epsilon) ,
\end{aligned}
\tag{4.77}
$$

of $\underset{s}{\mathbf{T}}$ to lowest order in ϵ. To $O(1)$, (4.77) imply

$$
\underset{s}{T}_{xx} - \underset{s}{T}_{zz} = \underset{s}{\mathscr{A}}\, \underset{s}{\alpha}\, \underset{s}{\nu}_{,z}\, \underset{s}{\nu}_{,z} + O(\epsilon)
\tag{4.78}
$$

for the non-zero normal stress difference in the granular solid. Since this is a static term, there are, to lowest order, no normal dynamic stress effects influencing the development of the debris flow. Likewise,

$$
\begin{aligned}
\underset{f}{T}_{xx} &= -\underset{f}{\nu} p + O(\epsilon) , \\
\underset{f}{T}_{xz} &= \underset{f}{\mu}\, \underset{f}{v}_{x,z} + O(\epsilon^2) , \\
\underset{f}{T}_{zz} &= -\underset{f}{\nu} p + O(\epsilon) ,
\end{aligned}
\tag{4.79}
$$

is obtained from (4.70) and (4.75) for the components of $\underset{f}{\mathbf{T}}$. Note that, of the two fluid viscosities, only the fluid shear viscosity $\underset{f}{\mu}$ is important to $O(1)$.

Substituting (4.77) and (4.79) into (4.64) and (4.65), respectively, we obtain finally the forms

$$
\begin{aligned}
\underset{s}{\Sigma}_{xz,z} &= -\frac{1}{[\varrho]}\, \underset{s}{\varrho}\, \underset{s}{\nu} \sin\gamma - \mathscr{M}_{\mathrm{D}}\, m_{\mathrm{D}}\, \underset{sf}{v}_x + O(\epsilon) , \\
-\underset{s}{\nu} p_{,z} + \underset{s}{\Sigma}_{zz,z} &= \frac{1}{[\varrho]}\, \underset{s}{\varrho}\, \underset{s}{\nu} \cos\gamma + \underset{s}{\mathscr{P}}\, \bar{\xi}\, \underset{s}{\beta}_1\, \underset{s}{\nu}_{,z} + \underset{s}{\mathscr{P}}\, \underset{s}{\nu}\, \underset{s}{\beta}_{1,z} + \underset{s}{\mathscr{A}} \left\{ \underset{s}{\alpha}_1\, \underset{s}{\nu}_{,z}\, \underset{s}{\nu}_{,z} \right\}_{,z} + O(\epsilon) ,
\end{aligned}
\tag{4.80}
$$

and

$$
\begin{aligned}
\underset{f}{\Sigma}_{xz,z} &= -\frac{1}{[\varrho]}\, \underset{f}{\varrho}\, \underset{f}{\nu} \sin\gamma + \mathscr{M}_{\mathrm{D}}\, m_{\mathrm{D}}\, \underset{sf}{v}_x + O(\epsilon) , \\
-\underset{f}{\nu} p_{,z} &= \frac{1}{[\varrho]}\, \underset{f}{\varrho}\, \underset{f}{\nu} \cos\gamma + \underset{s}{\mathscr{P}} (1 - \bar{\xi})\, \underset{s}{\beta}_1\, \underset{s}{\nu}_{,z} + O(\epsilon) ,
\end{aligned}
\tag{4.81}
$$

of the solid and fluid momentum balance relations, respectively, to lowest order, *i.e.*, to $O(1)$, via (4.68) and (4.76), as well as (4.70) and (4.75), respectively.

On the basis of the constitutive relations (4.5), $(4.76)_{2,3}$ and $(4.75)_2$, the $O(1)$ solid (4.80) and fluid (4.81) momentum balances represent 7 first-order differential equations in 7 unknowns $\underset{s}{\nu}$, p, $\underset{s}{v}_x$, $\underset{f}{v}_x$, $\underset{f}{\Sigma}_{xz}$, $\underset{s}{\Sigma}_{xz}$ and $\underset{s}{\Sigma}_{zz}$. So, once the constitutive functions $\psi = \hat{\psi}_1(\underset{s}{\nu}, \underset{s}{\nu}_{,z}^2)$, $\underset{s}{\lambda} = \hat{\underset{s}{\lambda}}(\underset{s}{\nu}, \underset{s}{II}_{\mathbf{D}})$, $\underset{s}{\mu} = \hat{\underset{s}{\mu}}(\underset{s}{\nu}, \underset{s}{II}_{\mathbf{D}})$, $\underset{s}{\eta} = \hat{\underset{s}{\eta}}(\underset{s}{\nu}, \underset{s}{II}_{\mathbf{D}})$ and $\underset{f}{\mu} = \hat{\underset{f}{\mu}}(\underset{s}{\nu})$, as well as the appropriate boundary conditions at the base and free surface are specified (see Sect. 6), solution of the above system is possible. Since, as indicated by the result (4.78), no dynamic normal stress effects influence the material behaviour of the granular solid, we could in fact neglect $\underset{s}{\eta} = \hat{\underset{s}{\eta}}(\underset{s}{\nu}, \underset{s}{II}_{\mathbf{D}})$.

The above thin-layer approximation has been carried out in the context of a planar flow geometry and is appropriate for channel-flow regime, or very wide, uni-directional flows. For the fan-flow regime, however, the debris flow spreads sideways (*i.e.*, in the y-direction), at which point we can no longer treat the flow as two-dimensional. Analogous to the above formulation, however, we can, assuming the fan-flow is thin compared to its width and length, scale the components of the three dimensional constituent momentum balance using two aspect ratios $[y]/[x]$ and $[z]/[x]$, and proceed as above, which we do not do here. In the context of avalanches, this was done by Hutter et al. (1993).

4.5 Takahashi's model

A number of existing models for debris flows can be obtained as special cases of the above formulation. In particular, the heuristic two-dimensional saturated mixture model proposed by Takahashi (1991) for a debris flow can be obtained rigorously from the thin-layer model formulated in the previous section as a special case, as we show in this section. We emphasize that he did not (at least explicitly) utilize the thin-layer approximation in his model; his results, however, can be derived rigorously only in this context when we introduce the additional assumptions he made beyond those in the last section.

To obtain Takahashi's (1991) first model momentum relation (2.1.5), we add $(4.80)_1$ and $(4.81)_1$ together, which yields

$$(\underset{s}{\Sigma}_{xz} + \underset{f}{\Sigma}_{xz})_{,z} + \frac{1}{[\varrho]}\,\varrho\,\sin\gamma = O(\epsilon) \quad . \tag{4.82}$$

To obtain his remaining model relations, we must assume

$$\underset{s}{\mathscr{P}} \text{ and } \underset{s}{\mathscr{A}} \text{ are } O(\epsilon) \quad , \tag{4.83}$$

such that the corresponding terms in $(4.80)_2$ and $(4.81)_2$ drop out. Indeed, in this case, the sum of $(4.80)_2$ and $(4.81)_2$ yields

$$-p_{,z} + \underset{s}{\Sigma}_{zz,z} - \frac{1}{[\varrho]}\,\varrho\,\cos\gamma = O(\epsilon) \quad , \tag{4.84}$$

representing (2.1.6) in Takahashi (1991), while $(4.81)_2$ reduces to

$$-p_{,z} - \frac{1}{[\varrho]}\,\underset{f}{\varrho}\,\cos\gamma = O(\epsilon) \quad , \tag{4.85}$$

i.e., (2.1.7) in Takahashi (1991), which can be combined with (4.84) to obtain the field relation

$$\underset{s}{\Sigma}_{zz,z} - \frac{1}{[\varrho]}\,\underset{sf}{\varrho}\,\underset{s}{\nu}\,\cos\gamma = O(\epsilon) \tag{4.86}$$

for $\underset{s}{\Sigma}_{zz}$ with $\underset{sf}{\varrho} = \underset{s}{\varrho} - \underset{f}{\varrho}$, corresponding to (2.1.8) in Takahashi (1991). Beyond these simplifications, the coefficient of $\underset{s}{\mathbf{w}}$ in (4.61) reduces to its classical form p, while $\underset{s}{S}_{xx}$ and $\underset{s}{S}_{zz}$ in (4.68) reduce to $-\underset{s}{\nu}\,p$, again to $O(1)$. In essence, then, Takahashi (1991) tacitly neglects the effects of sharp spatial variations in the solid distribution in his model, something that may play a role for example in phenomena such as inverse grading, as well as the initiation of debris flows, the subject of future work.

5 On numerical implementation

In this section we discuss how a debris flow model, once this model has been chosen, can be put into practice, *i.e.*, numerically integrated. We emphasize that the final model will consist of the discretized equations that are numerically solved, and that judgement of the suitability will involve both physical and computational aspects. A numerical code should clearly make visible, first, on which physical model it is based upon, second, what analytical transformations or simplifications are made to make it amenable to computational handling and what discretization techniques are used to generate the numerical code. Often, the last two steps are interwoven.

We shall give examples for all these.

5.1 Hierarchy of models

Debris flow models can vary in their complexity because of the different *physical processes* that are incorporated. None of the models listed below is general enough to be able to describe inverse grading or the riding of large boulders on the free surface. The following models have been proposed and are at our disposal:

Two phase flow model

A detailed study of this model is given in Sect. 4 and based on Svendsen and Hutter (1995). In a somewhat reduced form it has been used by Takahashi (1991), but because of the peculiarities mentioned before it is probably advantageous not to use Takahashi's reduction and to start from the Svendsen and Hutter (1995) equations. These equations – as opposed to the two phase models proposed earlier – are based on sound thermodynamic principles. In two phase models, one uses balance laws of mass and momentum for both components. Because constitutive relations for the solid and the fluid phase are needed, these models have the potential to incorporate the *grain-inertia regime* (via a nonlinear viscous relationship for the solid stresses) as well as the *macro-viscous regime* via a Newtonian type stress relation for the fluid (with the viscosity accordingly adjusted if the fluid is a slurry). The dispersive pressure can equally be incorporated. If it is expressed in terms of the solids concentration, a further equation for the fluctuation energy may not need to be incorporated, if it is expressed in terms of such a fluctuation energy, an equation for its evolution is needed. This energy equation then quantifies the degree of intensity of turbulence or particle collision.

The two phase model also needs a constitutive equation for the interaction force between the solid particles and the fluid. It has the structure

$$\underset{s}{\mathbf{m}} = c_1 \underset{sf}{\mathbf{v}} + c_2 \underset{s}{\mathbf{w}} \tag{5.1}$$

(compare with (4.33)), where the coefficients $c_{1,2}$ are not necessarily constant. Classically, c_1 is the Darcy permeability and c_2 the fluid pressure p, but it was shown by Jöhnk et al. (1993) and Hutter et al. (1994) that this led to singular behaviour in two phase gravity flow and needed amendment, see Svendsen et al. (1995), Svendsen and Hutter (1995).

If a two phase model is used, only this corrected model should be used; it is physically sound and allows deposition and erosion processes to be incorporated at the bed.

Two-component diffusive mixture model

These models are characterized by employing two balance laws for the mass of the sediment and water[8], but only one balance law of momentum for the mixture as a whole, the material points of which move with the barycentric velocity. Thus, it is supposed that the grains and the interstitial fluid have approximately the same velocities. The difference between these can be accounted for in the mass balance of the grains by postulating a diffusive sediment mass flux leading to a sediment mass balance of the form

$$\underset{s}{\bar{\xi}}_{,t} + (\nabla \underset{s}{\bar{\xi}}) \cdot \mathbf{v} = D\,(\nabla^2 \underset{s}{\bar{\xi}}) \quad , \tag{5.2}$$

where D is a diffusivity, but proposed equations ignore this Fickian diffusion (Takahashi et al. 1992). This changes equation (5.2) from parabolic to hyperbolic. Numerically this is more problematic because diffusion generally stabilizes the equations.

Without the balance law of the solid's mass the model equations (of mass and momentum and constitutive relations of stress) would not contain a dynamic equation through which the grain concentration distribution could function as a prognostic variable; it would at most play a diagnostic role; this is the major reason for incorporating the solid's mass balance. However with diffusion, (5.2) permits only dilution of the sediments (this follows from the second law, $D > 0$). This is why physically one wishes $D = 0$, so that concentration is frozen[9] to the body points. Addition or subtraction of mass can still be incorporated through the erosion and deposition processes at the debris flow bed. This is most easily seen if the balance laws of mass of the mixture as a whole and that of the grains, *i.e.*, (5.2), with $D = 0$, is depth-integrated[10]

$$h_{,t} + Q_{,x} = \underset{s}{\iota} \,, \quad c_{,t} + (cQ)_{,x} = \underset{sw}{\iota} \,, \tag{5.3}$$

[8] In practice the balance laws of mass for the mixture as a whole and for the grains is formulated, but this equivalent.

[9] If the mass balance would be formulated for the water and Fickian diffusion used for its flux, then thermodynamics would require dilution of the water in obvious contradiction with the above. Thus, diffusion models are only meaningful for tracer diffusion of very low concentration. This makes neglection of the diffusive mass flux compelled in these models when concentrations are not very small.

[10] These equations are strictly only correct, when the mixture is volume preserving and the concentration does not vary with depth, two conditions, which generally cannot simultaneously be satisfied.

in which h, Q, c, ι_s and ι_{sw} are the debris flow depth, the volume flux (discharge), the depth-integrated form of the sediment mass concentration or fraction $\bar{\xi}_s$, the erosion rate of sediments plus water, and the erosion rate of the sediments at the bed, respectively, and must be constitutively prescribed.

Computationally, (5.3) is less critical than the local variant (5.2). Moreover, because field variables like c and Q are integrated quantities over debris flow depth, these equations are also less critical from a physical point of view than their local counterparts (5.2). As such, *two-component diffusive mixture models are most likely to be physically appropriate only when being used in a hydraulic type formulation.* When local features of particle concentrations are of interest, the two phase models may also be relevant.

Table 1. Hierarchy of debris flow models

	Two-component model	Diffusive model	Single-component model
Balance relations	• solid mass fluid mass • solid momentum fluid momentum	• solid mass mixture mass • mixture momentum	• mixture mass • mixture momentum [• mixture fluctuation energy]
Constitutive relations	• solid stress (nonlinear viscous) or plastic) fluid stress (Newtonian) • interaction force (Darcy relation)	• diffusive mass flux (Fickian diffusion) for grains, but ignored • mixture stress (nonlinear viscous or plastic)	• mixture stress (nonlinear viscous or plastic) [• Fluctuation energy flux (Fourier-type relation) • Fluctuation energy annihilation rate]
Remarks	• Permits utmost flexibility. Erosion and deposition can be incorporated. • Includes constitutive relation for the fluid-solid interaction rate density, see Svendsen & Hutter (1995). • Model appropriate when density variations over depth are significant.	• Thermodynamics implies model is appropriate only in a hydraulic-type formulation. • In full 3-D formulation, particle concentration not realistically modeled. • Erosion and deposition can be incorporated.	• Mixture means here solid + fluid or solid + air, where air is treated as massless. • Full 3-D formulation allows determination of velocity and particle concentration fields. • Deposition and erosion processes allow no separate treatment of water and sediments, and should then be excluded from model.
Possible extensions	• Incorporation of fluctuation energy balance relations for fluid and solid. • Incorporation of *particle size distribution function evolution relation* to model inverse grading.	• Incorporation of fluctuation energy balance relation for mixture. • Incorporation of *particle size distribution function evolution relation.*	• Incorporation of fluctuation energy balance relation for mixture. • Incorporation of *particle size distribution function evolution relation.*

Single component model

This is by far the most popular description; it consists of the balance laws of mass and momentum of the mixture, *i.e.*, grains plus water, and constitutive relations for the mixture stress tensor. Due to the saturation condition (the water fills all pores), and incompressibility assumption for both components, the mixture mass balance reduces to the evolution relation

$$(\underset{s}{\varrho} - \underset{f}{\varrho})[\underset{s}{\nu}_{,t} + \underset{s}{\mathbf{w}} \cdot \mathbf{v}] + \varrho\,(\operatorname{div}\mathbf{v}) = 0 \tag{5.4}$$

for the sediment volume fraction $\underset{s}{\nu}$. Deposition and erosion processes at the bed are not accounted for in this case, however, and the exchange of water and sediment with the bed takes place at the value of the solids fraction that the model produces at the base. It follows that *such a model is not likely to model particle segregation in a physically correct fashion.*

With these limitations, model equations are capable of predicting debris flow processes and two types of models are thinkable, *i.e.*, (i), ones without an evolution equation for the fluctuations (see Chen (1987), Takahashi (1971), and others), and (ii), those including evolution equation(s) for the fluctuation energy due to particle collisions and turbulence of the interstitial fluid (Jenkins and Askari 1994). Inclusion of the fluctuation energy allows the processes of erosion[11] and deposition to be more realistically modeled. Table 1 provides a summary.

5.2 Differentiation of models according to the complexity in computational performance

Once a mathematical model that describes debris flow with a certain complexity has been selected, there is the need to find solutions to these equations for certain physically relevant well posed initial boundary value problems. We briefly discuss here three, *i.e.*, (1), simple hydraulic models, (2), higher-order hydraulic models, and (3), full scale three-dimensional models. Actually, among non-specialists and engineers, hydraulic models would not be considered as a form of computational preparation of a mathematical-physical formulation. We wish to do so for reasons that will become apparent as we proceed.

Simple hydraulic models

Consider any formulation of a debris flow with its field equations and boundary and initial conditions and assume that they form a well posed mathematical initial value problem. A hydraulic model reduces the spatial dimension of this problem from 3 to 2 or 1 and replaces some of the field variables by others that can be viewed as depth or cross sectional averages of the former. The thesis is that the true variation of the field variables over depth or cross section is not important, or that by replacing these by the corresponding averaged quantities, the essential physics is preserved, so that it suffices to describe the flow as a whole. This yields *channel hydraulic models* if integrations are over cross sections (reduction to one dimension), or *depth-integrated hydraulic models* (reduction to two dimensions).

It is important that the governing equations are referred to the appropriate coordinates prior to the depth or cross sectional integration, for otherwise curvature effects of the topography may not properly be taken into account.

Depth-averaged models

If one uses Cartesian coordinates with the x-axis pointing in the direction of steepest descent then this is tantamount to allow only small deviations of the base over which the debris mass flows from a flat bed. In

[11] Especially the erosion from the sediment bed is largely governed by the amount of turbulence or kinetic fluctuation energy that is present close to the bed. Deposition, on the other hand, is mainly governed by (negative) buoyancy and thus requires careful modelling of the variation of the solid's fraction. A review of this is given in Hutter (1995) and is a major topic in dust and powder flow.

the only presently existing hydraulic model (O'Brien et al. 1993) this is assumed despite the fact that abupt deviations from a flat bed occurred in the model[12] used by them.

The next step is to introduce a curvature dependence of the bed in the direction of steepest descent (this curve may be assumed to have vanishing torsion) while the direction perpendicular remains flat. If the basal topography deviates only slightly from this cylindrical surface, then its level lines only deviate slightly from straight lines. Such a model is appropriate when curvature effects in the downhill direction are appreciable. Deviations of the basal topography from the cylindrical surface must in general be small (Wieland 1995).

A further generalization, *e.g.*, introduction of sidewise curvature effects, or a curve for the direction of steepest descent with non-vanishing torsion stretches the applicability of depth averaged hydraulic models beyond their applicability. All this follows from detailed models of granular flows performed by Hutter and associates (Greve et al. 1994; Koch et al. 1994; Koch 1994; Hutter 1995; Wieland 1995).

Channel models

Such models are appropriate for flows through corries or in case of straight plane deformation and should be used only if the information needed must not be very detailed, and errors can be large. The height and a mean velocity are the main variables; this means the free surface is represented as a horizontal straight line across the channel width. Its transverse deformation or inclination cannot be determined.

All formulations we have seen use Cartesian metric and simply integrate equations in the two transverse directions (y, z-coordinates). What emerges are *Boussinesq-type equations* that are valid when the path along which the debris flow proceeds is straight. If it is curved in a vertical plane, then the chute flow can be treated along a flow route that is substantially curved in the downhill direction. Such a formulation may provide the possibility that a single debris mass separates into two or more smaller masses as it moves downhill. For analyses of granular flows and comparison with experiments, see Hutter and Koch (1991), Greve (1991), Greve and Hutter (1993), Savage and Hutter (1991), Hutter et al. (1993a).

Channel models, in which the channel axis is spatially curved (non-vanishing torsion) have so far not been proposed. Such a formulation, however, is physically probably inappropriate because sidewise centripetal forces generate in curved corries transverse inclination of the free surface. Higher order channel models are needed to this end.

In the process of deduction of the hydraulic equations, mathematical simplifications are introduced which are somewhat hidden and often forgotten. These are:

1. **Averaging and multiplication are assumed to commute**. This means that if $\langle f \rangle$ denotes the average of some quantity f, then $\langle fg \rangle = \langle f \rangle \langle g \rangle$ is assumed. This is actually correct only if f and g are uniformly distributed over the depth (or cross section). This problem arises for instance in the evaluation of the convective acceleration terms:

$$\langle v_i v_j \rangle_{,j} = [\lambda_{(ij)} \langle v_i \rangle \langle v_j \rangle]_{,j} \quad , \tag{5.5}$$

   where $\lambda_{(ij)}$ are the so-called Boussinesq coefficients, for which values must be prescribed[13]. To set $\lambda_{(ij)} = 1$ amounts to ignoring bulk variations of $\mathbf{v}$.

2. **The basal shear stress and pressure must via the constitutive relations and assumed depth (or cross sectional) distributions of the field variables be related to the average variables**. This requires ad hoc guesses for these distributions which are in general not consistent with the model. Furthermore, these are distributions often independently introduced from the Boussinesq coefficients, but, of course, both are interdependent.

[12] Assuming that equations have the form based on a Cartesian metric even though the coordinate system may be orthogonal but curved requires small mean and Gaußian curvature of the basal topography, small relative to a characteristic length of the debris flow parallel to the base, see Hutter (1995), Greve, Koch and Hutter (1994).

[13] Classically, in fact, one introduces only one Boussinesq coefficient and writes $\langle v_i v_j \rangle = \lambda \langle v_i \rangle \langle v_j \rangle$. If in a depth-integrated model $\mathbf{v}$ is assumed to be parabolic, then $\lambda \approx 1.2$. There are also terms others than the convective acceleration terms where such approximations must be made.

Higher-order hydraulic models

Such models are better suited to *fanned, shallow debris flows* than channelized situations; so we restrict considerations to the former. In numerical analysis, these models are better known as *semi-spectral models*; the method itself is sometimes called the *Kantorovich technique*, and consists of a function expansion in the depth direction. What emerges is a set of evolution equations of which the spatial independent variables are those perpendicular to the depth.

Let x, y, z be a set of orthogonal curvilinear coordinates, the surface $z = 0$ being close to the basal topography over which the debris material is flowing. Let, moreover, $z = h_{\mathrm{f}}(x, y, t)$ and $z = h_{\mathrm{b}}(x, y)$ denote the free surface and the basal surface, respectively. Then,

$$\sigma = \frac{z - h_{\mathrm{b}}}{h_{\mathrm{f}} - h_{\mathrm{b}}} , \qquad \sigma \in [0, 1] \tag{5.6}$$

maps the domain $z \in [h_{\mathrm{b}}, h_{\mathrm{f}}]$ onto the unit closed interval $[0, 1]$. Let $u(x, y, z, t)$ be any one of the field variables, $\{\phi_\alpha(\sigma) \mid \alpha = 1, 2, 3, \ldots\}$ a known infinite complete function set on $[0, 1]$, and $\{u^\alpha(x, y, t) \mid \alpha = 1, 2, 3, \ldots\}$ an analogous set of unknown functions. Then $u(x, y, z, t)$ can be represented in the form

$$u(x, y, z, t) = \sum_{\alpha=1}^{N} \phi_\alpha(\sigma)\, u^\alpha(x, y, t) \quad , \tag{5.7}$$

where the ϕ_α-functions describe the depth variation of u, and the unknown u^α depend on the remaining variables. The representation (5.7) amounts to a separation of u into a known set $\{\phi_\alpha \mid \alpha = 1, 2, 3, \ldots\}$ and an unknown set $\{u^\alpha \mid \alpha = 1, 2, 3, \ldots\}$. The summation in (5.7) should be performed over an infinite number of shape functions, *i.e.*, for $N = \infty$; in practice, however, N is finite, and $N = 1$ corresponds to the simple hydraulic models.

A higher order hydraulic model corresponds to a set of field equations for the set $\mathbf{u} = \{u^\alpha \mid \alpha = 1, 2, 3, \ldots\}$ of all unknown field variables, and the N describes the order of approximation. The corresponding equation set is derived by the *method of weighted residuals*. The idea behind this method is the following: let $\mathbf{f}$ be a set of M equations describing a system in a domain Ω with an unknown solution $\bar{\mathbf{u}}$ and a known forcing term $\mathbf{b}$, such that

$$\mathbf{f}(\bar{\mathbf{u}}) - \mathbf{b} = \mathbf{0} \tag{5.8}$$

holds. Now, in general, an arbitrary field $\mathbf{u}_{\mathrm{a}}$ does not satisfy (5.8), but rather produces an error

$$\mathbf{r}(\mathbf{u}_{\mathrm{a}}) = \mathbf{f}(\mathbf{u}_{\mathrm{a}}) - \mathbf{b} \neq \mathbf{0} , \tag{5.9}$$

called the residue. The method of weighted residuals forces the average of the residuals to be zero by setting as many weighted integrals of the residuals equal to zero over Ω as is necessary to determine the approximate solution

$$\int_\Omega \mathbf{r}(\mathbf{u}_{\mathrm{a}}) \cdot \mathbf{v}_\beta \, d\Omega = 0 , \qquad \beta = 1, \ldots, M \quad , \tag{5.10}$$

where $\{\mathbf{v}_\beta \mid \beta = 1, \ldots, M\}$ is a set of linearly independent weighting functions, and M is the number of weighted integrals of the residuals necessary to determine the approximate solution $\mathbf{u}_{\mathrm{a}}$. Generally, $\mathbf{u}_{\mathrm{a}}$ is postulated as a product of two truncated sets in the form

$$\mathbf{u}_{\mathrm{a}} = \sum_{\alpha=1}^{N} \phi_\alpha \mathbf{u}^\alpha \quad , \tag{5.11}$$

where the $\{\phi_\alpha \mid \alpha = 1, \ldots, N\}$ generate a set of linearly independent known functions, called shape, basis or trial functions, and the $\{\mathbf{u}^\alpha \mid \alpha = 1, \ldots, N\}$ constitute a set of unknown functions.

For the limit as N tends to infinity, the ϕ_α should form a complete set of functions to ensure that every possible admissible function may be represented. This set may be constructed from a set of polynomials such as Legendre or Tshebyshev polynomials (which are complete in $[0, 1]$ and derivable from an eigenvalue problem, such that the spectral method is appropriate). The representation (5.10) allows interpretation of the method of weighted residuals as a procedure for ensuring that the error caused by the linear independent set of shape functions in the operator describing the problem is minimized by making it orthogonal to another (or the same) set of linearly independent weighting functions. This shows that the method of weighted residuals is a *projection method.*

The method has been extensively used by Hutter and his associates. Raggio (1980) and Raggio and Hutter (1982a,b,c) used it to derive extended, higher order Chrystal models for surface seiches in lakes. Particularly successful was the method in the development of analytical-numerical solutions to the topographic wave operator, see Stocker (1987), Stocker and Hutter (1986, 1987b,c,d, 1988) and Stocker (1988). For debris flows, no higher order hydraulic models have been derived so far. The operator $\mathbf{f}(\mathbf{u})$ is given by the chosen model equations as surmised in Table 1 and $\mathbf{u}$ are the field variables arising in these. Whether these equations are referred to Cartesian or curvilinear coordinates must be decided in advance. The number of shape functions that are needed, *i.e.*, the order of the hydraulic model, is expected to be small except perhaps where the curvature of the basal topography changes relatively abruptly and also close to the snout. The final model equations are still continuous operator equations and are solved either by FD or FE discretization methods.

Full scale three dimensional models

The hydraulic models discussed above are, from a computational point of view nothing else than a first step in a discretization technique to solve the proposed three dimensional model equations. Their semi-spectral form is particularly suited for debris flows, since the dominant flow direction is essentially orthogonal to the direction of spectral expansion.

One can, of course, dispense with this approach and directly pass to a 3D, FD- or FE-implementation, again using the principle of weighted residuals to deduce the finite element matrix equation

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u} + \mathbf{b} \quad , \tag{5.12}$$

and proceeding in the temporal integration with the method of line or any other method. Because of the parabolicity the governing equations will likely exhibit conditional stability; implicit temporal integration is likely necessary to avoid numerical instabilities from being developed.

5.3 Remarks on integration procedures

Debris flows belong mathematically to the *free boundary value problems* in which the domain geometry must be determined along with the field quantities that are defined by the equations governing their evolution in these domains. Numerically, this fact introduces additional complexities and also makes integration routines, if they are not properly formulated, prone to numerical instabilities.

Experience shows that free boundary value problems are computationally more easily handled and discretized integration routines are more stable if prior to this numerical implementation a fixed-domain mapping is introduced, that reduces all equations to the same fixed domain for all time (or for a certain time interval).

In continuum mechanics, the free boundary value problem arises when equations are referred to the present configuration, and hence equations are written in the *Eulerian formulation.* The fixed domain mapping is effected by the *"pull-back" operation* of these equations to the reference configuration, *i.e.*, the transformation to the *Lagrangian formulation.*

Ideally, one would refer all equations of a debris flow problem to the initial configuration, which is fixed and then solve these pulled-back equations. This has been done in the debris flow context only by Hungr (1994) in a limited application and by referring to experiences gained in avalanche flows by Savage and Hutter (1989). Indeed, in the granular avalanche work of Hutter and associates[14], the Lagrangian numerical integration procedure is systematically used. All Eulerian integration schemes that were tried, failed to be numerically stable.

[14] Hutter and Savage (1991), Hutter and Koch (1991), Greve and Hutter (1993), Greve et al. (1994), Koch et al. (1994), Koch (1994), Wieland (1995).

Because of the large distances travelled by the material particles in a debris flow, deformations are generally also large and, consequently, deformations of a FD- or FE-net are substantial. Koch (1994) found that his FD-code became increasingly unstable especially when originally approximately equilateral triangular grids were severely stretched. Such situations are likely to always occur in realistic debris flow situations. It may therefore be advantageous to introduce intermediate configurations and use a so-called *adapted or advected FD- or FE-formulation*, in which the equations are not pulled-back to the initial configuration but successively to the intermediate configurations. At each intermediate configuration a new grid is generated. Such intermediate configurations need probably not be introduced at each time step, but only when deformations become large.

It is recommended that computational schemes are designed with careful considerations of these facts.

6 Boundary conditions

Debris flows possess two distinct surfaces that bound the domain of the moving material, *i.e.*, (i), the free surface, and (ii), the bed. Boundary conditions that must be formulated at these surfaces are of *kinematic and dynamic nature*, and their complexity depends on the complexity of the theory that is employed as well as on the physical processes one intends to include at these boundaries. We ignore situations in which the material may become unsaturated in subregions, and thus assume that the *phreatic surface* is coincident with the free surface. Generally, when erosion of material from or deposition to the bed is considered the basal surface is changing with time, and this change ought to be determined along with that of the free surface.

6.1 Kinematic and dynamic boundary conditions

Kinematic conditions

Let $F(\mathbf{x}(t)) = 0$ be the equation describing either the free or basal surface. Then, since $F = 0$ for all time, the time rate of change of F following the surface must vanish, *i.e.*,

$$\dot{F} = F_{,t} + (\nabla F) \cdot \mathbf{w} = 0 \quad , \tag{6.1}$$

where $\mathbf{w}$ denotes the velocity with which the surface moves. Note that this velocity need not coincide with the grain or fluid velocity of the particles instantaneously sitting upon the surface, *i.e.*, $\underset{\mathfrak{a}}{\mathbf{v}}^{\pm}$ for $\mathfrak{a} = \mathrm{s}$ or $\mathfrak{a} = \mathrm{f}$, where $f^{\pm}$ denotes the limit of some field f as we approach the surface from the $\pm$ side, where the + side corresponds to the outside part of the boundary with respect to the debris flow, and $-$ to the debris flow side. In terms of this $\underset{\mathfrak{a}}{\mathbf{v}}^{\pm}$, (6.1) can be written as

$$F_{,t} + (\nabla F) \cdot \underset{\mathfrak{a}}{\mathbf{v}}^{\pm} = -|\nabla F| \underset{\mathfrak{a}}{a}^{\pm} \quad , \tag{6.2}$$

with

$$\underset{\mathfrak{a}}{a}^{\pm} = (\mathbf{w} - \underset{\mathfrak{a}}{\mathbf{v}}^{\pm}) \cdot \mathbf{n} \tag{6.3}$$

the *component volume flux relative to the surface into the debris flow*, and

$$\mathbf{n} = \frac{\nabla F}{|\nabla F|} \tag{6.4}$$

the outward unit normal vector pointing into the + region into which the surface propagates (with $F = 0$ accordingly defined), which corresponds by convention to the atmosphere at the free surface, and to the ground at the base. The kinematic relation (6.2) yields the result

$$[\![\underset{\mathfrak{a}}{\mathbf{v}} \cdot \mathbf{n}]\!] = -[\![\underset{\mathfrak{a}}{a}]\!] \quad , \tag{6.5}$$

where $[\![f]\!] = f^+ - f^-$ represents the jump of f across the surface $F = 0$. The result (6.5) provides a useful interpretation of the kinematic surface condition; if $[\![\underset{\alpha}{a}]\!] = 0$, the difference of the velocities on the two sides is tangential to the surface, *i.e.*, $[\![\underset{\alpha}{\mathbf{v}} \cdot \mathbf{n}]\!] = 0$. Then, with

$$\begin{array}{rcccll} F_{\mathrm{f}}(x,y,z,t) & = & -z_{\mathrm{f}}(x,y,t) + z & = & 0 & \text{free surface}\ , \\ F_{\mathrm{b}}(x,y,z,t) & = & z_{\mathrm{b}}(x,y,t) - z & = & 0 & \text{base}\ , \end{array} \tag{6.6}$$

(6.2) takes the forms

$$\begin{array}{rcll} z_{\mathrm{f},t} + z_{\mathrm{f},x}\underset{\alpha}{v}{}_x^{\pm} + z_{\mathrm{f},y}\underset{\alpha}{v}{}_y^{\pm} - \underset{\alpha}{v}{}_z^{\pm} & = & N_{\mathrm{f}}\,\underset{\alpha}{a}{}_{\mathrm{f}}^{\pm} & \text{at } z = z_{\mathrm{f}}(x,y,t)\ , \\ z_{\mathrm{b},t} + z_{\mathrm{b},x}\underset{\alpha}{v}{}_x^{\pm} + z_{\mathrm{b},y}\underset{\alpha}{v}{}_y^{\pm} - \underset{\alpha}{v}{}_z^{\pm} & = & -N_{\mathrm{b}}\,\underset{\alpha}{a}{}_{\mathrm{b}}^{\pm} & \text{at } z = z_{\mathrm{b}}(x,y,t)\ , \end{array} \tag{6.7}$$

with $N_{\mathrm{f}} = \sqrt{1 + (z_{\mathrm{f},x})^2 + (z_{\mathrm{f},y})^2}$, and the same for N_{b}. These are the kinematic boundary conditions when a Cartesian coordinate system is used.

Often the kinematic relations (6.5) and (6.7) are combined through a depth integration of the mass balance to form a global mass balance statement, however, all these forms are only justifiable when the velocity field $\underset{\alpha}{\mathbf{v}}$ is solenoidal,

$$\begin{array}{rcl} \operatorname{div} \underset{\alpha}{\mathbf{v}} & = & 0\ , \\ \underset{\alpha}{v}{}_{x,x} + \underset{\alpha}{v}{}_{y,y} + \underset{\alpha}{v}{}_{z,z} & = & 0\ . \end{array} \tag{6.8}$$

Integrating this equation from $z = z_{\mathrm{b}}$ to $z = z_{\mathrm{f}}$, and using (6.5) and (6.7) yields

$$(z_{\mathrm{f}} - z_{\mathrm{b}})_{,t} + Q_{x,x} + Q_{y,y} = N_{\mathrm{f}}\,\underset{\alpha}{a}{}_{\mathrm{f}}^{\pm} - N_{\mathrm{b}}\,\underset{\alpha}{a}{}_{\mathrm{b}}^{\pm}\ , \tag{6.9}$$

with

$$Q_i = \int_{z_{\mathrm{b}}}^{z_{\mathrm{f}}} \underset{\alpha}{v}{}_i(\cdot, z)\, dz\ . \tag{6.10}$$

Only models with constant density (constant solids fraction) have solenoidal velocity fields. Thus, (6.9) is not applicable[15] without separate justification.

Dynamic conditions

The dynamic conditions emerge from the balance laws when these are formulated for an infinitely thin "pill-box" surrounding the singular boundary surface. Assuming no mass or momentum exchange takes place between the components on the surface, these are given by

$$\begin{array}{rcl} [\![\underset{\alpha}{\bar{\varrho}}\,\underset{\alpha}{a}]\!] & = & 0\ , \\ \underset{\alpha}{\bar{\varrho}}{}^{\pm}\underset{\alpha}{a}{}^{\pm}\,[\![\underset{\alpha}{\mathbf{v}}]\!] + [\![\underset{\alpha}{\mathbf{t}}]\!] & = & \mathbf{0}\ , \end{array} \tag{6.11}$$

on $F = 0$ for mass and momentum with $\underset{\alpha}{\mathbf{t}} = \underset{\alpha}{\mathbf{T}}\,\mathbf{n}$. If balance laws of the fluctuation energy for the fluid and/or the grains are also introduced, then jump conditions for these would also have to be formulated. Note that, if $\mathbf{w} = \underset{\alpha}{\mathbf{v}}$, the surface is material with respect to the corresponding component, and $\underset{\alpha}{a} = 0$. In that case, $(6.11)_1$ is trivial, and $(6.11)_2$ reduces to the traction continuity relation $[\![\underset{\alpha}{\mathbf{T}}]\!]\mathbf{n} = \mathbf{0}$.

[15] Takahashi (1991), *e.g.*, uses (6.9). The equation is not applicable when, *e.g.*, density variations across the debris flow depth are ignored but variations in the direction parallel to the bed are permitted.

6.2 Boundary conditions at the free surface

Two-component model. Define the free surface to be material with respect to the solid particles; thus $\underset{s}{a} = 0$, and $(6.7)_1$ yields

$$\left.\begin{aligned} z_{f,t} + z_{f,x}\underset{s}{v}_x^\pm + z_{f,y}\underset{s}{v}_y^\pm - \underset{s}{v}_z^\pm &= 0 \\ z_{f,t} + z_{f,x}\underset{w}{v}_x^\pm + z_{f,y}\underset{w}{v}_y^\pm - \underset{w}{v}_z^\pm &= N_f\,\underset{wf}{a}^\pm \end{aligned}\right\} \text{ at } z = z_f(x,y,t) \quad . \tag{6.12}$$

The first of these serves as the evolution equation for $z = z_f$, and the second as an equation for the determination of $(\underset{w}{\mathbf{v}}^\pm - \underset{s}{\mathbf{v}}^\pm)\cdot\mathbf{n}$, *i.e.*,

$$(\underset{w}{\mathbf{v}}^\pm - \underset{s}{\mathbf{v}}^\pm)\cdot\mathbf{n} = -\underset{wf}{a}^\pm \quad . \tag{6.13}$$

The two-phase model is only meaningful when $\underset{wf}{a}^\pm \le 0$, in which case there is water flow on the surface of the debris flow, and $\underset{wf}{a}^+$ can be identified with the precipitation intensity.

The dynamic boundary conditions on the free surface can be deduced from (6.11) and the assumption $\mathbf{w} = \underset{a}{\mathbf{v}}$. If one further assumes that[16] $[\![\underset{w}{\mathbf{v}}]\!] = \mathbf{0}$, and that the traction from the atmospheric side vanishes, then

$$\underset{s}{\mathbf{t}}^- = \mathbf{0}\ , \qquad \underset{w}{\mathbf{t}}^- = \mathbf{0} \tag{6.14}$$

follows from $(6.11)_2$.

Two-component diffusive mixture model. Here, only the barycentric velocity is defined, and thus there is only a single kinematic equation. Therefore the free surface must be defined to be material with respect to the barycentric velocity, implying

$$\left.\begin{aligned} z_{f,t} + z_{f,x}v_x^\pm + z_{f,y}v_y^\pm - v_z^\pm &= 0 \\ \mathbf{T}^-\mathbf{n} &= \mathbf{0} \end{aligned}\right\} \text{ at } z = z_f(x,y,t) \tag{6.15}$$

as the kinematic equation and the boundary condition of stress. Here again the traction from the atmospheric side is assumed to vanish.

Single-component model. Here, the boundary conditions are the same as (6.15), if the free surface is assumed to be material and the traction from the atmospheric side vanishes.

Remark: When density gradients (or gradients of the solids fraction) enter the stresses as an independent variable, and/or balance laws of fluctuation energy are added to the model then further boundary conditions need to be added involving these variables. For granular media Hutter and Rajagopal (1994) provide a discussion on the peculiarities that may be accompanied with these.

6.3 Boundary conditions at the base

Structurally, these are the same as on the free surface, however, dependent on which model is used sedimentation or erosion of particles as well as drainage of water can be incorporated.

Two-component model. If sedimentation/erosion as well as water drainage processes are incorporated, then the basal surface is non-material relative to both phases, and $(6.7)_2$ take the forms

$$\left.\begin{aligned} z_{b,t} + z_{b,x}\underset{s}{v}_x^\pm + z_{b,y}\underset{s}{v}_y^\pm - \underset{s}{v}_z^\pm &= -N_b\,\underset{sb}{a}^\pm \\ z_{b,t} + z_{b,x}\underset{w}{v}_x^\pm + z_{b,y}\underset{w}{v}_y^\pm - \underset{w}{v}_z^\pm &= -N_b\,\underset{wb}{a}^\pm \end{aligned}\right\} \text{ at } z = z_b(x,y,t) \quad , \tag{6.16}$$

in which $\underset{sb}{a}^\pm$ represents the sediment erosion (deposition) rate when greater than or equal to (less than or equal to) zero, and $\underset{wb}{a}^\pm$ the water drainage velocity when greater than or equal to zero, representing a water

[16] This means that the water layer that might exist on the free surface moves with the same speed as the water just inside.

Table 2. Boundary conditions at the free surface

	Two-component model	Diffusive model	Single-component model
Boundary conditions	• free surface is material for solid particles: $z_{f,t} + z_{f,x} \underset{s}{v}{}_x^{\pm} + \cdots = 0$, $(\underset{w}{v}{}^{\pm} - \underset{s}{v}{}^{\pm}) \cdot n = -\underset{wf}{a}{}^{\pm}$. • free surface is stress-free: $\underset{w}{t}{}^{\pm} = \mathbf{0}$, $\underset{s}{t}{}^{\pm} = \mathbf{0}$.	• free surface is material for mixture: $z_{f,t} + z_{f,x} v_x^{\pm} + \cdots = 0$. • free surface is stress-free: $t^{\pm} = \mathbf{0}$.	• free surface is material for mixture: $z_{f,t} + z_{f,x} v_x^{\pm} + \cdots = 0$. • free surface is stress-free: $t^{\pm} = \mathbf{0}$.
Prescribed quantities	$\underset{wf}{a}{}^{\pm}$ Precipitation intensity.	• Precipitation could be incorporated, but model is then likely to be inaccurate.	• Precipitation could be incorporated, but model is then likely to be inaccurate.
Remarks	• If density gradients are incorportated, additional boundary conditions are required. • If fluctuation energy is incorportated, a corresponding flux relation is required.	• If density gradients are incorportated, additional boundary conditions are required. • If fluctuation energy is incorportated, a corresponding flux relation is required.	• If density gradients are incorportated, additional boundary conditions are required. • If fluctuation energy is incorportated, a corresponding flux relation is required.

source for the debris flow from the ground water. While the drainage velocity function $\underset{w}{a}{}_b^{\pm}$ is mainly governed by ground hydrology, the erosion/deposition velocity $\underset{s}{a}{}_b^{\pm}$ of the sediment depends on both the compaction, cohesion, and so on, of the bed as well as the turbulence intensity and density structure of the flow. Both functions must be constitutively prescribed. Takahashi et al. (1992) give propositions for these; see also Hutter (1995) and Gauer (1995).

The dynamic boundary conditions (6.11) are not needed because they would only give information on the tractions immediately below the basal surface. Instead a sliding law relating the tangential component of the velocities $\underset{a}{\mathbf{v}}{}^{\pm}$ and the shear tractions must be prescribed. One could propose the no-slip condition for both phases, but that is only appropriate when $\underset{s}{a}{}_b^{\pm} = 0 = \underset{w}{a}{}_b^{\pm}$, in which case one has

$$\underset{s}{\mathbf{v}}{}^{\pm} = \mathbf{0} = \underset{w}{\mathbf{v}}{}^{\pm} \quad . \tag{6.17}$$

If the sediment is neither depositing nor eroding, then $z = z_b$ is material for the grains and a no-slip condition $\underset{s}{\mathbf{v}}{}^{\pm} = \mathbf{0}$ probably makes sense for these, but the water would then need a sliding law

$$\underset{w}{\mathbf{t}}{}_{\|}^{-} = \underset{w}{\mathbf{t}}{}^{-} - (\mathbf{n} \cdot \underset{w}{\mathbf{t}}{}^{-})\mathbf{n} = \underset{w}{C}\, \underset{w}{\mathbf{v}}{}_{\|}^{-} \quad , \tag{6.18}$$

where $\underset{w}{C}$ is a sliding coefficient that could depend on local water pressure, shear traction, and so on, where $\underset{w}{C} = 0$ corresponds to perfect sliding, and $\underset{w}{C} = \infty$ to no-slip.

In the general case when the erosion/deposition and drainage processes operate both equations (6.16) apply and sliding laws require that

$$\underset{a}{\mathbf{t}}{}_{\|}^{-} = \underset{a}{C}\, \underset{a}{\mathbf{v}}{}_{\|}^{-} \tag{6.19}$$

for $\alpha = \text{s}, \text{f}$, where $\underset{\text{s}}{C}$ and $\underset{\text{w}}{C}$ must be prescribed. Instead of doing this, one may alternatively prescribe $\underset{\text{s}}{C}$ and the difference in tangential velocity

$$\{(\underset{\text{s}}{\mathbf{v}}^- - \underset{\text{w}}{\mathbf{v}}^-) - [(\underset{\text{s}}{\mathbf{v}}^- - \underset{\text{w}}{\mathbf{v}}^-) \cdot \mathbf{n}]\mathbf{n}\} = \mathbf{d} \quad , \tag{6.20}$$

which is more like a Darcy-type relation. A first guess would certainly be $\mathbf{d} = \mathbf{0}$.

Table 3. Boundary conditions at the base

	Two-component model	Diffusive model	Single-component model
Boundary conditions	1) $\underset{\text{s}}{v}^\pm = \mathbf{0}$, $\underset{\text{w}}{v}^\pm = \mathbf{0}$, (no-slip), with no deposition or erosion of sediments, nor drainage of water. or 2) $\underset{\text{s}}{v}^\pm = \mathbf{0}$ (no slip), $z_{\text{b},t} + z_{\text{b},x}\, \underset{\text{w}}{v_x^\pm} + \cdots = -N_\text{b}\, \underset{\text{w}}{a_\text{b}^\pm}$, with no deposition or erosion of sediments, nor drainage of water, $\underset{\text{w}}{t}_{\parallel}^{\pm} = \underset{\text{w}}{C}\, \underset{\text{w}}{v}_{\parallel}^{\pm}$. or 3) $z_{\text{b},t} + z_{\text{b},x}\, \underset{\text{s}}{v_x^\pm} + \cdots = -N_\text{b}\, \underset{\text{s}}{a_\text{b}^\pm}$, $z_{\text{b},t} + z_{\text{b},x}\, \underset{\text{w}}{v_x^\pm} + \cdots = -N_\text{b}\, \underset{\text{w}}{a_\text{b}^\pm}$, $\underset{\text{s}}{t}^{\pm} = \underset{\text{s}}{C}\, \underset{\text{s}}{v}_{\parallel}^{\pm}$, $\underset{\text{w}}{t}_{\parallel}^{\pm} = \underset{\text{w}}{C}\, \underset{\text{w}}{v}_{\parallel}^{\pm}$.	1) $v^\pm = \mathbf{0}$ (no slip), provided there is neither deposition or erosion of sediments nor drainage of water. or 2) $\underset{\text{s}}{v}^\pm = \mathbf{0}$ (no slip), $z_{\text{b},t} + z_{\text{b},x}\, v_x^\pm + \cdots = -N_\text{b}\, a_\text{b}^\pm$, with no deposition or erosion of sediments, nor drainage of water, $t_{\parallel}^{\pm} = C\, v_{\parallel}^{\pm}$.	1) $v^\pm = \mathbf{0}$ (no slip), provided there is neither deposition or erosion of sediments nor drainage of water. or 2) $\underset{\text{s}}{v}^\pm = \mathbf{0}$ (no slip), $z_{\text{b},t} + z_{\text{b},x}\, v_x^\pm + \cdots = -N_\text{b}\, a_\text{b}^\pm$, with no deposition or erosion of sediments, nor drainage of water, $t_{\parallel}^{\pm} = C\, v_{\parallel}^{\pm}$.
Prescribed quantities	$\underset{\text{s}}{a_\text{b}^\pm}$ depositional/ erosional velocity. $\underset{\text{w}}{a_\text{b}^\pm}$ drainage/ water velocity.	$a_\text{b}^\pm$ depositional/ erosional, drainage velocity.	$a_\text{b}^\pm$ depositional/ erosional, drainage velocity.
Remarks	• If density gradients are incorporated, additional boundary conditions are needed. • If fluctuation energy and turbulence are included, boundary conditions are needed for these. • Deposition and erosion processes should not be modeled without evolution relation for fluctuations.		

Diffusion model. Since only the barycentric velocity is defined in these models, the kinematic condition (6.7) takes the form

$$z_{b,t} + z_{b,x} v_x^{\pm} + z_{b,y} v_y^{\pm} - v_z^{\pm} = -N_b \, a_b^{\pm} \quad , \tag{6.21}$$

where $a_b^{\pm}$ is the combined erosion deposition-drainage velocity. Water and gravel are not separable here, as a constitutive relation for both together must be postulated.

The other condition is a sliding law, now expressed for the mixture stress, *i.e.*,

$$\mathbf{t}_{\parallel}^{-} = C \, \mathbf{v}_{\parallel}^{-} \quad . \tag{6.22}$$

The no-slip condition is again only meaningful when $a_b^{\pm} = 0$, in which case (6.21) and (6.22) reduce to

$$\mathbf{v}^{\pm} = \mathbf{0} \qquad \text{at } z = z_b \quad . \tag{6.23}$$

The boundary conditions for the single-component model are conceptually no different than those of the two-component model, so (6.21)–(6.23) apply.

Tables 1 and 2 summarize the boundary conditions formulated at the free surface and base of the debris flow.

When density gradients or concentration gradients are independent variables arising in the constitutive relations and/or balance laws for the fluctuation energy are part of the theory, then additional boundary conditions might enter the formulation. The form of these depends upon how the additional variables are introduced in the theory, see Hutter (1995).

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