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Debye-Hückel Limit of Quantum Coulomb Systems

I. Pressure and Diagonal Reduced Density Matrices

J. R. Fontaine*

Institut de physique théorique, EPFL-PHB – Ecublens, CH-1015 Lausanne, Switzerland

Abstract. In this paper, we consider charge symmetric quantum Coulomb systems with Boltzmann statistics. We prove that the theory of screening of Debye and Hückel is a combined classical and mean field limit of these quantum Coulomb systems.

Introduction

Quantum Coulomb systems are known to be stable: the thermodynamic functions of these systems, as well as their convexity properties have been obtained from the principles of statistical mechanics by Lieb and Lebowitz [1]. The microscopic properties of these systems are however far from being understood. For instance, nothing is known about the clustering properties of their reduced density matrices (R.D.M.).

In this paper, we show that the classical theory of screening of Debye and Hückel (see for instance [2, p. 275]) is an exact classical and mean field limit of quantum Coulomb systems. For technical reasons, we have to impose restrictions on the class of systems we consider. We restrict ourself to charge symmetric systems in the Grand Canonical Ensemble; for a two component system (which is the case we consider for simplicity), this means that the activities $\underset{\sim}{z}$, masses m, and absolute value of charge e of both species have to be the same. Moreover, we only deal with quantum systems with Boltzmann statistics (we shall have to add short range forces to insure stability). For such models Fröhlich and Park have been able to prove the existence of the thermodynamic limit of the reduced density matrices [3]. These systems are described by three parameters: $\beta=e^2(kT)^{-1}$, $\alpha=\hbar^2(me^2)^{-1}$, $z=\underset{\sim}{z}(\alpha\beta 2\pi)^{-3/2}$. From these 3 parameters only 2 are independent. Indeed, because of the scaling properties of the Coulomb potential, the system described by the parameters (β,α,z) is equivalent to the one with parameters $(\beta/\ell,\alpha/\ell,z/\ell^3)$, where ℓ is any non-zero positive number which represents a change of length-scale.

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The aim of this paper is to find a regime of temperature and fugacity for which the quantum system in the thermodynamic limit can be approximated by the classical Debye Hückel theory with exponentially decaying correlation functions. There are two conditions to be satisfied:

A) The quantum Coulomb system has to be close to a classical Coulomb system.

B) That classical model has to be close to the mean field theory of Debye Hückel.

In order to express quantitatively conditions A and B, it is useful to introduce three other length scales: the Debye length $\ell_D = (\beta z)^{-1/2}$, the de Broglie length $\lambda_d = (\alpha\beta)^{1/2}$, and the "ion sphere radius" a defined by $z^{-1} = 4/3\pi a^3$. Since we shall be in a regime of parameters for which the density $\varrho \approx z$, a is the mean distance between point charges. From general textbooks (see for instance [4, p. 213]), we expect that a quantum system of particles with two body potential V (with Bose-Einstein or Fermi-Dirac statistics) converges to its classical limit if

A1) $\lambda_d a^{-1} \ll 1$,

A2) $\lambda_d r_0^{-1} \ll 1$, where r_0 is the range of the two body potential V.

A1 essentially ensures that the effects of statistics are negligible, and A2 ensures that the non-commutativity of $-\Delta$ and V is negligible. For our system A1 does not come in, since we have chosen Boltzmann statistics and A2 will be replaced by $\eta = \lambda_d \ell_D^{-1} \ll 1$; that is we replace the infinite range of the Coulomb potential by the "screening" length ℓ_D. Condition B is satisfied if the plasma parameter $\varepsilon = \beta \ell_D^{-1}$ is small. The main result of this paper is to prove that the pressure and the diagonal reduced density matrices (DRDM) converge to the function predicted by the classical Debye-Hückel theory when ε and η both go to zero[1]. The non-diagonal reduced density matrices have been handled similarly by Oguey [5]. Because of our choice of the Boltzmann statistics, for the system to be stable, we have to add short range forces. This is equivalent to regularize the Coulomb potential at short distances. Actually we choose the following two-body potential:

$$V(q_1, q_2) = e_1 e_2 \frac{1 - \exp(-\mu \ell_D^{-1} |x_1 - x_2|)}{4\pi |x_1 - x_2|};$$

here q_i denotes a charge e_i $(= \pm e)$ at position x_i, and μ is a positive real number. Since the Debye-Hückel theory does not contain any short range forces, we shall have to remove them at the same time as we take $\beta \to 0$.

To prove the result, we first express the system as a field theory using a combination of the Ginibre representation and of the Sine-Gordon transformation. This is also what Fröhlich and Park did to prove the existence of the thermodynamic limit. We then get a field theory with a cosine interaction. The Debye-Hückel theory essentially corresponds to a quadratic approximation of the cosine. In order to extract this contribution out of the cosine, we use the method of complex translation of a Gaussian measure invented by McBryan and Spencer [6] and used in [7, 8] for similar problems. Once we have the Gaussian contribution to the D.R.D.M., the remainder is estimated using correlation inequalities. The

1 We take at the same time a classical and a mean field limit

correlation inequalities we use are simple generalization of those found by Kennedy [8] and are only known to hold for charge symmetric systems.

The paper is organized as follows: Sect. 1 contains a precise definition of the system and its representation in terms of field theory. We state the main results in Sect. 2. In Sect. 3, we generalize Kennedy's correlation inequalities. Section 3 contains the proofs of our results.

Caution. In the whole paper c stands for a constant which can take different values at different places.

1. Definition of the Model

We consider quantum Coulomb systems with Boltzmann statistics. In order to insure stability, we therefore have to add a short range potential to the usual Coulomb force.

In a specific way $q=(e,x)$ denotes a charge of type e at position $x\in\mathbb{R}^3$; $e=\pm 1$. All particles have the same mass m.

The two body potential is:

$$V(q_1,q_2)=e_1e_2\frac{1-\exp(-\mu\ell_D^{-1}|x_1-x_2|)}{4\pi e^2|x_1-x_2|}, \tag{1}$$

$\mu>0$ and ℓ_D, the Debye length will be specified later.

The N-particle Hamiltonian is defined on the Hilbert space

$$\mathscr{H}_\Lambda^N=\bigotimes_{j=1}^N L^2(\Lambda,d^3x_j), \qquad \Lambda\subseteqq\mathbb{R}^3$$

by:

$$H_\Lambda^N((q)_N)=-\sum_{i=1}^N\frac{\hbar^2}{2me^2}\Delta_i^\Lambda+U((q)_N).$$

Where Δ^Λ denotes the Laplacian with Dirichlet boundary conditions on $\partial\Lambda$, $(q)_N\equiv(q_1,\dots,q_N)$,

$$U((q)_N)=\sum_{1\leqq i<j\leqq N}V(q_i,q_j).$$

As usual the Grand Canonical partition function is defined by:

$$\Xi_\Lambda(\beta,z)=\sum_{N=0}^\infty\frac{\underset{\sim}{z}^N}{N!}\sum_{(e)_N}\int d(x)_N[\exp-\beta H_\Lambda((q)_N)]((x)_N,(x)_N),$$

where $\underset{\sim}{z}\in\mathbb{R}^+$.

The *pressure* is given by:

$$\begin{aligned}P_\Lambda(\beta,\underset{\sim}{z})&=|\Lambda|^{-1}\log\Xi_\Lambda(\beta,\underset{\sim}{z}),\\ P(\beta,\underset{\sim}{z})&=\lim_{|\Lambda|\to\infty}P_\Lambda(\beta,\underset{\sim}{z}).\end{aligned} \tag{2}$$

The *reduced density matrices* (R.D.M.) are:

$$\begin{aligned}\varrho_\Lambda(\beta,\underset{\sim}{z};(x)_N,(y)_N)=\Xi_\Lambda(\beta,\underset{\sim}{z})^{-1}\underset{\sim}{z}^N\sum_{M=0}^\infty\frac{\underset{\sim}{z}^M}{M!}\sum_{(e)_M}\int d(x')_M\\ \cdot[\exp-\beta H_\Lambda]((x)_N(x')_M,(y)_N(x')_M).\end{aligned} \tag{3}$$

We shall mainly consider the diagonal R.D.M. ≡ D.R.D.M.:

$$\varrho_\Lambda(\beta, \underset{\sim}{z}; (x)_N, (x_N)) \equiv \varrho_\Lambda(\beta, \underset{\sim}{z}, (x)_N),$$
$$\varrho(\beta, \underset{\sim}{z}, (x)_N) = \lim_{|\Lambda| \to \infty} \varrho_\Lambda(\beta, \underset{\sim}{z}; (x)_N).$$

1.1. The Ginibre Representation [9]

Let $d\underset{\sim}{\mathbf{P}}^{\alpha,\Lambda}_{xy}(\omega)$ be the conditional Wiener measure associated to the kernel $\exp(-\alpha\Delta^\Lambda)$. It is defined on paths $\omega(s)$ such that:

$$\omega(s=0)=x, \quad \omega(s=\alpha)=y, \quad \omega(s) \in \Lambda \subset \mathbb{R}^3,$$
$$d\underset{\sim}{\mathbf{P}}^{\alpha,\Lambda}_{(x)_N(y)_N} \equiv \prod_{j=1}^{N} d\underset{\sim}{\mathbf{P}}^{\alpha,\Lambda}_{x_jx_j}(\omega_j).$$

The *Feynman-Kac* formula yields [9] $(\alpha = \hbar^2(2me^2)^{-1})$:

$$[\exp(-\beta H_\Lambda)]((x)_N, (x)_N) = \int d\underset{\sim}{\mathbf{P}}^{\alpha\beta,\Lambda}_{(x)_N(y)_N} \exp\left[-\frac{1}{\alpha}\int_0^{\beta\alpha} ds\, U((e)_N(\omega(s)_N)\right]. \qquad (4)$$

1.2. Scaling Properties of the System

$\Omega = \underset{s \in [0\lambda]}{\times} \dot{\mathbb{R}}^s$, $\dot{\mathbb{R}}$ is the one point compactification of $\mathbb{R}$.

Proposition 1.3.1. *Let $f \in c(\Omega)$, then*

$$\int d\mathbf{P}^\lambda_{\mu x \mu x}(\omega) f(\omega(s))\mu^2 = \int d\mathbf{P}^{\lambda\mu^{-2}}_{xx} f(\mu\omega(\mu^{-2}s)), \qquad \mu \in \mathbb{R}_+.$$

Proof. On cylindric functions this follows from a change of variable. □

Proposition 1.3.1 will be used to establish the scaling properties of our Coulomb system. It is convenient to introduce the normalized measure

$$d\mathbf{P}^\lambda_{xx}(\omega) = (2\pi\lambda)^{3/2} d\underset{\sim}{\mathbf{P}}^\lambda_{xx}(\omega)$$

and $z = \underset{\sim}{z}\alpha^{-3/2}\beta^{-3/2}(2\pi)^{-3/2}$.

Proposition 1.3.2. *The system described by the parameters Λ, β, α, z, $V(x-y)$ is equivalent to the system with parameters $\Lambda\ell^{-3}$, $\beta\ell^{-1}$, $\alpha\ell^{-1}$, $z\ell^3$, $\ell V((x-y)\ell)$.*

Proof. Consider the case of the partition function; for the correlation functions this is similar,

$$\Xi(\Lambda) = \sum_n \frac{z^n}{n!} \int_{\Lambda^n} d(x)_n d\mathbf{P}^{\alpha\beta,\Lambda}_{(x)_n(x)_n} \exp\left[-\frac{1}{\alpha}\int_0^{\alpha\beta} U((\omega(s))_n)ds\right]$$
$$= \sum_n \frac{z^n}{n!} \int_{\left(\frac{\Lambda}{\ell^3}\right)^n} \ell^{3n} d(x)_n d\mathbf{P}^{\alpha\beta,\Lambda}_{\ell(x)_n\ell(x)_n} \exp\left[-\frac{1}{\alpha}\int_0^{\alpha\beta} U((\omega(s))_n)ds\right].$$

Using Proposition 1.3.1, this can be written as:

$$= \sum_n \frac{z^n\ell^{3n}}{n!} \int_{\left(\frac{\Lambda}{\ell^3}\right)^n} d(x)_n dP^{\alpha\beta\ell^{-2}}_{(x)_n(x)_n} \exp\left[-\frac{\ell}{\alpha}\int_0^{\alpha\beta\ell^{-2}} \ell U(\ell(\omega(s))_n)ds\right] \chi_{\Lambda/\ell^3}((\omega)_n). \quad □$$

The way we recover the classical Debye-Hückel theory from our quantum system is by letting both ε and η go to zero. Actually (ε, η) can approach $(0,0)$ through any curve of the type $\eta = \varepsilon^p$, where p is any strictly positive number. According to the different values of p, we have the following regimes (of closeness to Debye-Hückel); high temperature and moderate density, moderate temperature and low density, low temperature and low density....

Because of Proposition 1.3.2, we shall always work in units where $\ell_D = 1$. With these units the Debye-Hückel limit corresponds to $\beta \to 0$ and $\lambda_d = c\beta^p$. To have easier notations, $\lambda_d^2 \equiv \lambda$.

1.3. The Sine-Gordon Representation

Consider the Schwartz space

$$\mathscr{S} \subset L^2(\mathbb{R}^\nu \otimes [o\beta], d^3x\,ds) \subset \mathscr{S}'.$$

Here $[o\beta]$ is a circle and ds is the corresponding Haar measure. Define

$$\mathbb{U}(x, s; y, t) \equiv \alpha^{-1} V(x-y)\delta(s-t).$$

$d\mu$ denotes the Gaussian measure on $\mathscr{S}'$ of mean 0 and covariance $\mathbb{U}$:

$$\int \phi(f)\phi(g)d\mu = (f, \mathbb{U}g), \qquad (\phi \in \mathscr{S}',\, f \text{ and } g \in \mathscr{S}).$$

The Wick-ordered quantities are:

$$:\exp[i\phi(f)]: = \exp[1/2(f, \mathbb{U}f)]\exp[i\phi(f)],$$
$$:\cos\phi(f): = \exp[1/2(f, \mathbb{U}f)]\cos[\phi(f)].$$

The fundamental identity which yields the Sine-Gordon representation is:

$$\int \prod_{j=1}^{N} :\exp\left[ie_j \int_0^\lambda \phi(\omega_j(s), s)ds\right]: d\mu = \exp\left[-\int_0^\lambda ds\, U((e)_N, (\omega(s))_N)\right],$$

$$\mathbb{C}_\Lambda^\lambda \equiv \int_\Lambda d^3x \int d\mathbf{P}_{xx}^{\Lambda;\lambda}(\omega) :\cos\int_0^\lambda ds\phi(\omega(s), s)ds:.$$

We can now write

$$\Xi_\Lambda(\beta, z) = \int d\mu \exp[z\mathbb{C}_\Lambda^\lambda],$$

$$\varrho_\Lambda(\beta, z; (e)_N(x)_N) = \Xi_\Lambda^{-1}(\beta, z)z^N \int \prod_{j=1}^{N} d\mathbf{P}_{x_jx_j}^{\Lambda;\lambda}(\omega_j)$$
$$\cdot :\exp\left[i\int_0^\lambda ds\phi(\omega_j(s), s)ds\right]: \exp[z\mathbb{C}_\Lambda^\lambda]. \tag{6}$$

It is convenient to absorb the Wick ordering as a multiplicative constant of the activity: we introduce

$$z_0 = ze(\beta V(0)). \tag{7}$$

1.4. The Gaussian Approximation

In this section we compute the covariance of the Gaussian theory obtained by replacing the cosine by its quadratic approximation.

$d\mu_G$ is formally defined by:

$$d\mu_G=\frac{1}{\text{Normalisation}}\,d\mu_0\exp\left[-\frac{z}{2}\int d^3x\int d\mathbf{P}^{\lambda}_{xx}\left(\int_0^{\lambda}\phi(\omega(s),s)ds\right)^2\right],$$

$$\int d\mu_G=1\,,$$

$$\int d^3x\int d\mathbf{P}^{\lambda}_{xx}\left(\int_0^{\lambda}ds\,\phi(\omega(s),s)\right)^2$$
$$=\int d^3x\int d\mathbf{P}^{\lambda}_{xx}\int h_\omega(x',s')h_\omega(x'',s'')\quad\phi(x',s')\phi(x'',s'')d^3x'd^3x''ds'ds''\,,$$

where

$$h_\omega(x,s)=\delta(x-\omega(s))\,.$$

After performing the dx and $d\mathbf{P}_{xx}(\omega)$ integration we get:

$$=(2\pi)^{-3}\int_0^{\lambda}ds'\int_0^{\lambda}ds''\int d^3x'd^3x''\phi(x',s')\phi(x'',s'')H(x'-x'',s'-s'')$$

with

$$H(x,s)=(2\pi\lambda)^{3/2}(2\pi(\lambda-s))^{-3/2}(2\pi s)^{-3/2}\exp[-\lambda x^2(2s(\lambda-s))^{-1}]\,,$$
$$H(x'-x'',s'-s'')=\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)h_\omega(x',s')h_\omega(x'',s'')\,. \tag{8}$$

$H(x,s)$ is periodic in s; one can take the Fourier transform:

$$\tilde{H}(k,k_0)=(2\pi)^{-3}\int dx\int_0^{\lambda}ds\exp(-ikx-ik_0s')H(x,s)$$
$$=\int_0^{\lambda}ds\exp(-ik_0s)\exp[-k^2s(\beta-s)(2\lambda)^{-1}]\,,$$
$$k_0=2\pi n\lambda^{-1}\,,\qquad n\in\mathbb{Z}\,,\qquad \tilde{H}(k,k_0)\in\mathbb{R}\,.$$

It is now easy to compute the covariance

$$\int\phi(x,s)\phi(y,t)d\mu_G=G(x-y,t-s)\,,$$

with

$$\tilde{G}(k,k_0)=[\alpha\tilde{V}(k)^{-1}+z\tilde{H}(k,k_0)]^{-1}\,. \tag{9}$$

Remark that

$$\int d\mu_G\int_0^{\lambda}\phi(x,s)\int_0^{\lambda}\phi(y,t)=\lambda G_0(x-y)\,,$$
$$\tilde{G}_0(k)=[\alpha\tilde{V}(k)^{-1}+z\tilde{H}(k,0)]^{-1}\,, \tag{10}$$

$$\tilde{H}(k,0)=\lambda\int_0^1\exp[-k^2\mu(1-\mu)\lambda]d\mu\,. \tag{11}$$

2. The Results

We express the results in units where $\ell_D=1$. As mentioned above the Debye-Hückel limit corresponds to $\lambda\to 0$ and $\beta\to 0$ through the curves $\lambda=c\beta^p$. Since the Debye-Hückel theory does not contain any short range forces, we shall have to remove them at the same time as we take $\beta\to 0$. In fact we take $\mu=\lambda^{-1/20}$; with this choice of μ, p will have to be smaller than 5.

We summarize our convention for the rest of the paper:

$$\begin{array}{|c|}\hline \ell_D=1\,, \\ \beta\to 0\,, \\ \lambda=c\beta^p\,, \quad p<5\,, \\ \mu=c\lambda^{-1/20}\,. \\ \hline\end{array}$$

Remark that for β small $\tilde{V}(k)\simeq k^2$ and $\tilde{H}(k,0)\simeq\lambda$; this shows that $\tilde{G}_0(k)=(k^2+1)^{-1}$, which is the Debye-Hückel propagator. Using the analyticity properties of $G_0(k)$, it is easy to show that for λ sufficiently small,

i) $$|G_0(x)|\leqq c\exp(-\tfrac{1}{2}|x|)\,|x|^{-1}\,, \tag{12}$$

where c is λ and μ independent

ii) $$|G_0(x)|\leqq \int\frac{\mu^2 d^3k}{k^2(k^2+\mu^2)}\leqq c\mu \quad \text{as} \quad \mu\to\infty\,. \tag{13}$$

Note also that

$$V(0)=\int\frac{\mu^2 d^3k}{k^2(k^2+\mu^2)}\leqq c\mu \quad \text{as} \quad \mu\to\infty\,. \tag{14}$$

This implies that

$$\exp(\beta V(0))\approx\exp\beta\mu c\to 1 \quad \text{as} \quad \beta\to 0\,. \tag{15}$$

Theorem 2.1.

$$\beta p(\beta,z)=2z+\frac{1}{12\pi}+zo(\beta)\,.$$

Theorem 2.2. *Let $q,\dots,q_N$ be distinct (non-coïncident) charges in $\mathbb{R}^3$. Then the correspondent diagonal reduced density matrices $\varrho(\beta,z,(q)_N)$ are asymptotic to the classical Debye-Hückel correlation functions:*

$$\varrho(\beta,z,(q)_N)=z^N\left[1+\frac{\beta N}{8\pi}-\beta\sum_{1\leqq i<j\leqq N}e_ie_j(-\Delta+1)^{-1}(x_j,x_i)+o(\beta)\right].$$

Remark. Theorem 2.2 applied to the density yields

$$\varrho(q)=z+\frac{1}{8\pi}+zo(\beta)\,.$$

3. Correlation Inequalities

In this section we generalize Kennedy's correlation inequalities to quantum systems [8].

Notation.

$$h_\omega(x,s)=\delta(x-\omega(s)),$$

$\langle\,\rangle_{z=0}$ is defined by (6) with $z=0$.

Proposition 3.1. *For any g such that $(g, Vg)<\infty$, we have:*

i) $$\langle\cosh\phi(g)\rangle \leqq \langle\cosh\phi(g)\rangle_{z=0},$$

ii) $$\langle\phi^{2n}(g)\rangle \leqq d_n(g, \mathbb{U}g)^n,$$

with

$$d_n=\frac{(2n)!\,e^n}{2^n n^n}, \quad n\neq 1,$$
$$=1, \quad n=1.$$

Remark. If we choose $g=h_\omega$,

ii) $$\Rightarrow \int dP^\lambda_{xx}(\omega)\langle\phi^{2n}(h_\omega)\rangle \leqq d_n[\beta V(0)]^n,$$

with

$$V(0)\leqq c\mu.$$

Proposition 3.1. (ii) *For $n=1$ is a result by Fröhlich and Park* [3].

Proof. (i) It is sufficient to prove

$$\begin{aligned}&\frac{z^n}{n!}\int d\mu\cosh\phi(g)\int_\Lambda dx_1\dots dx_n dP^\lambda_{x_1x_1}(\omega_1)\dots d\mathbf{P}^\lambda_{x_nx_n}(\omega_n)\\&\quad\cdot\exp i[\phi(h_{\omega_1})+\dots+\phi(h_{\omega_n})]\\&\leqq[\int d\mu\cosh\phi(g)]-\frac{z^n}{n!}\int d\mu\int_\Lambda dx_1\dots dx_n d\mathbf{P}^\lambda_{x_1x_1}\dots d\mathbf{P}^\lambda_{x_nx_n}\\&\quad\cdot\exp[i\phi(h_{\omega_1})+\dots+i\phi(h_{\omega_n})].\end{aligned}$$

But this is implied by the inequality

$$\int d\mu\cosh\phi(\varrho)\exp[i\phi(\varrho')]\leqq\int d\mu\cosh\phi(\varrho)\int d\mu\exp[i\phi(\varrho')],$$

which is immediate by explicit computation.

(ii) It is a direct consequence of (i) see [8].

4. The Proofs

4.1. The Pressure: Proof of Theorem 2.1

We essentially follow [8]; we define an interpolating function

$$\begin{aligned}&\Xi(u)=\exp(z|\Lambda|)\int d\mu\\&\quad\cdot\exp\left\{\int d^3x\int d\mathbf{P}^{\Lambda;\lambda}_{xx}(\omega)\frac{z_0}{u^2}\left[\cos\left(u\int_0^\lambda ds\,\phi(\omega(s),s)ds\right)-1\right]\right\},\\&\qquad\Xi(1)=\Xi(\Lambda,\beta),\end{aligned}$$

$$\lim_{u\to 0} \Xi(u) = \exp(z_0|\Lambda|) \int d\mu \cdot \exp\left[-\tfrac{1}{2}z_0 \int d^3x \int d\mathbf{P}_{xx}^{\Lambda,\lambda}\left(\int_0^\lambda ds\,\phi(\omega(s),s)\right)^2\right].$$

We shall estimate (uniformly in Λ)

$$|\Lambda|^{-1}\left[\log\Xi(1) - \log\left(\lim_{u\to 0}\Xi(u)\right)\right] = |\Lambda|^{-1}\int_0^1 du \frac{d}{du}\log\Xi(u).$$

Now

$$\begin{aligned}&\Xi'(u)\Xi^{-1}(u)\\ &= \left\langle \frac{d}{du} z_0 u^{-2} \int_\Lambda d^3x \int d\mathbf{P}_{xx}^{\Lambda,\lambda}(\omega)\left[\cos u \int_0^\lambda \phi(\omega(s),s)\,ds - 1\right]\right\rangle_{zu^{-2}}\\ &= z_0 \int_\Lambda d^3x \sum_{n=2}^\infty \frac{2n-2}{(2n)!} u^{2n-3} \int dP_{xx}^{\Lambda,\lambda}(-1)^n \left\langle\left(\int_0^\lambda \phi(\omega(s),s)\,ds\right)^{2n}\right\rangle_{zu^{-2}}.\end{aligned}$$

Using Proposition 3.1 (remark)

$$\leqq z_0|\Lambda| \sum_{n=2}^\infty (2n-2)\beta^n \frac{e^n}{2^n n^n} c^n \mu^n \leqq |\Lambda| c z_0 \beta^2\mu^2 \leqq |\Lambda| c\beta\mu^2$$

$$\Rightarrow \beta p = z_0 - \lim_{\Lambda\to\infty} |\Lambda|^{-1} \log \int d\mu \exp\left[-\tfrac{1}{2}z_0 \int_\Lambda d^3x \int d\mathbf{P}_{xx}^{\Lambda,\lambda}\left(\int_0^\lambda ds\,\phi(\omega(s),s)\right)^2\right],$$

$$z_0 = z + \tfrac{1}{2}\beta z V(0) + zO(\beta^2).$$

We therefore get:

$$\beta p = z + D + O(\beta),$$

$$D = [2(2\pi)^3]^{-1} \int d^3k \left[k^{-2} - (k^2+\mu^2)^{-1} - \sum_n \log(1 + \tilde{V}(k) z\alpha^{-1}\tilde{H}(k,n))\right],$$

$$\begin{aligned}\tilde{H}(k,n) &= \lambda \int_0^1 ds \exp(2i\pi ns)\exp(-k^2 s(1-s)\lambda/2)\\ &\equiv \lambda\tilde{H}_0(k,n)\,(\in \mathbb{R}),\end{aligned}$$

$$\begin{aligned}D &= [2(2\pi)^3]^{-1} \int d^3k[k^{-2} - (k^2+\mu^2)^{-1} - \log(1+\tilde{V}(k)z\beta\tilde{H}_0(k,0))]\\ &\quad + [2(2\pi)^3]^{-1} \int d^3k \sum_{n\neq 0} \log(1+\tilde{V}(k)z\beta\tilde{H}_0(k,n))\\ &\equiv A + B.\end{aligned}$$

i) By a dominated convergence argument

$$\lim_{\beta\to 0} A = [2(2\pi)^3]^{-1} \int d^3k[k^{-2} - \log(1+k^{-2})].$$

ii) It is easy to realize that for each $n \neq 0$,

$$\lim_{\beta\to 0} H_0(k,n) = 0.$$

From $\ln(1+x) \leqq 2|x|$, $x \geqq -\frac{1}{2}$, we have for β small:

$$|B| \leqq [2(2\pi)^3]^{-1} \int d^3k \sum_{n \neq 0} |\tilde{V}(k)|\, |\tilde{H}_0(k,n)|$$

$$= 2\sqrt{\lambda}\mu^2(2\pi)^2 \int_0^\infty \frac{dk}{k^2+\mu^2\lambda} \sum_{n \neq 0} |F(n,k)|, \tag{16}$$

where

$$F(n,k) = \int_0^1 ds \exp(2i\pi ns) \exp(-k^2 s(1-s)/2).$$

The left-hand side of (16) is written as $B_2 + B_1$, where in B_2, the k integration is from 0 to 1.

Estimate of B_1.

$$\sum_{n \neq 0} |F(n_1 k)| = \sum_n f(n)\, |n|\, |F(n,k)|,$$

$$f(n) = |n|^{-1}, \quad n \neq 0,$$
$$= 0, \quad n = 0.$$

By Schwartz inequality and the Parceval identity,

$$\sum_{n \neq 0} |F(n,k)| \leqq \left[\sum_{n \neq 0} \frac{1}{n^2}\right]^{1/2} \left[\int_0^1 ds (4\pi)^2 (s-\tfrac{1}{2})^2 k^4 \exp(-k^2 s(1-s))\right]^{1/2}$$
$$\leqq ck^2.$$

Therefore

$$|B_1| \leqq c\sqrt{\lambda}\mu^2.$$

Estimate of B_2.

i) Bounds on $F(n,k)$

a)
$$|2\pi n F(n,k)| \leqq \int_0^1 ds |s-\tfrac{1}{2}| k^2 \exp(k^2(s^2-s)/2)$$
$$\leqq c \int_0^{1/2} ds k^2 \exp(k^2(s^2-s))$$
$$\leqq c \int_0^{1/2} ds k^2 \exp(-k^2 s/8).$$

We used that on $[o\frac{1}{2}]$, $s^2 - s \leqq -\frac{1}{4}s$,

$$= c \int_0^{1/2} du \exp(-u/8) \leqq c.$$

b) Similarly one gets

$$|4\pi^2 n^2 F(n,k)| \leqq c \int_0^{1/2} k^2 \exp\left(-\frac{k^2}{8}s\right) + c \int_0^{1/2} k^4 \exp\left(-\frac{k^2}{8}s\right)$$
$$\leqq c_1 + c_2 k^2.$$

Doing the geometric mean of a and b

$$|n^{3/2}F(n,k)| \leqq c|k| \quad \text{and} \quad |n^{5/4}F(n,k)| \leqq c|k|^{1/2} (k \in [1,\infty]).$$

We therefore get the bounds

$$|F(n,k)| \leqq c|k|^{-1/2} n^{-5/4}$$

and

$$|B_2| \leqq \sqrt{\lambda}\mu^2 \int_1^\infty dk k^{-3/2} \sum_n n^{-5/4}$$
$$\leqq c\sqrt{\lambda}\mu^2 .$$

We finally get $B \leqq c\sqrt{\lambda}\mu^2$, and $\lim_{\beta\to 0} B = 0$. In the Debye-Hückel limit the pressure becomes:

$$\beta p = z + [2(2\pi)^3]^{-1} \int d^3k [k^{-2} - \log(1 + k^{-2})] = z + \frac{1}{12\pi}. \quad \square$$

4.2. The Correlation Functions

We shall consider in detail the case of the two-point function. The general case is similar. We start by giving a first approximation to the full correlation functions.

Notations.

$$\left.\begin{array}{l} h_i \equiv \delta(y - \omega_i(s))\chi_{0\lambda}(s) \\ h_i' \equiv \delta(y - x_i)\chi_{0\lambda}(s) \end{array}\right\} \quad i = 1, 2,$$

$$A \equiv z^2 \int d\mathbf{P}^\lambda_{x_1x_1}(\omega_1) d\mathbf{P}^\lambda_{x_2x_2}(\omega_2) \langle :\exp(i\phi(h_1))::\exp(i\phi(h_2)): \rangle ,$$

$$B \equiv z^2 \langle :\exp(i\phi(h_1'))::\exp(i\phi(h_2')): \rangle .$$

Lemma 4.2.1. $|A - B| \leqq c z^2 \beta \mu^{3/2} \lambda^{1/4}$ *uniformly in* x_1 *and* x_2.

Proof.

$$\begin{aligned} |A - B| &= z_0^2 \int d\mathbf{P}^\lambda_{x_1x_1}(\omega_i) d\mathbf{P}^\lambda_{x_2x_2}(\omega_2) \langle \cos(\phi(h_1 + h_2)) - \cos(\phi(h_1' + h_2')) \rangle \\ &\leqq z_0^2 \int d\mathbf{P}^\lambda_{x_1x_1}(\omega_1) d\mathbf{P}^\lambda_{x_2x_2}(\omega_2) \\ &\quad \cdot \langle |\phi((h_1' - h_1') + (h_2 - h_2'))| \, |\phi((h_1 + h_1') + (h_2 + h_2'))| \rangle \\ &\leqq z_0^2 \int d\mathbf{P}^\lambda_{x_1x_1}(\omega_1) d\mathbf{P}^\lambda_{x_2x_2}(\omega_2) \langle \{\phi[(h_1 - h_1') + (h_2 - h_2')]\}^2 \rangle_{z=0}^{1/2} \\ &\quad \cdot \langle \{\phi[(h_1 + h_1') + (h_2 + h_2')]\}^2 \rangle_{z=0}^{1/2} . \end{aligned}$$

We have used Schwartz inequality and correlation inequality (3.1).

Using again Schwartz inequality and the fact that $d\mathbf{P}^\lambda_{xx}(\omega)$ is a normalized measure we get

$$\begin{aligned} &\leqq z_0^2 [\int d\mathbf{P}^\lambda_{x_1x_1}(\omega_1) d\mathbf{P}^\lambda_{x_2x_2}(\omega_2) \langle \{\phi[(h_1 - h_1') + (h_2 - h_2')]\}^2 \rangle_{z=0}]^{1/2} \\ &\quad \cdot [\int d\mathbf{P}^\lambda_{x_1x_1}(\omega_1) d\mathbf{P}^\lambda_{x_2x_2}(\omega_2) \langle \{\phi[(h_1 + h_1') + (h_2 + h_2')]\}^2 \rangle_{z=0}]^{1/2} . \end{aligned} \tag{17}$$

(17) can now be estimated by explicit computation:

$$\int d\mathbf{P}^{\beta}_{x_1x_1}(\omega_1)h_1 = \lambda^{3/2}(\lambda-s)^{-3/2}s^{-3/2}\exp-[(x_1-y)^2\lambda(2s(\lambda-s))^{-1}]$$
$$\equiv g(x_1-y,s),$$

$$\int d\mathbf{P}^{\lambda}_{x_1x_1}(\omega_1)d\mathbf{P}^{\lambda}_{x_2x_2}(\omega_2)\,[\{(h_1-h'_1)+(h_2-h'_2)\},\mathbb{U}\{(h_1-h'_1)+(h_2-h'_2)\}]$$
$$=4\beta V(0)-4\alpha^{-1}\int V(x_1-y)g(x_1-y,s)d^3y ds+\beta V(x_1-x_2)$$
$$+\int g(x_1-y,s)\alpha^{-1}V(y-y')\delta(s-s')g(x_2-y',s')d^3yd^3y'dsds'$$
$$+\int g(x_1-y,s)\alpha^{-1}V(y-x_2)\delta(s-s')d^3ydsds'$$
$$-\int g(x_2-y,s)\alpha^{-1}V(y-x_1)\delta(s-s')d^3ydsds'$$
$$=4\int_0^{\lambda} ds\int d^3k[1-\exp(-(\lambda-s)s\lambda^{-1}k^2)]\tilde{V}(k)\alpha^{-1}$$
$$+\int_0^{\lambda} ds\int d^3k e^{ik(x_1-x_2)}\alpha^{-1}\tilde{V}(k)[1+\exp(-2(\lambda-s)s\lambda^{-1}k^2)$$
$$-2\exp(-(\lambda-s)sk^2\lambda^{-1})]\,.$$

Now

$$\int_0^{\lambda} ds\int d^3k[1-\exp(-(\lambda-s)sk^2\lambda^{-1})]\mu^2k^{-2}(k^2+\mu^2)^{-1}\alpha^{-1}$$
$$=\beta\mu^2\int_0^1 ds\sqrt{\lambda}\int_0^{\infty} dk(k^2+\lambda\mu^2)^{-1}\,[1-\exp(-s(1-s)k^2)]$$
$$\leqq c\beta\sqrt{\lambda}\mu^2$$
$$\Rightarrow \int d\mathbf{P}^{\lambda}_{x_1x_1}d\mathbf{P}^{\lambda}_{x_2x_2}\langle\{\phi[(h_1-h'_1)+(h_2-j'_2)]\}^2\rangle_{z=0}\leqq\beta\mu^2\sqrt{\lambda}c\,.$$

The last estimate is obtained by splitting the k integration as in (4.1). A similar computation yields the estimate

$$\int d\mathbf{P}^{\lambda}_{x_1x_1}d\mathbf{P}^{\lambda}_{x_2x_2}\langle\{\phi[(h_1+h'_1)+(h_2+h'_2)]\}^2\rangle_{z=0}\leqq\beta\mu c\,. \quad \square$$

4.2.2. Remark. For n-point correlation functions we would get the bound

$$|A-B|\leqq cz^n\beta\lambda^{1/4}\mu^{3/2}\,.$$

In order to extract the Debye-Hückel part of the full correlation functions we shall use the method of "complex translation of a Gaussian measure" introduced in [6] and used in [7] for the low fugacity expansion of the dipoles gas.

We first recall the basic formula [10].

4.2.3. Complex Translation. Let $d\mu(\phi)$ be a Gaussian measure with smooth covariance $V(x,y)$ and $g(x)\in\mathscr{S}(\mathbb{R}^3)$. We have:

$$\int d\mu(\phi)F(\phi)=\int d\mu(\phi)F(\phi+ig)\exp[\tfrac{1}{2}(g,V^{-1}g)-i(\phi,V^{-1}g)]\,.$$

Using Lemma 4.2.1, we see that it is enough to consider correlations of the type

$$\varrho'(e_1,x_1;e_2,x_2)=z^2\left\langle :\exp\left(ie_1\int_0^{\lambda}\phi(x_1,s)ds\right): :\exp\left(ie_2\int_0^{\lambda}\phi(x_2,s)ds\right):\right\rangle\,. \quad (18)$$

In order to avoid complicated notations we take $e_1 = e_2 = 1$,

$$(18) = z_0^2 \left\langle \exp\left(i \int_0^\lambda [\phi(x_1, s) + \phi(x_2, s)] ds \right) \right\rangle .$$

Define

$$\psi_r(x,t) = (2\pi)^{-3} \int \exp(ik(x - x_r)) \cdot [\alpha \tilde{V}^{-1}(k) + z\tilde{H}(k,0)]^{-1} \otimes \mathbb{1}(t), \quad r = 1, 2,$$
$$\psi(x,t) = \psi_1(x,t) + \psi_2(x,t).$$

Using the analyticity properties of $a\tilde{V}^{-1}(k) + z\tilde{H}(k,0)$, it is easy to get the bounds $(\mu \to \infty)$, (c is β, z, μ-independent)

$$\begin{aligned} \psi_r(x,t) &\leqq c\alpha^{-1} \exp(-\tfrac{1}{2}|x - x_r|), \quad |x - x_r| > 1 \\ \psi_r(x,t) &\leqq \mu\alpha^{-1} c . \end{aligned} \tag{18b}$$

Lemma 4.2.4.

i) $(\psi, \alpha V^{-1}\psi) = \beta\psi(x_1) + \beta\psi(x_2) - \int dx \int d\mathbf{P}_{xx}^\lambda(\omega) \left(\int_0^\lambda \phi(\omega(s), s) ds \right)^2 .$

ii) $(\phi, \alpha V^{-1}\psi) = \int_0 [\phi(x_1, s) + \phi(x_2, s)] ds - z \int dx \int d\mathbf{P}_{xx}^\lambda(\omega)$

$$\cdot \left(\int_0^\lambda \phi(\omega(s), s) ds \right) \left(\int_0^\lambda \psi(\omega(s), s) ds \right).$$

Proof.

i) $\alpha V^{-1}\psi_r = \delta(x - x_r) - \int \exp(ik(x - x_r)) z\tilde{H}(k,0) [\alpha V^{-1}(k) + z\tilde{H}(k,0)]^{-1}$,

$(\psi, \alpha V^{-1}\psi) = \beta\psi(x_1) + \beta\psi(x_2) - 2\lambda \int (1 + \cos(k(x_1 + x_2)) z\tilde{H}(k,0) [\alpha V^{-1}(k) + z\tilde{H}(k,0)]^{-2}$

$$= z \int dx \int dP_{xx}^\lambda(\omega) \left(\int_0^\lambda \psi(\omega(s)) ds \right)^2 .$$

The last equality follows from

$$\int d^3x \int dP_{xx}^\lambda(\omega) \int_0^\lambda ds'' \delta(x' - \omega(s')) \int_0^\lambda \delta(x'' - \omega(s'')) ds''$$
$$= \lambda \int_0^\lambda ds H(x' - x'', s) \quad [\text{see (8)}].$$

ii) $(\phi, \alpha V^{-1}\psi) = \int_0^\lambda [\phi(x_1, s) + \phi(x_2, s)] ds$

$$\begin{aligned} &- \int_0^\lambda ds \int d^3x \phi(x,s) \int \exp(ik(x - x_1)) + \exp(ik(x - x_2)) \\ &\cdot \tilde{H}(k,0) [\alpha V^{-1}(k) + z\tilde{H}(k,0)]^{-1} d^3k . \end{aligned} \tag{19}$$

Again, using (8), the second term of the left-hand side of (19) can be written as:

$$-z \int d^3x \int d\mathbf{P}_{xx}^\lambda(\omega) \left(\int_0^\lambda \phi(\omega(s), s) ds \right) \left(\int_0^\lambda \psi(\omega(s)) ds \right). \quad \square$$

We now want to evaluate (18) using a complex translation by the function $i\psi(x,t)$:

$$z_0^2\left\langle \exp\left(i\int_0^\lambda(\phi(x_1,s)+\phi(x_2,s))ds\right)\right\rangle$$

$$=z_0^2\Xi^{-1}(\Lambda)\int d\mu\left\{\exp\left[i\int_0^\lambda(\phi(x_1,s)+\phi(x_2,s))ds\right]\right.$$

$$\cdot\exp[-\lambda(\psi(x_1)+\psi(x_2))]\exp[\tfrac{1}{2}\lambda(\psi(x_1)+\psi(x_2))]$$

$$\cdot\exp\left[-\int d^3x\int dP_{xx}^\lambda(\omega)\left(\int_0^\lambda ds\,\psi(\omega(s))\right)^2\right]$$

$$\cdot\exp\left[-i\int_0^\lambda(\phi(x_1,s)+\phi(x_2,s))ds\right]$$

$$\cdot\exp\left[iz_0\int d^3x\int d\mathbf{P}_{xx}^\lambda\left(\int_0^\lambda\phi(\omega(s),s)ds\right)\left(\int_0^\lambda\psi(\omega(s))ds\right)\right]$$

$$\cdot\exp\left[iz_0\int d^3x\int d\mathbf{P}_{xx}^\lambda\left(\cos\int_0^\lambda\phi(\omega(s),s)ds\right)\left(\cosh\int_0^\lambda\psi(\omega(s))ds\right)\right]$$

$$\left.\cdot\exp\left[iz_0\int d^3x\int d\mathbf{P}_{xx}^\lambda\left(\sin\int_0^\lambda\phi(\omega(s),s)ds\right)\left(\sinh\int_0^\lambda\psi(\omega(s))ds\right)\right]\right\}.$$

As in [8] we want to rewrite (19) as

$$\varrho'(x_1x_2)=z_0^2\exp(A+S)\langle\exp(R(\phi)+iI(\phi))\rangle,$$

with $A=-\frac{1}{2}\lambda(\psi(x_1)+\psi(x_2))$,

$$S=z_0\int d^3x\int d\mathbf{P}_{xx}^\lambda(\omega)\left[\cosh\int_0^\lambda\psi(\omega(s))ds-1-\tfrac{1}{2}\left(\int_0^\lambda\psi(\omega(s))ds\right)^2\right],$$

$$R(\phi)=z_0\int d^3x\int d\mathbf{P}_{xx}^\lambda(\omega)\left[\left(\cos\int_0^\lambda\phi(\omega(s),s)ds\right)\left(\cosh\int_0^\lambda\psi(\omega(s))ds-1\right)\right],$$

$$I(\phi)=-z_0\int d^3x\int d\mathbf{P}_{xx}^\lambda(\omega)$$

$$\cdot\left[\sin\int_0^\lambda\phi(\omega(s),s)ds\sinh\int_0^\lambda\psi(\psi(s)ds-\int_0^\lambda\phi(\omega(s),s)ds\int_0^\lambda\psi(\omega(s))ds\right].\qquad(20)$$

Remark. In order to be precise, the way to get (20) is to first consider finite volume expectation values and do a translation by the function

$$\psi_{r,\Lambda}(x,t)=\int_0^\lambda dt(\mathbb{U}^{-1}+zH_\Lambda)^{-1}(x_r,t_r;x,t),$$

where

$$H_\Lambda(x'',s'';x',s')=\int_\Lambda d^3x\int d\mathbf{P}_{xx}^{\lambda;\Lambda}(\omega)h_\omega(x's')h_\omega(x'',s''),$$

and then take the limit $\Lambda\uparrow\infty$ to get (20).

Lemma 4.2.5. $|S|\leqq C\beta^3\mu^2$.

Proof. Using Taylor's theorem we have the bound

$$|S| \leqq z_0 \int dx \int d\mathbf{P}^{\lambda}_{xx}(\omega) \frac{1}{4!} \cdot \left(\int_0^{\lambda} \psi(\omega(s)) ds \right)^4 \cosh\left[\xi \int_0^{\lambda} \psi(\omega(s)) ds \right], \quad 0 < \xi < 1 .$$

The bounds (18b) imply

$$\begin{aligned} |S| &\leqq c z_0 \beta^2 \mu^2 \int d^3x \int d\mathbf{P}^{\lambda}_{xx}(\omega) \left(\int_0^{\lambda} \psi(\omega(s)) ds \right)^2 \\ &\leqq c z_0 \beta^2 \mu^2 \lambda^2 \int |\tilde{\psi}(k)|^2 \tilde{H}_0(k) d^3k . \end{aligned}$$

Let us recall that

$$\tilde{H}_0(k, 0) = \int_0^1 ds \exp(-\lambda k^2 s(1-s)/2) ,$$

and that

$$\begin{aligned} &\tilde{\psi}(k) = \exp(ikx_1 + ikx_2) [\alpha k^2 \mu^{-2}(k^2 + \mu^2) + z_0 \beta \alpha \tilde{H}_0(k)]^{-1} \\ &\Rightarrow |S| \leqq c z \beta^2 \mu^2 \lambda^2 \alpha^{-2} . \qquad (21) \\ &|S| \leqq c \beta^3 \mu^2 . \quad \square \end{aligned}$$

Lemma 4.2.6.

$$\langle |\exp R(\phi) - 1| \rangle \leqq c \beta^2 \mu .$$

Proof. i) Let us first prove $|R(\phi)| \leqq c\beta$. Using Taylor's theorem we have:

$$\begin{aligned} &\left| z_0 \int d^3x \int d\mathbf{P}^{\lambda}_{xx}(\omega) \left(\cos \int_0^{\lambda} \phi(\omega(s), s) ds - 1 \right) \left(\cosh \int_0^{\lambda} \psi(\omega(s)) ds - 1 \right) \right| \\ &\leqq c z_0 \int d^3x \int d\mathbf{P}^{\lambda}_{xx}(\omega) \left(\int_0^{\lambda} \psi(\omega(s)) ds \right)^2 \\ &\leqq c z_0 \lambda^2 \int |\tilde{\psi}(k)|^2 |H_0(k)| d^3k \leqq c\beta . \end{aligned}$$

ii) Again a double application of Taylor's theorem yields:

$$\begin{aligned} &\langle |\exp R(\phi) - 1| \rangle \leqq \langle |R(\phi)| \exp(c\beta) \rangle \\ &= z_0 \int d^3x \, d\mathbf{P}^{\lambda}_{xx}(\omega) \left\langle \left| \cos \int_0^{\lambda} \phi(\omega(s), s) ds - 1 \right| \right\rangle \cdot \left| \cosh \int_0^{\lambda} \psi(\omega(s)) - 1 \right| \\ &\leqq z_0 \left(\int d^3x \, d\mathbf{P}^{\lambda}_{xx}(\omega) \left(\int_0^{\lambda} \psi(\omega(s)) ds \right)^2 \left\langle \left(\int_0^{\lambda} \phi(\omega(s), s) ds \right)^2 \right\rangle \right) \\ &\leqq c z_0 \beta^3 \mu \quad \text{by Proposition 3.1.} \quad \square \end{aligned}$$

Lemma 4.2.7.

$$\langle |\cos I(\phi) - 1| \rangle \leqq c(\beta^3 \mu^3 + \beta^{3/2} \mu^{1/2}) .$$

Proof. Write $I(\phi)=I_1(\phi)+I_2(\phi)$, with:

$$I_1(\phi)=z_0\int dx\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left[\int_0^{\lambda}\phi(\omega(s),s)ds-\sin\int_0^{\lambda}\phi(\omega(s),s)ds\right]\cdot\sinh\int_0^{\lambda}\psi(\omega(s))ds\,,$$

$$I_2(\phi)=z_0\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left[\int_0^{\lambda}\psi(\omega(s))ds-\sinh\int_0^{\lambda}\psi(\omega(s))ds\right]\cdot\int_0^{\lambda}\phi(\omega(s),s)ds\,.$$

Now,

$$|\cos I(\phi)-1|\leqq|I_2(\phi)|+\tfrac{1}{2}(I_1(\phi))^2\,.$$

i)
$$\begin{aligned}\langle|I_2(\phi)|\rangle&\leqq z_0\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left(\int_0^{\lambda}\psi(\omega(s))ds\right)^2\cdot\left\langle\left|\int_0^{\lambda}\phi(\omega(s),s)ds\right|\right\rangle\\&\leqq z_0\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left(\int_0^{\lambda}\psi(\omega(s))ds\right)^2\cdot\left\langle\left(\int_0^{\lambda}\phi(\omega(s),s)ds\right)^2\right\rangle^{1/2}.\end{aligned}$$

Using correlation inequality (3.1),

$$\leqq z_0\sqrt{\beta\mu}\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left(\int_0^{\lambda}\psi(\omega(s))ds\right)^2$$

$\Rightarrow\langle|I_2(\phi)|\rangle\leqq cz_0\beta^{5/2}\mu^{1/2}$ as in (21).

ii) Bounds on

$$\langle\tfrac{1}{2}(I_1(\phi))^2\rangle\,,$$

$$|I_1(\phi)|\leqq cz_0\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left|\int_0^{\lambda}\phi(\omega(s),s)ds\right|^3\left|\int_0^{\lambda}\psi(\omega(s))ds\right|,$$

$$\begin{aligned}\langle(I_1(\phi))^2\rangle=z_0^2\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)&\left|\int_0^{\lambda}\psi(\omega(s))ds\right|\\\cdot\int d^3y\int d\mathbf{P}^{\lambda}_{yy}(\omega')&\left|\int_0^{\lambda}\psi(\omega'(s))ds\right|\\\cdot&\left\langle\left(\int_0^{\lambda}\phi(\omega(s),s)ds\right)^6\right\rangle.\end{aligned}$$

Using Proposition 3.1, we have:

$$\langle(I_1(\phi))^2\rangle\leqq cz_0^2\beta^3\mu^3\left(\int d^3x\int d\mathbf{P}^{\lambda}_{xx}(\omega)\left|\int_0^{\lambda}\psi(\omega(s))ds\right|\right)^2.$$

But

$$\int d^3x \int d\mathbf{P}^{\lambda}_{xx}(\omega) \left|\int_0^{\lambda} \psi(\omega(s))ds\right|$$
$$= \int_0^{\lambda} ds \int d^3x' |\psi(x')| \int d^3x \int d\mathbf{P}^{\lambda}_{xx}(\omega)\delta(x'-\omega(s)) \leqq c\beta$$
$$\Rightarrow \langle (I_1(\phi))^2 \rangle \leqq z_0^2\beta^3\mu^3\beta^2 c\,. \quad \square$$

Proof of Theorem 2.1.2. Using (20) we have:

$$|\varrho(e_1,x_1;e_2,x_2) - z_0^2 e^A| \leqq \varrho(e_1,x_1;e_2,x_2) - \varrho'(e_1,x_1;e_2,x_2)|$$
$$+ |z_0^2 \exp(A+S) \langle \exp R(\phi) \cos I(\phi) \rangle - z_0^2 \exp A|\,. \tag{22}$$

In order to estimate the second term of the right-hand side of (22), we write it as:

$$|z_0^2 \exp A[(\exp S - 1) + 1] \langle [(\exp R(\phi) - 1) + 1]$$
$$\cdot [(\cos I(\phi) - 1) + 1] \rangle - z_0^2 \exp A|$$
$$\leqq z_0^2 \exp A[[(\exp S - 1) + 1] \langle |\exp R(\phi) - 1| + |\cos I(\phi) - 1|$$
$$+ |\exp R(\phi) - 1|\,|\cos I(\phi) - 1| + 1 \rangle - 1]$$
$$\leqq z_0^2 o(\beta)\,.$$

We used Lemmas 4.2.5., 4.2.6., 4.2.7. It is now easy to check that:

$$z_0^2 \exp A = z^2 + \frac{\beta z^2}{4\pi} - \beta z^2(-\Delta+1)^{-1}(x_1,x_2) + z^2 o(\beta)\,. \quad \square$$

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