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# DECAY EXPONENTS AND DIFFRACTION COEFFICIENTS FOR SURFACE WAVES ON SURFACES OF NON-CONSTANT CURVATURE

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## Abstract

The diffraction of a plane scalar wave by a hard elliptic cylinder is investigated theoretically. The field is obtained and expanded asymptotically for incident wavelengths small compared with the dimensions of the generating ellipse. The method of obtaining the asymptotic expansion of the diffracted field parallels the methods of reference [9]. However, additional terms in the asymptotic expansion are obtained. In reference [9] it was shown that the asymptotic expansion of the diffracted field was in agreement with the geometrical theory of diffraction as presented in references [4] and [5]. The additional terms in the field obtained in this paper we interpret geometrically as higher order corrections to the decay exponents and diffraction coefficients as given in reference [5]. Finally, we obtain additional terms to those given in reference [8] for the asymptotic expansion of the field diffracted by a parabolic cylinder. We then show that these higher order corrections have the same geometrical interpretation as in the case of the elliptic cylinder. The determination of these corrections permits the geometric theory to be extended to longer wavelengths than could be treated previously. Similar results are obtained for soft cylinders. Then the field on a hard convex cylinder of arbitrary shape is determined asymptotically by a quite different method - that of asymptotically solving an integral equation. The result is found to coincide with the generalization based upon the solution for the elliptic cylinder.

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## 1. Introduction

When a wave is incident upon an opaque object large compared to the incident wavelength a shadow is formed. Some radiation penetrates into the shadow. The first quantitative analysis of this penetration effect for the case of a smooth object was that of G. N. Watson [1]. He showed that the field in the shadow of a sphere consists of a sum of modes. Each mode decays exponentially with increasing distance from the shadow boundary into the shadow. Numerous authors have pursued Watson's analysis, considering spheres which are not opaque or which are surrounded by non-uniform media. Many of these investigaton are described by H. Bremmer [2]. Independently W. Franz and K. Depperman [3] discovered the existence of an exponentially decaying wave travelling around a circular cylinder. They also observed that this wave continues travelling into the illuminated region. These results, as well as those referred to above pertain to bodies of constant curvature. What are the corresponding results for objects of non-constant curvature?

This question is answered by the geometrical theory of diffration introduced by J. B. Keller [4], which predicted that radiation travels along surface rays. These rays are geodesics on the surface of any object which originate at the shadow boundary. They continually shed diffracted rays which irradiate the shadow and also enter the illuminated region. A quantitative theory of the field diffracted by a cylinder of arbitrary convex cross-section was constructed with the aid of these rays [5]. In this theory certain decay exponents and diffraction coefficients were introduced. The decay exponents determine the rate of decay of the various field modes along a surface ray.. The diffraction coefficients determine the amplitudes of the various modes on a surface ray, and the amplitude of the field on the shed diffracted rays. It was assumed that the decay exponents and diffraction coefficients depend only upon local properties of the ray and the surface. By comparing the predictions of this theory with the results of W. Franz [6] for the circular cylinder, the decay exponents and diffraction coefficients were determined. A similar analysis was performed for three dimensional curved objects by B. R. Levy and J. B. Keller [7].

The results of the geometrical theory of diffraction have been tested by comparing them with the exact solutions of certain diffraction problems involving objects of non-constant curvature. To make this comparison it was necessary to expand the exact solution asymptotically for wavelength small compared to the dimensions of the object. This has been done for the field diffracted by a parabolic cylinder by S. O. Rice [8], an elliptic cylinder by B. R. Levy [9] and by R. K. Ritt and N. D. Kazarinoff [10], and for an ellipsoid of revolution by J. B. Keller and B. R. Levy [11] and by R. K. Ritt and N. D. Kazarinoff [12]. In all cases the leading term in the asymptotic expansion agreed precisely with the results of the geometrical theory.

We now propose to improve the geometrical theory of diffraction by an arbitrary cylinder so that it will also yield the next term in the asymptotic expansion. To this end we must determine the next terms in the expressions for the decay exponents and the diffraction coefficients. The previously determined terms involve the radius of curvature of the cylinder. The new terms will involve the derivative of the radius with respect to arclength along the cross-sectional curve. To find the new terms we shall examine the next term in the asymptotic expansion of the exact expression for the field diffracted by an elliptic cylinder. We shall express terms of local geometrical quantities such as the radius of curvature and its derivative. Then we shall assume that the final geometrical expression is correct for an arbitrary cylinder. As a first test of this result, we shall show that it correctly yields the next term for the field diffracted by a parabolic cylinder. Of course, it also yields the correct term in the case of a circular cylinder. The results are also obtained by asymptotically solving the integral equation for the cylinder current. These results coincide with those obtained by generalizing the results obtained for the elliptic cylinder.

The determination of these new corrections permits us to use our theory for longer wavelengths than could have been treated previously . The improvement resulting from the correction to the decay exponent is shown in [7].

## 2. Diffraction by an elliptic cylinder

Let us consider the field u produced by a line source parallel to the generators of an elliptic cylinder. Then u is the solution of the following problem

$$(\Delta + k^2)u = \delta(\xi-\xi_o)\,\delta(\eta) \tag{1}$$

$$\frac{\partial u(a,\eta)}{\partial \xi} = 0 \tag{2}$$

$$\lim_{r \to \infty} r(iku - u_r) = 0 \quad . \tag{3}$$

For simplicity the source has been taken to lie in the plane containing the major axis of the ellipse. The elliptic coordinates $(\xi,\eta)$ are related to cartesian coordinates by the equations

$$x = h \cosh \xi \cos \eta \tag{4}$$

$$y = h \sinh \xi \sin \eta \tag{5}$$

In (4) and (5) h denotes one half the interfocal distance of the ellipses $\xi$ = constant, of which $\xi = a$ is the cross-section of the cylinder.

In reference [9] it is shown that on the cylinder the solution of (1) - (3) can be written in the form

$$u(a,\eta) = (kh)^2 \sum_{n=1}^{\infty} b_n \frac{C_n(\eta-\pi)}{C_n'(\pi)} \frac{V_n^{(1)}(\xi_o)}{\partial/\partial b\, V_n^{(1)'}(a)} \tag{6}$$

The functions $C_n$ and $V_n^{(1)}$ are defined and asymptotically expanded for large kh in reference [9]. A breif review of the pertinent properties of these functions follows.

The function $V_n^{(1)}$ is the outgoing solution of the equation

$$\frac{d^2V_n^{(1)}}{d\xi^2} - (kh)^2 (b_n^2 - \cosh^2\xi)V_n^{(1)} = 0 . \tag{7}$$

For large kh it has the asymptotic expansion

$$V_n^{(1)}(\xi) \sim \zeta^{1/4}\, 3^{1/3}\, \pi^{-1} (b_n^2 - \cosh^2\xi)^{-1/4} A(3^{1/3} e^{-i\pi/3} (kh)^{2/3} \zeta) \tag{8}$$

Here $b_n$ is defined by

$$V_n^{(1)'}(a) = 0 . \tag{9}$$

The functions $\zeta$ and A are defined by

$$\frac{2}{3}\zeta^{3/2} = - \int_{\cosh^{-1}b_n}^{\xi} (b_n^2 - \cosh^2x)^{1/2}\, dx \tag{10}$$

$$A(t) = \int_0^{\infty} \cos(z^3 - tz)dz \quad . \tag{11}$$

The function $C_n(\eta)$ is the even solution of the equation

$$C_n'' + (kh)^2(b_n^2 - \cos^2\eta)C_n = 0 \ . \tag{12}$$

For large kh it has the asymptotic expansion

$$C_n \sim \cos\left[kh \int_0^{\eta} (b_n^2 - \cos^2\eta)^{1/2}d\eta\right] \left\{(b_n^2 - 1)/(b_n^2 - \cos^2\eta)\right\}^{1/4} . \tag{13}$$

We now specialize (6) to the case of plane wave incidence. To do this we multiply (6) by $e^{3\pi i/4}2^{3/2}\pi^{1/2}(kh \cosh \xi_o)^{1/2}\exp\left[-ikh \cosh \xi_o\right]$ and let $\xi_o \to \infty$. Then we obtain

$$u(a,\eta) = E \sum_{n=1}^{\infty} \frac{b_n C_n(\eta-\pi)}{C_n'(\pi)} \ \frac{\exp\left[-ikh \int_0^{\pi/2}(b_n^2 - \cos^2\eta)^{1/2}d\eta\right]}{\partial/\partial b \ V_n^{(1)'}(a)} \ . \tag{14}$$

Here $E = e^{5\pi i/6} \ 2^{1/2} \ (kh)^{7/3}$ . Upon expanding the $C_n$ function we find that (14) becomes

$$u(a,\eta) \sim \frac{iE}{kh} \sum_{n=1}^{\infty} b_n \left\{(b_n^2-1)(b_n^2-\cos^2\eta)\right\}^{-1/4}$$

$$\times \ \frac{\exp\left[ikhG(\pi/2,\eta)\right] + \exp\left[ikhG(\eta,3\pi/2)\right]}{\partial/\partial b \ V_n^{(1)'}(a)} \left\{1-\exp\left[2ikhG(0,\pi)\right]\right\}^{-1} \tag{15}$$

† The details of the evaluation of the limit are to be found in reference [n], p. 14.

In (15) G is defined by

$$G(\alpha,\beta) = \int_{\alpha}^{\beta} (b_n^2 - \cos^2\eta)^{1/2}\, d\eta .$$

In reference [9] the leading term in the asymptotic expansion of each of the summands in (15) was computed. In order to carry out this calculation it was found necessary to compute two terms in the asymptotic expansion of the eigenvalue $b_n$. We shall now compute a further term in the asymptotic expansion of each of the terms in (15). In order to do this we shall first compute another term in the asymptotic expansion of $b_n$. To do this we first observe that the leading terms in the asymptotic expansion of $V_n^{(1)'}(\xi)$ are obtained by differentiating (8). The leading term in the asymptotic expansion of $V_n^{(1)'}(\xi)$ comes from differentiating the Airy function A. Therefore $\zeta_n$ will be nearly equal to the result obtained in [9] so we write it in the form

$$\zeta_n = 3^{-1/3}\, e^{i\pi/3}(kh)^{-2/3}(1+\delta_n)q_n \quad . \tag{17}$$

Here $\zeta_n = \zeta(a)$, $q_n$ is the nth root of the equation $A'(q_n) = 0$, and $\delta_n$ is an as yet undetermined correction which is small compared to unity.

We now set $b_n = \cosh a + \epsilon_n$ and insert this expression into (10) which determines $\zeta$. In this way we obtain

$$\frac{2}{3}\,\zeta_n^{3/2} = \frac{2^{3/2}}{3}\,\frac{(\cosh a)^{1/2}}{\sinh a}\;\epsilon_n^{3/2} - \frac{2^{1/2}\epsilon_n^{5/2}(\cosh^2 a+7)}{30\,\sinh^3 a(\cosh a)^{1/2}} + O(\epsilon_n^{7/2}) . \tag{18}$$

Now we insert (17) and the above form of $b_n$ into (9) and obtain the following result for $\delta_n$

$$\delta_n = \frac{3^{2/3} e^{i\pi/3} (\cosh^2 a + 15 \sinh^2 a + 7)}{80\ 2^{1/3} (\sinh a \cosh a)^{4/3} (kh)^{2/3}} + O((kh)^{-4/3}) \ . \tag{19}$$

By comparing (18) and (17), and using (19) for $\delta_n$. we determine $\epsilon_n$. Then $b_n$ is given by

$$b_n = \cosh a + \frac{\tau_n (\sinh a)^{2/3}}{(\cosh a)^{1/3} (kh)^{2/3}} + \frac{\tau_n^{\ 2} (\cosh^2 a + 7)}{30 (\sinh a)^{2/3} (\cosh a)^{5/3} (kh)^{4/3}} \tag{20}$$

$$- \frac{(2 \cosh^2 a - 1)}{20 \tau_n (\sinh a)^{2/3} (\cosh a)^{5/3} (kh)^{4/3}} + O((kh)^{-2}) \ .$$

In (20), the quantity $\tau_n$ is defined in terms of $q_n$ by

$$\tau_n = \frac{q_n e^{i\pi/3}}{6^{1/3}} \ . \tag{21}$$

Upon substituting (20) into (16) and asymptotically expanding the the integral, we obtain

$$ikhG(\alpha,\beta) = ikh \int_\alpha^\beta (\cosh^2 a - \cos^2 \eta)^{1/2} d\eta + i(kh)^{1/3} (\sinh a \cosh a)^{2/3} \tag{22}$$

$$\times \int_\alpha^\beta (\cosh^2 a - \cosh^2 \eta)^{-1/2}\, d\eta + \frac{i\tau_n^{\ 2}}{30 (kh)^{1/3} (\cosh a \sinh a)^{2/3}}$$

$$\times \int_{\alpha}^{\beta} (\cosh^2 a - \cos^2 \eta)^{-3/2} \Big\{ (\cosh^2 a + 7)(\cosh^2 a - \cos^2 \eta)$$

$$- 15 \sinh^2 a \cos^2 \eta \Big\} d\eta - \frac{i(2\cosh^2 a - 1)}{20 \tau_n (kh)^{1/3} (\sinh a \cosh a)^{2/3}}$$

$$\times \int_{\alpha}^{\beta} (\cosh^2 a - \cosh^2 \eta)^{-1/2} d\eta + O((kh)^{-1}) \; .$$

In reference [9] it was shown that the first two integrals in (22) have simple geometric interpretations in terms of the arclength s along the ellipse. To show this we let $s_1$ and $s_2$ be the values of s corresponding to $\eta = \alpha$ and $\eta = \beta$ respectively. Then we find that the first term on the right of (22) is just ik times the arclength

$$ik \int_{s_1}^{s_2} ds \; . \tag{23}$$

Similarly, the second term on the right of (22) is

$$ik^{1/3} \tau_n \int_{s_1}^{s_2} b^{-2/3} \, ds \; . \tag{24}$$

Here b(s) denotes the radius of curvature of the ellipse.

We shall now express the third and fourth terms appearing on the right side of (22) in geometric terms.

The third term can be written as

$$\frac{i\tau_n^{\,2}}{30\ k^{1/3}} \int_{s_1}^{s_2} b^{-4/3}\left(1 + \frac{16}{9}\ b_s^{\,2} - \frac{8bb_{ss}}{3}\right) ds\ .$$

The fourth term is equal to

$$- \frac{i}{20\tau_n k^{1/3}} \int_{s_1}^{s_2} b^{-4/3}\left(2+ \frac{2}{9}\ b_s^{\,2} - \frac{bb_{ss}}{3}\right) ds\ .$$

The second derivatives in (25) and (26) can be eliminated by integrating by parts . Then (25) and (26), respectively become

$$\frac{i\tau_n^{\,2}}{30k^{1/3}} \left\{ - \frac{8}{3}\ \frac{b_s}{b^{1/3}}\ \Big]_{s_1}^{s_2} + \int_{s_1}^{s_2} b^{-4/3}\ \left(1 + \frac{8}{9}\ b_s^{\,2}\right) ds\ . \right. \qquad (27)$$

$$- \frac{i}{20\tau_n k^{1/3}} \left\{ - \frac{1}{3}\ \frac{b_s}{b^{1/3}}\ \Big]_{s_1}^{s_2} + \int_{s_1}^{s_2} b^{-4/3}\ \left(2 + \frac{1}{9}\ b_s^{\,2}\right) ds\ . \right. \qquad (28)$$

Let us now insert (22) into the expression (15) for $u(a,\eta)$. In doing so we shall utilize the geometric forms (23), (24), (27) and (28) for the integrals in (22). We must also evaluate $\partial V_n'(a)/\partial b$ which we find, by the methods of reference [9], to be given by

$$\frac{\partial}{\partial b} V_n'(a) \sim \pi^{-1}(kh)^{4/3}\, 2^{1/2}(\sinh a)^{-1/2}(\cosh a)^{1/2} e^{-2i\pi/3} \tag{29}$$

$$\times\, q_n A(q_n) + O\left((kh)^2\right) \quad .$$

When all these expressions are inserted into (15), we finally obtain the following asymptotic formula for u:

$$u(a,\eta) = \frac{\pi(\cosh a)^{1/2}}{(\cosh^2 a - \cos^2 \eta)^{1/4}} \sum_{n=o}^{\infty} \left\{q_n A(q_n)\right\}^{-1} \left(\exp\left[ikt_1 + \int_{Q_1}^{P} \beta_n ds\right] \right. \tag{30}$$

$$\left. + \exp\left[ikt_2 + \int_{Q_2}^{P} \beta_n ds\right]\right) \left\{ \gamma_n + O(kh)^{-2/3} \right\}$$

$$\times \left\{ 1 + \exp\left[ikT - \int_{o}^{T} \beta_n ds\right] \right\}^{-1} \quad .$$

In (30) the distances $t_1$ and $t_2$ and the points $Q_1$, $Q_2$ and P are as shown in figure 1. The distance T is the total arclength of the ellipse. The quantities $\beta_n$ and $\gamma_n$ are defined by

$$\beta_n = i\tau_n k^{1/3} b^{-2/3} + \frac{i\tau_n^2}{30\, k^{1/3}}\, b^{-4/3}\left(1 + \frac{8}{9}\, b b_s^2\right) \tag{31}$$

$$- \frac{i}{20\tau_n k^{1/3}}\, b^{-4/3}\left(2 + \frac{1}{9}\, b_s^2\right) \quad .$$

$$\gamma_n = \exp\left[\frac{e^{i\pi/6} 6^{1/3} b s(P)}{k^{1/3} b^{1/3}(P)} \left(\frac{q_n^{\,2}}{45} + \frac{1}{60 q_n}\right)\right] . \tag{32}$$

We shall now relate (30) to the geometric theory of diffraction presented in reference [5]. When this theory is applied to the present case it yields

$$u^d(P) = 2^{1/2} 6^{1/3} \pi^{-1/2} k^{1/6} e^{-i\pi/12} b^{-1/3}(P) \sum_{n=o}^{\infty} A(q_n)\, B_n\,(P) \tag{33}$$

$$\left\{B_n(Q_1)\exp\left[ikt_1 + \int_{Q_1}^{P} \beta_n ds\right] + B_n(Q_2)\exp\left[ikt_2 + \int_{Q_2}^{P} \beta_n\, ds\right]\right\}$$

$$\left\{1 - \exp\left[ikT - \int_{o}^{T} \beta_n\, ds\right]\right\}^{-1} .$$

Here $\beta_n$ and $B_n$ are respectively the decay exponent and diffraction coefficient of the nth mode. The leading terms $\beta_n^{\,o}$ and $B_n^{\,o}$ in the expressions for these quantities, as given in reference [5], are

$$\beta_n^{\,o} = i\tau_n\, k^{1/3}\, b^{-2/3} \tag{34}$$

$$B_n^{\,o}(P) = \pi^{3/4}\, 2^{-1/4}\, 6^{-1/6}\, k^{-1/2}\, [q_n^{\,1/2}\, A(q_n)]^{-1}\, e^{i\pi/24} b^{1/6}(P) . \tag{35}$$

Upon comparing (33) and (30) we see that they are identical provided that the decay exponent $\beta_n$ is given by (31) and the diffraction $B_n(P)$ by

$$B_n(P) = B_n^{\circ}(P)\,\gamma_n(P) \tag{36}$$

The new results (31) and (36) for $\beta_n$ and $B_n$ agree with the previous results (34) and (35) to the lowest order in $k^{-1}$. The new value of $\beta_n$ is valid to $O(k^{-2/3})$ and the new value of $B_n$ to $O(k^{-3/4})$. Thus they contain corrections to the previous results.

The preceding results pertain to a hard elliptic cylinder - i.e one on which $\partial u/\partial n = 0$. We have performed a similar calculation for a soft elliptic cylinder - i.e. one on which u=o. In this case we also find corrections to the decay exponents and to the diffraction coefficients. For the decay exponents we obtain

$$\beta_n = i\tau_n k^{1/3} b^{-2/3} + \frac{i\tau_n^{\;2}\, b^{-4/3}}{30\,k^{1/3}}\left(1 + \frac{8}{9}\,b_s^{\;2}\right). \tag{37}$$

Here $\tau_n = 6^{-1/3} q_n e^{i\pi/3}$ and $q_n$ is the nth root of the equation $A(q_n) = 0$. For the diffraction coefficients we find

$$B_n(P) = B_n^{\circ}(P)\,\exp\left[\frac{e^{i\pi/6}6^{1/3}b_s(P)\,q_n^{\;2}}{45\,k^{1/3}b^{1/3}(P)}\right]. \tag{38}$$

Here $B_n^{\circ}(P)$ is the lowest order result for the diffraction coefficient, given in reference [5], as

$$B_n^{\circ}(P) = \pi^{3/4}\,2^{1/4}\,6^{-2/3}\,k^{-1/12}\left[A'(q_n)\right]^{-1} e^{i\pi/24}\,b^{1/6} \tag{39}$$

The new results (37) and (39) contain corrections to the previous results, as in the hard case.

We now assume that the results (31) and (36) apply to any hard cylinder and that (37) and (38) apply to any soft cylinder. Of course the cylinder must have a smooth cross section. As a first check on these results we see that when $b_s = 0$ (31) and (37) agree with the results (A17a) and (A17b) of W. Franz [6] for a circular cylinder.

3. Diffraction by a parabolic cylinder.

As a check on the higher order corrections to the decay exponents and diffraction coefficients which were derived in Section 2 we now consider the problem of diffraction by a parabolic cylinder. Our solution will closely parallel that of S. O. Rice [8]. However, we find it more convenient to use parabolic cylinder functions which differ from his and hence we will rederive his results. We again consider the problem of evaluating the field on the surface of a hard parabolic cylinder due to an incident plane wave. For convenience we first consider the diffraction problem for an incident cylindrical wave and then obtain the plane wave result by a limiting procedure.

To formulate the diffraction problem we take the z axis of an (x,y,z) rectangular coordinate axis to be parallel to the generators of the parabolic cylinder. In the (x,y) plane we introduce parabolic coordinates $(\xi,\eta)$ through

$$\begin{aligned} x &= \xi\eta \\ y &= \tfrac{1}{2}(\eta^2-\xi^2) \quad . \end{aligned} \tag{1}$$

Here $\eta > 0$ and $-\infty < \xi < \infty$. The parabolic cylinder is defined by $\eta$ = constant = $\eta_o$. The line source is located at $y = 0$, $x = x_o$, i.e., $\xi = \eta = a = x_o^{1/2}$. The wave function $u(\xi,\eta)$ then satisfies the equation

$$u_{\xi\xi} + u_{\eta\eta} + k^2(\xi^2 + \eta^2)u = \delta(\xi-a)\delta(\eta - a) . \tag{2}$$

In addition u satisfies the boundary condition

$$u_\eta(a,\xi) = 0, \tag{3}$$

and the Sommerfeld radiation condition

$$\lim_{r\to\infty} r(iku - u_r) = 0 .$$

Now to find u we first note that the product $\phi(\xi)\psi(\eta)$ satisfies (2) with the delta functions replaced by zero if $\phi$ and $\psi$ satisfy the ordinary differential equations

$$\psi'' - k^2(b^2 - \eta^2)\psi = 0 \tag{5}$$

$$\phi'' + k^2(b^2 + \xi^2)\phi = 0. \tag{6}$$

Here b is an arbitrary separation constant. We next note that for an infinite set $b_n$ of values of b, there exist solutions of (5), $\psi_n(\eta)$, which are 'outgoing†' and for which

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† Since the polar coordinate variable r is equal to $\xi^2+\eta^2/2$ we take the outgoing condition on $\psi$ to mean that as $\eta \to \infty$, $\psi \to Ae^{ik\eta^2/2}$. Here and in the following A will denote a generic amplitude function.

$$\psi_n'(a) = 0 \ .$$

We next assume that the $\psi_n(\eta)$ are complete and express u as

$$u(\xi,\eta) = \sum_{n=0}^{\infty} \Phi_n(\xi)\ \psi_n(\eta). \tag{7}$$

By exactly the same calculation as was carried out in reference [9] it is easy to show that

$$\int_{\eta_0}^{\infty} \psi_n(\eta)\psi_m(\eta) = -\ \delta_{nm}(2k^2 b_n)^{-1}\ \psi_n(\eta_0)\ \frac{\partial}{\partial b}\ \psi_n'(\eta_0)\ \ . \tag{8}$$

Here $\delta_{nm}$ is the Kronecker delta and $\partial/\partial b\ \psi_n'(\eta_0)$ is the value of $\partial/\partial b \psi'(\eta)$ evaluated at $b = b_n$ and $\eta = \eta_0$. Thus upon substituting (7) into (2) multiplying by $\psi_n(\eta)$, integrating from $\eta_0$ to $\infty$, and making use of (8) we find that $\Phi_n$ satisfies

$$\Phi_n'' + k^2(b^2 + \xi^2)\ \Phi_n = \frac{-2k^2 b_n \psi_n(a)}{\psi_n(\eta_0)\partial/\partial b\ \psi_n'(\eta_0)}\quad \delta(\xi - a)\ . \tag{9}$$

To solve (9) we first characterize the solutions of (6) by means of their asymptotic expansions as $k \to \infty$. As $k \to \infty$ there exist solutions $\phi^{(1)}(\xi)$ and $\phi^{(2)}(\xi)$ having the following asymptotic expansions

$$\phi^{(1)}(\xi) \sim (b^2 + \xi^2)^{-1/4}\ \exp\left[ik \int_0^{\xi} (b^2 + \xi^2)^{1/2}\ d\xi\right] \tag{10}$$

$$\phi^{(2)}(\xi) \sim (b^2 + \xi^2)^{-1/4} \exp\left[-ik \int_0^{\xi} (b^2 + \xi^2)^{1/2} d\xi\right] . \tag{11}$$

A simple calculation shows that as $|\xi| \to \infty$

$$\phi^{(1)} \sim A \exp\left[ik\xi|\xi|/2\right] \tag{12}$$

$$\phi^{(2)} \sim A \exp\left[-ik\xi|\xi|/2\right] . \tag{13}$$

It is thus apparent that as $\xi \to \infty$, $\phi^{(1)}$ is the outgoing solution of (6), while as $\xi \to -\infty$, $\phi^{(2)}$ is the outgoing solution of (6). Since the variable $\xi$ takes on both positive and negative values, we see that for $\xi > \xi_o$ the solution of (9) is proportional to $\phi^{(1)}(\xi)$, while for $\xi < \xi_o$ the solution of (9) is proportional to $\phi^{(2)}(\xi)$. These conditions together with the jump conditions imposed by the delta function allow a unique determination of $\Phi_n(\xi)$. We then find that for $\xi < \xi_o$

$$u(\xi,\eta) = ik \sum_{n=o}^{\infty} b_n \psi_n(\eta)\phi_n^{(2)}(\xi) \frac{\phi_n^{(1)}(a)\psi_n(a)}{\psi_n(\eta_o)\ \partial/\partial b\ \psi_n'(\eta_o)} . \tag{14}$$

Now to pass to plane wave excitation we multiply by

$$c = e^{3\pi i/4}\, 2^{3/2}\, \pi^{1/2}\, k^{1/2}\, a\, e^{-ika^2} \tag{15}$$

and let $a \to \infty$. In order to evaluate this limit we require the asymptotic expans on of the function $\psi_n(a)$. Using the methods of Olver [13] as in Section II we find

$$\psi_n(\eta) \sim \zeta^{1/4}\, 3^{1/3}\, \pi^{-1}(b^2 - \eta^2)^{-1/4}\, A(3^{1/3} e^{-i\pi/3}\, k^{2/3}\, \zeta) \tag{16}$$

Here

$$\frac{2}{3}\zeta^{3/2} = \int_b^\eta (b^2 - \eta^2)^{1/2}\, d\eta. \tag{17}$$

When $\eta > b$, (16) becomes

$$\psi_n(\eta) \sim e^{i\pi/12}\, k^{-1/6}\, 2^{-1}\, \pi^{-1/2}(\eta^2 - b^2)^{-1/4} \exp\left[ik \int_b^\eta (\eta^2 - b^2)^{1/2} d\eta\right]. \tag{18}$$

Now a simple calculation shows that as $\eta \to \infty$

$$\int_b^\eta (\eta^2 - b^2)^{1/2}\, d\eta \sim -\frac{b^2}{2} \log \frac{2\eta}{b} + \frac{\eta^2}{2}. \tag{19}$$

Also as $\xi \to \infty$

$$\int_0^\xi (\xi^2 + b^2)^{1/2}\, d\xi \sim \frac{b^2}{2} \log \frac{2\xi}{b} + \frac{\xi^2}{2}. \tag{20}$$

Upon using (19) in (18) and (20) in (10) we find

$$\lim_{a \to \infty} c\phi_n^{(1)}(a)\, \psi_n(a) = 2^{1/2}\, k^{1/3}\, e^{5\pi i/6} \tag{21}$$

Thus for $\eta = \eta_o$ and for an incident plane wave (14) becomes

$$u(\xi,\eta_o) = k^{4/3}\, e^{4\pi i/3}\, 2^{1/2} \sum_{n=0}^{\infty} b_n \frac{\phi_n^{(2)}(\xi)}{\partial/\partial b\, \psi_n^{(1)'}(\eta_o)} \quad . \tag{22}$$

To obtain the asymptotic expansion of (22) as $k \to \infty$ we proceed exactly as in the case of the elliptic cylinder. We first find the following three term asymptotic expansion of $b_n$ from the condition that $\psi_n'(\eta_o) = 0$.

$$b_n = \eta_o + \frac{\tau_n}{k^{2/3}\eta_o^{1/3}} - \frac{7}{30}\,\frac{\tau_n^2}{k^{4/3}\eta_o^{5/3}} - \frac{1}{20\tau_n k^{4/3}\eta_o^{5/3}} + O(k^{-2}) \ . \tag{23}$$

Then upon using (23) in (16) and (11) we find

$$\partial/\partial b\, \psi_n'(\eta_o) = \pi^{-1}\, e^{-2i\pi/3} q_n A(q_n) 2^{1/2}\, \eta_o^{1/2}\, k^{4/3} + O(k^2) \tag{24}$$

$$\phi_n^{(2)}(\xi) = (\eta_o^2 + \xi^2)^{-1/4} \exp\Big[-ik \int_0^{\xi} (\eta_o^2 + \xi^2)^{1/2} d\xi - ik^{1/3}\tau_n\eta_o^{2/3} \tag{25}$$

$$\times \int_0^{\xi} (\eta_o^2 + \xi^2)^{-1/2} d\xi - \frac{i\tau_n^2\, \eta_o^{4/3}}{30\, k^{1/3}} \int_0^{\xi} (\eta_o + \xi^2)^{-3/2}$$

$$\times\ (8\xi^2/\eta_o^2 - 7)d\xi + \frac{i}{20\tau_n k^{1/3}\eta_o^{2/3}} \int_0^{\xi} (\eta_o^2 + \xi^2)^{-1/2} d\xi\Big]$$

Upon substituting (24), (25) and (23) into (22) we find

$$u(\xi,\eta_o) = \pi\eta_o^{1/2}(\eta_o^2 + \xi^2)^{-1/4} \sum_{n=o}^{\infty} \left[q_n A(q_n)\right]^{-1} \Bigg\{ \exp\Bigg[-ik\int_o^{\xi}(\eta_o^2+\xi^2)^{1/2}d\xi \tag{26}$$

$$- ik^{1/3}\tau_n\eta_o^{2/3}\int_o^{\xi}(\eta_o^2+\xi^2)^{-1/2}d\xi - \frac{i\tau_n^2\eta_o^{4/3}}{30\,k^{1/3}}\int_o^{\xi}(\eta_o^2+\xi^2)^{-3/2}$$

$$\times\ (8\xi^2/\eta_o^2 - 7)d\xi + \frac{i}{20\tau_n k^{1/3}\eta_o^{2/3}}\int_o^{\xi}(\eta_o^2+\xi^2)^{-1/2}d\xi\Bigg] + O(k^{-2/3})\Bigg\} .$$

In the case of the parabolic cylinder the incident rays are parallel to the x axis and the diffracted ray to the point $(\xi,\eta_o)$ follows the path QP as shown in Figure 2.

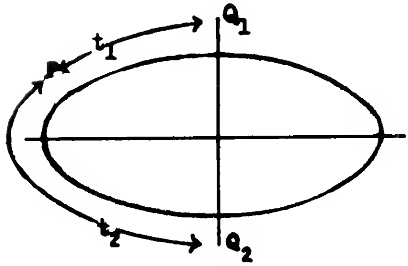


Figure 1

A cross section of the elliptic cylinder showing the points $Q_1$ and $Q_2$ at which two incident rays are tangent to it. The incident field is a plane wave coming from the right. The tangent rays produce diffracted rays which travel distances $t_1$ and $t_2$ to a point P on the surface.

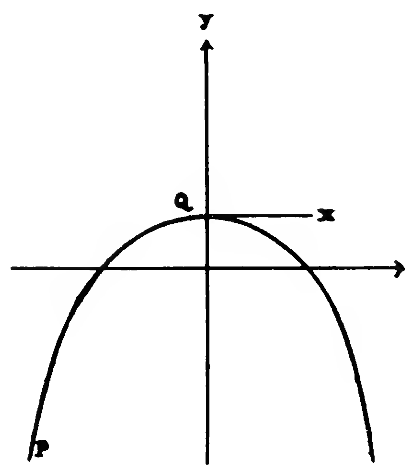


Figure 2

A cross section of the parabolic cylinder. The incident field is a plane wave coming from the right. The diffracted ray follows the parabolic path QP.

A simple calculation shows that the element of arclength along the parabola, ds, is given by

$$ds = (\xi^2 + \eta_o^2)^{1/2} \, |d\xi| \tag{27}$$

Thus the first term in the exponent in (26) is ikt, since $\xi$ is negative. Similarly, we find that the radius of curvature of the parabola, b, is given by

$$b = \eta_o^{-1} (\eta_o^2 + \xi^2)^{3/2} \tag{28}$$

Upon using (27) and (28) a simple calculation shows that the exponent in (26) can be written as

$$\gamma_n \exp\left[ikt + \int_Q^P \beta_n \, ds\right] . \tag{29}$$

Here $\beta_n$ and $\gamma_n$ are defined by (31) and (32) of Section II. Upon applying the geometric theory of reference [5] to the present case it is easy to show that the geometric construction agrees with (26) to lowest order in $k^{-1}$. Again we have in (26) higher order corrections to the diffraction coefficients $B_n^o(P)$ and the decay exponents $\beta_n^o$ as given by equations (34) and (35) of Section II. These corrections are identical with those given by equations (31) and (36) of Section II.

## 4. Integral equation method.

We will now derive the asymptotic expansion of each mode of the diffracted field on an arbitrary convex cylinder by a different method. In this method we begin with an integral equation and obtain a formal asymptotic solution of it. This asymptotic solution coincides with the expression for a mode given by the geometric theory of diffraction, with the corrected decay exponents and diffraction coefficients found in section 2. This independent derivation, which follows the procedure used by W. Franz and K. Depperman [3] in the case of a circular cylinder, confirms our previous result.

We consider the two dimensional problem of finding a function u(x,y) satisfying the following equations

$$(\nabla^2 + k^2)u = 0 \qquad \text{in } D \tag{1}$$

$$\partial u/\partial n = 0 \qquad \text{on } C \tag{2}$$

$$\lim_{r \to \infty} r^{1/2}\left(\frac{\partial u}{\partial r} - iku\right) = 0 \,. \tag{3}$$

Here C is a given simple smooth convex curve with a piecewise continuous second derviative. If C is closed, D denotes its exterior. If C is open and extends to infinity, D denotes the non-convex portion of the plane, bounded by C.

From (1) - (3) it follows that on C, u satisfies the following integral equation

$$u(s) = -\frac{i}{2} \int_C u(s') \frac{\partial}{\partial n'} H_0^{(1)} \left[kr(s,s')\right] ds' \quad . \tag{4}$$

Here s denotes arclength along C measured from some fixed point, u(s) is the value of u at the point s on C, r(s,s') is the distance between the points s and s', and the normal n' points into D.

If C is closed, the only single-valued solution of (4) 's $u \equiv 0$. If C is open, presumably the only bounded solution is also $u \equiv 0$. Therefore if u is to represent a mode, it must be multi-valued in the former case, or unbounded in the latter case. Consequently we assume that on C a single mode u has the following asymptotic expansion for large values of k

$$u(s) \sim \exp\left[iks + \sum_{n=-1}^{\infty} v_n(s) k^{-n/3}\right] . \tag{5}$$

The coefficients $v_n(s)$ are to be determined by requiring (5) to satisfy (4) asymptotically.

Before inserting (5) into (4), we note that for large values of k the function $\partial H_o^{(1)}[kr(s,s')]/\partial n'$ has the asymptotic expansion

$$\frac{\partial H_o^{(1)}[kr(s,s')]}{\partial n'} \sim \frac{\partial r}{\partial n'} \left(\frac{2k}{\pi r}\right)^{1/2} e^{i(kr + \pi/4)} \sum_{m=o}^{\infty} \frac{(-1)^m (0,m)}{(2ikr)^m} \left[1 - \frac{m + 1/2}{ikr}\right] . \tag{6}$$

The symbol (0,m) is defined by

$$(0,m) = \Gamma\left(\tfrac{1}{2} + m\right)/m!\, \Gamma\left(\tfrac{1}{2} - m\right). \tag{7}$$

Now we insert (5) and (6) into (4) and then divide by the left hand side of the resulting equation. In this way we obtain the equation

$$1 \sim e^{-i\pi/4}(k/2\pi)^{1/2} \int_C \frac{\partial r}{\partial n'} r^{-1/2} \sum_{m=0}^{\infty} (-1)^m (0,m)(2ikr)^{-m}\left[1 - \frac{m+1/2}{ikr}\right]$$

$$\times \exp[ik(r-s-s')] \exp\left[\sum_{n=-1}^{\infty} k^{-n/3}(v_n(s') - v_n(s)\right]ds' . \qquad (8)$$

In order to determine the $v_n(s)$ from (8) we first expand the integral in (8) asymptotically for large values of k. We perform this expansion by using the concept of stationary phase. The derivative of the phase of the integrand is $1 + dr/ds'$, which vanishes if $dr/ds' = -1$. This condition is satisfied only at $s' = s$, and then only if $dr/ds'$ denotes the one sided derivative computed with $s' < s$. Thus to evaluate the integral we expand the integrand in the one sided neighborhood $s' < s$ of the point $s' = s$. For this purpose we use the following expansions which are derived in the appendix

$$r = \sum_{n=1}^{\infty} c_n(s)(s-s')^n \qquad (9)$$

$$r^{-1/2} \partial r/\partial n' = - \frac{\kappa(s)}{2} (s-s')^{1/2} \sum_{n=0}^{\infty} \rho_n(s)(s-s')^n \qquad (10)$$

$$\sum_{m=0}^{\infty} (-1)^m (0,m)(2ikr)^{-m} \left[1 - \frac{m+1/2}{ikr}\right] = \sum_{n=-\infty}^{\infty} \beta_n(s,k)(s-s')^n \qquad (11)$$

Here $\kappa(s)$ denotes the curvature of C. The first few of the coefficients $c_n$, $\rho_n$ and $\beta_n$ are listed in Table I.

We now insert (9) - (11) into (8), making use of the explicit values of $\rho_o$, $c_1$, $c_2$ and $c_3$. We also expand $v_{-1}(s')$ in a power series about the point $s' = s$. Then (8) assumes the form

$$1 \sim e^{3\pi i/4} \kappa(s)(k/8\pi)^{1/2} \int_{-\infty}^{s} (s-s')^{1/2} \exp\left[- \frac{ik\kappa^2(s)}{24} (s-s')^3 \right.$$

$$\left. - k^{1/3}\dot{v}_{-1}(s)(s-s')\right] F(k,s,s')ds' \quad . \tag{12}$$

The function $F(k,s,s')$ appearing in (12) is defined by

$$F(k,s,s') = \exp\left[ik \sum_{n=4}^{\infty} c_n(s)(s-s')^n + \sum_{n=-1}^{\infty} \sum_{m=1}^{\infty} k^{-n/3} \frac{v_n^{(m)}(s)(s'-s)^m}{m!}\right.$$

$$\left. - k^{1/3}\dot{v}_{-1}(s)(s'-s)\right] \sum_{n=o}^{\infty} \rho_n(s)(s-s')^n \sum_{n=-\infty}^{\infty} \beta_n(s,k)(s-s')^n. \tag{13}$$

In (13) $v_n^{(m)}(s)$ denotes the m-th derivative of $v_n(s)$.

To complete the asymptotic evaluation of the integral we introduce the new variable t by means of the definition

$$s-s' = e^{-i\pi/6}\left(\frac{24}{\kappa^2(s)k}\right)^{1/3} t \quad . \tag{14}$$

When (14) is used in (13), it shows that $F(k,s,s')$ has an expansion of the form

$$F(k,s,s') \sim 1 + \sum_{n=1}^{\infty} k^{-n/3}\, b_n(t,s) \,. \tag{15}$$

We next define $\alpha(s)$ by the equation

$$\dot{v}_{-1}(s) = \alpha(s)(\kappa(s))^{2/3}(24)^{-1/3}\, e^{i\pi/6} \,. \tag{16}$$

Finally we insert (14) - (16) into (12), which becomes

$$1 \sim i(3/\pi)^{1/2} \int_0^{\infty} t^{1/2}\, e^{-\alpha t - t^3}\, (1 + \sum_{n=1}^{\infty} k^{-n/3}\, b_n(t,s))dt. \tag{17}$$

Upon comparing coefficients of the various powers of k in the asymptotic form of the integral equation (17), we obtain the following set of equations

$$1 - i(3/\pi)^{1/2} \int_0^{\infty} t^{1/2}\, e^{-\alpha t - t^3}\, dt = 0 \tag{18}$$

$$\int_0^{\infty} t^{1/2}\, e^{-\alpha t - t^3}\, b_n(t,s)dt = 0, \qquad n = 1,\ 2,\ \ldots \tag{19}$$

From these equations we shall determine the coefficients $v_n(s)$.

W. Franz [6] has shown that the left side of (18) can be rewritten in terms of the Airy function A defined in equation (11) of section 2. Thus (18) becomes

$$\frac{12}{\pi} \, e^{-i\pi/6} \, A' \left( - \frac{e^{i\pi/3}}{4^{1/3}} \, \alpha \right) \, A \left( - \frac{e^{-i\pi/3}}{4^{1/3}} \, \alpha \right) = 0 \, . \tag{20}$$

The appropriate value of $\alpha$ is determined by the vanishing of the A' factor in (20). If we denote by $q_n$ the roots of the equation $A'(q_n)=0$, then the values $\alpha_n$ of $\alpha$ are given by

$$\alpha = \alpha_n = -e^{i\pi/3} \, 4^{1/3} \, q_n \, . \tag{21}$$

It will be useful to introduce the function $h(\alpha)$ defined by

$$h(\alpha) = \int_0^\infty t^{-1/2} \, e^{-\alpha t - t^3} \, dt \, . \tag{22}$$

Franz [6] has shown that

$$h(\alpha) = 4^{5/6} \sqrt{3/\pi} \, A \left( - \frac{e^{i\pi/3} \alpha}{4^{1/3}} \right) \, A \left( - \frac{e^{-i\pi/3} \alpha}{4^{1/3}} \right) , \tag{23}$$

and that h satisfies

$$h'''(\alpha) = - \frac{1}{6} \, h(\alpha) - \frac{\alpha}{3} \, h'(\alpha) \, . \tag{24}$$

To determine the consequences of (19) we must first compute the $b_n$. We shall calculate only $b_1$ and $b_2$. To do so we substitute (14) and (16) into (13), expand the exponential functions and multiply together the resulting series in powers of $k^{-1/3}$. In this way we obtain

$$b_1 = e^{-i\pi/6}\,(24)^{1/3}\,\frac{\dot{\kappa}}{\kappa^{5/3}}\left\{\left(\frac{-2}{3} - \dot{v}_0\,\frac{\kappa}{\dot{\kappa}}\right)t + \frac{\alpha}{3}\,t^2 + t^4\right\} \tag{25}$$

$$b_2 = \frac{3e^{2i\pi/3}\,\kappa^{2/3}}{8(24)^{1/3}t} - \frac{e^{-i\pi/6}(24)^{1/3}}{\kappa^{2/3}}\,\dot{v}_1 t + e^{-i\pi/3}\,(24)^{2/3}$$

$$\times\left\{\frac{t^2}{\kappa^{4/3}}\left(\frac{\ddot{\kappa}}{3\kappa} - \frac{\kappa^2}{48} + \frac{\dot{\kappa}^2}{24\kappa^2}\right) + \frac{2\alpha t^3}{3\kappa^2}\left(-\frac{\ddot{\kappa}}{6\kappa^{1/3}} - \frac{13\dot{\kappa}^2}{36\kappa^{4/3}}\right)\right. \tag{26}$$

$$\left. + \frac{\alpha^2\dot{\kappa}^2 t^4}{13\kappa^{10/3}} + \frac{24t^5}{\kappa^{10/3}}\left(-\frac{\kappa\ddot{\kappa}}{80} + \frac{\kappa^4}{1920} - \frac{33\dot{\kappa}^2}{720}\right) + \frac{\alpha\dot{\kappa}^2 t^6}{3\kappa^{10/3}} + \frac{\dot{\kappa}^2\,t^8}{2\kappa^{10/3}}\right\}.$$

When (25) is inserted into (19) an equation for $\dot{v}_0$ is obtained. This equation contains integrals of the form

$$\int_0^\infty t^{n-1/2}\,e^{-\alpha t - t^3}\,dt = (-1)^n h^{(n)}(\alpha). \tag{27}$$

In (27) the integral has been expressed in terms of the n-th derivative of $h(\alpha)$ which is defined by (22). Thus from (19) we find that the right hand side of (25) must vanish when $t^n$ is replaced by $(-1)^{n+1}h^{(n+1)}(\alpha)$. This yields

$$\left(\frac{-2}{3} - \dot{v}_o \frac{\kappa}{\dot{\kappa}}\right) h'' - \frac{\alpha}{3} h'''(\alpha) - h^{(V)}(\alpha) = 0 . \tag{28}$$

By using (24) we find that

$$h^{(IV)}(\alpha) = - \frac{1}{2} h' - \frac{\alpha}{3} h'' \tag{29}$$

$$h^{(V)}(\alpha) = \frac{\alpha}{18} h + \frac{\alpha^2}{9} h' - \frac{5}{6} h'' \tag{30}$$

When (24) and (30) are used in (28), the following expression for $\dot{v}_o$ results

$$\dot{v}_o = \frac{1}{6} \frac{\dot{\kappa}}{\kappa} . \tag{31}$$

Upon integrating (31) we finally obtain for $v_o$ the expression

$$v_o = \log \kappa^{1/6} + \delta . \tag{32}$$

Here $\delta$ is an integration constant.

The analysis of (19) for the case n = 2 proceeds in exactly the same way. In this case we obtain the condition that the right hand side of (26) vanishes when $t^n$ is replaced by $(-1)^{n+1} h^{(n+1)}(\alpha)$[†]. In order to simplify the resulting expression we must express the sixth through ninth derivatives of h in terms of h, h', and h''. Upon doing this we find

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[†] In order to avoid writing cumbersome equations we denote by $b_2^*$ the right hand side of (26) with $t^n$ replaced by $(-1)^{n+1} h^{(n+1)}(\alpha)$.

$$h^{(VI)} = \frac{7}{36} h + \frac{5\alpha}{9} h' + \frac{\alpha^2}{9} h'' \tag{33}$$

$$h^{(VII)} = - \frac{\alpha^2}{54} h + \frac{3}{4} h' - \frac{\alpha^3}{27} h' + \frac{7}{9} \alpha h'' \tag{34}$$

$$h^{(VIII)} = - \frac{\alpha}{6} h - \frac{7}{18} \alpha^2 h' + \frac{55}{36} h'' - \frac{\alpha^3}{27} h'' \tag{35}$$

$$h^{(IX)} = \frac{\alpha^3}{162} h - \frac{91}{216} h - \frac{157}{108} \alpha h' + \frac{\alpha^4}{81} h' - \frac{\alpha^2}{2} h'' . \tag{36}$$

We also note that

$$h''(\alpha) = - \frac{\alpha}{6} h(\alpha). \tag{37}$$

This result follows upon differentiating (23) twice, then using the equation satisfied by $A(x)$

$$A'' + \frac{x}{3} A = 0 , \tag{38}$$

and finally noting that when $\alpha$ is defined by (21)

$$A' \left( \frac{-e^{i\pi/3} \alpha}{4^{1/3}} \right) = 0. \tag{39}$$

We now insert the preceding relations for the derivatives of $h$, together with (31) and (37) into $b_2^*$. We then find that the coefficient of $h'$ vanishes and that the equation $b_2^* = 0$ may be solved to yield

$$\dot{v}_1 = 6(24)^{1/3}\, \kappa^{2/3}\, \alpha^{-1}\, e^{-i\pi/6} \left\{ \frac{\kappa^{2/3}}{64} - \frac{1}{6\kappa^{4/3}} \left( \frac{1}{3}\frac{\ddot{\kappa}}{\kappa} - \frac{1}{48}\kappa^2 + \frac{1}{24}\frac{\dot{\kappa}^2}{\kappa^2} \right) \right.$$

$$+ \frac{\alpha^3}{27\kappa^2} \left( \frac{\ddot{\kappa}}{6\kappa^{1/3}} + \frac{13}{36}\frac{\dot{\kappa}^2}{\kappa^{4/3}} \right) + \frac{7\alpha^3}{648}\frac{\dot{\kappa}^2}{\kappa^{10/3}} + \frac{24}{\kappa^{10/3}} \left( \frac{7}{36} - \frac{\alpha^3}{54} \right) \tag{40}$$

$$\left. \times \left( \frac{\kappa\ddot{\kappa}}{80} - \frac{\kappa^4}{1920} + \frac{33}{720}\dot{\kappa}^2 \right) - \frac{4\alpha^3}{81}\frac{\dot{\kappa}^2}{\kappa^{10/3}} + \frac{1}{2}\frac{\dot{\kappa}^2}{\kappa^{10/3}} \left( \frac{29\alpha^3}{324} - \frac{91}{216} \right) \right\}.$$

We next make use of (21) of this section and (21) of section 2 and set $\kappa^{-1} = b$ = the radius of curvature of C. Then a straightforward calculation shows that (40) may be written as

$$\dot{v}_1 = \frac{i\tau_n^2}{30} \left( b^{-4/3} + \frac{16}{9} b^{-4/3} b_s^2 - \frac{8b^{-1/3}}{3} b_{ss} \right)$$

$$- \frac{i}{20\tau_n} \left( 2b^{-4/3} + \frac{2}{9} b^{-4/3} b_s^2 - \frac{b^{-1/3}}{3} b_{ss} \right). \tag{41}$$

Let us now combine our results (16), (21), (32), and (41). By inserting them into (5) we obtain the asymptotic expansion of $u(s)$ on the cylinder C up to and including terms in $k^{-1/3}$.

$$u \sim E\, b^{-1/6}(s) \exp \left[ iks + i k^{1/3} \tau_n \int^s b^{-2/3}(s) ds \right.$$

$$+ \frac{ik^{-1/3}}{30} \tau_n^2 \int^s \left( b^{-4/3} + \frac{16}{9} b^{-4/3} b_s^2 - \frac{8b^{-1/3}}{3} b_{ss} \right) ds \tag{42}$$

$$\left. - \frac{ik^{-1/3}}{20\tau_n} \int^s \left( 2b^{-4/3} + \frac{2}{9} b^{-4/3} b_s^2 - \frac{b^{-1/3}}{3} b_{ss} \right) ds + \dots \right].$$

Here E denotes an arbitrary constant.

Let us now compare the result (42) with the expression for a mode given by the geometric theory of diffraction [5]. That theory yields for u a single term of the sum in (33) of section 2. Let us insert into that equation the improved decay exponents (31) and diffraction coefficients (36). Then we find that each term of (33) coincides with (42) provided that the product of the constant coefficients in (33) is equated to the constant E in (42). This agreement between the two methods of obtaining the improved decay exponents and diffraction coefficients again confirms the results of section 2. The method of the present section can also be modified to apply to soft cylinders, on which u = 0.

## Appendix

In order to calculate the quantity $r(s,s')$ in the neighborhood of $s = s'$ we first observe that if $\vec{x}(s)$ is the position vector to the curve C, then

$$r^2 = \left(\vec{x}(s) - \vec{x}(s')\right)^2 \quad . \tag{1}$$

By Taylor's theorem

$$\vec{x}(s') - \vec{x}(s) = \sum_{n=1}^{\infty} (s'- s)^n \, \frac{\vec{x}^{(n)}(s)}{n!} \quad . \tag{2}$$

Thus, upon taking the dot product of (2) with itself we find

$$r^2 = \sum_{\rho=2}^{\infty} (s'- s)^{\rho} \, b_{\rho} \quad . \tag{3}$$

Here

$$b_{\rho} = \sum_{n=1}^{\rho-1} \frac{\vec{x}^{(n)}(s)\cdot\vec{x}^{(\rho-n)}(s)}{n!\,(\rho-n)!} \quad . \tag{4}$$

Since s is arclength along C we have

$$\dot{\vec{x}}(s) \cdot \dot{\vec{x}}(s) = 1, \tag{5}$$

and

$$\ddot{\vec{x}}(s) \cdot \dot{\vec{x}}(s) = 0. \tag{6}$$

From the Frenet equations of differential geometry we have

$$\dddot{\vec{x}}(s) = \dot{\kappa}\kappa^{-1}\ddot{\vec{x}} - \kappa^2 \dot{\vec{x}} . \tag{7}$$

Upon using (5), (6), and (7) recursively to obtain the higher derivatives of $\vec{x}(s)$ in terms of $\dot{\vec{x}}(s)$ and $\ddot{\vec{x}}(s)$ we see that $b_\rho$ can be expressed in terms of $\kappa$ and its derivatives. In this way we find

$$b_2 = 1; \quad b_3 = 0; \quad b_4 = \frac{-\kappa^2}{12}; \quad b_5 = \frac{-\kappa\dot{\kappa}}{12}; \quad b_6 = \frac{-1}{45}\dot{\kappa}^2 - \frac{1}{40}\kappa\ddot{\kappa} + \frac{\kappa^4}{360} . \tag{8}$$

Then upon taking the square root of the right hand side of (3) we find

$$r = \sum_{n=1}^{\infty} c_n(s-s')^n . \tag{9}$$

Here

$$c_1 = b_2; \quad c_2 = b_3; \quad c_3 = \frac{b_4}{2}; \quad c_4 = \frac{-b_5}{2}; \quad c_5 = \left(\frac{b_6}{2} - \frac{b_4^2}{8}\right) . \tag{10}$$

From (8) and (10) the entries for $c_n$ in Table I are obtained.

In order to calculate $r^{-1/2}\, \partial r/\partial n'$ we note that the unit normal to C at s' is $\kappa^{-1}(s')\, \ddot{\vec{x}}(s') = \vec{v}_2(s')$ and hence

$$\frac{\partial r}{\partial n} = \nabla(\vec{r}) \cdot \vec{v}_2(s') = \frac{\vec{r}\cdot\vec{v}_2(s')}{r} . \tag{11}$$

Thus by making use of (2) and the Taylor expansion of $\vec{v}_2(s')$ about s = s' we obtain

$$r\frac{\partial r}{\partial n'} = \sum_{\rho=1}^{\infty} (s'-s)^{\rho} f_{\rho}. \tag{12}$$

Here

$$f_{\rho} = \sum_{k=1}^{\rho} \frac{\vec{x}^{(k)}(s)\cdot\vec{v}_2^{(\rho-k)}(s)}{k!\ (\rho-k)!} . \tag{13}$$

Again upon using the Frenet equations recursively the coefficients $f_{\rho}$ may be easily evaluated to obtain

$$f_1 = 0; \quad f_2 = -\frac{\kappa}{2}; \; f_3 = -\frac{\dot{\kappa}}{3}; \; f_4 = -\frac{\ddot{\kappa}}{8} + \frac{\kappa^3}{24} . \tag{14}$$

Now by applying the binomial theorem to (3) we find

$$r^{-3/2} = (s-s')^{-3/2}\left(b_2 - \frac{3}{4} b_4(s-s')^2 + \frac{3}{4} b_5(s-s')^3 + \ldots\right). \tag{15}$$

Thus upon multiplying (15) and (12) we find

$$r^{-1/2}\frac{\partial r}{\partial n'} = -\frac{\kappa(s)}{2}(s-s')^{1/2}\sum_{n=o}^{\infty} \rho_n(s)(s-s')^n , \tag{16}$$

where

$$\rho_o = 1; \quad \rho_1 = \frac{2f_3}{\kappa}; \quad \rho_2 = -\frac{2f_4}{\kappa} - \frac{3}{4} b_4 . \tag{17}$$

Thus from (8), (14), and (17) we obtain the values of $\rho_n$ given in Table I. Upon using (9) and the values of (0,m) given in section 4 it is a simple matter to calculate the values of $\beta_o$ and $\beta_{-1}$ as given in Table I and to conclude that $\beta_n(s-s')^n = o(k^{-2/3})$ for $n \neq 0, -1$ and $s-s' = O(k^{-1/3})$.

Table I

| n | $\beta_n$ | $\rho_n$ | $c_n$ |
| --- | --- | --- | --- |
| -1 | 3i/8k | | |
| 0 | 1 | 1 | |
| 1 | - | $-\frac{2}{3}\ \frac{\dot{\kappa}}{\kappa}$ | 1 |
| 2 | - | $\frac{\ddot{\kappa}}{4\kappa} - \frac{\kappa^2}{12} + \frac{\kappa^2}{16}$ | 0 |
| 3 | - | - | $-\frac{\kappa^2}{24}$ |
| 4 | - | - | $\frac{\kappa\dot{\kappa}}{24}$ |
| 5 | - | - | $-\frac{\dot{\kappa}^2}{90} - \frac{\kappa\ddot{\kappa}}{80} + \frac{\kappa^4}{1920}$ |

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