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ERRATUM

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Erratum to: Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors

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In [1, Sect. 4.3, pp. 1029–1033], the construction of a global Poincaré map with suitable properties, for a flow containing a singular hyperbolic attractor, was presented and summarized in [1, Theorem 5] as follows:

Theorem 1 [1, Theorem 5, Sect. 4, p. 1021] *For an open and dense subset of C^2 vector fields X having a singular hyperbolic attractor Λ on a 3-manifold, there exists a finite family Ξ of cross sections and a global (n-th return) Poincaré map $R : \Xi_0 \to \Xi$, $R(x) = X_{\tau(x)}(x)$ such that*

(1) the domain $\Xi_0 = \Xi \backslash \Gamma$ is the entire cross sections with a family Γ of finitely many smooth arcs removed, and $\tau : \Xi_0 \to [\tau_0, +\infty)$ is a smooth function bounded away from zero by some uniform constant $\tau_0 > 0$.

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(2) *We can choose coordinates on* Ξ *so that the map* R *can be written as* $F : \tilde{Q} \to Q$, $F(x, y) = (T(x), G(x, y))$, *where* $Q = \mathbb{I}\times\mathbb{I}$, $\mathbb{I} = [0, 1]$, *and* $\tilde{Q} = Q\backslash\Gamma_0$ *with* $\Gamma_0 = \mathcal{C}\times\mathbb{I}$ *and* $\mathcal{C} = \{c_1, \ldots, c_n\} \subset \mathbb{I}$ *a finite set of points.*

(3) *The map* $T : \mathbb{I}\backslash\mathcal{C} \to \mathbb{I}$ *is* $C^{1+\alpha}$ *piecewise monotonic with* $n + 1$ *branches defined on the connected components of* $\mathbb{I}\backslash\mathcal{C}$ *and has a finite set of a.c.i.m.,* μ^i_T. *Also* $\inf |T'| > 1$ *where it is defined,* $1/|T'|$ *has universal bounded p-variation, and then* $d\mu^i_T/dm$ *has bounded p-variation.*

(4) *The map* $G : \tilde{Q} \to \mathbb{I}$ *preserves and uniformly contracts the vertical foliation* $\mathcal{F} = \{\{x\}\times\mathbb{I}\}_{x\in\mathbb{I}}$ *of* Q*: There exists* $0 < \lambda < 1$ *such that* $\operatorname{dist}(G(x, y_1), G(x, y_2)) \le \lambda\cdot|y_1 - y_2|$ *for each* $y_1, y_2 \in \mathbb{I}$. *In addition, the map* G *satisfies a type of bounded variation regularity condition; see [1, Sect. 3].*

(5) *The map* F *admits a finite family of physical probability measures* μ^i_F *which are induced by* μ^i_T *in a standard way. The Poincaré time* τ *is integrable both with respect to each* μ^i_F *and with respect to the two-dimensional Lebesgue area measure of* Q.

(6) *Moreover if, for all singularities* $\sigma \in \Lambda$, *we have the eigenvalue relation* $-\lambda_2(\sigma) > \lambda_1(\sigma)$, *then the second coordinate map* G *of* F *has a bounded partial derivative with respect to the first coordinate, i.e., there exists* $C > 0$ *such that* $|\partial_x G(x, y)| < C$ *for all* $(x, y) \in (\mathbb{I}\backslash\{c_1, \ldots, c_n\}) \times \mathbb{I}$.

This construction was a modification of a similar construction in [2]. The modification was done to provide injectivity. Injectivity of the global Poincaré map is important for the arguments in [1] where the physical measure is shown to be exact dimensional.

As pointed to the authors by Fan Yang, the construction presented in the paper has a problem: Although the resulting global Poincaré map R is injective, the smoothness domains of this global Poincaré map might not be strips, that is, union of stable leaves crossing the cross section. This is needed for the rest of the arguments in the section, which rely on quotienting the dynamics of R over the stable leaves to obtain a one-dimensional piecewise expanding map of class $C^{1+\alpha}$, for some $\alpha > 0$.

The problem does not affect the first part of the paper, namely [1, Theorem A] on the decay of correlations of maps preserving a contracting foliation, but some changes are necessary for the application to singular hyperbolic flows in [1, Corollaries 1 & 2, p. 1006].

Corollary 1 *There exists an open dense set* $\mathcal{A}$ *of vector fields (satisfying a nonresonance condition) in* $SH^2(M^3)$ *such that, for each* $X \in \mathcal{A}$, *we can find a finite family* Ξ *of cross sections to the flow* X_t *of* X *such that an iterate of the Poincaré first return map* $F : dom(F) \subset \Xi \to \Xi$ *has a finite set of SRB measures* μ^i_F, *each of them has exponential decay of correlations with respect to Lipschitz observables: There are* $C, \Lambda \in \mathbb{R}^+$, $\Lambda < 1$ *satisfying for every pair* $f, g : \Xi \to \mathbb{R}$ *of Lipschitz functions*

$$\left|\int f\cdot(g\circ F^n)\,d\mu^i_F - \int g\,d\mu^i_F\int f\,d\mu^i_F\right| \le C\Lambda^n||g||_{Lip}||f||_{Lip}, \quad n \ge 1.$$

Corollary 2 *If* X_t *is a flow over a singular hyperbolic attractor in the setting of Corollary 1 and if, in addition, the eigenvalues of every equilibrium point* σ *of* X *in* Λ *satisfy* $\lambda_1(\sigma) + \lambda_2(\sigma) < 0$ *(which includes the classical Lorenz system of ODEs), then for each regular point* $x_0 \in \Lambda$ *such that* $d_{\mu_X}(x_0)$ *exists, we have*

$$\lim_{r\to 0}\frac{\log \tau_r^{X_t}(x, x_0)}{-\log r} = d_{\mu_X}(x_0) - 1$$

for μ_X*-almost each* $x \in \Lambda$.

In this note, we explain how the construction of the global Poincaré map R can be adjusted, in a similar way to the one originally presented in [2], to obtain [1, Corollary 1] (where the n-th return map F considered in the statement is replaced by the map R defined here) and [1, Corollary 2], which are the main results of the second part of [1].

1 Adjusting the construction of the global Poincaré map and recovering Corollary 1

To adjust the construction of the global Poincaré map, we keep Steps 1–3 in [1, Sect. 4.3], where we obtained a family Ξ_0 of adapted cross sections to the flow with flow-boxes giving an open cover of the attractor so that the Poincaré first return time between elements of Ξ_0 is bigger than some uniform positive constant. Consequently, we also have the properties stated in [1, Remark 12, p. 1031].

Now we change [1, Definition 7, p. 1032] of the global Poincaré map to be the same as in [2]:

$$R(z) = X_{\tau_0(X_T(z))}(X_T(z)), \quad z \in \Xi, \tag{1.1}$$

where $\tau_0(x)$ is the first hitting time of x to a cross section of Ξ, and $T > 0$ is a large threshold time (defined in the same Sect. 4.3 of [1]) ensuring Poincaré maps after this time are hyperbolic (more precisely, Proposition 7 and Lemma 6 from [1] simultaneously hold for R). The function

$$\tau(z) = \tau_0(X_T(z)) + T$$

is the global Poincaré time replacing the definitions in equation (4.19) of [1, p. 1032].

In this way, as shown in [2, Sect. 5], the global Poincaré map R satisfies all the properties in [1, Theorem 5] with the exception of R being a n-th return map, which is enough to prove [1, Corollary 1] using [1, Theorem A], as already written there.

2 Recovering Corollary 2

However, this adjustment does not guarantee that R is injective in general, since the orbit segment $[z, X_T(z)]$ can intersect several cross sections of Ξ, and this is the only way for injectivity to fail.

If injectivity does not hold, then the Steinberger relation from [1, Theorem 9, p. 1043] cannot be applied. Consequently, the proof of exact dimensionality of the singular hyperbolic flows presented in [1, Sect. 6, pp. 1042–1043] would not hold.

Moreover, the application to the results given in [1, Sect. 7] has the following problem: The almost everywhere existence of local dimension for the invariant measure might fail. We note, however, that [1, Corollary 2] deals only with target points x_0 where the local dimension exists.

We now show how to change [1, Sect. 7] adapting it to the return map which is considered here, in a way that [1, Corollary 2] is proved in the same form as it is stated.

The first step of the strategy implemented in [1, Sect. 7, p. 1043] is to obtain the logarithm law for the global first return map P of the cross section from the logarithm law of the (long) return map R.

We know that this latter logarithm law holds by applying [1, Proposition 11] since R has superpolynomial decay of correlations. We recall the statement of this result which is a general criteria to obtain the Logarithm law. In what follows, let $\tau_R(x, B_r(\overline{x_0}))$ be the number of R iterations needed for the orbit of x to enter in the target $B_r(\overline{x_0})$.

Proposition 1 [1, Proposition 11, p. 1044] *For each* x_0

$$\limsup_{r\to 0} \frac{\log \tau_R(x, B_r(x_0))}{-\log r} \geq \overline{d}_{\mu_R}(x_0)\ , \quad \liminf_{r\to 0} \frac{\log \tau_R(x, B_r(x_0))}{-\log r} \geq \underline{d}_{\mu_R}(x_0) \qquad (2.1)$$

hold for μ_R*-almost every* x.

Moreover, if the system has superpolynomial decay of correlations under Lipschitz observables and $d_\mu(x_0)$ *exists, then for* μ*-almost every* x*, it holds*

$$\lim_{r\to 0} \frac{\log \tau_R(x, B_r(x_0))}{-\log r} = d_{\mu_R}(x_0). \qquad (2.2)$$

This first step was done in [1, Remark 17, p. 1044], where the properties of the long return map are important. We separate our corrections into three parts, which apply for the definition of R here presented in (1.1) and lead to the same goal of the above-mentioned Remark 17 in [1].

2.1 Invariant measures associated with the first return map P and to R

Let $\Xi = \cup_i \Sigma_i$ be the global cross section and $P : \Xi \to \Xi$ be the its first return map.

Consider the suspension flow S_t^P over the cross section Ξ with roof function τ_0 (first return time function) and base transformation P with invariant measure μ_P such that the measure $\widetilde{\mu_P} = (\mu_P \times \mathrm{Leb})/\mu_P(\tau_0)$ on the suspension is conjugated to the physical measure μ of the original flow X_t.

More precisely, in the space $\Xi \times [0, +\infty)$ with the distance given by the maximum of the distances in Ξ (induced by the Riemannian distance from M) and in $\mathbb{R}$ (the Euclidean distance), we consider the relation $(x, \tau_0(x)) \sim_P (P(x), 0)$, extend it to be symmetric and transitive, and on the quotient space, we define $S_t^P(x, s) = (x, s + t), t \in \mathbb{R}$. Now, $\Phi_P : \Xi \times [0, +\infty)/ \sim_P \to M, (x, s) \mapsto X_s(x)$ is a diffeomorphism with its image such that $\Phi_P(S_t^P(x, s)) = X_t(\Phi_P(x, s))$. If μ_X is the physical measure for the flow, then we define $\widetilde{\mu_P} = (\Phi_P^{-1})_*\mu_X$ which is an S_t^P-invariant and ergodic probability measure. This measure induces a P-invariant ergodic probability μ_P such that $\widetilde{\mu_P} = (\mu_P \times \mathrm{Leb})/\mu_P(\tau_0)$.

Consider also the suspension flow S_t^R over the same space Ξ with roof function τ and associated return map R as base transformation with the physical invariant measure μ_R. On this suspension, the measure $\widetilde{\mu_R} = (\mu_R \times \mathrm{Leb})/\mu_R(\tau)$ is semiconjugated to the physical measure μ_X of the flow X_t, that is, $\mu_X = (\Phi_R)_*\widetilde{\mu_R}$. Indeed, $\Phi_R(S_t^R(x, s)) = X_t(\Phi_R(x, s))$ where $\Phi_R : \Xi \times [0, +\infty)/ \sim_R \to M$ has the same expression as before, the only difference being the equivalence relation $(x, \tau(x)) \sim_R (R(x), 0)$, which identifies $(R(x_1), 0) \sim_R (R(x_2), 0) \sim_R (x_1, \tau(x_1)) \sim_R (x_2, \tau(x_2))$ for x_1, x_2 in different cross sections with $R(x_1) = R(x_2)$. Hence, Φ_R is in general not injective: Points may have finitely many pre-images, as observed before.

From [1, Remark 12, item (2)], we know that Ξ can be constructed in a way that there exists $\varepsilon_1 > 0$ such that

$$\underline{\tau} = \inf\{t > 0 : X_t(x) \in \Xi, x \in \Xi\} \geq \varepsilon_1.$$

that is, the minimum time needed to go from one cross section $\Sigma_i \in \Xi$ to another $\Sigma_j \in \Xi$ by the flow is bounded away from zero. Consequently, $0 < \varepsilon_0 < \inf \tau_0 < \tau_0 < \tau$.

Hence, fixing $0 < t < \varepsilon_0$ and a measurable subset $A \subset \Xi$, the set $A \times [0, t]$ satisfies

$$\widetilde{\mu_P}(A \times [0,t]) = \mu_X\big(\Phi_P(A \times [0,t])\big) = \mu_X\big(X_{[0,t]}(A)\big)$$
$$= \widetilde{\mu_R}\Big(\Phi_R^{-1}\big(X_{[0,t]}(A)\big)\Big) \geq \widetilde{\mu_R}\big(A \times [0,t]\big)$$

and the inequality above is due to the lack of injectivity of R (and consequently of Φ_R). This means that

$$\frac{\mu_P(A)\cdot t}{\mu_P(\tau_0)} \geq \frac{\mu_R(A)\cdot t}{\mu_R(\tau)} \quad \text{thus} \quad \mu_P(A) \geq \mu_R(A)\frac{\mu_P(\tau_0)}{\mu_R(\tau)}.$$

Because A was arbitrary, we see that $\mu_P \geq \kappa \cdot \mu_R$ for some constant $\kappa > 0$.

2.2 Consequences for local dimension

From this, a P-invariant subset with full μ_R-measure must have full μ_P-measure. Indeed, let $A \subset \Xi$ be P-invariant such that $\mu_R(A) = 1$. Then, $\mu_P(A) > 0$ and $\mu_P(A) = 1$ since μ_P is P-ergodic. Also, from the inequality obtained in the last remark, we have that for all sufficiently small $\delta > 0$

$$\frac{\log(\mu_R(B_\delta(x)))}{\log\delta} \geq \frac{\log(\mu_P(B_\delta(x))/\kappa)}{\log\delta}$$

and so $\overline{d}_{\mu_R}(x) \geq \overline{d}_{\mu_P}(x)$ and $\underline{d}_{\mu_R}(x) \geq \underline{d}_{\mu_P}(x)$ for all $x \in \Xi$.

2.2.1 Local dimensions of μ_X, μ_P and μ_R

Let $x_0 \in M$ be such that $d_{\mu_X}(x_0) = \lim_{r\to 0} \frac{\log\mu_X(B_r(x_0))}{\log r}$ be well defined, and let $(x, s) \in \Xi \times [0, +\infty)/\sim_P$ be such that $\Phi_P(x, s) = x_0$ with $0 \leq s < \tau_0(x)$. Since both $\overline{d}_{\mu_X}$ and $\underline{d}_{\mu_X}$ are X_t-invariant, then the points where d_{μ_X} exists form another invariant subset. Hence, we can assume without loss that $s > 0$.

Lemma 1 *If d_{μ_X} exists at some point x_0 for the flow, then both d_{μ_R} and d_{μ_P} exist at some point $\overline{x_0}$ in the intersection of the orbit of x_0 with Ξ and are equal to $d_{\mu_X}(x_0) - 1$.*

Proof Since locally the distance in $\Xi \times [0, +\infty)/\sim_P$ is the maximum of the distances in Ξ and in $\mathbb{R}$, then we get $B_\delta(x, s) = B_\delta(x) \times (s - \delta, s + \delta)$ and

$$\widetilde{\mu_P}(B_\delta(x,s)) = \frac{1}{\mu_P(\tau_0)} \cdot (2\delta) \cdot \mu_P(B_\delta(x)).$$

Note that the distance in Ξ is induced by the distance in the manifold. Moreover, $\mu_X(B_r(x)) = \widetilde{\mu_P}\big(\Phi_P^{-1}(B_r(x))\big)$ and Φ_P is injective and locally a diffeomorphism. Hence, we can find $\kappa > 0$ such that

$$B_{r/\kappa}(x,s) \subset \Phi_P^{-1}(B_r(x)) \subset B_{\kappa r}(x,s)$$

for all small enough $r > 0$ (we can take $\kappa = \|D\Phi_P(x,s)\| + \|D\Phi_P(\Phi_P(x,s))^{-1}\|$). This ensures that

$$d_{\mu_X}(x_0) \geq \limsup_{r\to 0} \frac{\log\big((2\kappa r)\cdot\mu_P(B_{\kappa r}(x))/\mu_P(\tau_0)\big)}{\log r} \geq 1 + \overline{d}_{\mu_P}(x);$$

and also

$$d_{\mu_X}(x_0) \le \liminf_{r\to 0} \frac{\log\left((2r/\kappa)\cdot \mu_P(B_{r/\kappa}(x))/\mu_P(\tau_0)\right)}{\log r} \le 1 + \underline{d}_{\mu_P}(x).$$

Hence, $d_{\mu_P}(x) = d_{\mu_X}(x_0) - 1$ where x is the first hit of x_0 to Ξ in negative time.

Therefore, if $d_{\mu_X}(x_0)$ exists, then $d_{\mu_P}(x)$ exists with the value $d_{\mu_X}(x_0) - 1$ for all points $x \in \Xi$ in the same orbit of x_0.

Now for μ_R: since $\Phi_R : \Xi \times [0, +\infty)/\sim_R \to M$ is a finite-to-one local diffeomorphism almost everywhere, there are $i = i(x_0) \in \mathbb{Z}^+$, $(x_j, s_j) \in \Xi \times (0, +\infty)/\sim/R$ such that $\Phi_R(x_j, s_j) = x$ and $0 < s_j < \tau(x_j)$; $j = 1, \dots, i$.

Hence, we can find $\kappa' > 0$ such that we obtain the disjoint unions

$$\cup_{j=1}^{i} B_{r/\kappa'}(x_1, t_j) \subset \Phi_R^{-1}(B_r(x)) \subset \cup_{j=1}^{i} B_{\kappa' r}(x_1, t_j)$$

for all small enough $r > 0$ (we can take $\kappa' = \sum_{j=1}^{i} \|D\Phi_R(x_j, s_j)\| + \|D\Phi_R(\Phi_R(x_j, s_j))^{-1}\|$) implying the inequality

$$d_{\mu_X}(x_0) \ge \limsup_{r\to 0} \frac{\log\left(\sum_{j=1}^{i}(2\kappa' r)\cdot \mu_R(B_{\kappa' r}(x_j))/\mu_R(\tau)\right)}{\log r} \ge 1 + \min_{1\le j\le i} \overline{d}_{\mu_R}(x_j).$$

But, as already observed, we also have $\overline{d}_{\mu_R}(x_j) \ge \overline{d}_{\mu_P}(x_j)$ and $\underline{d}_{\mu_R}(x_j) \ge \underline{d}_{\mu_P}(x_j)$ for all $j = 1, \dots, i$. Hence, there exists $1 \le h \le i$ such that

$$\overline{d}_{\mu_R}(x_h) = \min_{1\le j\le i} \overline{d}_{\mu_R}(x_j) \le \underline{d}_{\mu_R}(x_h)$$

allowing us to conclude $\overline{d}_{\mu_R}(x_h) = \overline{d}_{\mu_P}(x_h) = \underline{d}_{\mu_R}(x_h) = \underline{d}_{\mu_P}(x_h)$. □

2.3 The logarithm law for the first return map

We will now prove a logarithm law for the first return map P for an exact dimensional target point.

2.3.1 The logarithm law for R implies the logarithm law for P

Let us suppose that $\overline{x_0}$ is a point as in the statement of Lemma 1 at the previous Sect. 2.2, for which the local dimension of the measures μ_R and μ_P coincides. Applying [1, Proposition 11], we know that there exists a full (μ_R and μ_P)-measure subset $A \subset \Xi$ such that

$$\limsup_{r\to 0} \frac{\log(\tau_R(x, B_r(\overline{x_0})))}{-\log r} \le d_{\mu_R}(\overline{x_0}), \quad x \in A.$$

Let also $\tau_P(x, B_r(\overline{x_0}))$ be the number of P iterations needed for the orbit of x to enter in the target $B_r(\overline{x_0})$. Note that if $y = R^n(x)$, then $y = P^m(x)$ with $m \le n\left\lceil \frac{T}{\tau} \right\rceil$ (one iteration of R corresponds at most to $\left\lceil \frac{T}{\tau} \right\rceil$ iterations of P, where $\lceil z \rceil$ is the least integer equal or greater than $z \in \mathbb{R}$). Then,

$$\tau_P(x, B_r(\overline{x_0})) \le \left\lceil \frac{T}{\tau} \right\rceil \tau_R(x, B_r(\overline{x_0})) \quad \text{and} \quad \limsup_{r\to 0} \frac{\log(\tau_P(x, B_r(\overline{x_0})))}{-\log r} \le d_{\mu_R}(\overline{x_0}).$$

By Sect. 2.1, this holds for μ_P-almost each x, and by the choice of $\overline{x_0}$, we have $d_{\mu_R}(\overline{x_0}) = d_{\mu_P}(\overline{x_0})$. Then, the logarithm law also holds for P.

2.4 The logarithm law for the flow

After the changes described in the previous Sects. 2.1, 2.2 and 2.3 of this erratum, we can continue the construction as in [1, Sect. 7, p. 1044]: By applying [1, Proposition 12] and [1, Proposition 13], we get the logarithm law for the flow for each target point where the local dimension exists, leading to [1, Corollary 2].

References

1. Araujo, V., Galatolo, S., Pacifico, M.J.: Decay of correlations for maps with uniformly contracting fibers and logarithm law for singular hyperbolic attractors. Math. Zeitschrift **276**(3–4), 1001–1048 (2014)
2. Araújo, V., Pujals, E.R., Pacifico, M.J., Viana, M.: Singular-hyperbolic attractors are chaotic. Trans. A.M.S. **361**, 2431–2485 (2009)