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Decay of Solutions of Wave Equation in a Star-Shaped Domain with Nonlinear Boundary Feedback

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Abstract

We study the uniform stabilization of the wave equation by means of a nonlinear dissipative boundary feedback. We consider a Neumann condition on the whole boundary, and the observation is the boundary displacement and velocity. Extending a result of E. Zuazua, we obtain, in a nonlinear framework, estimates of the decay, for any displacement. We establish a similar result for the one-dimensional wave equation with a variable coefficient.

Stabilisation de l'Equation des Ondes dans un Domaine Etoilé au Moyen d'un Contrôle Frontière non Linéaire

Résumé

On étudie la stabilisation uniforme de l'équation des ondes au moyen d'un feedback frontière dissipatif non linéaire. On considère une condition de Neumann sur tout le bord, l'observation étant la vitesse et le déplacement. On généralise un résultat de E. Zuazua, en obtenant, dans le cadre non linéaire, des estimations de décroissance quelle que soit l'amplitude du déplacement. On établit un résultat analogue pour l'équation des ondes monodimensionnelle, à coefficient variable.

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1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^n$, strictly star-shaped with respect to $x_0 \in \Omega$ and having smooth boundary $\Gamma = \partial\Omega$ of class C^2. We define the mapping m: $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ by

(1.1) $m(x) = x - x^0$.

For all $x \in \Gamma$, we have $m(x).\nu(x) > 0$, where $\nu(x)$ is the outer unit normal vector to Γ at $x \in \Gamma$, and " . " denotes the scalar product in $\mathbb{R}^n$. Since Ω is bounded with smooth boundary Γ, there is a positive constant $\delta > 0$, suth that

(1.2) $m(x).\nu(x) > \delta^{-1}$, $\forall x \in \Gamma$.

We denote by $\dfrac{\partial}{\partial\nu}$ the normal derivative and by " ' " the time-derivative $\dfrac{d}{dt}$.

Let us consider the following wave equation

(1.3) $y'' - \Delta y = 0$ in $\Omega \times (0, \infty)$,

(1.4) $\dfrac{\partial y}{\partial\nu} + \alpha\, y = -\, g(y')$ on $\Gamma \times (0, \infty)$,

(1.5) $y(0) = y_0 \in H^1(\Omega)$, $y'(0) = y_1 \in L^2(\Omega)$,

where $\alpha \in L^\infty(\Gamma)$ satisfies $\alpha(x) \geq \alpha_0 > 0$, $\forall x \in \Gamma$, and g: $\mathbb{R} \longrightarrow \mathbb{R}$ is a continuous nondecreasing function such that

(1.6) $g(0) = 0$, $g(s)s > 0$, $|g(s)| \leq C(1 + |s|)$, for all $s \neq 0$.

It is well-known that for any initial data $(y_0, y_1) \in H^1(\Omega) \times L^2(\Omega)$, the equations (1.3)-(1.5) have a unique weak solution y such that

(1.7) $y \in C^0(\mathbb{R}^+, H^1(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega))$.

We define the energy associated with a solution y of the equations (1.3)-(1.5) as follows:

(1.8) $E(t) = \dfrac{1}{2} \int_\Omega (\,|\nabla y|^2 + |y'|^2)dx\ + \dfrac{1}{2}\int_\Gamma \alpha y^2 d\Gamma$.

By Green's formula, we obtain (at least formally at this stage)

(1.9) $\dfrac{d}{dt} E(t) = E'(t) = -\int_\Gamma g(y')y' d\Gamma \leq 0$.

The assumption (1.6) implies that the energy E(t) is nonincreasing and defines a Lyapunov function. It is easy to prove the strong stabilization by applying Lasalle's invariance principle (see [5], [17]). Indeed, in the more general cases for which g is a maximal monotone graph, the strong asymptotic stabilization has also been proved ([4], [13], [14]). Our aim in the present paper is to estimate the rate of decay of the energy E(t), when the nonlinear feedback function g satisfies suitable growth conditions.

The problem of estimating the rate of decay of the energy E(t) has been studied extensively by many authors, among which we mention Lagnese and Chen. Recently, Zuazua [19] using a method of multipliers, established estimates in the case where $\alpha > 0$ is a small constant. In the case of a linear feedback, he also proved the uniform exponential decay for any $\alpha > 0$ by a compactness-uniqueness argument [18] (see also [12]).

We obtain here an improvement of these results in the sense that we prove the estimates *for any positive* $\alpha \in L^\infty(\Gamma)$ *in the case of nonlinear feedback* g. Also, in the one dimensional case, we establish the estimates for the wave equation with *variable coefficient*.

Our two main results are as follows:

Theorem 1. Let Ω be a bounded domain of $\mathbb{R}^n$ strictly star-shaped with respect to $x_0 \in \Omega$ and having smooth boundary $\Gamma = \partial\Omega$ of class C^2. Assume that g is a continuous nondecreasing function such that (1.6) holds. Then for every solution y of the equations (1.3)-(1.5), we have:

(i) If there exist some positive constants C_1, C_2 such that

(1.10) $C_1 |s| \leq |g(s)| \leq C_2 |s|$, $\forall s \in \mathbb{R}$,

then given any constant M > 1, there exists a constant $\lambda > 0$ such that
(1.11) $E(t) \leq M E(0) e^{-\lambda t}$, $\forall t \geq 0$.

(ii) If there exist some positive constants C_1, C_2 and p > 1 such that

(1.12) $C_1 Min(|s|, |s|^p) \leq |g(s)| \leq C_2 |s|$, $\forall s \in \mathbb{R}$,

then given any constant M > 1, there exists a constant $\mu > 0$ depending on E(0) such that
(1.13) $E(t) \leq M E(0) (1 + \mu t)^{\frac{-2}{p-1}}$, $\forall t \geq 0$.

Theorem 2. Let Ω be a bounded domain of $\mathbb{R}^n$ strictly star-shaped with respect to $x_0 \in \Omega$ and having smooth boundary $\Gamma = \partial\Omega$ of class C^2. Assume that g is a continuous nondecreasing function such that (1.6) holds. Let y be any solution of the equations (1.3)-(1.5). If there exist some positive constants C_1, C_2 and $p < 1$ such that

$$(1.14) \qquad C_1|s| \leq |g(s)| \leq C_2 \mathrm{Max}(|s|, |s|^p), \quad \forall\, s \in \mathbb{R},$$

then given any constant $M > 1$, there exists a constant $\mu > 0$ depending on E(0) such that

$$(1.15) \qquad E(t) \leq M\, E(0)\, (1 + \mu t)^{\frac{-2p}{1-p}}, \qquad \forall\, t \geq 0.$$

2. Proof of the main results

For the sake of completeness, we start with a brief outline of the well-posedness of the equations (1.3)-(1.5) by the semi-group approach.

Let us define

(2.1) $\qquad H = H^1(\Omega) \times L^2(\Omega)$

with the scalar product

(2.2) $\qquad <(y,\hat{y}),(z,\hat{z})> = \int_\Omega (\nabla y \nabla z + \hat{y}\hat{z})dx + \int_\Gamma \alpha yz d\Gamma$

We introduce the nonlinear operator A on H :

(2.3) $\qquad D(A) = \{(y, \hat{y}) \in H^1(\Omega) \times H^1(\Omega); \Delta y \in L^2(\Omega); \dfrac{\partial y}{\partial \nu} + \alpha\, y = - g(\hat{y})$ on $\Gamma\}$

which is dense in H, and for any $(y, \hat{y}) \in D(A)$,

(2.4) $\qquad A(y, \hat{y}) = (-\hat{y}, -\Delta y\,).$

Remark 2.1 Here Δy is defined in the sense of distributions. We see that the normal derivative $\dfrac{\partial y}{\partial \nu}$ is not defined by the usual trace theorem. The equality $\dfrac{\partial y}{\partial \nu} + \alpha y = - g(\hat{y})$ on Γ means that

(2.5) $\qquad \int_\Omega \Delta y\phi dx + \int_\Omega \nabla y \nabla\phi dx + \int_\Gamma \alpha y\phi d\Gamma + \int_\Gamma g(\hat{y})\phi d\Gamma = 0, \forall\phi \in H^1(\Omega).$

However, when both definitions of the traces make sense, they coincide.

By means of the operator A, we transform the equations (1.3)-(1.5) into the following operational form:

(2.6) $\qquad (y, \hat{y}\,)' + A(y, \hat{y}) = 0, \quad (y(0), \hat{y}(0)\,) = (y_0, y_1\,) \in H.$

It is easy to prove that A is a maximal monotone operator on H. Using the methods of Brezis [2], Haraux [8] or Lions and Magenes [16], we deduce that for any initial data (y_0, y_1) $\in D(A)$, the equation (2.6) has a unique strong solution y such that

$$(2.7) \qquad y \in W^{1,\infty}(\mathbb{R}^+, H^1(\Omega)), \quad \Delta y \in L^\infty(\mathbb{R}^+, L^2(\Omega)),$$

$$(2.8) \qquad \|(y(t), \hat{y}(t))\|_H \leq \|(y_0, y_1)\|_H, \quad \forall\, t \geq 0.$$

Furthermore, since D(A) is dense in H for the strong topology, we can extend the mapping

$$(2.9) \qquad (y_0, y_1) \to (y(t), \hat{y}(t))$$

to a strongly continuous semi-group of contractions $(S(t))_{t\geq 0}$ in H. Thus for every initial data $(y_0, y_1) \in H$, we can define the unique weak solution of the equation (2.6) by the following formula:

$$(2.10) \qquad (y(t), \hat{y}(t)) = S(t)\,(y_0, y_1), \quad \forall\, t \geq 0$$

with the regularity

$$(2.11) \qquad y \in C^0(\mathbb{R}^+, H^1(\Omega)) \cap C^1(\mathbb{R}^+, L^2(\Omega)).$$

By the density of D(A) in H and the fact that $(S(t))_{t\geq 0}$ is a strongly continuous semi-group of contractions in H, it is sufficient to prove Theorems 1 and 2 for smooth initial data $(y_0, y_1) \in D(A)$. Therefore we assume in the sequel that the regularity (2.7) holds.

Let us recall some technical results:

Lemma 2.2 Let Ω be a bounded domain of $\mathbb{R}^n$, with smooth boundary $\Gamma = \partial\Omega$ of class C^2. Then there exists a constant $\beta > 0$ suth that for any $y \in H^1(\Omega)$, the solution ϕ of the equation

$$(2.12) \qquad \begin{cases} -\Delta\phi = 0 & \text{in } \Omega, \\ \phi = y & \text{on } \Gamma, \end{cases}$$

satisfies the following estimates:

(2.13) $\int_\Omega |\phi|^2 dx \leq \beta^2 \int_\Gamma |y|^2 d\Gamma,$

(2.14) $\int_\Omega \nabla\phi\nabla y dx \geq 0.$

Proof. Inequality (2.13) is standard in the theory of elliptic equations. In order to prove (2.14), it is enough to multiply (2.12) respectively, by ϕ and y. By comparison, it follows that

$$\int_\Omega \nabla\phi\nabla y \, dx = \int_\Omega |\nabla\phi|^2 \, dx \ .$$

Remark 2.3 It is clear that problem (2.12) is equivalent to the following minimization problem:

(2.15) $|\nabla\phi|_{L^2(\Omega)} = \underset{\substack{\psi\in H^1(\Omega), \\ \psi = y \text{ on } \Gamma}}{\text{Inf}} |\nabla\psi|_{L^2(\Omega)}$

So ϕ is the function which takes the same values as y on the boundary Γ and has least norm $|\nabla\phi|_{L^2(\Omega)}$. This function plays an important role in the proof of Theorem 1 (see Remark 2.6).

Lemma 2.4 Let Ω be a bounded domain of $\mathbb{R}^n$ strictly star-shaped with respect to $x_0 \in \Omega$ and having smooth boundary $\Gamma = \partial\Omega$ of class C^2. Then the following inequality holds:

(2.16) $2 \int_\Omega \Delta y(m.\nabla y)dx \leq (n-2) \int_\Omega |\nabla y|^2 dx + R^2\delta \int_\Gamma |v|^2 d\Gamma$

for any function $y\in H^1(\Omega)$ such that $\Delta y\in L^2(\Omega)$ and $\dfrac{\partial y}{\partial v} = v \in L^2(\Gamma)$. In (2.16), we have set R $= \|m\|_{L^\infty(\Gamma)}$.

Proof. First, we assume $\dfrac{\partial y}{\partial v} = v \in H^{\frac{1}{2}}(\Gamma)$. In this case, we know that y $\in$ H²(Ω), and the following identity due to Rellich (see for instance [12]) holds :

(2.17) $2 \int_\Omega \Delta y(m.\nabla y)dx = (n-2) \int_\Omega |\nabla y|^2 dx$

$$+ 2 \int_\Gamma \frac{\partial y}{\partial v}(m.\nabla y)d\Gamma - \int_\Gamma (m.v) |\nabla y|^2 d\Gamma, \quad \text{for all } y \in H^2(\Omega).$$

By the geometrical condition (1.2), we deduce from (2.17) and Young's inequality applied to $\int_\Gamma \frac{\partial y}{\partial v}(m.\nabla y)d\Gamma$ that

$$(2.18) \qquad 2 \int_\Omega \Delta y(m.\nabla y)dx \leq (n-2) \int_\Omega |\nabla y|^2 dx + R^2\delta \int_\Gamma |v|^2 d\Gamma \quad \text{for all } y \in H^2(\Omega).$$

Now, consider $v \in L^2(\Gamma)$. By density, there exists a sequence $v_k \in H^{\frac{1}{2}}(\Gamma)$ such that $v_k \to v$ in $L^2(\Gamma)$. Let y_k be the solution of the equation

$$\begin{cases} y_k - \Delta y_k = y - \Delta y & \text{in } \Omega, \\ \dfrac{\partial y_k}{\partial v} = v_k & \text{on } \Gamma. \end{cases}$$

Then (2.18) is valid for (v_k, y_k). By the classical regularity theory, we know that

$\|y - y_k\|_{H^1(\Omega)} \leq C \|v - v_k\|_{H^{-1/2}(\Gamma)}$. So $y_k \to y$ in $H^1(\Omega)$ and since $\Delta y_k \to \Delta y$ in $L^2(\Omega)$, we obtain (2.16) for (v, y) by passing to the limit in (2.18).

Now, let us introduce the functional

$$(2.19) \qquad \rho(t) = 2 \int_\Omega y'(m.\nabla y)dx + (n-1) \int_\Omega y'ydx + C_0 \int_\Omega y'\varphi dx$$

where φ is the solution of equation (2.12) and C_0 is a positive constant to be determined later on (see (2.30)).

We note that the functional $\rho(t)$ is a slight, but essential modification of the one introduced in Zuazua ([19]), by adding the last integral term. The idea of the proof of the following lemma is to estimate the integral $\int_\Omega y'\varphi'dx$. We show that with a suitable estimate

of this term, we can overcome the difficulty caused by the nonlinear feedback and by the amplitude of the function α. This point seems to be new.

We have the following important result:

Lemma 2.5 Under the assumptions of Theorem 1, there exist some positive constants C_3, C_4 and C_5 such that the following estimates hold:

$$(2.20) \qquad |\rho(t)| \le C_3 E(t)$$

$$(2.21) \qquad \rho'(t) \le - E(t) + C_4 \int_\Gamma |y'|^2 d\Gamma + C_5 \int_\Gamma g^2(y') d\Gamma$$

for any solution of equations (1.3)-(1.5).

Proof First of all, it is easy to deduce (2.20) from (2.19) by applying the Cauchy–Schwarz inequality and (2.13). The constant C_3 is given by

$$(2.22) \qquad C_3 = 2R + (n-1)\gamma + \frac{C_0 \beta}{\sqrt{\alpha_0}}$$

where γ is the best constant such that

$$(2.23) \qquad \int_\Omega \psi^2 dx \le \gamma^2 \left\{ \int_\Omega |\nabla \psi|^2 dx + \int_\Gamma \alpha |\psi|^2 d\Gamma \right\}, \quad \forall \psi \in H^1(\Omega).$$

Next, we calculate the derivative $\rho'(t)$

$$(2.24) \qquad \rho'(t) = -C_0 \int_\Omega \nabla y \nabla \varphi dx + C_0 \int_\Omega y' \varphi' dx + 2 \int_\Omega \Delta y (m.\nabla y) dx - (n-1) \int_\Omega |\nabla y|^2 dx$$

$$- \int_\Omega |y'|^2 dx + (n-1+C_0) \int_\Gamma \frac{\partial y}{\partial \nu} y d\Gamma + \int_\Gamma (m.\nu) |y'|^2 d\Gamma.$$

We deduce from (2.14) that the first term of the right-hand side of (2.24) is negative ($C_0 \ge 0$). For the second term of (2.24), by applying Young's inequality, we deduce from (2.13) that

$$(2.25) \quad C_0 \int_\Omega y'\varphi' dx \le \frac{1}{2}\int_\Omega |y'|^2 dx + \frac{C_0^2}{2} \int_\Omega |\varphi'|^2 dx \le \frac{1}{2}\int_\Omega |y'|^2 dx + \frac{(C_0\beta)^2}{2} \int_\Gamma |y'|^2 d\Gamma.$$

In order to apply Lemma 2.4 to estimate the third term of the right–hand side of (2.24), we need to verify that $-\{\alpha y + g(y')\} \in L^2(\Gamma)$. This is obvious, since $y' \in L^\infty(\mathbb{R}^+, H^1(\Omega))$ and $|g(s)| \le C\{1 + |s|\}$ (see(1.6) and (2.7)).

Combining (2.16), (2.24) and (2.25), we deduce that

$$(2.26) \quad \rho'(t) \le -\frac{1}{2}\int_\Omega (|\nabla y|^2 + |y'|^2)dx + \left(R + \frac{(C_0\beta)^2}{2}\right) \int_\Gamma |y'|^2 d\Gamma$$

$$+ \theta R^2\delta \int_\Gamma |\frac{\partial y}{\partial v}|^2 d\Gamma + (n-1+C_0) \int_\Gamma \frac{\partial y}{\partial v} y\, d\Gamma$$

where $\theta \ge 1$ is a constant to be determined later on (see(2.29)).

On the other hand, taking into account the boundary condition (1.4), we have

$$(2.27) \quad \int_\Gamma |\frac{\partial y}{\partial v}|^2 d\Gamma \le -\|\alpha\| \int_\Gamma \alpha |y|^2 d\Gamma - 2\|\alpha\| \int_\Gamma \frac{\partial y}{\partial v} y\, d\Gamma + \frac{\|\alpha\|}{\alpha_0} \int_\Gamma g^2(y')d\Gamma$$

where we have set $\|\alpha\| = \|\alpha\|_{L^\infty(\Gamma)}$.

Plugging (2.27) into (2.26), we obtain

$$(2.28) \quad \rho'(t) \le -\frac{1}{2}\int_\Omega (|\nabla y|^2 + |y'|^2)dx - \|\alpha\|\theta R^2\delta \int_\Gamma \alpha |y|^2 d\Gamma + \theta R^2\delta\frac{\|\alpha\|}{\alpha_0} \int_\Gamma g^2(y')d\Gamma$$

$$+ \left(R + \frac{(C_0\beta)^2}{2}\right) \int_\Gamma |y'|^2 d\Gamma + (n-1+C_0 -2\theta R^2\delta\|\alpha\|) \int_\Gamma \frac{\partial y}{\partial v} y\, d\Gamma.$$

Finally, with the following choices of constants θ and C_0:

$$(2.29) \quad \theta = \text{Max}\left\{ 1, \frac{1}{2R^2\delta\|\alpha\|}, \frac{n-1}{2R^2\delta\|\alpha\|}\right\}$$

$$(2.30) \quad C_0 = 2\theta R^2\delta\|\alpha\| - (n-1) \ge 0$$

we deduce from (2.28) that

$$(2.21) \qquad \rho'(t) \leq - E(t) + C_4 \int_\Gamma |y'|^2 d\Gamma + C_5 \int_\Gamma g^2(y')d\Gamma$$

where $C_4 = R + \dfrac{(C_0\beta)^2}{2}$, and $C_5 = \theta R^2 \delta \dfrac{\|\alpha\|}{\alpha_0}$.

The proof of Lemma 2.5 is now completed.

Remark 2.6 The proof of Lemma 2.5 consists of two essential steps:

(i) Estimation of the term $\ - C_0\int_\Omega \nabla y \nabla \varphi dx$.

If, as in the standard procedure, we apply the Cauchy-Schwarz inequality to the integral term $- C_0\int_\Omega \nabla y \nabla \varphi dx$, we have to control this term by $C_0\int_\Omega |\nabla y|^2 dx$. But we find immediately that this estimation requires the smallness of the constant C_0. This is just what we want to avoid. Fortunately, we can omit the term $-C_0\int_\Omega \nabla y \nabla \varphi dx$, which is negative by virtue of (2.14). Furthermore, we know that $\int_\Omega \nabla y \nabla \varphi dx \leq (\int_\Omega |\nabla y|^2 dx)^{1/2}.(\int_\Omega |\nabla \varphi|^2 dx)^{1/2} \leq \int_\Omega |\nabla y|^2 dx$, because φ is the function which has the least norm and takes the same values as y on the boundary Γ (see Remark 2.3).

(ii) Estimation of the term $\ C_0\int_\Omega y'\varphi' dx$.

Using Young's inequality, this term can be dominated by the sum of the two terms $\dfrac{1}{2}\int_\Omega |y'|^2 dx + \dfrac{C_0^2}{2}\int_\Omega |\varphi'|^2 dx$. The first one with a suitable coefficient (i.e. 1/2) can be easily handled. Therefore, we have to affect to the second one an arbitrary coefficient (i.e. $C_0^2/2$). By means of (2.13), we can transform $\int_\Omega |\varphi'|^2 dx$ into a integral term on the boundary $\int_\Gamma |\varphi'|^2 d\Gamma$. In other words, for estimating the term $\int_\Omega y'\varphi' dx$, we distinguish the roles of the functions y' and φ', and we evaluate $\int_\Omega |\varphi'|^2 dx$ by $\int_\Gamma |\varphi'|^2 d\Gamma$. If we replace the function φ by y in (2.19), instead of $-C_0\int_\Omega \nabla y \nabla \varphi dx$, we obtain $-C_0\int_\Omega |\nabla y|^2 dx$ which is also negative. There is no problem in step (i). However, in place of $C_0\int_\Omega y'\varphi' dx$, we have to consider $C_0\int_\Omega |y'|^2 dx$. We meet again the troublesome problem of the smallness of C_0.

Now, we see clearly why we have been able to control the term $\int_\Omega y'\varphi' dx$: we have chosen the function φ as the second multiplier which allows us to evaluate the domain integral $\int_\Omega |\varphi'|^2 dx$ by the boundary integral $\int_\Gamma |y'|^2 d\Gamma$. ·

Remark 2.7 We have introduced the constant $\theta \geq 1$ for a technical reason. In fact, if we take $\theta = 1$, then for any α such that $\|\alpha\| \geq (n-1)/2\delta R^2$, we have $C_0 \geq 0$. Thus Lemma 2.5 is true. In the opposite case: $0 < \|\alpha\| < (n-1)/2\delta R^2$, Theorem 1 has been proved by Zuazua in [19].

Proof of Theorem 1

From now on, the method which we will adopt is analogous to the one used in [19]. Let us recall some essential steps for the sake of completeness.

First, by the conditions (1.10) and (1.12), we deduce from (2.21) that

$$(2.31) \qquad \rho'(t) \leq - E(t) + C_6 \int_\Gamma |y'|^2 d\Gamma$$

with the constant $C_6 = C_4 + C_2^2 C_5$.

Next, for $\varepsilon > 0$, we introduce a perturbed energy E_ε

$$(2.32) \qquad E_\varepsilon(t) = E(t) + \varepsilon (E(t))^{\frac{p-1}{2}} \rho(t)$$

Since the energy is nonincreasing, given any $M > 1$, we show easily that the following inequalities hold

$$(2.33) \qquad M^{\frac{-1}{2}} (E_\varepsilon(t))^{\frac{p+1}{2}} \leq (E(t))^{\frac{p+1}{p}} \leq M^{\frac{1}{2}} (E_\varepsilon(t))^{\frac{p+1}{2}}$$

provided ε is chosen such that

$$(2.34) \qquad \varepsilon \leq C_3^{-1} (E(0))^{\frac{1-p}{2}} (1 - M^{-\frac{1}{p+1}}) = C_7 (E(0))^{\frac{1-p}{2}}$$

Now, we calculate the derivative of the perturbed energy $E_\varepsilon(t)$

$$(2.35) \qquad E_\varepsilon'(t) = E'(t) + \varepsilon \frac{p-1}{2} (E(t))^{\frac{p-3}{2}} E'(t)\rho(t) + \varepsilon(E(t))^{\frac{p-1}{2}} \rho'(t).$$

Combining (1.9), (2.20), (2.31) and (2.35), we get:

$$(2.36) \quad E_\varepsilon'(t) \le \left(-1 + \varepsilon \frac{p-1}{2} C_3 (E(0))^{\frac{p-1}{2}}\right) \int_\Gamma g(y')y'd\Gamma + \varepsilon C_6 (E(t))^{\frac{p-1}{2}} \int_\Gamma |y'|^2 d\Gamma - \varepsilon(E(t))^{\frac{p+1}{2}}.$$

On the other hand, by (1.10) and (1.12), we deduce that

$$(2.37) \qquad \varepsilon C_6 (E(t))^{\frac{p-1}{2}} \int_{\{|y'|\ge 1\}} |y'|^2 d\Gamma \le \left(1 - \varepsilon \frac{p-1}{2} C_3 (E(0))^{\frac{p-1}{2}}\right) \int_{\{|y'|\ge 1\}} g(y')y'd\Gamma$$

if we choose $\varepsilon > 0$ such that

$$(2.38) \qquad \varepsilon \le C_1 (E(0))^{\frac{1-p}{2}} \left(C_1 C_3 \frac{p-1}{2} + C_6\right)^{-1} = C_8 (E(0))^{\frac{1-p}{2}}.$$

From (2.36) and (2.37), it follows:

$$(2.39) \qquad E_\varepsilon'(t) \le \left(-1 + \varepsilon \frac{p-1}{2} C_3 (E(0))^{\frac{p-1}{2}}\right) \int_{\{|y'|\le 1\}} g(y')y'd\Gamma +$$

$$\varepsilon C_6 (E(t))^{\frac{p-1}{2}} \int_{\{|y'|\le 1\}} |y'|^2 d\Gamma - \varepsilon(E(t))^{\frac{p+1}{2}}.$$

Now we distinguish the cases $p = 1$ and $p > 1$.

(i) case $p = 1$. In this case (1.10), (2.33), (2.38) and (2.39) yield:

$$(2.40) \qquad E_\varepsilon'(t) \le -\varepsilon E(t) \le -\varepsilon M^{\frac{-1}{2}} E_\varepsilon(t),$$

consequently, we get:

$$E_\varepsilon(t) \le E_\varepsilon(0) \, e^{-\varepsilon M^{\frac{-1}{2}} t}, \qquad\qquad \forall\, t \ge 0,$$

which, together with (2.33), implies:

$$(2.41) \qquad E(t) \le M\, E(0)\, e^{-\varepsilon M^{\frac{-1}{2}} t} = M\, E(0)\, e^{-\lambda t}, \qquad \forall\, t \ge 0,$$

where we have set $\lambda = \varepsilon\, M^{-\frac{1}{2}}$.

From (2.34) and (2.38), we observe that the constant ε does not depend on E(0), hence the constant λ defined above does not depend on E(0) too. This will not be the case when $p > 1$ (see Remark 2.9).

(ii) case $p > 1$. From (2.39) and by Young's inequality, we have:

$$(2.42) \qquad E_\varepsilon'(t) \le \left(-1 + \varepsilon \frac{p-1}{2} C_3(E(0))^{\frac{p-1}{2}}\right) \int\limits_{\{\,|y'|\le 1\}} g(y')y'\,d\Gamma \;+$$

$$\varepsilon(2C_6)^{\frac{p-1}{2}} \left(\int\limits_{\{\,|y'|\le 1\}} |y'|^2 d\Gamma\right)^{\frac{p+1}{2}} - \frac{\varepsilon}{2}(E(t))^{\frac{p+1}{2}}.$$

We apply Hölder's inequality with the exponents $\alpha = \dfrac{p+1}{p-1}$, $\alpha' = \dfrac{p+1}{2}$ to the second term of the right-hand side of (2.42), it follows:

$$\left(\int\limits_{\{\,|y'|\le 1\}} |y'|^2 d\Gamma\right)^{\frac{p+1}{2}} \le \left(\int\limits_{\{\,|y'|\le 1\}} d\Gamma\right)^{\frac{p-1}{2}} \cdot \int\limits_{\{\,|y'|\le 1\}} |y'|^{p+1} d\Gamma \le \left(\text{meas } \Gamma\right)^{\frac{p-1}{2}} \cdot \int\limits_{\{\,|y'|\le 1\}} |y'|^{p+1} d\Gamma.$$

The above estimate, together with (1.12) and (2.42), implies the following inequality:

$$E_\varepsilon'(t) \le \left(-1 + \varepsilon \frac{p-1}{2} C_3(E(0))^{\frac{p-1}{2}}\right) \int\limits_{\{\,|y'|\le 1\}} g(y')y'\,d\Gamma \;+$$

$$\varepsilon C_1^{-1}(2C_6)^{\frac{p-1}{2}} \left(\text{meas } \Gamma\right)^{\frac{p-1}{2}} \int\limits_{\{\,|y'|\le 1\}} g(y')y'\,d\Gamma \; - \frac{\varepsilon}{2}(E(t))^{\frac{p+1}{2}}.$$

We have thus proved

(2.43) $E_\varepsilon'(t) \le -\frac{\varepsilon}{2} E(t)^{\frac{p+1}{2}}$, $\forall \, t \ge 0$,

provided ε satisfies:

(2.44) $\varepsilon \le C_1\left(C_1 C_2\frac{p-1}{2}(E(0))^{\frac{p-1}{2}} + (2C_6)^{\frac{p-1}{2}}\left(\text{meas } \Gamma\right)^{\frac{p-1}{2}}\right)^{-1} =$

$$(E(0))^{\frac{1-p}{2}}\left(C_9 + C_{10}(E(0))^{\frac{1-p}{2}}\right)^{-1}.$$

Combining (2.33) and (2.43), we get:

(2.45) $E_\varepsilon'(t) \le -\frac{\varepsilon}{2} M^{-\frac{1}{2}}(E_\varepsilon(t))^{\frac{p+1}{2}}$

Finally, solving the differential inequality (2.45) and using (2.33), we obtain

(2.46) $E(t) \le M \, E(0) \left\{ 1 + t\frac{\varepsilon}{4}\frac{p-1}{p} M^{p+1}(E(0))^{\frac{-p}{2}} \right\}^{\frac{p-1}{-2}} = M \, E(0) \left\{ 1 + \mu t\right\}^{\frac{-2}{p-1}}.$

The proof of Theorem 1 is thus completed.

Remark 2.9 The constant ε is chosen such that the restrictions (2.34) (2.38) and (2.44) are satisfied. In fact, we can choose ε as follows:

(2.47) $\varepsilon = \text{Min}\left(C_7 , \, C_8 , \left(C_9 + C_{10}(E(0))^{\frac{1-p}{2}}\right)^{-1}\right)(E(0))^{\frac{1-p}{2}}$

with constants C_7, C_8, C_9 and C_{10} independent of $E(0)$. In particular, we conclude that ε
and consequently $\mu = \frac{\varepsilon(p-1)}{4} M^{-\frac{p}{p+1}} (E(0))^{\frac{p-1}{2}}$, depend continuously on $E(0)$.

Proof of Theorem 2
First, by the conditions (1.14) and (2.21), we have

(2.48) $\rho'(t) \le - E(t) + C_6 \int_\Gamma g^2(y')d\Gamma$

with the constant $C_6 = C_5 + \dfrac{C_4}{C_1^2}$.

Next, for $\varepsilon > 0$, we introduce the perturbed energy E_ε

$$(2.49) \qquad E_\varepsilon(t) = E(t) + \varepsilon \, (E(t))^{\frac{1-p}{2p}} \, \rho(t).$$

Then, for any $M > 1$, we have

$$(2.50) \qquad M^{\frac{-1}{2}} (E_\varepsilon(t))^{\frac{p+1}{2p}} \le (E(t))^{\frac{p+1}{2p}} \le M^{\frac{1}{2}} (E_\varepsilon(t))^{\frac{p+1}{2p}}$$

provided ε satisfies

$$(2.51) \qquad \varepsilon \le C_3^{-1} (E(0))^{\frac{p-1}{2p}} (1 - M^{\frac{p-1}{2p}}) = C_7 (E(0))^{\frac{p-1}{2p}}.$$

We have also

$$(2.52) \qquad E_\varepsilon'(t) = E'(t) + \varepsilon \frac{1-p}{2p} (E(t))^{\frac{1-3p}{2p}} E'(t)\rho(t) + \varepsilon(E(t))^{\frac{1-p}{2p}} \rho'(t).$$

Combining (1.9), (2.20), (2.48), and (2.52), we get

$$(2.53) \qquad E_\varepsilon'(t) \le \left\{ -1 + \varepsilon \frac{1-p}{2p} C_3 (E(0))^{\frac{1-p}{2p}} \right\} \int_\Gamma g(y')y' d\Gamma +$$

$$\varepsilon \, C_6 \, (E(t))^{\frac{1-p}{2p}} \int_\Gamma g^2(y') d\Gamma - \varepsilon(E(t))^{\frac{1+p}{2p}}.$$

By the conditions (1.14) we have

$$(2.54) \qquad |g(s)|^2 \le C_2 g(s)s, \qquad\qquad \text{for } |s| \ge 1$$

$$(2.55) \qquad |g(s)|^2 \le (C_2)^{\frac{2}{p+1}} (g(s)s)^{\frac{2p}{p+1}}, \qquad \text{for } |s| < 1.$$

We deduce from (2.54)

$$(2.56) \qquad \varepsilon \, C_6 \, (E(t))^{\frac{1-p}{2p}} \int_{\{|y'|\ge 1\}} g^2(y') d\Gamma \le \left\{ 1 - \varepsilon \frac{1-p}{2p} C_3 (E(0))^{\frac{1-p}{2p}} \right\} \int_{\{|y'|\ge 1\}} g(y')y' d\Gamma$$

provided ε is chosen such that

$$(2.57) \qquad \varepsilon \leq (E(0))^{\frac{p-1}{2p}} \left\{ C_2 C_6 + \frac{1-p}{2p} C_3 \right\}^{-1} = C_8 (E(0))^{\frac{p-1}{2p}}.$$

From (2.53) and (2.56), it follows

$$(2.58) \qquad E_\varepsilon'(t) \leq \left\{ -1 + \varepsilon \frac{1-p}{2p} C_3 (E(0))^{\frac{1-p}{2p}} \right\} \int\limits_{\{|y'| \leq 1\}} g(y')y' d\Gamma +$$

$$\varepsilon C_6 (E(t))^{\frac{1-p}{2p}} \int\limits_{\{|y'| \leq 1\}} g^2(y') d\Gamma - \varepsilon(E(t))^{\frac{1+p}{2p}}.$$

From (2.55) and by Young's inequality, we have

$$(2.59) \qquad \varepsilon C_6 (E(t))^{\frac{1-p}{2p}} \int\limits_{\{|y'| \leq 1\}} g^2(y') d\Gamma \leq \frac{\varepsilon}{2}(E(t))^{\frac{1+p}{2p}} + \varepsilon(2C_6)^{\frac{1+p}{2p}} \left(\int\limits_{\{|y'| \leq 1\}} g^2(y') d\Gamma \right)^{\frac{1+p}{2p}}$$

$$\leq \frac{\varepsilon}{2}(E(t))^{\frac{1+p}{2p}} + \varepsilon(2C_6)^{\frac{1+p}{2p}} (C_2)^p \left\{ \int\limits_{\{|y'| \leq 1\}} (g(y')y')^{\frac{2p}{1+p}} d\Gamma \right\}^{\frac{1+p}{2p}}.$$

We apply Hölder's inequality with exponents $\alpha = \frac{p+1}{2p}$, $\alpha' = \frac{p+1}{1-p}$ to the last term of the right-hand side of (2.59), to obtain

$$(2.60) \qquad \left\{ \int\limits_{\{|y'| \leq 1\}} (g(y')y')^{\frac{2p}{1+p}} d\Gamma \right\}^{\frac{1+p}{2p}} \leq \int\limits_{\{|y'| \leq 1\}} g(y')y' d\Gamma \cdot \left\{ \int\limits_{\{|y'| \leq 1\}} d\Gamma \right\}^{\frac{1-p}{2p}}$$

$$\leq \int\limits_{\{|y'| \leq 1\}} g(y')y' d\Gamma \cdot \{ \text{meas } \Gamma \}^{\frac{1-p}{2p}}.$$

Inserting (2.59), (2.60) into (2.58) yields

$$(2.61) \qquad E_\varepsilon'(t) \leq \left\{ -1 + \varepsilon \frac{1-p}{2p} C_3 (E(0))^{\frac{1-p}{2p}} \right\} \int\limits_{\{|y'| \leq 1\}} g(y')y' d\Gamma$$

$$+ \ \varepsilon(2C_6^{\frac{1+p}{2p}} (C_2)^p (\text{meas } \Gamma)^{\frac{1-p}{2p}} \int_{\{|y'|\leq 1\}} g(y')y'd\Gamma - \frac{\varepsilon}{2}(E(t))^{\frac{1+p}{2p}}.$$

By choosing ε so that

$$(2.62) \qquad \varepsilon \leq \{ \frac{1-p}{2p}C_3(E(0))^{\frac{1-p}{2p}} + (2C_6)^{\frac{1+p}{2p}}(C_2)^p(\text{meas } \Gamma)^{\frac{1-p}{2p}} \}^{-1}$$

we deduce from (2.61) that

$$(2.63) \qquad E_\varepsilon'(t) \leq - \frac{\varepsilon}{2}(E(t))^{\frac{1+p}{2p}}.$$

Finally, solving the differential inequality (2.63), using (2.50), we obtain

$$(2.64) \qquad E(t) \leq M \ E(0) \ \{ 1 + t\frac{\varepsilon}{4}\frac{1-p}{p}M^{p+1}(E(0))^{\frac{1-p}{2p}} \}^{\frac{-2p}{1-p}} = M \ E(0) \ \{ 1 + \mu t \}^{\frac{-2p}{1-p}}.$$

The proof of Theorem 2 is thus completed.

Remark 2.10. One can also consider the problem where the boundary condition

$$(1.4) \qquad \frac{\partial y}{\partial \nu} + \alpha \ y = - g(y') \qquad \text{on } \Gamma$$

is replaced by the conditions

$$\frac{\partial y}{\partial \nu} + \alpha \ y = - g(y') \qquad \text{on } \Gamma_0 = \{x \in \Gamma; \ m.\nu > 0\}$$

$$y = 0 \qquad \text{on } \Gamma_* = \Gamma \backslash \Gamma_0 ,$$

as in [19]. Then, with the assumption (1.2) on Γ_0, which implies $\overline{\Gamma}_0 \cap \overline{\Gamma}_* = \varnothing$, we have the same results as before, for any $\alpha \geq \alpha_0 > 0$. In [19], the assumption (1.2) is not necessary (and in fact not true), but α has to be small. Thus for this variant, the results of [19] are not the same as ours.

However, in the case of Dirichlet boundary conditions on part of the boundary, the results are true without any α [19]. The interest in considering an α in that case is in the gain of robustness, as shown by Zuazua [20].

§3 Wave equation with a variable coefficient

In arbitrary space dimension, some recent results about the stabilization problem for the wave equation with variable coefficients can be found in [9], [10]. There the author considers regular coefficients satisfying a specific coercivity condition.

In this section, taking a more general multiplier than $(m.\nabla y)$, we show that the results of the previous section are still valid in the one dimensional case *for a large class of coefficients*. However the extension of this method to the case of arbitrary space dimension seems difficult.

Let us consider the following equation:

$$(3.1) \qquad y'' - (ay_x)_x = 0 \qquad \text{in } (0, 1)\times (0 \, \infty),$$

$$(3.2) \qquad \begin{cases} ay_x(0, t) - \alpha y(0, t) = g(y'(0,t)) & t > 0, \\ ay_x(1, t) + \alpha y(1, t) = -g(y'(1,t)) & t > 0, \end{cases}$$

$$(3.3) \qquad y(0) = y_0 \in H^1(0, 1), \, y'(0) = y_1 \in L^2(0, 1)$$

where α is a positive constant, and $g: \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous nondecreasing function such that

$$(3.4) \qquad g(0) = 0, \quad g(s)s > 0, \quad \text{for all } s \neq 0.$$

We assume that the function a belongs to the space $H^1(0, 1)$ and that

$$(3.5) \qquad 0 < \underline{a} \leq a(x), \qquad 0 \leq x \leq 1.$$

Under the assumptions (3.4) and (3.5), it is easy to see that the equations (3.1)-(3.3) have a unique strong solution y such that

$$(3.6) \qquad y \in W^{1,\infty}(\mathbb{R}^+, H^1(0,1)) \cap L^\infty(\mathbb{R}^+, H^2(0,1))$$

for every initial data $(y_0, y_1) \in W$, where W denotes the subset of $H^1(0,1)\times L^2(0,1)$

$$(3.7) \qquad W = \left\{ (y, \hat{y}) \in H^2(0,1)\times H^1(0,1) ; (ay_x)(0) - \alpha y(0) = g(\hat{y}(0)), \right.$$
$$\left. (ay_x)(1) + \alpha y(1) = - g(\hat{y}(1)) \right\}.$$

Furthermore, we can extend the mapping $(y_0, y_1) \longrightarrow (y(t), y'(t))$ to a strongly continuous semi-group of contractions $(S(t))_{t \geq 0}$ in $H^1(0,1) \times L^2(0,1)$. Thus for every initial data $(y_0, y_1) \in H^1(0,1) \times L^2(0,1)$ there exists a unique weak solution of the equations (3.1)-(3.3) which is given by the following formula:

(3.8) $(y(t), y'(t)) = S(t)(y_0, y_1), \quad \forall \ t \geq 0$

with the regularity

(3.9) $y \in C^0(\mathbb{R}^+, H^1(0, 1)) \cap C^1(\mathbb{R}^+, L^2(0, 1)).$

As in the previous section, we assume the regularity (3.6), since the subset W is dense in the space $H^1(0,1) \times L^2(0,1)$ and $(S(t))_{t \geq 0}$ is a strongly continuous semi-group of contractions.

The strong asymptotic stabilization can be proved by a standard argument (see [1], [5], [17]). As in the previous section, we consider here the problem of estimating the rate of decay.

For any solution y of the equations (3.1)-(3.3), we define the associated energy

(3.10) $E(t) = \dfrac{1}{2}\displaystyle\int_0^1 (a|y_x|^2 + |y'|^2)dx \ + \ \dfrac{\alpha}{2}\left(y^2(0,t) + y^2(1,t)\right)$

Theorem 3. Let g be a continuous nondecreasing function such that (3.4) holds. Assume that $a \in H^1(0,1)$ satisfies (3.5). Then we have:

(i) If there exist some positive constants C_1, C_2 such that

(3.11) $C_1|s| \leq |g(s)| \leq C_2|s|, \qquad \forall \ s \in \mathbb{R},$

then given any $M > 1$, there exists a constant $\lambda > 0$ such that

(3.12) $E(t) \leq ME(0)e^{-\lambda t}, \qquad \forall \ t \geq 0.$

(ii) If there exist some positive constants C_1, C_2 and $p > 1$ such that

(3.13) $C_1 \mathrm{Min}\{|s| , |s|^p\} \leq |g(s)| \leq C_2|s| , \qquad \forall \ s \in \mathbb{R},$

then given any $M > 1$, there exists a constant $\mu > 0$ depending on $E(0)$ such that

(3.14) $E(t) \leq ME(0) (1+\mu t)^{-\frac{2}{p-1}}, \qquad \forall \ t \geq 0.$

Sketch of the proof

First, we replace the functional $\rho(t)$, introduced in the previous section, by the following functional

$$(3.15) \qquad \rho_1(t) = 2\int_0^1 y'\varphi y_x dx + C_0\int_0^1 y'\psi dx$$

where C_0 is a positive constant and ψ is the solution of the equations

$$(3.16) \qquad (a\psi_x)_x = 0, \qquad\qquad 0 < x < 1,$$

$$(3.17) \qquad \begin{cases} \psi(0,t) = y(0,t), \\ \psi(1,t) = y(1,t). \end{cases}$$

Solving the equations (3.16)-(3.17), we have:

$$(3.18) \qquad \psi(x,t) = \frac{\displaystyle\int_0^x a^{-1}(s)ds}{\displaystyle\int_0^1 a^{-1}(s)ds}\, (y(1,t) - y(0,t)) + y(0,t)$$

We verify that the following inequalities hold

$$(3.19) \qquad \int_0^1 a\psi_x y_x dx = \left(\int_0^1 a^{-1}(s)ds\right)^{-1}(y(1,t) - y(0,t))^2 \geq 0,$$

$$(3.20) \qquad \int_0^1 \psi^2 dx \leq 5\left(y^2(0,t) + y^2(1,t)\right).$$

The function φ is chosen so that

(3.21) $\qquad \varphi \in H^1(0,1), \qquad \varphi_x \geq 1, \quad a\left(\frac{\varphi}{a}\right)_x \geq 1,$

(3.22) $\qquad \varphi(0) < 0, \quad \varphi(1) > 0,$

(3.23) $\qquad \varphi(0)a(1) + \varphi(1)a(0) = 0.$

An example of such a function will be given explicitly (see (3.36)).

Next, it is easy to verify that

(3.24) $\qquad |\rho_1(t)| \leq C_3 E(t), \qquad \forall \, t \geq 0,$

with

$$C_3 = \left\| \frac{\varphi}{\sqrt{a}} \right\|_{L^\infty(0,1)} + C_0 \sqrt{\frac{5}{\alpha}} \; .$$

We calculate the derivative of the functional $\rho_1(t)$

$$\rho_1'(t) = \left[ay_x^2 \varphi \right]_0^1 + \left[(y')^2 \varphi \right]_0^1 - \int_0^1 \left(ay_x^2 \left(\frac{\varphi}{a} \right)_x a + |y'|^2 \varphi_x \right) dx + C_0 [ay_x \psi]_0^1 - C_0 \int_0^1 ay_x \psi_x dx + C_0 \int_0^1 y' \psi' dx.$$

Taking into account (3.18), (3.19), (3.21) and (3.22), we obtain

(3.25) $\qquad \rho_1'(t) \leq \theta \left[ay_x^2 \varphi \right]_0^1 + \left[|y'|^2 \varphi \right]_0^1 - \int_0^1 (ay_x^2 + |y'|^2) \, dx$

$$+ C_0 [ay_x y]_0^1 + C_0 \left(\int_0^1 |y'|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |\psi'|^2 dx \right)^{\frac{1}{2}}$$

where θ is an arbitrary constant satisfying

(3.26) $\qquad \theta \geq 1.$

We apply Young's inequality to the last term of the right-hand side of (3.25), using (3.20)

$$(3.27) \qquad C_0 \left(\int_0^1 |y'|^2 dx \right)^{\frac{1}{2}} \left(\int_0^1 |\psi'|^2 dx \right)^{\frac{1}{2}} \leq \frac{1}{2} \int_0^1 |y'|^2 dx + \frac{C_0^2}{2} \int_0^1 |\psi'|^2 dx$$

$$\leq \frac{1}{2} \int_0^1 |y'|^2 dx + \frac{5C_0^2}{2} \left(|y'(0,t)|^2 + |y'(1,t)|^2 \right)$$

Inserting (3.27) into (3.25) yields

$$(3.28) \qquad \rho_1'(t) \leq \theta \left[a y_x^2 \varphi \right]_0^1 + \left[|y'|^2 \varphi \right]_0^1 - \frac{1}{2} \int_0^1 \left(a y_x^2 + |y'|^2 \right) dx \right) + C_0 \left[a y_x y \right]_0^1$$

$$+ \frac{5C_0^2}{2} \left(|y'(0,t)|^2 + |y'(1,t)|^2 \right)$$

On the other hand, using the boundary conditions (3.2), a straightforward computation shows that

$$(3.29) \qquad \left[a y_x^2 \varphi \right]_0^1 = -\alpha^2 \left[\frac{\varphi}{a} y^2 \right]_0^1 + \left[\frac{g^2(y')}{a} \varphi \right]_0^1 - 2\alpha (\varphi(0) y_x(0,t) y(0,t) + \varphi(1) y_x(1,t) y(1,t))$$

Eliminating the term $\left[a y_x^2 \varphi \right]_0^1$ appearing in (3.28) and (3.29), using (3.22) and (3.23), we get

$$(3.30) \qquad \rho_1'(t) \leq -\frac{1}{2} \int_0^1 \left(a y_x^2 + |y'|^2 \right) dx - \theta \alpha^2 C_4 \left(|y(0,t)|^2 + |y(1,t)|^2 \right)$$

$$+ \left(C_5 + \frac{5C_0^2}{2} \right) \left(|y'(0,t)|^2 + |y'(1,t)|^2 \right) + C_5 \theta \left(\frac{g^2(y'(0,t))}{a(0)} + \frac{g^2(y'(1,t))}{a(1)} \right)$$

$$+ \left(C_0 - 2\alpha \theta \frac{\varphi(1)}{a(1)} \right) \left[a y_x y \right]_0^1$$

where we have set

$$(3.31) \quad \begin{cases} C_4 = \text{Min}\left\{ -\dfrac{\varphi(0)}{a(0)}, \ \dfrac{\varphi(1)}{a(1)} \right\}, \\[3mm] C_5 = \text{Max}(-\varphi(0), \ \varphi(1)). \end{cases}$$

Now, if we choose the constants θ and C_0 as follows

$$(3.32) \quad \theta = \text{Max}\left\{ 1, \ \dfrac{1}{2\alpha C_4} \right\},$$

$$(3.33) \quad C_0 = 2\alpha\theta\dfrac{\varphi(1)}{a(1)} > 0,$$

we deduce from (3.30) that

$$(3.34) \quad \rho_1'(t) \le -E(t) + C_6\left(|y'(0,t)|^2 + |y'(1,t)|^2\right) + C_7\left(g^2(y'(0,t) + g^2(y'(1,t))\right)$$

with the constants C_6 and C_7 given by

$$(3.35) \quad \begin{cases} C_6 = C_5 + \dfrac{5C_0^2}{2}, \\[3mm] C_7 = \text{Max}\left(\dfrac{1}{a(0)}, \ \dfrac{1}{a(1)}\right) C_5\theta. \end{cases}$$

We see that the estimate (3.34) will play the same role than inequality (2.21). The remaining part of the proof is quite analogous to the proof of Theorem 1, using now (3.11) or (3.13).

In order to complete the proof of Theorem 3, we give an example of function φ satisfying (3.21)-(3.23).

$$(3.36) \quad \varphi(x) = \begin{cases} \exp(\displaystyle\int_x^0 |\dfrac{a_x}{a}|d\sigma)\cdot\displaystyle\int_{x_0}^x \exp(\displaystyle\int_0^\sigma |\dfrac{a_x}{a}|d\tau)d\sigma, & 0 \le x \le x_0, \\[4mm] \exp(\displaystyle\int_1^x |\dfrac{a_x}{a}|d\sigma)\cdot\displaystyle\int_{x_0}^x \exp(\displaystyle\int_\sigma^1 |\dfrac{a_x}{a}|d\tau)d\sigma, & x_0 \le x \le 1, \end{cases}$$

where $0 < x_0 < 1$ is the unique solution of the following transcendental equation

$$(3.37) \qquad a(1) \int_{x_0}^{0} \exp\left(\int_{0}^{\sigma} \frac{a_x}{a}d\tau\right)d\sigma + a(0)\int_{x_0}^{1} \exp\left(\int_{\sigma}^{1} \frac{a_x}{a}d\tau\right)d\sigma = 0$$

which is just the condition (3.23).

Remark 3.1 Similarly, we can also consider the sub-linear case. If we assume there exist some positive constants C_1, C_2 and $p < 1$ such that

$$(3.38) \qquad C_1|s| \leq |g(s)| \leq C_2 \mathrm{Max}(|s|, |s|^p), \quad \forall \, s \in \mathbb{R},$$

then given any constant $M > 1$, there exists a constant $\mu > 0$ depending on $E(0)$ such that the following "symetrical" estimate holds

$$(3.39) \qquad E(t) \leq ME(0)\,(1+\mu t)^{-\frac{2p}{1-p}}, \qquad\qquad \forall \, t \geq 0.$$

Starting from estimate (3.34), which is similar to (2.21), we have just to copy then the proof of Theorem 2.

Remark 3.2 Obviously, we can prove the same estimates when (3.2) is replaced by the slightly more general condition (for α and β positive constants)

$$\begin{cases} ay_x(0, t) - \alpha y(0, t) = g(y'(0,t)) & t > 0, \\ ay_x(1, t) + \beta y(1, t) = -g(y'(1,t)) & t > 0. \end{cases}$$

Remark 3.3 We can also replace the condition (3.2) by

$$(3.2)' \qquad \begin{cases} (ay_x)(0,t) = 0, & t > 0, \\ (ay_x)(1,t) + \alpha y(1,t) = -g(y'(1,t)), & t > 0. \end{cases}$$

In that case, instead of the functional $\rho_1(t)$, we consider

$$(3.40) \qquad \hat{\rho}(t) = 2\int_{0}^{1} y'\hat{\phi}y_x dx + \hat{C}_0 y(1,t)\int_{0}^{1} y' dx.$$

where $\hat{C}_0$ is a positive constant and the function $\hat{\phi}$ is defined hereafter by (3.44).

Accordingly, we consider the following energy

$$(3.41) \qquad \hat{E}(t) = \frac{1}{2} \int_0^1 \left(a |y_x|^2 + |y'|^2 \right) dx + \frac{\alpha}{2} y^2(1,t).$$

A straightforward computation shows that

$$(3.42) \qquad |\hat{p}_1(t)| \leq \hat{C}_3 \hat{E}(t),$$

$$(3.43) \qquad \hat{p}_1'(t) \leq -\hat{E}(t) + \hat{C}_6 |y'(1,t)|^2 + \hat{C}_7 g^2(y'(1,t)),$$

where $\hat{C}_6$, $\hat{C}_7$ are constants depending only on the variable coefficient a and $\hat{\phi}$ is given by

$$(3.44) \qquad \hat{\phi}(x) = \exp\left(\int_1^x \frac{a_x}{a} d\sigma \right) \cdot \int_0^x \exp\left(\int_\sigma^1 \frac{a_x}{a} d\tau \right) d\sigma.$$

Thus, we have proved that the estimates (3.12), (3.14) and (3.39) are still valid for any solution y of the equations (3.1), (3.2)' and (3.3).

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