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# Decay of Supercurrents in Condensates in Optical Lattices

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In this paper we discuss decay of superfluid currents in boson lattice systems due to quantum tunneling and thermal activation mechanisms. We derive asymptotic expressions for the decay rate near the critical current in two regimes, deep in the superfluid phase and close to the superfluid-Mott insulator transition. The broadening of the transition at the critical current due to these decay mechanisms is more pronounced at lower dimensions. We also find that the crossover temperature below which quantum decay dominates is experimentally accessible in most cases. Finally, we discuss the dynamics of the current decay and point out the difference between low and high currents.

**KEY WORDS:** Optical lattices; condensates; decay.

## 1. INTRODUCTION

Some of the most intriguing questions in low-temperature physics concern the ways in which superconductors lose their superconducting properties, because of thermal or quantum fluctuations. Mike Tinkham has long been fascinated with these issues, and has done much to advance our understanding of the subject.

An early contribution in this area was the work of Newbower *et al.* on effects of fluctuations on the superconducting transition of tin whisker crystals [1]. Experimental data were compared with theories of thermally activated phase slips, both in the linear regime and in the nonlinear regime of finite current flows. More recently, Tinkham and collaborators studied the loss of superconductivity in very thin wires of MoGe, deposited on carbon nanotubes, where quantum fluctuations are involved [2–4]. Related work from Tinkham's laboratory, in recent years, has elucidated the breakdown of superconductivity in ultrasmall metallic grains, measured by the even–odd alternation of Coulomb-blockade energies [5,6], vortex motion, and resistance in high-temperature superconductors [7,8], and critical currents in frustrated arrays of Josephson junctions [9].

The decay of supercurrents in liquid $^4$He and in Bose-Einstein condensates of ultracold atoms has much in common with the decay of superconductivity. Concepts of flux-line motion, and of phase slips due to thermal or quantum fluctuations, appear in both cases. A new dimension has been added to the subject by recent experimental advances, where cold atoms have been trapped in a region that contains a spatially periodic potential, produced by optical standing waves (see for example [10]). The ability to vary continuously the parameters of the system, by changing the strength of the periodic potential, as well as by varying the number of trapped atoms and the shape of the overall confining potential, allows one to explore new regimes of parameters and to make more precise confrontations between theory and experiment. In turn, these developments give added urgency to the theoretical study of supercurrent decay.

In the present paper, we discuss similarities and differences between the decay of supercurrents in superconductors and systems of trapped atoms, and we present some new results for the latter. Specifically

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we consider certain experimental procedures which have become standard in systems of ultra cold atoms. In the first scheme, a condensate is prepared on a lattice with a specified intensity, when the lattice is suddenly accelerated to a finite velocity. In other words, a moving condensate is prepared in the lattice frame, essentially fixing the phase gradient. A similar experiment in superconductors would involve threading a flux through a closed superconducting loop. Such sudden lattice boosts were applied by several groups to demonstrate a dynamical instability of the superfluid when the imposed phase gradient exceeds $\pi/2$ per unit cell [11]. A related experiment involves tilting the lattice, thus subjecting the atoms to a linear potential. This is equivalent to imposing a constant voltage on a superconductor. The technique was used, for example, to demonstrate Bloch oscillations of a condensate [12]. In a third experimental sequence, one can prepare a moving condensate, then continuously increase the depth of the lattice toward the superfluid-Mott insulator transition.

The response of the atomic system to the perturbations, can be measured, by direct observation of the time evolution. Decay of the current, for example, is observed by repeated experiments, where atoms are released from the trap after varying waiting periods. The phase gradient in the superfluid at the time of release may be inferred from a *time of flight* measurement of the momentum distribution. This should be contrasted with superconductors, where measurements probe I–V characteristics.

Besides the differences in the experimental observation procedures, there are unique features of trapped atom systems which influence the physics of supercurrent decay. First, to a very good approximation such systems can be considered perfectly clean. Supercurrents decay only due to breaking of Galilean invariance by the periodic potential.

A second feature that distinguishes the dynamics of ultra cold atoms is their nearly perfect isolation from the environment. Strictly speaking they are always underdamped. However, we are usually interested in the dynamics of a subset of the system degrees of freedom, such as the super-current. How much the dynamics of the the interesting variables is damped, depends solely on the remaining system degrees of freedom rather than on external dissipation sources. In superconductors, effects of quenched disorder, phonons, fermion degrees of freedom, and coupling to a substrate can complicate the situation greatly, and the order parameter dynamics is most frequently overdamed.

## 2. CRITICAL CURRENT IN THE SUPERFLUID PHASE

Ultra cold atoms in an optical lattice, confined to the lowest Bloch band are described by the well known Bose-Hubbard Hamiltonan:

$$H = -J \sum_{\langle ij \rangle} (a_i^\dagger a_j + \text{h.c.}) + \frac{U}{2} \sum_i n_i (n_i - 1), \qquad (1)$$

where $J$ and $U$ are the hopping amplitude and the on-site repulsive interaction, $\langle ij \rangle$ denotes pairs of nearest neighbor sites. Another implicit parameter in this Hamiltonian is the average number of bosons per site, $N$. In this paper we shall be primarily concerned with the case where $N$ is a large integer. We shall address two separate regimes: the first is defined by the conditions $UN^2 \gg JN \gg U$, while the second regime corresponds to the superfluid near the transition to a Mott insulator ($UN^2 \gg JN \sim U$).

If the condition $UN \gg J$ holds, then the interactions are sufficiently strong to suppress amplitude fluctuations of the order parameter, and (1) can be mapped to the quantum rotor model:

$$H = -JN \sum_{\langle ij \rangle} \cos(\varphi_i - \varphi_j) - \frac{U}{2} \sum_i \left( \frac{\partial}{\partial \varphi^i} \right)^2 \qquad (2)$$

The additional condition $JN \gg U$ ensures that the system is far from the superfluid-insulator transition, and facilitates a semiclassical approximation because fluctuations in $\varphi$ as well as in the density, are small. In the classical limit the boson creation and annihilation operators can be treated as complex numbers subject to discrete Gross-Pitaevskii equations [13]:

$$i \frac{d\psi_j}{dt} = -J \sum_{k \in O} \psi_k + U |\psi_j|^2 \psi_j, \qquad (3)$$

where the set $O$ contains the nearest neighbors of site $j$. In the quantum rotor limit $UN \gg J$ the number fluctuations can be integrated out leaving us with only the equations of motion for the phase $\phi_j = \arg \psi_j$:

$$\frac{d^2 \phi_j}{dt^2} = -2UJN \sum_{k \in O} \sin(\phi_k - \phi_j). \qquad (4)$$

Alternatively Eq. (4) immediately follows from the Hamiltonian (2). Both Eqs. (3) and (4) can support stationary current carrying states, $\psi_j \propto \exp(ipx_j)$. A simple linear stability analysis of (3) or (4), shows [14,15] that these states become unstable toward small perturbations when the phase twist exceeds a critical value of $\pi/2$ per unit cell. The onset of this instability is signaled by appearance of imaginary frequencies. This instability was recently observed experimentally [12].

In principle, one can identify another type of instability, characterized by appearance of negative frequencies, in systems described by Eq. (3) [14]. In general this occurs at a phase twist $p^* < \pi/2$. However, in the quantum rotor limit $UN \gg J$, where we work, the two instabilities coincide.

While the modulational instability occurs precisely at $p = \pi/2$ for $JN \gg U$, we expect that the current can decay at smaller momenta due to either quantum or thermal fluctuations (see also Ref. 16). We envision the following experimental scheme to observe this. The condensate is either boosted to a state with a certain phase gradient or gradually accelerated. Following the boost or while the system is accelerating we probe the evolution of the phase gradient. If the system is sufficiently close to the modulational instability, i.e. $p$ is slightly below $\pi/2$, the coherent motion of the condensate is expected to decay. The larger the phase gradient, the faster this decay will occur.

The other regime we shall address, is that of the superfluid close to the quantum phase transition to a Mott insulator at commensurate filling (i.e. $JN \sim U$, and for simplicity we still assume that $N \gg 1$). Now the phase fluctuations are large, and (2) cannot be treated semiclassically. However, one can use the semiclassical description after coarse graining the system. Since the coherence length $\xi$ diverges at the transition, one can use a continuum description of the static and dynamic properties of the condensate. At commensurate filling the appropriate quantum action written in terms of the superfluid order parameter reads [17,18]:

$$S = C \int d^d x \, dt \left\{ \left| \frac{d\psi}{dt} \right|^2 - |\nabla \psi|^2 + |\psi|^2 - \frac{1}{2}|\psi|^4 \right\}, \tag{5}$$

where length is measured in units of $\xi$ and time in units of $\xi/c$, with $c$ the sound velocity, $C$ is a numerical prefactor. The bare parameters $\xi$, $c$, and $C$ can be found using a mean-field approximation [19]. For the cubic d-dimensional lattice they read:

$$\xi = \frac{1}{\sqrt{2d(1-u)}}, \quad c = 2JN\sqrt{2d},$$

$$C = \frac{1}{2(2d)^{d/2}}(1-u)^{\frac{3-d}{2}}, \tag{6}$$

where we introduced the dimensionless interaction $u = U/U_c$ with $U_c = 8JNd$ being the critical interaction strength in the mean field approximation. The action (5) correctly describes low-energy dynamics of the system in the vicinity of the phase transition, only if the couplings $\xi$, $c$, $U_c$, and $C$ are properly renormalized. While in three dimensions the effects of such renormalization should be weak, in two- and especially one-dimensional cases they strongly modify the couplings and the critical exponents. The bare mean field parameters can then be used only as an estimate. Note that the dimensionless part of the action (5) is general, and so are the conclusions we reach in this paper, once the renormalized, rather than mean field parameters are used. The action (5) is obviously extremized by stationary current-carrying states: $\psi_p(x) = \sqrt{1-(p\xi)^2}\, e^{ipx\xi}$. It is easy to check [19] that these states are stable with respect to small fluctuations for $p < p_c = 1/(\xi\sqrt{3})$. Since $\xi$ diverges at the phase transition, the critical phase twist vanishes at that point as it should.

A possible experimental procedure to measure the decay rate at low currents follows. A condensate with a specified phase gradient is prepared in a weak lattice (small $u$). Then, the lattice potential is gradually increased in time, driving the system closer to the Mott phase. This, in turn, results in the increase of the correlation length $\xi$ and in decrease of the critical momentum $p_c$. As $p_c$ approaches $p$ the superfluid current is expected to decay either due to quantum or thermal fluctuations.

## 3. DECAY OF THE SUPERFLUID CURRENT

In this section we describe how the superfluid current decays in a lattice when $p$ is below $p_c$. We shall consider first the Gross-Pitaevskii regime $JN \gg U$ in the quantum rotor limit $UN \gg J$ and then turn to the situation in the vicinity of the superfluid insulator transition $JN \sim U$. In each case we shall address the effects of both quantum and thermal fluctuations.

### 3.1. Gross-Pitaevskii Regime

#### *3.1.1. Quantum Decay*

The action corresponding to the quantum rotor model (2) is given by

$$S = \int d\tau \left[ \sum_j \frac{1}{2U} \left( \frac{d\phi_j}{d\tau} \right)^2 - \sum_{\langle j,j' \rangle} 2JN \cos(\phi_j - \phi_{j'}) \right], \tag{7}$$

or after the rescaling $\tau \to \tau/\sqrt{UNJ}$:

$$S = (JN/U)^{1/2} s, \tag{8}$$

where

$$s = \int d\tau \left[ \sum_j \frac{1}{2} \left( \frac{d\phi_j}{d\tau} \right)^2 - \sum_{\langle j,j' \rangle} 2\cos(\phi_j - \phi_{j'}) \right]. \tag{9}$$

To leading order in $\sqrt{U/JN}$, which plays the role of the effective Planck's constant for this problem [20], the tunneling rate depends on the action $S_b$, associated with the bounce solution of the classical equations of motion in the inverted potential [21]:

$$\Gamma \propto e^{-S_b}, \tag{10}$$

Clearly the action should vanish at $p = \pi/2$, since at this point the spectrum becomes unstable and the tunneling barrier disappears. Deep in the superfluid regime $U/JN \ll 1$, the tunneling is effective only if $p$ is close to $\pi/2$, where the product $s(JN/U)^{1/2}$ is not too large. In this case one can make further progress in calculating the tunneling action by expanding (9) up to cubic terms in phase differences $\phi_j - \phi_{j'}$:

$$s \approx \sum_{j,\mathbf{k}} \int d\tau \left[ \frac{1}{2} \left( \frac{d\phi_{j\mathbf{k}}}{d\tau} \right)^2 + \cos(p)(\phi_{j+1,\mathbf{k}} - \phi_{j,\mathbf{k}})^2 + (\phi_{j,\mathbf{k}+1} - \phi_{j,\mathbf{k}})^2 - \frac{1}{3}(\phi_{j+1,\mathbf{k}} - \phi_{j,\mathbf{k}})^3 \right]. \tag{11}$$

Here we explicitly split the site index into longitudinal ($j$) and transverse ($\mathbf{k}$) components. Also, for convenience, we shifted the phase $\phi_{j,\mathbf{k}} \to \phi_{j,\mathbf{k}} + pj$ so that the metastable state now corresponds to $\phi_{j,\mathbf{k}} = 0$. Note that at $p \to \pi/2$ only longitudinal modes become soft, due to the prefactor $\cos p$ in front of the quadratic term in the action. This implies that we can safely apply a continuum approximation for the phases along the transverse directions. Then instead of (11) we derive:

$$s \approx \sum_j \int d\tau d^{d-1}x \left[ \frac{1}{2} \left( \frac{d\phi_j}{d\tau} \right)^2 + \left( \frac{d\phi_j}{d\mathrm{x}} \right)^2 + \cos(p)(\phi_{j+1} - \phi_j)^2 - \frac{1}{3}(\phi_{j+1} - \phi_j)^3 \right]. \tag{12}$$

In this equation $\mathbf{x}$ denotes transverse coordinates which reside in a $d-1$ dimensional space. Upon rescaling

$$\phi = \cos(p)\tilde{\phi}, \quad \tau = \frac{\tilde{\tau}}{\sqrt{\cos(p)}}, \quad x = \frac{\tilde{x}\sqrt{2}}{\sqrt{\cos p}}, \tag{13}$$

the action (12) simplifies further:

$$s \approx (\pi/2 - p)^{\frac{6-d}{2}} s_d, \tag{14}$$

where

$$\tilde{s}_d = 2^{\frac{d-1}{2}} \sum_j \int d^d\xi \left[ \frac{1}{2} \left( \frac{d\tilde{\phi}_j}{d\xi} \right)^2 + (\tilde{\phi}_j - \tilde{\phi}_{j+1})^2 - \frac{1}{3}(\tilde{\phi}_j - \tilde{\phi}_{j+1})^3 \right] \tag{15}$$

is just a number, which is determined only by the dimensionality $d$. We will provide its detailed variational derivation elsewhere [19] and here just quote the results: $\tilde{s}_1 \approx 7, \tilde{s}_2 \approx 25, \tilde{s}_3 \approx 90$. From the scaling (13) it is obvious that the characteristic transverse dimension of the instanton $x$ scales as $(\pi/2 - p)^{-1/2} \gg 1$, justifying the continuum approximation. Above $d = 6$ the tunneling action would experience a discontinuous jump at $p = \pi/2$. However, since we deal with $d \le 3$, the action always continuously vanishes at $p \to \pi/2$. In this way we derive the asymptotic decay rate of a uniform current state near the modulation instability:

$$\Gamma \propto \exp\left[ -\tilde{s}_d (JN/U)^{1/2} (\pi/2 - p)^{\frac{6-d}{2}} \right] \tag{16}$$

### *3.1.2. Thermal Decay*

To calculate the exponent characterizing the thermal decay rate, one has to compute the difference of energies of the metastable state and the saddle-point which separates two adjacent metastable minima [22–24]. Both saddle-point and metastable configurations are the stationary solutions of the equations of motion (4):

$$\sum_{k \in O} \sin(\phi_k - \phi_j) = 0. \tag{17}$$

The metastable state corresponds to the uniform phase twist: $\phi_{j,\mathbf{k}} = jp$. The saddle-point state relevant for the current decay can be easily found in one dimension:

$$\phi_j = \begin{cases} p'j, & j < 0 \\ \pi + p'(j-2), & j \ge 1, \end{cases} \tag{18}$$

where $p' \approx p - (\pi - 2p)/M$ if we use periodic boundary conditions for the system with $M$ sites. The energy difference between the two states in the limit $M \to \infty$ is

$$\Delta E = 2JN(2\cos p - \sin p(\pi - 2p)). \tag{19}$$

Correspondingly, the decay rate is proportional to

$$\Gamma \propto e^{-\beta\Delta E} = e^{-2JN\beta(2\cos p-(\pi-2p)\sin p)}. \quad (20)$$

In particular when $p \to \pi/2$ we have:

$$\Gamma \propto e^{-\frac{4}{3}NJ\beta(\pi/2-p)^3}. \quad (21)$$

In higher dimensions we cannot find the energy of the saddle-point explicitly for all values of $p$. However, in analogy with the quantum case, at $p$ close to $\pi/2$ we can expand the energy functional up to cubic terms in phase differences.

$$E_d \approx JN\sum_j \int d^{d-1}x\left[\left(\frac{d\phi_j}{dx}\right)^2 + \cos(p) \times (\phi_{j+1}-\phi_j)^2 - \frac{1}{3}(\phi_{j+1}-\phi_j)^3\right], \quad (22)$$

where $\phi_j(x)$ is the nontrivial solution of the corresponding Euler-Lagrange equations vanishing at $x \to \infty$. We again shifted the phase $\phi_j \to \phi_j + p_j$. After rescaling $\phi_j = \cos(p)\tilde{\phi}_j$ and $x = \tilde{x}\sqrt{2}/\sqrt{\cos p}$ we find:

$$E_d \approx JN 2^{\frac{d-1}{2}}(p_c-p)^{\frac{7-d}{d}}\sum_j \int d^{d-1}\tilde{x}\left[\frac{1}{2}\left(\frac{d\tilde{\phi}_j}{d\tilde{x}}\right)^2 + (\tilde{\phi}_{j+1}-\tilde{\phi}_j)^2 - \frac{1}{3}(\tilde{\phi}_{j+1}-\tilde{\phi}_j)^3\right]. \quad (23)$$

Note that the integral in the expression above coincides with $\tilde{s}_{d-1}$ up to a number $2^{\frac{d-2}{2}}$. So we immediately conclude that

$$E_d \approx \tilde{s}_{d-1}JN\sqrt{2}(\pi/2-p)^{\frac{7-d}{2}}. \quad (24)$$

Note that the activation energy characterizing the thermal decay vanishes faster than the tunneling action as $p \to \pi/2$. It implies that thermal fluctuations become increasingly important and dominate the decay of superfluid current as the system approaches the dynamical instability. Comparing the ratio $E_d/T$ and the tunneling action in (16) we obtain the crossover temperature:

$$T^* \approx c\left(\frac{\tilde{s}_{d-1}}{\tilde{s}_d}\right)\sqrt{\pi/2-p} \quad (25)$$

at which the quantum and thermal decay rates coincide. Here $c = \sqrt{2UJN}$ is the sound speed in equilibrium (i.e. $p = 0$). Alternatively, we can fix the temperature to obtain the momentum crossover scale $p^\star$ at which thermal and quantum decay rates coincide:

$$\pi/2 - p^* \approx \left(\frac{\tilde{s}_d}{\tilde{s}_{d-1}}\right)^2\left(\frac{T}{c}\right)^2. \quad (26)$$

At phase gradients larger than $p^*$, thermal decay dominates. The tunneling action in (16), at this value of momentum is given by

$$S_d^* = \tilde{s}_d\left(\frac{\tilde{s}_d}{\tilde{s}_{d-1}}\right)^{6-d}\sqrt{\frac{JN}{U}}\left(\frac{T}{c}\right)^{6-d}. \quad (27)$$

If $S_d^* \gg 1$, then at the crossover momentum the current decay is exponentially suppressed and will be nonzero only at $p$ closer to $\pi/2$. Then the thermally activated phase slips will dominate the decay process and quantum tunneling can be ignored. In the opposite limit $S_d^* \ll 1$ the current will decay at $p < p^*$ due to quantum process and the temperature effects are unimportant. The characteristic crossover temperature separating quantum and thermal decay regimes is thus:

$$T_q \approx A\, c\, \tilde{s}_d^{-\frac{7-d}{6-d}}\tilde{s}_{d-1}\left(\frac{U}{JN}\right)^{\frac{1}{2(6-d)}}, \quad (28)$$

where $A$ is a numerical constant of the order of one. Note that for all relevant dimensions $d \leq 3$ the last multiplier is always of the order of one because of the small exponent $1/(12-2d)$. Therefore, the crossover temperature $T_q$ is of the order of the sound velocity (or equivalently, the Josephson energy).

## 4. CURRENT DECAY IN THE VICINITY OF THE MOTT TRANSITION

Let us now address decay of supercurrents in the regime where $JN \sim U$ and large integer filling $N$. As we already argued, in the vicinity of the Mott-insulator phase transition the correlation length $\xi$ becomes large compared to the lattice constant. One can therefore use a continuum description of the problem (5). The relativistic dynamics of (5) is a special feature of the commensurate transition. Ordinary superfluids are described by a similar action, but with a kinetic term including first time derivative.

The Euler Lagrange-equations derived from (5) admit stationary current carrying solutions of the form

$$\psi = \sqrt{1-(p\xi)^2}e^{ip\xi x}, \quad (29)$$

We emphasize again that $x$ is measured in units of the correlation length $\xi$. The current state becomes unstable at $p > p_c = 1/(\xi\sqrt{3})$. Below $p_c$, the current can still decay due to quantum or thermal fluctuations.

For the thermal decay, only the static part of the action needs to be considered. Then there is no difference between our problem and the current decay in ordinary superfluids described by the Ginzburg-Landau free energy. In particular, in the context of super-conducting wires, the exponent characterizing the current decay rate in one dimension was computed by Langer and Ambegaokar [23], and the prefactor setting the time scale was later found by McCumber and Halperin [24]. In three dimensions, the asymptotic behavior of the corresponding exponent at $p \to 0$ was obtained by Langer and Fisher [25]. However, here we are interested in the opposite limit $p \to p_c$.

For both the thermal and the quantum cases we will use the scaling approach, successfully applied above for the quantum phase model in the Gross-Pitaevskii regime. We expand the action to cubic order in the amplitude ($\eta$) and phase ($\phi$) fluctuations, about the metastable minimum, and integrate out the gapped amplitude mode. After the rescaling:

$$x \to \frac{x}{2\,3^{1/4}\sqrt{\xi}\sqrt{p_c - p}}, \quad z \to \frac{z}{6\xi(p_c - p)},$$
$$\phi \to \phi 3^{3/4} : 2\sqrt{\xi}\sqrt{p_c - p} \tag{30}$$

the action to leading order in $p_c - p$ becomes:

$$S = C3^{9/4-d} : 2^d \xi^{5/2-d}(p_c - p)^{5/2-d} \int d\mathbf{z}\, dx (\nabla\phi)^2 + \left(\partial_x^2\phi\right)^2 - (\partial_x\phi)^3 \approx A_d(1-u)^{1/4}(p_c - p)^{5/2-d} \tag{31}$$

where $\mathbf{z}$ denotes all the transverse coordinates relative to the current direction, including time. $\nabla = (\partial_{\mathbf{z}}, \partial_x)$ is the gradient in $d+1$ dimensions. Accordingly, the quantum decay rate is given by $\Gamma_Q \propto \exp(-A_d(1-u)^{1/4}(p_c - p)^{5/2-d})$. A variational calculation [19], yields $A_1 \approx 18.4$ and $A_2 \approx 8.4$. As before, to calculate the thermal decay rate one simply has to substitute $d \to d-1$, so that

$$\Gamma_T(d) \propto \exp\left(-\frac{JN}{T}(2d)^{-3/4}(1-u)^{1/4} A_{d-1}(p_c - p)^{7/2-d}\right). \tag{32}$$

In the one-dimensional case the relevant constant $A_0 \approx 12.56$. It is interesting to contrast these results with the asymptotic decay rate (16), found in the Gross-Pitaevskii regime. First we observe that the tunneling action close to the Mott insulator vanishes as a smaller power of $p_c - p$. Moreover, for $d = 3$, the scaling hypothesis for the quantum decay rate breaks down, suggesting that $S$ is discontinuous at the critical current and is dominated by fluctuations of a finite (rather than diverging as $p \to p_c$) length scale. We therefore expect, that in three dimensions at zero temperature, the instability marks a sharp localization transition. At finite $T$, thermal fluctuations broaden this transition, because the activation energy barrier vanishes at $p_c$ for $d < 7/2$.

The quantum-to-thermal crossover for a given dimensionless interaction and phase gradient is found by comparing the two decay rates. In one and two dimensions we find

$$T^*(p) = \frac{JN}{(2d)^{3/4}} \frac{A_{d-1}}{A_d}(p_c - p). \tag{33}$$

In three dimensions $T^* = 0$ because the quantum decay is effectively suppressed. As discussed above for the quantum rotor model, there is a more useful, $p$-independent, crossover temperature scale. Using the same arguments as in the Gross-Pitaevskii limit, we can find the temperature separating the quantum and thermal decay regimes in one and two dimensions:

$$T_q \sim JNA_{d-1}A_d^{-\frac{7-2d}{5-2d}}(1-u)^{-\frac{1}{10-4d}}. \tag{34}$$

We see that near the Mott transition the crossover temperature strongly depends on interaction $u$. Thus as $u \to 1$, in one and two dimensions $T_q \to \infty$, and therefore the quantum decay always dominates over the thermal. In particular, in two dimensions we find $T_q \sim 0.03 JN/\sqrt{1-u}$, and in the one-dimensional case $T_q \sim 0.1 JN/(1-u)^{1/6}$; i.e., the crossover temperature is very high and the thermal decay is unimportant.

## 5. DYNAMICS OF THE CURRENT DECAY

We have seen that except for one case, corresponding to the three-dimensional relativistic model at $T = 0$, there is no sharp transition between the superfluid current-carrying state and the insulating state with no current. Indeed, in all other cases the tunneling action and the energy barrier vanish continuously as the system approaches the modulation instability. Thus, instead of a sharp transition boundary we can define a broad crossover region, defined roughly by $1 < S_d < 3$, which separates the superfluid phase with a relatively slow current decay and the insulating phase with a fast decay. The fact that the transition is broad does not imply, however, that within a single experiment a gradual current decay will be detected as the system is slowly tuned through the crossover region. The tunneling and thermal decay rate define a probability of creating a single phase slip per lattice site. The subsequent evolution, after a single phase slip has been created, can take one of two general routes. In the overdamped scenario, the phase slip rapidly dissipates its energy into phonon (Bogoliubov) modes and brings the system to the next metastable minimum with slightly lower current. In the second, underdamped scenario, the phase slip continues to unwind, triggering complete decay of the current in a single step.

In a closed system, i.e. with no coupling to the environment, these two regimes are well defined because the damping of the phase slip comes from the internal degrees of freedom, which are completely described by the equations of motion. Furthermore, near the critical current, whether the phase slip was thermally activated or induced by quantum tunneling, should make little difference for the dynamics that follow. This is because the energy barrier is very small, so the classically allowed motion following the tunneling event starts very close to the metsatable maximum, where it would start following thermal activation. To see what type of decay modes are realized in the Gross-Pitaevskii regime, we solve the classical equations of motion (3) numerically. We start from a uniform current state in a periodic lattice. To allow for current decay we add small fluctuations to the initial values of the classical fields $\psi_j(t=0)$. This mimics the effect of thermal fluctuations. In Fig. 1 we plot the computed current versus time for a one-dimensional array of $M = 200$ sites. Initially the system is assumed to be noninteracting ($U = 0$) and prepared in an eigenstate with a given phase gradient $p$ (specifically we consider $p = 2\pi/5$ and $p = \pi/10$). Then, the interaction is gradually increased in time reaching a constant value, and we follow the time evolution of the current. It is clear from the figure that the phase slips in the smaller current case ($p = \pi/10$) are overdamped leading to gradual decay. There are initially 10-phase twists in the system, and indeed, it is evident that each phase slip decreases the current by roughly 10%. On the other hand for the larger current ($p = 2\pi/5$) a single phase slip generates immediate current decay in the whole sample consistent with the underdamped regime. We will not attempt here to find the precise boundary between the two scenarios. However, we stress that near the instability the system is always in the underdamped regime. We checked that a similar overdamped to underdamped crossover occurs in other spatial dimensions. So if $p$ is not too small, then in a given experimental run, one will always see a sharp transition from the superfluid to the insulating regime. However the precise point, where the current decays will vary from run to run. The broadened transition below the critical current, which was the subject of this paper, will be evident from accumulated statistics of the point where the rapid decay occurred. On the other hand, in the absence of any fluctuations the transition would seem very sharp, and always occurs at $p = \pi/2$.

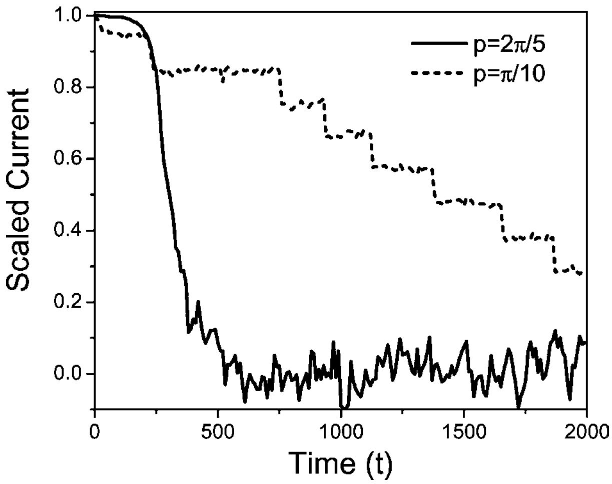


**Fig. 1.** Current (scaled to 1 at $t = 0$) versus time for a one-dimensional periodic array of 200 sites with two different initial phase gradianets. The evolution is determined solving equations of motion (3) with constant hopping amplitude $J = 1$ and interaction increasing in time $U = 0.01 \tanh 0.01t$ for $p = 2\pi/5$ and $U = \tanh 0.01\, t$ for $p = \pi/10$. To get the current decay we add small fluctuations to the initial values of the classical fields $\psi_j(t=0)$.

We did not carry out a similar numerical analysis for small currents close to the Mott transition. There are some reasons to anticipate that the decay will be overdamped in this case even close to the critical current. In particular, because the size of the phase slip in this case is large, it should be able to easily dissipate energy into phonon modes.

It is worth mentioning, that if the motion of phase slips is underdamped, then in a truly infinite system the current state is always unstable. Indeed the probability of a phase slip in the whole system is proportional to its size $M$. If a single phase slip triggers the current decay in the whole sample, then obviously a state with a uniform phase gradient cannot exist. However, in finite size systems these effects are not so crucial, because the decay probability depends exponentially on the couplings and current, but only linearly on the system size.

## 6. CONCLUSIONS

The modulational instability can be observed either by accelerating the condensate or by increasing the lattice potential and driving the system closer to the Mott transition, while the condensate is in motion. We showed that because of quantum or thermal effects, the current decays before the system becomes classically unstable. Therefore instead of a sharp transition, there is a crossover region where the decay rate grows from being exponentially small to large. The crossover region becomes narrower either deep in the superfluid regime (i.e. $JN \gg U$) or in higher dimensions. In particular, in the three-dimensional case we always expect a very sharp boundary separating the regions with very weak and very strong decay rates.

We found that deep in the superfluid regime the crossover temperature separating quantum and thermal decay is of the order of plasma frequency in all dimensions. At small currents, close to the Mott phase, the decay occurs predominantly through thermal fluctuations in three dimensions and through quantum tunneling in one and two dimensions. In the two-dimensional case the quantum tunneling becomes appreciable only at extremely low temperatures or very close to the Mott transition.

We argue that both overdamped and underdamped dynamics of the current decay can be realized in these systems. The underdamped regime corresponds to high currents close to $p = \pi/2$, while at low currents the dynamics is overdamped.

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