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Decentralized Observer with a Consensus Filter for Distributed Discrete-Time Linear Systems

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Abstract— **This paper presents a decentralized observer with a consensus filter for the state observation of a discrete-time linear distributed systems. In this setup, each agent in the distributed system has an observer with a model of the plant that utilizes the set of locally available measurements, which may not make the full plant state detectable. This lack of detectability is overcome by utilizing a consensus filter that blends the state estimate of each agent with its neighbors' estimates. We assume that the communication graph is connected for all times as well as the sensing graph. It is proven that the state estimates of the proposed observer asymptotically converge to the actual plant states under arbitrarily changing, but connected, communication and sensing topologies. As a byproduct of this research, we also obtained a result on the location of eigenvalues, the spectrum, of the Laplacian for a family of graphs with self-loops.**

I. Introduction

Decentralized estimation [1] has long been an active area of research with an increased recent interest in distributed systems. Here we focus on a decentralized estimation problem for a distributed system with multiple agents, where each agent estimates the state of the whole system. In this problem setup, each agent represents a physical entity such as a spacecraft or an aircraft in a formation. Some of the earlier research in decentralized estimation focused on combining the state estimates of a system with multiple agents into a single central estimate [2], [3], [4], where all the information is communicated to all agents in the system back and forth. This is a communication intensive approach and may not be appropriate for distributed systems with a large number of agents. The main idea behind these algorithms is to blend independently obtained state estimates into a single better state estimate, which has been the main idea behind the more recent algorithms as well. In the covariance intersection method [5], [6], the state estimates and their error covariance matrices are exchanged without the exact knowledge of correlation between the estimates of the different agents. The unknown correlation between the exchanged state estimates is bounded by a bound on the intersection of the error covariance matrices. This method ensures that the unknown correlations are accounted for, but it requires the computation of the error covariances and their inverses, which can be computationally demanding. In a recent approach to distributed system state estimation, Ref. [7] considers a fusion center that combines measurements or state estimates from the agents into a single estimate by using a Kalman filter with a particular structure. However we have to treat each agent as a fusion center if this approach is to be adapted, which considerably increases the complexity in the information routing problem.

A large number of recent research study the consensus problems in distributed systems in a graph theoretical framework [8], [9], [10], [11], [12], [13] to tackle the difficulties of the distributed estimation and control. The distributed Kalman filters with embedded consensus filters are studied in [14], [15]. Particularly [15] introduces a state estimator for continuous time linear time systems with a consensus filter that blends state estimates of neighboring agents, which motivated the particular observer structure we adapted in this paper. Our paper provides a stable observer with a consensus filter for a discrete-time distributed system with time varying communication topologies and measurement matrices. The synthesis of stable observers for discrete-time distributed linear time systems is non-trivial. In [15] a quadratic Lyapunov function of the estimation error is constructed by using the time-varying covariance matrix that is computed via a Riccati matrix differential equation. However this Lyapunov function has the right properties to be a valid candidate under non-trivial observability conditions on the linear system [16]. These conditions must be satisfied by the system matrices that are time-varying, which is non-trivial to verify. This happens in distributed systems due to constantly changing sensing topology, i.e., the set of measurements available to each agent changes as an unknown function of time.

This paper presents an observer with an embedded consensus filter for a class of discrete-time linear systems. Each agent utilizes its local measurements and its neighbors' communicated state estimates to update its own state estimate. This observer architecture makes the information routing problem straight-forward. The local measurement vectors are described linearly as a function of the plant state via time varying matrices. The local measurements do not provide the full state detectability that is required to have asymptotically convergent local observers. The consensus filters of each agent blends the state estimates with its neighbors' estimates to overcome this limitation by seeking consensus among neighbors' estimates. The consensus filters update their internal states more frequently then the local observers, that is, there are multiple consensus updates in between the observer state updates. This ensures that a sufficient level of consensus is reached for the stability of the observers. The number of consensus state updates between consecutive observer state

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updates is analytically determined ahead of time. The main contribution of this paper is the proof of the exponential stability of the observer error dynamics under time-varying communication and sensing topologies. We provide a method to compute quadratic Lyapunov functions that prove the exponential stability of the observer error dynamics. We also present a useful graph theoretic result, which is a byproduct of this research, on the smallest eigenvalue of the Laplacians for a class of undirected graphs with self-loops.

Notation

The following is a partial list of notation used in this paper: $Q = Q^T > (\geq) 0$ implies Q is a symmetric positive (semi-)definite matrix; $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ represents a finite graph with a set of vertices $\mathbf{V}$ and edges $\mathbf{E}$ with $(i, j) \in \mathbf{E}$ denoting that there is an edge between the vertices i and j; $\mathcal{L}(\mathbf{G})$ is the Laplacian matrix for the graph $\mathbf{G}$; $\mathbf{a}(\mathbf{G})$ is the algebraic connectivity of the graph G, which is the second smallest eigenvalue of $\mathcal{L}(\mathbf{G})$; $\mathbb{R}^n$ is the n dimensional real vector space; $\|v\|$ is the 2-norm of the vector v; I is the identity matrix of appropriate dimension and I_m is the identity matrix in $\mathbb{R}^{m\times m}$; $\mathbf{1}_m$ is a column vector of ones in $\mathbb{R}^m$; $\mathbf{e}_i$ is a vector with its ith entry $+1$ and the rest of entries zero; $\sigma(A)$ is the set of all eigenvalues of the matrix A and $\sigma_+(A)$ is the set of all of its positive eigenvalues; $\max(\sigma(P))$ and $\min(\sigma(P))$ are maximum and minimum eigenvalues of symmetric matrix P; "$\otimes$" is the Kronecker product; $(v_1, v_2, ..., v_m)$ represents a vector obtained by augmenting vectors $v_1, \ldots, v_m$ such that:

$$(v_1, v_2, ..., v_m) \equiv \begin{bmatrix} v_1^T & v_2^T & \ldots & v_m^T \end{bmatrix}^T.$$

E is the vertex-edge adjacency matrix, A adjacency matrix, and D is the diagonal matrix of node in-degrees for $\mathbf{G}$, then the following gives a relationship to compute the Laplacian matrix

$$\mathcal{L}(\mathbf{G}) = E^T E = D - A.$$

The Laplacian matrix is a symmetric matrix with non-negative diagonal and non-positive off-diagonal entries.

The following relationships are well known in the literature [17] and [18] for a connected undirected graph $\mathbf{G}$ with N vertices and without any *self-loops or multiple edges*

$$\mathbf{a}(\mathbf{G}) \geq 2(1 - \cos(\pi/N)) \quad (1)$$

$$\max(\sigma(\mathcal{L}(\mathbf{G}))) \leq 2\mathbf{d}(\mathbf{G}) \quad (2)$$

where $\mathbf{d}(\mathbf{G})$ is the maximum in-degree of $\mathbf{G}$. Indeed the inequality (2) is valid for any undirected graph without self-loops or multiple edges whether they are connected or not. Next we characterize the location of the Laplacian eigenvalues for a connected undirected graph $\mathbf{G}$ with self-loops. Having a self-loop does not change whether a graph is connected or not, that is, a graph with self-loops is connected if and only if the same graph with the self-loops removed is connected. Furthermore we define the Laplacian of an undirected graph with at least one self-loop as

$$\mathcal{L}(\mathbf{G}) = \mathcal{L}(\mathbf{G}_o) + \sum_{(i,i)\in\mathbf{E}} \mathbf{e}_i \mathbf{e}_i^T \quad (3)$$

where $\mathbf{G}_o$ is the largest subgraph of $\mathbf{G}$ with the self-loops removed, and

$$\mathcal{L}(\mathbf{G}_o) = \sum_{(i,j)\in\mathbf{E},\, i\neq j} (\mathbf{e}_i - \mathbf{e}_j)(\mathbf{e}_i - \mathbf{e}_j)^T. \quad (4)$$

II. System Description

We consider the problem of decentralized state observation for the following discrete-time linear system representing a group of N collaborative agents:

$$x_{k+1} = Ax_k \quad (5)$$

$$y_{i,k} = C_{i,k}x_k, \quad i = 1, \ldots, N \quad (6)$$

where $x_k \in \mathbb{R}^n$ is the state vector at time instance k and $y_{i,k} \in \mathbb{R}^{m_{i,k}}$ is the measurement vector of the ith agent at time instance k. In this scenario, each agent has its own measurements determined by the measurement matrix $C_{i,k}$ and it has direct communication links with a subset of other agents, which will be referred as the "neighbors". The set of communication links in between the agents determine the communication topology and an associated graph, $\mathbf{G}_{c,k}$, where each agent is represented by a vertex of $\mathbf{G}_{c,k}$, and each communication link is represented by an edge of $\mathbf{G}_{c,k}$. We assume that the graph $\mathbf{G}_{c,k}$ is a *undirected connected* graph [19] without self-loops or multiple edges for all times, which implies that [17] $\mathbf{a}(\mathbf{G}_{c,k}) > 0$ for all $k = 0, 1, \ldots$

We consider a "core" set of m measurements z_k,

$$z_k = \begin{bmatrix} z_{1,k} \\ \vdots \\ z_{m,k} \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 \\ \vdots \\ C_m \end{bmatrix}}_{C} x_k \quad \text{where} \quad z_{i,k} \in \mathbb{R}^p \ \forall\, i. \quad (7)$$

such that all locally available "actual" measurements for each agent can be formed as a linear combination of the core measurements as follows

$$y_{i,k} = (E_{i,k} \otimes I_p)\, z_k, \qquad i = 1, \ldots, N. \quad (8)$$

where $y_{i,k} \in \mathbb{R}^{m_{i,k}}$, $E_{i,k} \in \mathbb{R}^{q_{i,k}\times m}$, $i = 1, ..., N$, are "vertex-edge adjacency" matrices, hence $m_{i,k} = q_{i,k}p$, with p being the size of the local, core measurement vector, $z_{i,k}$. A vertex-edge adjacency matrix describes an edge between two vertices or a single vertex (for a self-loop) in a graph on each of its rows, whose entries corresponding to these vertices are $+1$ and -1 (it does not matter which entry is $+$ or $-$) and the rest of the entries are zeros. Note that if the edge described by a row is a self-loop then there is only one non-zero entry with $+1$.

The assumption that all actual measurements can be expressed in terms of the core measurements adds more structure to the problem at hand without losing much generality, and its use will become apparent in later sections.

Next we collect the set of all distinct local measurements into a global measurement vector y_k as follows

$$y_k = (E_k \otimes I_p)\, z_k = (E_k \otimes I_p)\, Cx_k, \quad (9)$$

where the vertex-edge adjacency matrix E_k contains all the distinct rows of all $E_{i,k},\ i = 1, ..., N$, that is, E_k is a vertex-edge adjacency matrix of a graph without multiple edges. Therefore, y_k is not necessarily an augmentation of all local measurements in general, that is, $y_k \neq (y_{1,k}, \ldots, y_{N,k})$ in general. Moreover a local measurement vector $y_{i,k}$ for any agent can simply be obtained by picking the right entries of the vector y_k. Consequently, a row of E_k can correspond to a measurement that belongs to multiple agents, that is, a measurement can be available to multiple agents. For each agent we will define a vector $h_{i,k} \in \mathbb{Z}^{q_{i,k}}$ that contains the positive integer numbers representing how many agents each measurement is available to. This implies that

$$E_k^T E_k = \sum_{i=1}^{N} E_{i,k}^T \left(\mathrm{diag}(h_{i,k})\right)^{-1} E_{i,k}. \qquad (10)$$

In summary, the sensing graph $\mathbf{G}_{s,k}$ is constructed with its vertices as the core set of measurements $z_{1,k}, \ldots, z_{m,k}$ and its edges represent the actual measurements at time instance k. Since a core measurement can also be an actual measurement, e.g., $y_{i,k} = z_{j,k}$, a sensing graph can have self loops, and in the case when all measurements are the core ones, the sensing graph can be completely disconnected in the usual sense. We introduce a concept of *pseudo-connected* graphs to capture useful properties of the sensing graphs that will be encountered (see Figure 8 for an example of a pseudo-connected graph).

Definition 1: An undirected graph $\mathbf{G}(\mathbf{V}, \mathbf{E})$ without multiple edges is defined to be pseudo-connected if every vertex is connected to itself and/or to another vertex and if every connected subgraph of $\mathbf{G}$ has at least one vertex with a self-loop.

Given the above definition, the following conditions are assumed to hold for the system defined by equations (5) and (6):

A1) $\mathbf{G}_{c,k}$ is a connected graph without self-loops or multiple edges $\forall\, k$.
A2) $\mathbf{G}_{s,k}$ is pseudo-connected without multiple edges $\forall\, k$.
A3) The pair (C, A) is detectable.
A4) Each agent knows $h_{i,k}$ at any given time instance k.

Assuming a pseudo-connected sensing graph implies that one or more of the core measurements are among the actual measurements at any given time. The assumption of having a connected communication graphs can be relaxed to, for example, having jointly connected communication graphs [20]. Such relaxations can lead to some generalizations of the forthcoming results, which is beyond the scope of this paper. The detectability of (C, A) pair ensures that an exponentially stable observer exists by utilizing only the core measurements. The last assumption of each agent having the information of the vectors $h_{i,k}$ simply means that each agent knows how many other agents have access to the information that it has. The $h_{i,k}$ vectors can be routed to each agent in real-time, or the distributed system at hand may have the working assumption that each measurement is known by a fixed number of agents at any given time.

III. Decentralized Observer with Consensus Filter

We propose the following local observers with a consensus filter that process both the locally available measurements and the neighbors' state estimates:

<u>*Local Observers with Consensus Filter*</u>

$$\begin{aligned}
\hat{x}_{i,k+1} &= A s_{i,k} + L_{i,k}(C_{i,k} s_{i,k} - y_{i,k}) && (11)\\
\xi_{i,l+1} &= \xi_{i,l} - \sum_{j \in S_{i,k}} \delta(\xi_{i,l} - \xi_{j,l}) && (12)\\
\xi_{i,0} &= \hat{x}_{i,k}, \qquad l = 1 \ldots r\\
s_{i,k} &= \xi_{i,r}, \qquad i = 1 \ldots N.
\end{aligned}$$

where r is the number of iterations, *consensus state updates*, per single time step, $\delta > 0$ is a design parameter, $S_{i,k}$ is the index set of neighbors for the agent i. The gain matrices $L_{i,k}$ are computed by using the matrix L, which is defined as the *core* observer gain matrix corresponding to the core measurement z_k, as follows

$$L_{i,k} = L\left(E_{i,k}^T \mathrm{diag}(h_{i,k})^{-1} \otimes I_p\right). \qquad (13)$$

The choices for the scalars r and δ will be explained later in the paper. Here we assume that the consensus filter can be iterated as many times as the integer r dictates during a single time step. Hence r can be seen as a design parameter that determines how fast the consensus dynamics need to be for the stability of this observation algorithm. Clearly the number r can be too large to be handled by the available communication hardware. Hence one of our design objectives is to determine the least conservative upper bound on the number of consensus updates r.

IV. Synthesis for a Stable Observer

In this section, we present a constructive proof of the exponential stability of the proposed decentralized observation algorithm. As a by-product of this proof, we obtain synthesis procedures to compute the observer gain matrix L and parameter r.

Let ξ_l be the overall (*stacked-up*) vector of $\xi_{i,k} \in \mathbb{R}^{nN}$ where N is the number of spacecraft and n is the number of states per agent.

$$\xi_l = (\xi_{1,l}, \xi_{2,l}, \ldots, \xi_{N,l})$$

Similarly, $\hat{x}_k$ and X_k (both in $\mathbb{R}^{nN}$) can be expressed as

$$\begin{aligned}
\hat{x}_k &:= (\hat{x}_{1,k}, \hat{x}_{2,k}, \ldots, \hat{x}_{N,k})\\
X_k &:= \mathbf{1}_N \otimes x_k = (x_k, x_k, \ldots, x_k).
\end{aligned}$$

From equation (12):

$$\begin{aligned}
\xi_{l+1} &= (I_{nN} - \delta\mathcal{L}_{c\otimes,k})\xi_l, \quad l = 1, \ldots, r\\
\Rightarrow \quad \xi_r &= (I_{nN} - \delta\mathcal{L}_{c\otimes,k})^r \hat{x}_k
\end{aligned}$$

where $\mathcal{L}_{c\otimes,k} = \mathcal{L}_{c,k} \otimes I_n$, and $\mathcal{L}_{c,k}$ is the Laplacian matrix of the communication graph $\mathbf{G}_{c,k}$ at time k. Let the observation error be defined as $e_{i,k} := \hat{x}_{i,k} - x_k$ and $e_k = (e_{1,k}, \ldots, e_{N,k})$, then using the equation (11), we have

$$\begin{aligned} e_{i,k+1} &= \hat{x}_{i,k+1} - x_{k+1} \\ &= As_{i,k} + L_{i,k}(C_{i,k}s_{i,k} - y_{i,k}) - Ax_k. \end{aligned}$$

Since $(\mathcal{L}_{c,k} \otimes I_n)(\mathbf{1}_N \otimes x_k) = \mathcal{L}_{c,k}\mathbf{1}_N \otimes x_k = 0$ and

$$(I_{nN} - \delta\mathcal{L}_{c\otimes,k})^r = I + (\ldots)\mathcal{L}_{c\otimes,k} + (\ldots)\mathcal{L}^2_{c\otimes,k} + \ldots,$$

we have

$$\begin{aligned} (I_{nN} - \delta\mathcal{L}_{c\otimes,k})^r X_k &= (I_{nN} - \delta\mathcal{L}_{c\otimes,k})^r(\mathbf{1}_N \otimes x_k) \\ &= (\mathbf{1}_N \otimes x_k). \end{aligned}$$

Then, the overall error dynamics, e_k can be expressed as:

$$\begin{aligned} e_{k+1} &= A_c(I_{nN} - \delta\mathcal{L}_{c\otimes,k})^r e_k \\ &= A_c[(I_N - \delta\mathcal{L}_{c,k})^r \otimes I_n]e_k \qquad (14) \end{aligned}$$

where

$$\begin{aligned} A_c &= \mathrm{diag}\{A + L_{i,k}C_{i,k};\ i=1,..,N\} \qquad (15) \\ &= \mathrm{diag}\left\{A + L(E^T_{i,k}\mathrm{diag}(h_{i,k})^{-1}E_{i,k} \otimes I_p)C;\right\} \\ &\quad \text{where} \quad i=1,..,N \end{aligned}$$

Theorem 1: Suppose that the sensing graph is pseudo-connected and communication graph is connected for all $k = 1, 2, \ldots$, and there exist some L, $P = P^T > 0$, and $\lambda \in [0, 1)$ such that the following inequality holds for the Laplacian, $\mathcal{L}_s$, of any pseudo-connected sensing graph $\mathbf{G}_s$,

$$\begin{aligned} &\lambda P - A_a(\mathcal{L}_s)^T P A_a(\mathcal{L}_s) \geq 0 \quad \text{where} \qquad (16) \\ &A_a(\mathcal{L}_s) := A + \frac{1}{N}L(\mathcal{L}_s \otimes I_p)C \end{aligned}$$

Let $\delta \in (0, 1/(N-1))$. Then there exists a large enough positive integer $r \geq 1$ such that the error dynamics of the observer given by the equation (14) are globally exponentially stable (GES), hence the observer given by equations (11) and (12) is GES and, for $i = 1, \ldots, N$,

$$\|\hat{x}_{i,k} - x_k\| \leq c_i\tilde{\lambda}^k_i\|\hat{x}_{i,k} - x_k\|, \quad \forall k = 0, 1, \ldots \qquad (17)$$

for some $c_i > 0$ and $\tilde{\lambda}_i \in (0, 1)$.

Proof: We consider the following Lyapunov function for the error dynamics

$$V(e_k) = e^T_k(I_\alpha \otimes P)e_k \quad \text{where} \quad I_\alpha = \begin{bmatrix} 1 & 0 \\ 0 & \alpha I_{N-1} \end{bmatrix} \qquad (18)$$

and $\alpha > 0$. The first step is to find an appropriate transformation that will split the error vector e_k into *agreement* and *disagreement* subspaces. These subspaces bring a geometrical insight to the observer synthesis, and help clarifying the roles of the measurement and communication feedback terms in the observer. Consider the following transformation

$$T = \underbrace{\left[\begin{array}{c|cccc} \frac{1}{\sqrt{N}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{N(N-1)}} \\ \hline & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{N(N-1)}} \\ & 0 & \frac{-2}{\sqrt{6}} & \cdots & \frac{1}{\sqrt{N(N-1)}} \\ \frac{\mathbf{1}}{\sqrt{N}} & \vdots & 0 & \ddots & \frac{1}{\sqrt{N(N-1)}} \\ & 0 & 0 & 0 & \frac{-(N-1)}{\sqrt{N(N-1)}} \end{array}\right]}_{T_c} \otimes I_n. \qquad (19)$$

It can easily be shown that columns of T and T_c has 2-norm equal to one and they are orthogonal to each other, hence T and T_c are orthogonal matrices such that $T^T T = TT^T = I_{nN}$ and $T_c T^T_c = T_c T^T_c = I_N$. Note that, for any graph without self loops or multiple edges $\mathbf{G}$, such as the graph of a communication topology, we have

$$\mathcal{L}(\mathbf{G}) = \left[\begin{array}{c|c} \mathbf{1}^T v & -v^T \\ \hline -v & V \end{array}\right] \qquad (20)$$

for some vector $v \geq 0$ and matrix $V = V^T$, which are related by $V\mathbf{1} = v$, and we can express matrix T_c as follows

$$T_c = \left[\begin{array}{c|c} 1/\sqrt{N} & w^T \\ \hline \mathbf{1}/\sqrt{N} & U \end{array}\right]$$

with appropriately defined vector w and matrix U. Now we can show that

$$T^T_c\mathcal{L}(\mathbf{G})T_c = \left[\begin{array}{c|c} 0 & \mathbf{0} \\ \hline \mathbf{0}^T & \mathcal{L}_p(\mathbf{G}) \end{array}\right] \qquad (21)$$

where $\mathcal{L}_p(\mathbf{G}) \in \mathbb{R}^{(n-1)\times(n-1)}$ is a symmetric matrix given by

$$\mathcal{L}_p(\mathbf{G}) = (1^T v)ww^T - wv^T U - U^T vw^T + U^T VU. \qquad (22)$$

This can be shown as follows:

$$T^T_c\mathcal{L}(\mathbf{G})T_c = \begin{bmatrix} 1 & \mathbf{1}^T \\ w & U^T \end{bmatrix} \begin{bmatrix} \underbrace{\mathbf{1}^T v - v^T\mathbf{1}}_{0} & 1^T vw^T - v^T U \\ \underbrace{-v + V\mathbf{1}}_{\mathbf{0}} & -vw^T + VU \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \underbrace{1^T vw^T - 1^T vw^T}_{\mathbf{0}^T} \underbrace{-v^T U + \overbrace{1^T V}^{v^T} U}_{\mathbf{0}^T} \\ 0 & \mathcal{L}_p(\mathbf{G}) \end{bmatrix}.$$

Note that T_c (hence T) is a universal transformation that does not depend on the graph at hand, and it generates $\mathcal{L}_p(\mathbf{G})$ (that is a function of the graph), which is symmetric. Furthermore, since T_c is used as a similarity transformation, for any connected graph $\mathbf{G}$ without self-loops or multiple edges, $\sigma(\mathcal{L}(\mathbf{G})) = \sigma(T^T_c\mathcal{L}(\mathbf{G})T_c)$. This implies that

$$\sigma(\mathcal{L}_p(\mathbf{G})) = \sigma(\mathcal{L}(\mathbf{G})) \backslash \{0\} \subset [2(1 - \cos(\pi/N)), 2\mathbf{d}(\mathbf{G})]. \qquad (23)$$

Define transformed error as $\tilde{e}_k \triangleq T^T e_k$. Then the equation (14) can be written as:

$$\begin{aligned} \tilde{e}_{k+1} &= T^T A_c[(I_N - \delta\mathcal{L}_{c,k})^r \otimes I_n]T\tilde{e}_k \\ &= \underbrace{T^T A_c T}_{:= \tilde{A}_c} \underbrace{T^T[(I_N - \delta\mathcal{L}_{c,k})^r \otimes I_n]T}_{:= \Lambda^r_k \otimes I_n} \tilde{e}_k \qquad (24) \end{aligned}$$

Next we derive an expression for Λ_k. Noting that $T = T_c \otimes I_n$, we have

$$T^T[(I_N - \delta\mathcal{L}_{c,k})^r \otimes I_n]T = [T_c^T(I_N - \delta\mathcal{L}_{c,k})^r T_c] \otimes I_n.$$

Here we have,

$$\begin{aligned} & T_c^T(I_N - \delta\mathcal{L}_{c,k})^r T_c \\ = \; & T_c^T I_N T_c + c_1 T_c^{-1}\mathcal{L}_{c,k}T_c + c_2 T_c^{-1}\mathcal{L}_{c,k}^2 T_c + \ldots \\ = \; & I_N + c_1\tilde{\mathcal{L}}_{c,k} + c_2\tilde{\mathcal{L}}_{c,k}^2 + \ldots = (I_N - \delta\tilde{\mathcal{L}}_{c,k})^r, \end{aligned}$$

where c_1 and c_2 etc., are some constants, and the newly defined $\tilde{\mathcal{L}}_{c,k}$ is

$$\tilde{\mathcal{L}}_{c,k} \triangleq T_c^T \mathcal{L}_{c,k} T_c = \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathcal{L}_{p,k} \end{bmatrix}$$

which is obtained by noting that the first column of T is in the null space of $\mathcal{L}_k$. Consequently

$$\begin{aligned} (I_N - \delta\tilde{\mathcal{L}}_{c,k})^r &= \left(\begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_{N-1} \end{bmatrix} - \delta \begin{bmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathcal{L}_{p,k} \end{bmatrix} \right)^r \\ &= \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & I_{N-1} - \delta\mathcal{L}_{p,k} \end{bmatrix}^r. \end{aligned}$$

This transformation allows us to project the overall estimation error vector $\tilde{e}_k$ into its components in the *agreement* subspace, ϵ_k, and the *disagreement* subspace, η_k:

$$\tilde{e}_k = \begin{bmatrix} \epsilon_k \\ \eta_k \end{bmatrix} \Rightarrow \begin{bmatrix} \epsilon_{k+1} \\ \eta_{k+1} \end{bmatrix} = \underbrace{\begin{bmatrix} A_{a,k} & F_k \Lambda_{p,k}^r \\ G_k & A_{d,k}\Lambda_{p,k}^r \end{bmatrix}}_{A_e} \begin{bmatrix} \epsilon_k \\ \eta_k \end{bmatrix}, \tag{25}$$

where $\Lambda_{p,k} = (I_{N-1} - \delta\mathcal{L}_{p,k}) \otimes I_n$ and

$$A_{a,k} = A + \frac{1}{N} L \left(\sum_{i=1}^N E_{i,k}^T E_{i,k} \otimes I_p \right) C.$$

$$G_k = \begin{bmatrix} \frac{L_1C_1 - L_2C_2}{\sqrt{2N}} \\ \frac{L_1C_1 + L_2C_2 - 2L_3C_3}{\sqrt{6N}} \\ \vdots \\ \frac{\sum_{i=1}^{N-1} L_iC_i - (N-1)L_NC_N}{\sqrt{N^2(N-1)}} \end{bmatrix},$$

$$F_k = \begin{bmatrix} \frac{L_1C_1 - L_2C_2}{\sqrt{2N}} & \ldots & \frac{\sum_{i=1}^{N-1} L_iC_i - (N-1)L_NC_N}{\sqrt{N^2(N-1)}} \end{bmatrix}.$$

$\Lambda_{p,k}$ is a symmetric matrix with $2(1 - \cos(\pi/N))I \leq \Lambda_{p,k} \leq 2\mathbf{d}(\mathbf{G}_{c,k})I$. Hence a choice of $\delta \in (0, 1/(N-1))$ renders the eigenvalues of the Laplacian of any connected communication graph inside the unit circle, i.e., $\sigma(\Lambda_{p,k}) < 1$. With the transformed state we can express the Lyapunov function in equation (18) $\tilde{V}(\tilde{e}_k)$ as follows

$$\begin{aligned} & \tilde{V}(\tilde{e}) = V(T_\alpha \tilde{e}_k) \\ = \; & \tilde{e}_k^T (I_\alpha^{-\frac{1}{2}} T_c^T I_\alpha^{-\frac{1}{2}} \otimes I_n)(I_\alpha \otimes P)(I_\alpha^{-\frac{1}{2}} T_c I_\alpha^{-\frac{1}{2}} \otimes I_n)\tilde{e}_k \\ = \; & \tilde{e}_k^T (I_\alpha^{-\frac{1}{2}} T_c^T I_\alpha^{-\frac{1}{2}} I_\alpha I_\alpha^{-\frac{1}{2}} T_c I_\alpha^{-\frac{1}{2}} \otimes P)\tilde{e}_k = \tilde{e}_k^T (I_\alpha^{-1} \otimes P)\tilde{e}_k. \end{aligned}$$

where I_α^{-1} is a positive definite matrix. Consider some $\gamma \in (\lambda, 1)$, we have

$$\begin{aligned} & \gamma\tilde{V}(\tilde{e}_k) - \tilde{V}(\tilde{e}_{k+1}) = \\ = \; & \tilde{e}_k^T \underbrace{(\lambda(I_\alpha^{-1} \otimes P) - A_e^T(I_\alpha^{-1} \otimes P)A_e)}_{:= S} \tilde{e}_k. \end{aligned} \tag{26}$$

Then we can express the matrix S as

$$\begin{aligned} S &= \begin{bmatrix} \gamma P & 0 \\ 0 & \gamma(\alpha^{-1} I_{N-1} \otimes P) \end{bmatrix} \\ &- \begin{bmatrix} A_{a,k} & F_k\Lambda_{p,k}^r \\ G_k & A_{d,k}\Lambda_{p,k}^r \end{bmatrix}^T \begin{bmatrix} P & 0 \\ 0 & \alpha^{-1} I_{N-1} \otimes P \end{bmatrix} \begin{bmatrix} A_{a,k} & F_k\Lambda_{p,k}^r \\ G_k & A_{d,k}\Lambda_{p,k}^r \end{bmatrix} \\ &= S_1 - S_2 \end{aligned}$$

where S_1 and S_2 are symmetric matrices defined by

$$S_1 = \begin{bmatrix} \gamma P - A_{a,k}^T P A_{a,k} - \alpha^{-1} G_k^T \hat{P} G_k & 0 \\ 0 & \alpha^{-1}\gamma\hat{P} \end{bmatrix}$$

$$S_2 = \begin{bmatrix} 0 & \alpha^{-1} G_k^T \hat{P} A_{d,k} \Lambda_{p,k}^r + A_{a,k} P F \Lambda_{p,k}^r \\ (\alpha^{-1} G_k^T \hat{P} A_{d,k} \Lambda_{p,k}^r + A_{a,k} P F \Lambda_{p,k}^r)^T & \Lambda_{p,k}^r (\alpha^{-1} A_{a,k}^T \hat{P} A_{a,k} + F^T P F) \Lambda_{p,k}^r \end{bmatrix}.$$

with $\hat{P} = I_{N-1} \otimes P$. By using the inequality (16), for any k

$$\begin{aligned} \gamma P - A_{a,k}^T P A_{a,k} &= \underbrace{\lambda P - A_{a,k}^T P A_{a,k}}_{\geq 0} + (\gamma - \lambda) P \\ \Rightarrow \gamma P - A_{a,k}^T P A_{a,k} &\geq (\gamma - \lambda) P > 0. \end{aligned}$$

Hence, $\alpha > 0$ can be chosen large enough such that $S_1 = S_1^T > 0$.

Next observe that $\Lambda_{p,k}$ is a symmetric matrix with all its eigenvalues in $(-1, 1)$. Hence $\lim_{r\to\infty} \Lambda_{p,k}^r = 0$. Since all nonzero blocks of S_2 contain $\Lambda_{p,k}$ as a multiplier, $\lim_{r\to\infty} S_2 = 0$. This implies that the spectral radius of the matrix S_2 can be made arbitrarily small by choosing r large enough for a given α. Consequently, since $S_1 = S_1^T > 0$ by a choice of large enough α, we can guarantee that $S = S_1 - S_2 > 0$, and then choosing r large enough, which then implies that

$$\gamma\tilde{V}(\tilde{e}_k) - \tilde{V}(\tilde{e}_{k+1}) = \tilde{e}_k^T S \tilde{e}_k \geq \tilde{\lambda}\tilde{V}(\tilde{e}_k) > 0 \text{ for } \tilde{e}_k \neq 0,$$

for some $\tilde{\lambda} \in (0, 1)$. Since $\tilde{V}$ is positive definite quadratic function of $\tilde{e}_k$, this implies the exponential stability of the error dynamics. The equation (17) is a direct consequence of the exponential stability of the error dynamics, which concludes the proof. ■

Next we will focus on how to design observer gain matrix L and a valid lower bound on the number of consensus iterations r so that the results of Theorem 1 apply.

A. *Computation of the Observer Gain Matrix L*

This section presents the result for the gain matrix L that quadratically stabilize the agreement subspace of the error dynamics. Note that the equation (25) implies that the error in the agreement subspace evolves, when $\eta_k = 0$, as follows,

$$\epsilon_{k+1} = A_{a,k}\epsilon_k, \quad \text{where} \quad A_{a,k} = A(\mathcal{L}_{s,k}).$$

As an immediate consequence of this observation, the condition (16) to hold for any pseudo-connected sensing graph implies that the error dynamics are quadratically stable in the agreement subspace.

Next we will construct the quadratic Lyapunov function of the condition (16). Since the sensing graph $\mathbf{G}_{s,k}$ is assumed to be pseudo connected, by using Theorem 3 together with equations (1) and (2), we have

$$\begin{aligned} 2\left(1-\cos\frac{\pi}{2N+1}\right)I &\leq \sum_{i=1}^{N} E_{i,k}^T \mathrm{diag}(h_{i,k})^{-1} E_{i,k} = \mathcal{L}_{s,k} \\ &\leq (2\mathbf{d}(\mathbf{G}_{s,o_k})+1)I \end{aligned} \quad (27)$$

where $\mathbf{G}_{s_k}^o$ is the sensing graph with the self-loops removed. Note that, since $\mathbf{G}_{s,k}$ has m vertices, $\mathbf{d}(\mathbf{G}_{s_k}^o) \leq m-1$. Let

$$\begin{aligned} \beta_1 &= 2\left(1-\cos\frac{\pi}{2N+1}\right) \\ \beta_2 &= \max_k 2\mathbf{d}(\mathbf{G}_{s_k}^o)+1 \leq 2m-1. \end{aligned}$$

Now we can express the matrix $A_{a,k} := A_a(\mathcal{L}_{s,k})$ defined in equation (16) for each time instance k as follows

$$\begin{aligned} A_{a,k} &= A+\frac{1}{N}L\left(\mathcal{L}_{s,k}\otimes I_p\right)C+\frac{\beta_1+\beta_2}{2N}LC-\frac{\beta_1+\beta_2}{2N}LC \\ &= A+\frac{1}{N}L\underbrace{\left(\mathcal{L}_{s,k}\otimes I_p-\frac{\beta_1+\beta_2}{2}I\right)}_{\Delta_k}C+\frac{\beta_1+\beta_2}{2N}LC \\ &= A+\frac{\beta_1+\beta_2}{2N}LC+\frac{1}{N}L\Delta_k C \end{aligned}$$

where $-\frac{\beta_2-\beta_1}{2}I \leq \Delta_k \leq \frac{\beta_2-\beta_1}{2}I$. Then the *agreement dynamics* for ϵ_k, when $\eta_k = 0$ for all k, are given by:

$$\begin{aligned} \epsilon_{k+1} &= \underbrace{\left(A+\frac{\beta_1+\beta_2}{2N}LC\right)}_{:=A_\epsilon}\epsilon_k+\frac{1}{N}Lp_k \quad (28) \\ p_k &= \Delta_k q_k \\ q_k &= C\epsilon_k \\ -\tilde{\beta}I &\leq \Delta_k = \Delta_k^T \leq \tilde{\beta}I \end{aligned}$$

where $\tilde{\beta} = \frac{\beta_2-\beta_1}{2} > 0$. We will establish the quadratic stability of the above system via a choice of the gain matrix L. The agreement dynamics given by equation (28) above, is known as the Norm-Bound Linear Differential Inclusion (NLDI) [21]. The standard form of this NLDI can be rewritten as follows,

$$p_k = \Delta_k q_k, \quad \|\Delta_k\| \leq \tilde{\beta} \quad \Leftrightarrow \quad p_k^T p_k \leq \tilde{\beta}^2 q_k^T q_k. \quad (29)$$

The expression (29) above can be rewritten as,

$$\begin{bmatrix} q_k \\ p_k \end{bmatrix}^T M \begin{bmatrix} q_k \\ p_k \end{bmatrix} \geq 0 \quad (30)$$

with M be given by

$$M = \begin{bmatrix} \hat{\alpha}\tilde{\beta}^2 I & 0 \\ 0 & -\hat{\alpha}I \end{bmatrix} \quad (31)$$

where $\hat{\alpha} > 0$ is a scalar variable introduced through S-procedure [21], [22].

Lemma 1: Consider the agreement dynamics given by (28) with the gain matrix L given by $L = P^{-1}S$, where $P = P^T > 0$ and S are obtained by solving the following ADLMI (*Agreement Dynamics LMI*) for some $\lambda \in [0,1)$, with solution variables P, S, and $\hat{\alpha}$,

$$\begin{bmatrix} \lambda P-\hat{\alpha}\tilde{\beta}^2 C^T C & 0 & A^T P+\frac{\beta_1+\beta_2}{2N}C^T S^T \\ 0 & \hat{\alpha}I & \frac{1}{N}S^T \\ PA+\frac{\beta_1+\beta_2}{2N}SC & \frac{1}{N}S & P \end{bmatrix} \geq 0. \quad (32)$$

Then the resulting agreement dynamics (28) are quadratically (hence globally exponentially) stable, and the condition (16) in Theorem 1 is satisfied with L, P, and λ.

Proof: For brevity, we defer the proof to [23]. ■

B. Extending the LMI

In Section IV-C, we compute the the bound on the number of iterations, r. As shown in equation (37), the bound on r will depend on the bound of P-norm of $A_{i,k} := A+L_{i,k}C_{i,k}$ as in equation (15). We propose the lemma which establishes a bound on $\bar{a}$, that is, $\|A_{i,k}\|_P \leq \bar{a}$ for all i and k. Note that, for all indices i and k,

$$\|A_{i,k}\|_P \leq \bar{a} \quad \Leftrightarrow \quad w^T A_{i,k} P A_{i,k} w \leq \bar{a}^2 w^T P w \quad \forall w. \quad (33)$$

Lemma 2: Given $A_{i,k}$ as in equation (15), then $\|A_{i,k}\|_P \leq \bar{a}$ for all i, k if the following matrix inequality is satisfied for some $\tilde{\alpha} > 0$,

$$\begin{bmatrix} \bar{a}^2 P - A^T P A & -A^T S - \tilde{\alpha}\beta_2 C^T & 0 \\ -SA - \tilde{\alpha}\beta_2 C & 2\tilde{\alpha}I & S^T \\ 0 & S & P \end{bmatrix} \geq 0, \quad (34)$$

where $\beta_2 = \max_k 2\deg(\mathbf{G}_{s,k}^o) + 1 (\leq 2m-1)$.

Proof: For brevity, we defer the proof to [23]. ■

C. Bound on Consensus Iterations

This section presents the main synthesis result of this paper, which gives a way to compute a bound on the number of consensus iterations r. In order to compute this bound for the convergence, we utilize the error dynamics,

$$\begin{aligned} e_{k+1} &= A_{c,k}(I_{nN}-\delta\mathcal{L})^r e_k \\ \Rightarrow \tilde{e}_{k+1} &= \tilde{A}_c \Lambda_k^r \tilde{e}_k \quad \text{where} \quad \tilde{e}_k = T^T e_k \end{aligned}$$

where $\tilde{A}_c$ and Λ are defined in equation (24). We also define θ, which is a bound on $\Lambda_{p,k} = (I_{N-1}-\delta\mathcal{L}_{p,k}) \otimes I_n$, where $\mathcal{L}_{p,k} := \mathcal{L}_p(\mathbf{G}_k)$ with $\mathcal{L}_p$ as defined in (22). To compute a nonconservative value for θ we can solve a simple optimization problem. Let $\underline{b}$ and $\bar{b}$ be the lower and upper bound on $\mathcal{L}_{p,k}$ as defined in equations (1) and (2), respectively as follows

$$\begin{aligned} \underline{b} &:= \min_{k=0,1,\ldots} \mathbf{a}(\mathbf{G}_{c,k}) \geq 2(1-\cos(\pi/N)) \quad \text{and} \\ \bar{b} &:= \max_{k=0,1,\ldots} \rho(\mathcal{L}_{c,k}) \leq 2(N-1). \end{aligned}$$

Then we define θ as follows:

$$\theta = \min_{\nu} \left\| (1 - \nu\bar{b},\ 1 - \nu\underline{b}) \right\|_{\infty}. \tag{35}$$

where δ is is the parameter in the consensus part of the observer, equation (12). This optimization problem, besides solving for the optimization variable θ, also provides us with the optimal value for δ in the consensus part of the observer in equation (12) as follows

$$\delta = \operatorname*{argmin}_{\nu} \left\| (1 - \nu\bar{b},\ 1 - \nu\underline{b}) \right\|_{\infty}. \tag{36}$$

Theorem 2: Suppose that there exist $\hat{\alpha} > 0$, $\tilde{\alpha} > 0$, $P = P^T > 0$, and S such that the matrix inequalities (32) and (34) are satisfied for some $\lambda \in [0, 1)$ and $\bar{a} > 0$. Then the overall error dynamics given by (14) are exponentially stable if the integer $r \geq 1$ is chosen such that $\rho(\Gamma(r)) < 1$, where

$$\Gamma(r) = \begin{bmatrix} \sqrt{\lambda} & \bar{a}\theta^r \\ \bar{a} & \bar{a}\theta^r \end{bmatrix} \quad \text{and} \quad \theta \text{ is given by (35).} \tag{37}$$

Proof: For brevity, we defer the proof to [23]. ■

We can choose r large enough such that the largest modulus eigenvalue of the matrix Γ is less than one, which ensures that the error dynamics is exponentially stable. This can be done via a simple line search on r as shown on an example case in Figure 1.

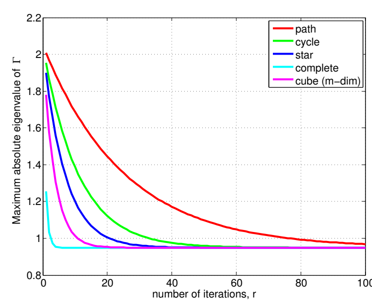

Fig. 1. Maximum absolute eigenvalue of the $\Gamma(r; L, \delta)$ matrix for the example of 8 agents and various topologies

V. Simulation Results

This section presents an application of the decentralized observer with the consensus filter on a formation flying spacecraft in Low Earth Orbit (LEO). The discrete time dynamics equations and the measurement equation are described in [24]. The state to be estimated is a relative translational state of all spacecraft with respect to the designated *leader* spacecraft, which we call the formation state. The size of the formation state vector is $6(N-1)$, where N is the number of spacecraft. The communication and sensing topologies are time-varying. The information is exchanged across the communication channel contains the locally computed formation state vector. The sensing links represent the measurement availability between the two spacecraft connected by the link, where the measurements are linear, position measurements. The information among the spacecraft is exchanged r times during a single time step, which allows for reaching the consensus between the estimates of different spacecraft. The observer gain L is pre-computed by solving for the LMIs in (32) and (34).

Again for brevity, in these simulations the parameters θ, δ and $\bar{a}$ are computed based on the procedure described in the upcoming journal paper [23]. Also, $h_{i,k} = 2$, $\forall i, k$, i.e., if there exist a relative measurement between two spacecraft, both spacecraft have access to it. Simulations show that the algorithm works even with only $r = 3$ even though the theory predicts higher number of iterations for certain topologies.

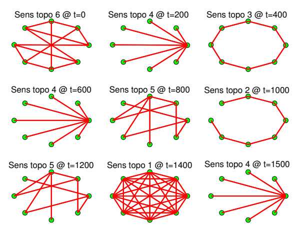

Fig. 2. Sensing, 8 agents

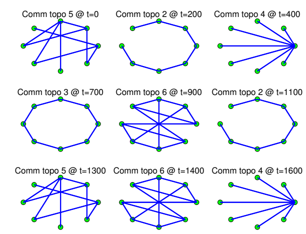

Fig. 3. Comm, 8 agents

The results are shown for the $\Delta t = 1sec$ with 8 spacecraft. The sensing and communication topologies change according to Figures 2, 3. These two topologies may or may not change at the same time. The time of topology change is indicated in the error-plots with the blue circle and red dot, for communication and sensing topology changes respectively.

Figures 4 and 5 show the performance of the algorithm for $\lambda = 0.90$ and $\lambda = 0.80$ respectively, with number of iterations set to $r = 11$. It is worth noting the improvement in convergence when $\lambda = 0.80$ by observing the transient behavior in both cases. On the other hand, in order to decrease the communication bandwidth and speed up the algorithm, we set $r = 3$. Figure 6 shows that the consensus has been reached and algorithm converges. In the transient period the consensus is not achieved as fast as in previous cases, which is expected with decreased number of iterations.

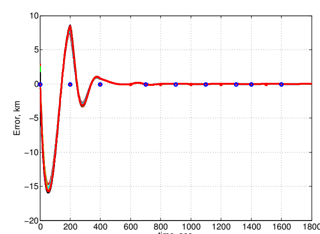

Fig. 4. 8 agents, $\Delta t = 1s$, $\lambda = 0.90$, $r = 11$

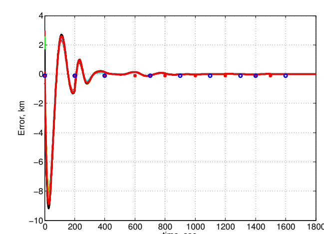

Fig. 5. 8 agents, $\Delta t = 1s$, $\lambda = 0.80$, $r = 11$

Decreasing the number of iterations too much can have negative effects, and in the case when $r = 1$ and $\lambda = 0.90$ the observer becomes unstable, as shown in Figure 7.

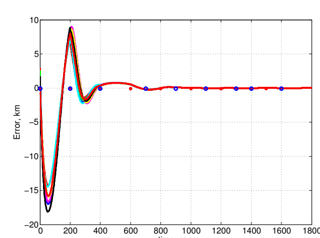

Fig. 6. 8 agents, $\Delta t = 1s$, $\lambda = 0.90$, $r = 3$

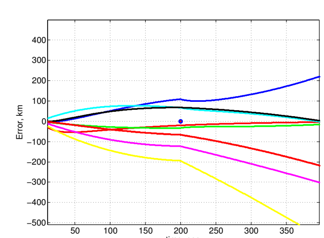

Fig. 7. 8 agents, $\Delta t = 1s$, $\lambda = 0.90$, $r = 1$

VI. Conclusions

The decentralized observer with consensus filter presented in this paper provides a technique for handling time-varying sensing and communication topologies in a distributed system in a computationally efficient way. Proof of convergence for the observer is also provided, which relies on having a sufficient number of consensus state updates between the measurement time instances. This number comes from a theoretical bound and its further refinement may be required to make the algorithm applicable to a large class of systems. Low communication requirements make it suitable for real systems with small communication bandwidth. A result on the eigenvalues of a Laplacian for a connected graph with self-loops is also presented.

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Appendix

Here we present useful results, without proofs for brevity, on the eigenvalues of undirected graphs with self-loops.

Lemma 3: The Laplacian of a pseudo-connected graph is positive definite.

Next we introduce the following definition of a *lifted graph*.

Definition 2: Given an undirected graph $\mathbf{G}(\mathbf{E},\mathbf{V})$ with N vertices and with at least one self-loop, its lifted graph $\hat{\mathbf{G}}(\hat{\mathbf{E}},\hat{\mathbf{V}})$ is a graph with $2N+1$ vertices and with no self-loops such that (Figure 8): For every vertex i in $\mathbf{G}$ there are vertices i and $i{+}N{+}1$ in $\hat{\mathbf{G}}$, $i=1,...,N$, and also a *middle vertex* $N{+}1$ with the following set of edges

$$(i,j)\in\mathbf{E} \Rightarrow (i,j)\in\hat{\mathbf{E}} \text{ and } (i+N+1,j+N+1)\in\hat{\mathbf{E}}$$
$$(i,i)\in\mathbf{E} \Rightarrow (i,N+1)\in\hat{\mathbf{E}} \text{ and } (N+1,i+N+1)\in\hat{\mathbf{E}}.$$

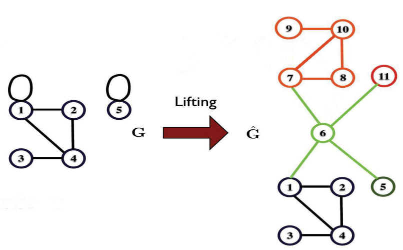

Fig. 8. Lifted graph of an undirected graph with self-loops

Theorem 3: For a finite undirected graph, $\mathbf{G}$, with self-loops but without multiple-edges:

$$\sigma\left(\mathcal{L}(\mathbf{G})\right)\subseteq\sigma\left(\mathcal{L}(\hat{\mathbf{G}})\right)\cap(0,2\mathbf{d}(\mathbf{G}_o){+}1], \qquad (38)$$

where $\mathbf{G}_o(\mathbf{V},\mathbf{E}_o)$ is a subgraph of $\mathbf{G}(\mathbf{V},\mathbf{E})$ where $\mathbf{E}_o\subset\mathbf{E}$ and $\mathbf{E}_o$ contains all the edges of $\mathbf{E}$ without the self-loops. Particularly if $\mathbf{G}$ is a pseudo-connected graph then

$$\sigma\left(\mathcal{L}(\mathbf{G})\right)\subseteq\sigma_+\left(\mathcal{L}(\hat{\mathbf{G}})\right)\cap(0,2\mathbf{d}(\mathbf{G}_o){+}1]. \qquad (39)$$

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