

Research Article

Decentralized Suboptimal State Feedback Integral Tracking Control Design for Coupled Linear Time-Varying Systems

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In this paper, a suboptimal state feedback integral decentralized tracking control synthesis for interconnected linear time-variant systems is proposed by using orthogonal polynomials. Particularly, the use of operational matrices allows, by expanding the subsystem input states and outputs over a shifted Legendre polynomial basis, the conversion of time-varying parameter differential state equations to a set of time-independent algebraic ones. Hence, optimal open-loop state and control input coefficients are forwardly determined. These data are used to formulate a least-square problem, allowing the synthesis of decentralized state feedback integral control gains. Closed-loop asymptotic stability LMI conditions are given. The proposed approach effectiveness is proved by solving a nonconstant reference tracking problem for coupled inverted pendulums.

1. Introduction

The decentralized control has given rise to increasing attention in the automatic control community. Obviously, studies are mainly related to the so-called large-scale interconnected dynamic system class. The latter family of plants finds its application in several fields such as mechanical systems [1–3], power-generating plants [4, 5], aircraft dynamics [6], aerospace transportation [7], and economic models [8]. The decentralized control of an interconnected system essence aims at making each subsystem being controlled using only its own local state variables. However, ensuring the global stability of the whole system still remains a challenging concern [1, 9, 10].

Over the past years, many research results have been dedicated to the decentralized control approach for linear [11, 12] and nonlinear large-scale systems [1, 10, 13]. We recall here some works in the literature such as decentralized optimal control using the successive approximation approach [14], state-dependent Riccati equation (SDRE) optimal control [10], feedback decentralized polynomial control for the multimachine power system [9], robust H_∞ decentralized observation and control [15], and Chebyshev

wavelet-based collocation scheme [16]. In most cases, particular classes of interconnected systems are handled, and specific mathematical conditions should be met at first in order to achieve the problem resolution.

On the contrary, linear time-variant (LTV) systems are known to be an immediate generalization of time-invariant systems. In fact, this class of linear systems is the result of nonlinear systems linearizing along a trajectory. Hence, it permits to cover a wider operating conditions of the systems to be studied, and as a consequence, ensures the applicability of the resulting control approaches. It is worth noting that the decentralized tracking problem of time-varying systems has not been addressed excessively in the literature. Using time-varying parameters for interconnected system modeling has been considered recently in order to solve various control problems. Among the latest contributions, one may cite [17], where authors combine an operator form of discrete-time linear systems with the classical Youla parameterization to characterize the set of stably realizable decentralized controllers for LTI, LTV, or even linear switched systems. Mohamed et al. [18] addressed the problem of the adaptive sliding mode observer for nonlinear interconnected systems with time-varying parameters.

Besides, the time-variation concept has concerned the delay modeling [19] or the output constraint modeling [20] of large-scale systems. For the tracking problem, we particularly recall the method in [1], where authors developed a decentralized tracking control for a class of time-varying systems based on the backstepping technique. The time-variant parameters in the global systems are unknown, bounded (that bounds need to be estimated and not literally using the time-varying parameters), and should verify some analytical assumptions in order to prove the stability of the closed-loop system.

Moreover, approaches based on similar mathematical tools (orthogonal functions, wavelets, and polynomials) used in this work have considered only the optimal control problem of time-varying systems. To our best knowledge, the decentralized scheme for LTV interconnected systems has not been treated yet. One may refer to [12, 21] and references therein that exhibit a review on orthogonal function approaches to solve the optimal control problem for time-varying systems. It is clear that most techniques are based on product operational matrices, and two formulations could be distinguished:

- (i) The Lagrange approach: Lagrange multipliers are introduced, and then a parameter optimization problem is formulated by giving the necessary conditions for optimization. Resolution could be done via a nonlinear optimization tool.
- (ii) The Riccati approach: here, the state transition matrix of the two-point boundary value problem (TPBVP) should be identified by the mean of an orthogonal basis that leads to multiple least-square problems.

However, in most works in the literature, as it is the case for the above described techniques, only open-loop optimal solutions are given, which is an important limitation since such control could not be efficiently implemented in practice.

On the other side, several orthogonal function-based contributions have been extensively proposed during the last three decades for the analysis, identification, optimal control, and model reduction of linear [22, 23] and some classes of nonlinear systems [24–27]. In the literature, there are several orthogonal function bases dedicated to solve the above cited problems arising in control theory. One may refer to research activities based on Legendre polynomials [22, 28], Chebyshev [12], Hermite polynomials [29], block pulse [25], Walsh functions [30], or even hybrid of piecewise and polynomials [31]. In our work, shifted Legendre polynomials are chosen to develop time-variant systems and manipulate related operational matrices, more specifically the Kronecker product operational matrix, to solve the posed optimal tracking problem.

In this paper, we aim to design a suboptimal decentralized state feedback integral tracking control technique applied for interconnected linear time-variant systems. This approach consists in expanding all variables involved in the considered dynamic subsystems over shifted Legendre basis.

It will be then possible to reduce the corresponding interconnected time-variant parameter differential equations into some coupled time-independent algebraic equations. Thus, computations become clearly more easier to be undertaken. More precisely, the proposed approach is achieved by following the below 2 steps :

- (1) Consider the global time-varying system and solve the related optimal control problem with a direct approach based on shifted Legendre polynomials. This permits to avoid the resolution of a high-order TPBVP with time-varying parameters and allows to obtain global state and control coefficients on the basis by just finding a direct relation between the coefficients and deducing the control from the approached expression of the criterion.
- (2) Once open-loop global state and control coefficients are obtained, we inject each subsystem state and control coefficients in the proposed decentralized state feedback with an integral action equation. That leads, by the mean of the integration operational matrix of the basis, to an algebraic equation where unknowns are only control gains. A formulated least-square problem is then solved.

Finally, we will be interested to study the stability of the overall system controlled with the proposed feedback action. As a consequence, an augmented system is constructed and LMI conditions are given, guaranteeing asymptotic stability of the closed loop.

This article is organized as follows: shifted Legendre polynomial properties and problem formulation are presented in Section 2. Open-loop optimal control synthesis and suboptimal state feedback integral control determination are given Section 3. Section 4 is reserved to the stability analysis. A numerical simulated example based on coupled inverted pendulums is provided in the Section 5, and it highlights the achieved developments.

2. Preliminaries and Problem Statement

2.1. Legendre Polynomials. The Legendre polynomials are orthogonal on the interval $[-1, 1]$, with a weight function $w(\tau) = 1$. The set of Legendre polynomials is obtained from the formula of Olinde Rodrigues [23]:

$$\ell_n(\tau) = \frac{1}{2^n n!} \frac{d^n (\tau^2 - 1)^n}{d\tau^n}. \quad (1)$$

This gives

$$\begin{aligned} \ell_0(\tau) &= 1, \\ \ell_1(\tau) &= \tau, \\ \ell_2(\tau) &= \frac{3\tau^2 - 1}{2}. \end{aligned} \quad (2)$$

These polynomials can also be obtained from the recursive relationship [23]:

$$(n+1)\ell_{n+1}(\tau) = (2n+1)\tau\ell_n(\tau) - n\ell_{n-1}(\tau), \quad (3)$$

with $\ell_0(\tau) = 1$ and $\ell_1(\tau) = \tau$.

2.2. Shifted Legendre Polynomials. In order to obtain orthogonal Legendre polynomials over the time interval $[0, t_f]$, which is more useful in control synthesis, we perform the following change of variable:

$$\tau = \frac{2t}{t_f} - 1 \text{ with } 0 \leq t \leq t_f. \quad (4)$$

The recursive relationship (3) becomes [32]

$$(n+1)s_{n+1}(t) = (2n+1)\left(\frac{2t}{t_f} - 1\right)s_n(t) - ns_{n-1}(t), \quad (5)$$

where $s_n(t)$ denotes an elementary shifted Legendre polynomial defined over $0 \leq t \leq t_f$, in which $s_0(t) = 1$ and $s_1(t) = (2t/t_f) - 1$.

The principle of orthogonality of the shifted Legendre polynomials (SLPs) is expressed by the following equation [33]:

$$\int_0^{t_f} s_i(t)s_j(t)dt = \frac{t_f}{2i+1}\delta_{ij}, \quad (6)$$

where δ_{ij} is the Kronecker symbol.

2.3. The Operational Matrix of Integration of Shifted Legendre Polynomials. In the case of shifted Legendre polynomials, the operational matrix of integration P_N is defined as follows [28]:

$$\int_0^t \mathcal{S}_N(\tau)d\tau \cong P_N \mathcal{S}_N(t), \quad (7)$$

where

$$P_N = \frac{t_f}{2} \begin{pmatrix} 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & \frac{1}{5} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \frac{1}{2N-3} & 0 & \frac{1}{2N-3} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -\frac{1}{2N-1} & 0 \end{pmatrix}, \quad (8)$$

is an $(N \times N)$ constant matrix and $\mathcal{S}_N(t) = [s_0(t), s_1(t), \dots, s_{N-1}(t)]^T$ denotes a shifted Legendre basis of dimension N .

2.4. The Integration of the Cross Product. The integration of the cross product of two shifted Legendre polynomial vectors can be obtained as [31]

$$C_p = \int_0^{t_f} \mathcal{S}_N(t)\mathcal{S}_N^T(t)dt = t_f \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \frac{1}{3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{2M-1} \end{bmatrix}, \quad (9)$$

where C_p is an $(N \times N)$ constant matrix.

2.5. Operational Matrix of the Kronecker Product. The product of two shifted Legendre polynomials $s_i(t)$ and $s_j(t)$ can be expressed by [32]

$$s_i(t)s_j(t) \cong \sum_{k=0}^{M-1} \psi_{ijk}L_k(t), \quad (10)$$

with

$$\psi_{ijk} = \frac{2k+1}{t_f} \int_0^{t_f} s_i(t)s_j(t)s_k(t)dt. \quad (11)$$

Then, we may write

$$s_j(t)\mathcal{S}_N(t) = \begin{bmatrix} \psi_{i00} \\ \psi_{i11} \\ \vdots \\ \psi_{i(N-1)(N-1)} \end{bmatrix} = K_s^i \mathcal{S}_N(t), \quad (12)$$

where K_s^i is an $N \times N$ constant matrix, and then it becomes

$$\mathcal{S}_N(t) \otimes \mathcal{S}_N(t) = \begin{bmatrix} K_s^0 \\ K_s^1 \\ \vdots \\ K_s^{N-1} \end{bmatrix} = K_{\mathcal{S}} \mathcal{S}_N(t), \quad (13)$$

where $K_{\mathcal{S}} \in \mathbb{R}^{N^2 \times N}$ is the Kronecker product operational matrix of shifted Legendre polynomials.

2.6. Problem Statement. Consider the following optimal control problem: find the optimal control $u_i^*(t)$, which minimizes the following quadratic performance index:

$$J = \frac{1}{2} \sum_i^M \int_{t_0}^{t_f} [e_i^T(t)Q_i e_i(t) + u_i^T(t)R_i u_i(t)]dt, \quad (14)$$

where M is the number of interconnected subsystems and $e_i(t)$ is the tracking error defined by

$$e_i(t) = y_i(t) - y_{ri}(t), \quad (15)$$

with $y_i(t) \in \mathbb{R}^{P_i}$ being the output of the i -th subsystem and $y_{ri}(t)$ being the reference submodel output.

Particularly, in this framework, we consider a global LTV system (S) consisting of M interconnected subsystems (S_i) described by the following state equation:

$$(S_i) \begin{cases} \dot{x}_i(t) = A_i(t)x_i(t) + B_i(t)u_i(t) + \sum_{\substack{j=1 \\ j \neq i}}^M H_{ij}(t)x_j(t), \\ y_i(t) = C_i x_i(t), \end{cases} \quad (16)$$

where $x_i(t) \in \mathbb{R}^{n_i}$, $u_i(t) \in \mathbb{R}^{m_i}$, and $y_i(t) \in \mathbb{R}^{p_i}$ are, respectively, the state vector, the control vector, and the output vector of the subsystem (S_i) and $A_i(t) = [a_{ij}(t)]$, $B_i(t) = [b_{ij}(t)]$, and C_i and $H_{ij}(t) = [h_{ij}(t)]$ are some time-dependent matrices characterising the subsystem (S_i) with respective dimensions $(n_i \times n_i)$, $(n_i \times m_i)$, $(p_i \times n_i)$, and $(n_i \times n_j)$.

We assume that each subsystem is assumed to be fully measurable and controllable, and its matrices have the following bounds:

$$\begin{aligned} \underline{A} &\leq a_{ij}(t) \leq \bar{A}, \\ \underline{B} &\leq b_{ij}(t) \leq \bar{B}, \\ \underline{H} &\leq h_{ij}(t) \leq \bar{H}, \end{aligned} \quad (17)$$

where notations $\underline{\lambda}$ and $\bar{\lambda}$ correspond, respectively, to the minimum and maximum of a time-dependent variable $\lambda_{ij}(t)$.

The linear time-invariant reference models to be used in this study are obviously stable and encompass all desired performances to be conferred to the controlled subsystems. These LTI models are described by the following state equations:

$$\begin{cases} \dot{x}_{ri}(t) = E_i x_{ri}(t) + F_i r_i(t), \\ y_{ri}(t) = G_i x_{ri}(t), \end{cases} \quad (18)$$

where $x_{ri}(t) \in \mathbb{R}^{\tilde{n}_{ri}}$ is the state vector of the i -th reference submodel, $r_i(t) \in \mathbb{R}^{\tilde{m}_{ri}}$ is a nonconstant input, and $y_{ri}(t) \in \mathbb{R}^{\tilde{p}_{ri}}$ its output vector generating, hence, a path to be tracked and E_i , F_i , and G_i are the chosen matrices characterising the reference model with respective dimensions $(\tilde{n}_{ri} \times \tilde{n}_{ri})$, $(\tilde{n}_{ri} \times \tilde{m}_{ri})$, and $(\tilde{p}_{ri} \times \tilde{n}_{ri})$.

In the sequel, we will be concerned with the synthesis of state feedback integral suboptimal controllers of the following form:

$$u_i(t) = \bar{N}_i r_i(t) + K_i x_i(t) + L_i \int_0^t (y_i(\tau) - r_i(\tau)) d\tau, \quad (19)$$

which are aimed to make each subsystem outputs track nonconstant inputs with respect to the corresponding reference submodel dynamics.

3. Optimal Tracking Control Synthesis

3.1. Criterion Approximation. A global criterion to be minimised could be defined as follows:

$$J = \frac{1}{2} \int_0^{t_f} (e^T(t) Q e(t) + u^T(t) R u(t)) dt, \quad (20)$$

where $e(t) \in \mathbb{R}^p$ and $u(t) \in \mathbb{R}^m$ denote the global system tracking error and the control input, respectively. Notice that this criterion is associated to the global system (S):

$$\begin{cases} \dot{x}(t) = A_g(t)x(t) + B_g(t)u(t), \\ y(t) = C_g x(t), \end{cases} \quad (21)$$

where global system matrices $A_g(t) \in \mathbb{R}^{n \times n}$, $B_g(t) \in \mathbb{R}^{m \times m}$, and $C_g \in \mathbb{R}^{p \times p}$ are given by

$$\begin{aligned} A_g(t) &= \text{diag}(A_i(t)) + H, \text{ with } H = [H_{ij}], H_{ii} = 0, \\ &\quad \forall i, j = 1, \dots, M, \\ B_g(t) &= \text{diag}(B_i(t)), \\ C_g &= \text{diag}(C_i). \end{aligned} \quad (22)$$

Applying the vec operator and related Kronecker product property [34] yields

$$\begin{aligned} \text{vec}(e(t)) &\approx (\mathcal{S}_N(t)^T(t) \otimes I_p) \text{vec}(e_N^T), \\ \text{vec}(u(t)) &\approx (\mathcal{S}_N(t)^T(t) \otimes I_m) \text{vec}(u_N^T), \end{aligned} \quad (23)$$

where I_p and I_m are, respectively, the $(p \times p)$ and $(m \times m)$ identity matrices and e_N^T and u_N^T are the coefficients obtained by developing, respectively, the error and the control input over the shifted Legendre basis.

With this approximation, the globalization of equation (2) gives

$$\text{vec}(e_N^T) \approx (I_N \otimes C_g) \text{vec}(x_N^T) - \text{vec}(y_{rN}^T), \quad (24)$$

where x_N^T and y_{rN}^T are the coefficients obtained by developing, respectively, the global system state and the global reference model output over the shifted Legendre basis.

Let us denote

$$\begin{aligned} z_x &= \text{vec}(x_N^T), \\ z_u &= \text{vec}(u_N^T), \\ Y_N &= \text{vec}(y_{rN}^T), \\ \Gamma_N &= \text{vec}(r_N^T). \end{aligned} \quad (25)$$

Exploiting the vec property (see Appendix), the global criterion to be minimised is expressed as follows:

$$J \approx [z_x^T (I_N \otimes C_g^T) - Y_N^T] \tilde{Q} [(I_N \otimes C_g) z_x - Y_N] + z_u^T \tilde{R} z_u, \quad (26)$$

where $\tilde{Q} = (C_p \otimes Q)$ and $\tilde{R} = (C_p \otimes R)$ with C_p the cross product matrix presented in the previous section.

The expression of Y_N in (26) depends on Γ_N and will be replaced by the following relation:

$$Y_N = (I_N \otimes G) (I_{nN} - P_N^T \otimes E) (P_N^T \otimes F) \Gamma_N, \quad (27)$$

where matrices $E = \text{diag}(E_i)$, $F = \text{diag}(F_i)$, and $G = \text{diag}(G_i)$ define the global reference model.

3.2. Dynamic Constraint Expansion over the Orthogonal Basis. The development of time-dependent matrices in (21), over shifted Legendre basis, yields

$$A_g(t) \approx \sum_{k=0}^{N-1} A_{gk} s_k(t) = \underbrace{\begin{bmatrix} s_0(t) \cdot I_n & & & 0 \\ & s_1(t) \cdot I_n & & \\ & & \ddots & \\ 0 & & & s_{N-1}(t) \cdot I_n \end{bmatrix}}_{\mathcal{S}_N^T(t) \otimes I_n} \cdot \begin{bmatrix} A_{g0} \\ \dots \\ A_{g1} \\ \dots \\ \vdots \\ \dots \\ A_{gN-1} \end{bmatrix} = (\mathcal{S}_N^T(t) \otimes I_n) \cdot \tilde{A}_g, \quad (28)$$

where $A_{gk} \in^{n \times n}$ and $\tilde{A}_g \in^{nN \times n}$.

Similarly, one may state

$$B_g(t) \approx (\mathcal{S}_N^T(t) \otimes I_n) \tilde{B}_g. \quad (29)$$

Now, the system expansion over the SLP base is given by

$$\begin{aligned} x_N^T \mathcal{S}_N(t) - x_{N0}^T \mathcal{S}_N(t) &= (\mathcal{S}_N^T(t) \otimes I_n) \tilde{A}_g x_N^T P_N \mathcal{S}_N(t) \\ &\quad + (\mathcal{S}_N^T(t) \otimes I_n) \tilde{B}_g u_N^T P_N \mathcal{S}_N(t), \end{aligned} \quad (30)$$

with x_{N0}^T being the initial state projection over the considered basis.

Applying the vec operator to (30) gives

$$\begin{aligned} &(\mathcal{S}_N^T(t) \otimes I_n) z_x - (\mathcal{S}_N^T(t) \otimes I_n) z_{x0} \\ &= [(\mathcal{S}_N^T(t) P_N) \otimes (\mathcal{S}_N^T(t) \otimes I_n) \tilde{A}_g] z_x \\ &\quad + [(\mathcal{S}_N^T(t) P_N) \otimes (\mathcal{S}_N^T(t) \otimes I_n) \tilde{B}_g] z_u. \end{aligned} \quad (31)$$

Notice that

$$\begin{aligned} &(\mathcal{S}_N^T(t) P_N) \otimes (\mathcal{S}_N^T(t) \otimes I_n) \tilde{A}_g \\ &= (\mathcal{S}_N(t) \otimes \mathcal{S}_N(t) \otimes I_n) (P_N^T \otimes \tilde{A}_g) \\ &= (\mathcal{S}_N(t) \otimes I_n) (K_S^T \otimes I_n) (P_N^T \otimes \tilde{A}_g). \end{aligned} \quad (32)$$

Hence, (31) becomes

$$z_x = Vz_u + Wz_{x0}, \quad (33)$$

where

$$\begin{aligned} W &= (I_n - (K_S^T \otimes I_n) (P_N^T \otimes \tilde{A}_g))^{-1}, \\ V &= W \cdot (K_S^T \otimes I_n) (P_N^T \otimes \tilde{B}_g). \end{aligned} \quad (34)$$

3.3. Optimal Open-Loop Control. Substituting (33) in (26) and setting the optimization condition $\partial J/z_u = 0$ yield the following optimal control:

$$z_u^* = -(\hat{Q}V + \tilde{R})^{-1} (z_{x0}^T W^T \hat{Q}V + Y_N^T \tilde{Q}C_g V), \quad (35)$$

where $\hat{Q} = C_g^T \tilde{Q} C_g$.

Finally, the optimal state coefficient z_x^* could be also recovered by injecting (35) in (33).

3.4. Suboptimal State Feedback Integral Synthesis. We are interested now, based on open-loop optimal results (z_u^*, z_x^*), to synthesize a suboptimal control of type (19).

It is possible now to capture, for each subsystem, optimal state x_{iN}^{*T} and input u_{iN}^{*T} coefficients from the optimal global system ones.

On the other hand, expansion of the decentralized state feedback integral control over SLP basis yields

$$\begin{aligned} u_{iN}^T \mathcal{S}_N(t) &= \bar{N}_i r_{iN}^T \mathcal{S}_N(t) + K_i x_{iN}^T \\ &\quad + L_i (y_{iN}^T - y_{riN}^T) P_N \mathcal{S}_N(t). \end{aligned} \quad (36)$$

Simplifying the basis and applying the vec operator with optimal coefficients, the synthesis of control parameters could be done by solving the following least-square problem:

$$\mathcal{A} \cdot \Delta = \mathcal{B}, \quad (37)$$

with

$$\mathcal{A} = \text{diag} \left(\left[r_{iN}^T \otimes I_{m_i} : x_{iN}^{*T} \otimes I_{m_i} : (C_i x_{iN}^{*T} P_N - r_{iN}^T P_N)^T \otimes I_{m_i} \right] \right),$$

$$\mathcal{B} = \begin{bmatrix} z_{u1}^* \\ z_{u2}^* \\ \vdots \\ z_{uM}^* \end{bmatrix},$$

(38)

where $z_{ui}^* = \text{vec}(u_{iN}^{*T})$ and

$$\Delta = \begin{bmatrix} \text{vec}(\bar{N}_1) \\ \text{vec}(K_1) \\ \text{vec}(L_1) \\ \dots \\ \text{vec}(\bar{N}_2) \\ \text{vec}(K_2) \\ \text{vec}(L_2) \\ \dots \\ \vdots \\ \dots \\ \text{vec}(\bar{N}_M) \\ \text{vec}(K_M) \\ \text{vec}(L_M) \end{bmatrix}. \quad (39)$$

4. Closed-Loop Stability Analysis

Consider the following augmented state space submodel:

$$\begin{bmatrix} \dot{x}_i(t) \\ \dot{x}_{ai}(t) \end{bmatrix} = \begin{bmatrix} A_i(t) & 0 \\ C_i & 0 \end{bmatrix} \begin{bmatrix} x_i(t) \\ x_{ai}(t) \end{bmatrix} + \begin{bmatrix} B_i(t) \\ 0 \end{bmatrix} u_i(t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix} r_i(t), \quad (40)$$

where $x_{ai} = \int_0^t (y_i(\tau) - r_i(\tau)) d\tau$. Hence, each controlled subsystems with input (19) may be written as

$$\dot{X}_i(t) = \bar{A}_i(t)X_i(t) + \bar{B}_i(t)r_i(t), \quad (41)$$

where $X_i(t) = \begin{bmatrix} x_i(t) \\ x_{ai}(t) \end{bmatrix}$ is the augmented state for each subsystem and

$$\begin{aligned} \bar{A}_i(t) &= \begin{bmatrix} A_i(t) + B_i(t)K_i & B_i(t)L_i \\ C_i & 0 \end{bmatrix}, \\ \bar{B}_i(t) &= \begin{bmatrix} B_i(t)\bar{N}_i \\ -1 \end{bmatrix}. \end{aligned} \quad (42)$$

The closed-loop global system may be constructed as follows:

$$\dot{X}_g(t) = \bar{A}_{cl}(t)X_g(t) + \bar{B}_{cl}(t)r(t), \quad (43)$$

where $\bar{A}_{cl} = \text{diag}(\bar{A}_i(t))$ and $\bar{B}_{cl} = \text{diag}(\bar{B}_i(t))$, for $i = 1, \dots, M$.

As a consequence, the linear time-varying model defined by equation (43) can be expressed in the following polytopic form such as $\mathbb{M} = [A_{cl} | \bar{B}_{cl}]$ belongs to a polytope of matrices \mathcal{M} defined by [35]

$$\begin{aligned} \mathcal{M} &= \left\{ \mathbb{M} = M(\theta) = [\bar{A}_{cl}(\theta) | \bar{B}_{cl}(\theta)] / M(\theta) \right. \\ &= \left. \sum_{i=1}^q (\theta_i [\bar{A}_{cli} | \bar{B}_{cli}]) \right\}, \end{aligned} \quad (44)$$

where $\theta \in \Theta$, the set of all barycentric coordinates:

$$\Theta = \left\{ \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_q \end{bmatrix} / \sum_{i=1}^q \theta_i = 1 \right\}. \quad (45)$$

The closed-loop system (43) is a mean square asymptotically stable with an H_∞ disturbance attenuation γ if and only if there exists a positive definite matrix $P \in \mathbb{R}^{(n+M) \times (n+M)}$ such that [36]

$$\begin{bmatrix} \bar{A}_{cli}^T P + P \bar{A}_{cli} & P \bar{B}_{cli} & C_g^T \\ \bar{B}_{cli} P & -\gamma I_m & 0 \\ C_g & 0 & -I_p \end{bmatrix} < 0, \quad \forall i = 1, \dots, M. \quad (46)$$

5. Application to Coupled Inverted Pendulums

The considered benchmark [1, 37] consists of two identical pendulums which are coupled through a moving spring and move in a plane (Figure 1). We assume that the pivot position of the moving spring is a function of time $a(t)$ which can change along the full length l of the pendulums.

The objective of the decentralized controller is to control each pendulum with mass m independently, such that each pendulum will follow its own desired (reference) trajectory, while the connected spring is moving. The linearized dynamic equations of the two pendulum systems (for small displacements about the equilibrium) are

$$\begin{aligned} \dot{x}_1 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{ka(t)^2}{ml^2} & 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)^2}{ml^2} & 0 \end{bmatrix} x_2, \\ \dot{x}_2 &= \begin{bmatrix} 0 & 1 \\ \frac{g}{l} - \frac{ka(t)^2}{ml^2} & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u_1 + \begin{bmatrix} 0 & 0 \\ \frac{ka(t)^2}{ml^2} & 0 \end{bmatrix} x_1, \end{aligned} \quad (47)$$

where $x_1 = [\theta_1, \dot{\theta}_1]^T$ and $x_2 = [\theta_2, \dot{\theta}_2]^T$, k and g are the spring and gravity constants, and u_1 and u_2 are the torque inputs applied at the pivot points.

It is clear that $a(t) \in [0, l]$, then we know that the displacement of the pendulums and the connected spring is bounded, so the constant bounds of matrices characterising the two subsystems can be obtained.

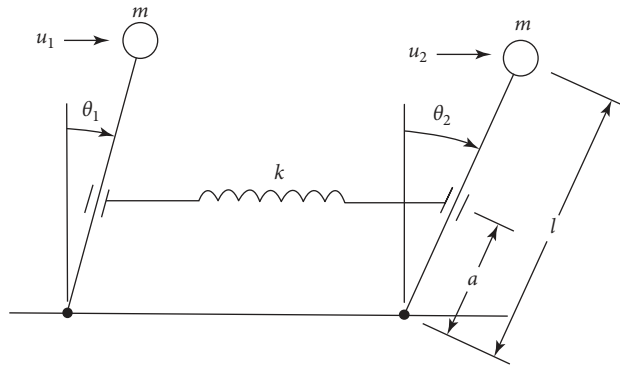


FIGURE 1: Two interconnected inverted pendulums.

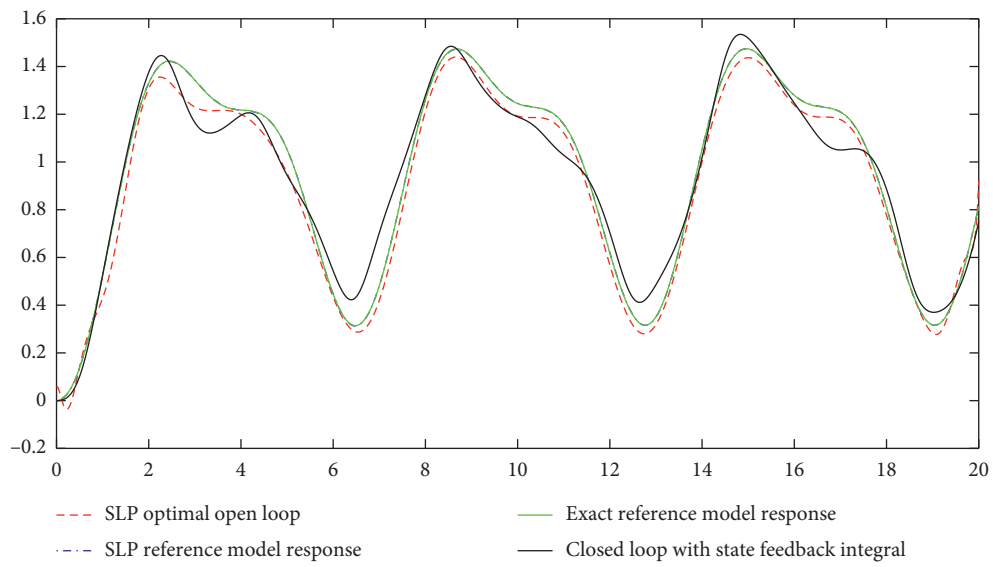


FIGURE 2: Optimal and suboptimal state trajectories of subsystem 1.

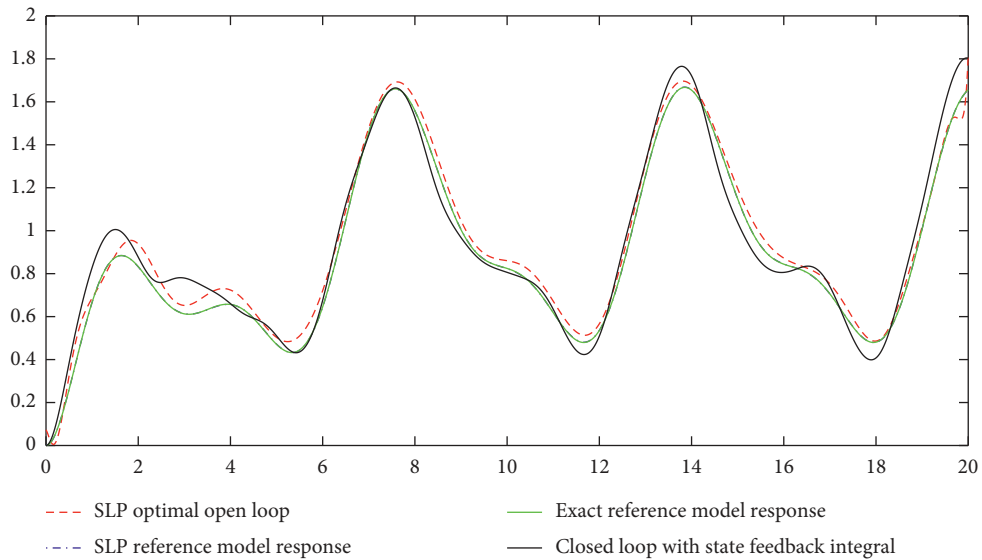


FIGURE 3: Optimal and suboptimal state trajectories of subsystem 2.

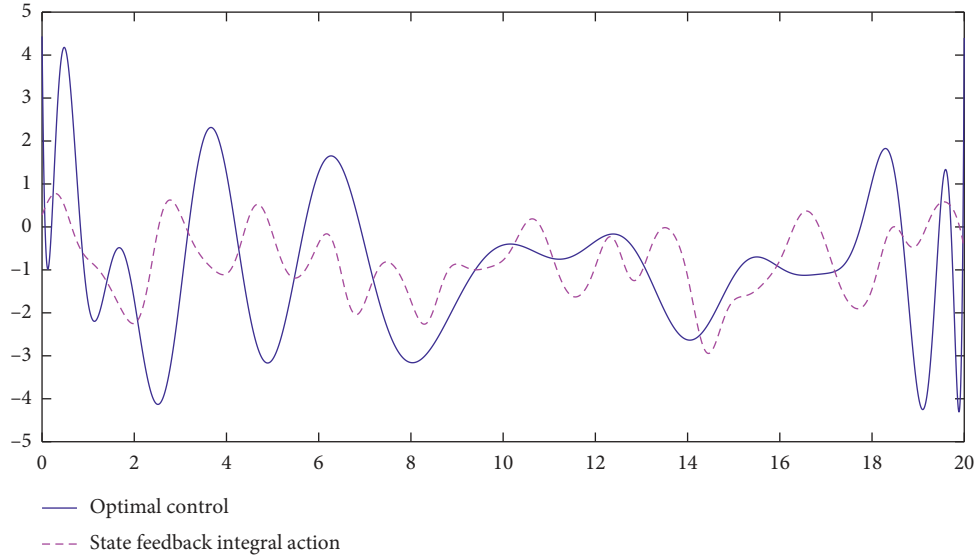


FIGURE 4: Optimal and suboptimal control signals of subsystem 1.

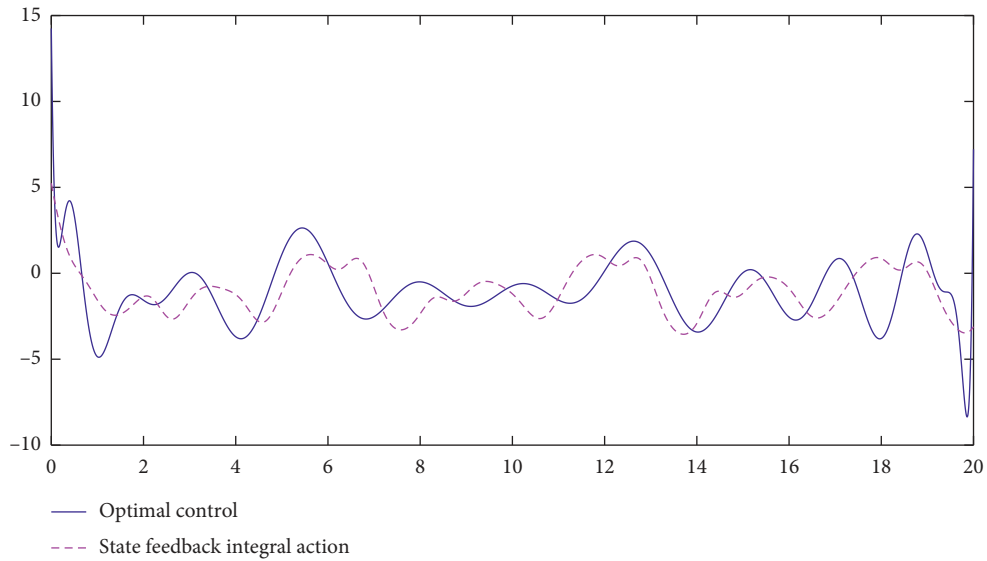


FIGURE 5: Optimal and suboptimal control signals of subsystem 2.

For simulations, we set the time-varying function $a(t) = (l/2)(1 + \sin(\omega t))$ and choose $g/l = 1$, $1/ml^2 = 1$, and let $kl^2/4 = 25$ and $\omega = \pi$.

The reference model (18) for each pendulum is chosen as

$$\begin{aligned} E_{1,2} &= \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \\ F_{1,2} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \\ G_{1,2} &= [1 \ 0]. \end{aligned} \quad (48)$$

The reference trajectories used in simulation were chosen as

$$\begin{aligned} r_1(t) &= 1 + \sin(t) + \sin(2t), \\ r_2(t) &= 1 + \cos(t) + \cos(2t). \end{aligned} \quad (49)$$

Optimal open-loop trajectories for both subsystems 1 and 2 are depicted, respectively, in Figures 2 and 3. Optimal tracking results are obtained for the horizon time $t_f = 20$ and $N = 26$ which is the shifted Legendre basis dimension. The tracking is ensured by minimising the quadratic criterion with $Q_{1,2} = \text{diag}(10^3)$ and $R_{1,2} = 1$.

The feedback approach applied to the interconnected pendulums leads to the following decentralized state feedback integral actions:

$$\begin{aligned}
 \bar{N}_1 &= 0.274, \\
 K_1 &= [-7.2635 - 3.6248], \\
 L_1 &= -2.9728, \\
 \bar{N}_2 &= 1.7476, \\
 K_2 &= [-8.3477 - 4.8091], \\
 L_2 &= -3.4561.
 \end{aligned} \tag{50}$$

Closed loop for both subsystems is also given in Figures 2 and 3. It is then clear that controlled states reproduce the shape of the optimal ones.

State feedback integral actions compared to optimal open-loop inputs are drawn in Figures 4 and 5. It appears that both optimal and suboptimal signals are in the same variation range.

The asymptotic stability with an H_∞ disturbance attenuation of the closed-loop system is verified by the feasible solution of the formulated LMI. The obtained LMI variables are

$$P = \begin{bmatrix} 2.9401 & 0.4237 & 0.7387 & -0.9920 & -0.1118 & -0.2182 \\ 0.4237 & 0.3752 & 0.2994 & 0.1285 & -0.0110 & -0.0081 \\ 0.7387 & 0.2994 & 2.6532 & -0.4524 & -0.0103 & -0.7878 \\ -0.9920 & 0.1285 & -0.4524 & 3.8961 & 0.6411 & 0.9556 \\ -0.1118 & -0.0110 & -0.0103 & 0.6411 & 0.4731 & 0.3503 \\ -0.2182 & -0.0081 & -0.7878 & 0.9556 & 0.3503 & 3.1039 \end{bmatrix},$$

$$\gamma = 0.3058. \tag{51}$$

6. Conclusion

In this paper, a new suboptimal decentralized control technique is designed by using orthogonal functions as an interesting tool of dynamical system approximation, more specifically shifted Legendre polynomials with operational matrices of integration and Kronecker product are exploited.

The main advantage of the proposed technique is its applicability to the class of time-varying interconnected systems. Hence, the suboptimal decentralized state feedback integral controller parameters are adjusted such that each subsystem has a specific desired performance of a chosen reference model by solving a time-independent least-square problem.

In the future work, we intend to extend the actual study to the synthesis of the optimal tracking control for interconnected nonlinear time-varying systems.

Appendix

Kronecker Product and vec(.) Function Property

For any matrices X , Y , and Z having appropriate dimensions, the following property of the Kronecker product is given [34]:

$$\text{vec}(XYZ) = (Z^T \otimes X)\text{vec}(Y), \tag{A.1}$$

where vec denotes the vectorization operator of a matrix [34] and \otimes stands for the Kronecker product.

Let A, B, C , and D be matrices with appropriate dimensions, and we recall the following property [34]:

$$(A \otimes B)(C \otimes D) = AC \otimes BD. \tag{A.2}$$

Data Availability

System parameters used for simulation in the paper are given in the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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