

# Decentralized Supply Chains with Competing Retailers Under Demand Uncertainty

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**I**n this paper, we investigate the equilibrium behavior of decentralized supply chains with competing retailers under demand uncertainty. We also design contractual arrangements between the parties that allow the decentralized chain to perform as well as a centralized one. We address these questions in the context of two-echelon supply chains with a single supplier servicing a network of (competing) retailers, considering the following general model: Retailers face random demands, the distribution of which may depend only on its own retail price (*noncompeting* retailers) or on its own price as well as those of the other retailers (*competing* retailers), according to general stochastic demand functions.

*Key words:* decentralized supply chains; coordination mechanisms; uncertain demands; inventory strategies

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## 1. Introduction

We investigate the equilibrium behavior of decentralized supply chains with competing retailers under demand uncertainty. In such chains, it is important to determine who controls which decisions and in which ways the different chain members are compensated. Of particular interest is the specification of contractual arrangements between the parties that allows the decentralized chain to perform as well as a centralized one, in which all decisions are made by a single entity maximizing chainwide expected profits. Such contracts are referred to as (*perfect*) *coordination* mechanisms. In this paper, we address these questions in the context of two-echelon supply chains with a single supplier servicing a network of (competing) retailers.

As part of the design of coordinating contracts, it is necessary to identify what *type* of contract is required to achieve specific beneficial outcomes. For example, when can coordination be achieved exclusively via a *linear* wholesale-pricing scheme—i.e., with a constant per-unit wholesale price? When does this constant wholesale price need to be differentiated by retailer, and if so, on what basis? When are *nonlinear* wholesale-pricing schemes required, with the per-unit wholesale price discounted on the basis of the retailer's order quantity or his annual sales volume? Many supply chains routinely use a variety of trade deals to provide incentives for retailers to reduce their prices and increase sales; see, e.g., Blattberg and Neslin (1990). Such trade deals include "billbacks" and "count-recounts," where the supplier reimburses

the retailers, in whole or in part, for discounts off its regular retail price for all units ordered or sold during a given period of time. The video rental industry has recently introduced "revenue-sharing" schemes, where the studios drastically reduce their wholesale prices to store chains such as Blockbuster in exchange for a given percentage of the rental revenues. Such schemes are believed to increase supply-chain-wide revenues by up to 30%; see, e.g., Shapiro (1998). To implement these schemes, the chain members find it worthwhile to retain a third party (e.g., Rentrak) and reward it with no less than 10% of its revenues, so as to monitor all rentals.<sup>1</sup> Trade deals and revenue-sharing programs can be viewed as mechanisms to share the risk between the supplier and the retailers. They increase in importance as the uncertainty about the product's sales increases. Other such risk-sharing mechanisms include buy-back agreements where the supplier commits to buy any unsold inventories back from the retailers at part or all of the original cost; see Pasternack (1985) and Padmanabhan and Png (1995, 1997).

We address the following general model: Each retailer faces a random demand volume during a given sales season, the distribution of which may depend only on its own retail price (*noncompeting* retailers) or on its own price as well as those of the other retailers (*competing* retailers) according to general stochastic

<sup>1</sup> Proper specification of the contract parameters continues, however, to be a challenge, threatening the continuation of these revenue-sharing schemes; see Peers (2001).

demand functions. All retailers order a one-time procurement from the supplier at the beginning of the season.

The simplest model has a single retailer facing demand with a known, exogenously given, distribution, as in the classical newsvendor problem, and with a supplier incurring linear production costs. Generalizing Spengler's (1950) seminal result on double marginalization, Pasternack (1985) showed that perfect coordination can be achieved with a simple constant wholesale price only, set equal to the supplier's per-unit procurement cost, i.e., provided double marginalization is avoided. This, of course, results in an unsatisfactory arrangement for the supplier, whose profits vanish. As an alternative, Pasternack proposes a buy-back arrangement under which a constant wholesale price, larger than the per-unit procurement cost, is combined with a constant partial refund for any unit that remains unsold. Coordination can be achieved via any one of a continuum of wholesale price/buy-back rate combinations. See Lariviere (1999), Cachon (1999), Tsay et al. (1999), and Taylor (2002) for an extensive treatment of this model as well as alternative types of coordination mechanisms.

As stated by Kandel (1996), the situation is considerably more complex when the retailer *chooses* his retail price and the demand distribution depends on this price. Lariviere (1999) quotes Kandel as claiming that no payment scheme with a constant per-unit wholesale price and buy-back rate induces perfect coordination, even though Emmons and Gilbert (1998) showed that under such buy-back contracts, both the supplier and the retailer may improve their profits. (The only exception, of course, is the trivial arrangement under which the wholesale price equals the supplier's per-unit procurement cost and her profits are eliminated.)

After verifying formally that, under an endogenously determined retail price, no payment scheme with a constant per-unit wholesale price (larger than the supplier's per-unit procurement cost) and a constant buy-back rate can achieve coordination for *any* problem instance, we show that a so-called linear "price-discount sharing" scheme does. This scheme is closely related to the more commonly known "bill-backs" and "count-recounts." Ailawadi et al. (1999) demonstrate the prevalence of this type of trade promotions as well as its advantages over other types of schemes. The payment scheme continues to be combined with the supplier's commitment to buy back any of the unsold units at a *constant* below the per-unit wholesale price, net of any subsidy.

We show that the same linear price discount sharing scheme coordinates the chain when dealing with a *network* of *noncompeting* retailers. Under retail *competition*, the choices any given retailer makes for

its price and stocking quantity impact not just its own profit, but that of each of its competitors as well. This applies even under the simplest possible contractual arrangements with the supplier, e.g., under constant per-unit wholesale prices. We characterize the equilibrium behavior of the decentralized chain under such payment schemes and show that coordination can, again, be achieved with a price discount sharing scheme, except that the supplier's share or subsidy in the retailer's discount (from its reference value) now fails to be proportional with the size of the discount. While the price discount sharing scheme is universally applicable, we identify a second scheme, with constant per-unit wholesale prices, that induces perfect coordination under a broad class of demand functions and distributions.

As mentioned, the literature on coordination mechanisms in decentralized supply chains with price setting or competing retailers, under demand *uncertainty* is sparse. This is in contrast to the literature on deterministic models in which, starting with the single-retailer, single-period model in the seminal paper by Spengler (1950), an understanding has been reached for general infinite-horizon settings with competing and noncompeting retailers; see Chen et al. (2001), Bernstein and Federgruen (2003), and the many references cited therein. (All of these assume that demands occur at a constant deterministic rate; some allow for more general cost structures.) Recall that almost all of the literature on competitive stochastic systems with endogenously determined retail prices has focused on models with a single retailer. Other than the above references, we mention Li and Atkins (2000), in which, at the beginning of the period, the supplier and the retailer simultaneously and without cooperation choose a production capacity level and retail price, respectively. The authors show that the decentralized chain can be coordinated with a quantity discount scheme combined with the retailer paying a fixed portion of the cost of unused capacity.

The study of oligopoly models with *uncertainty* with respect to demands (or certain cost parameters) was initiated in the economics literature. These models ignore the problem of mismatches between supply and demand and the associated costs of overstocking and understocking. (In other words, they assume that the retailers can make their procurement decisions after observing actual demands.) On the other hand, this stream of literature incorporates complications not addressed here, such as asymmetric information, information sharing, and signaling. See Vives (2000, Chapter 8) for an excellent survey.

A second related stream of papers considers competitive newsvendor problems in which the distribution of the *primary* demand for each of the firms' products, as well as the retailer prices, are *exogenously*

given. Interdependency and competition between the firms arise because some or all of the unmet demand at a given firm is redirected as a *secondary* demand stream towards one of the competitors offering a substitute product. Parlar (1988) and Wang and Parlar (1994) initiated this type of model, analyzing a two- and three-firm oligopoly, respectively. Lippman and McCardle (1997) and Netessine and Rudi (2003) consider models with an arbitrary number of firms and various joint distributions for the primary demand as well as various allocation rules for the substitution or secondary demands. They establish conditions for the existence of a unique Nash equilibrium. Anupindi and Bassok (1999) address a model with two firms, establishing the first (perfect) coordination result for a decentralized supply chain with uncertain demands and competing retailers.

Bryant (1980) appears to be the first published paper to address the setting of our paper, i.e., a competitive oligopoly model with stochastic demands, endogenously determined prices, and with retailer procurements and stocking levels determined in *advance* of demand realizations. In this model, the retailers simultaneously announce their prices and stocking levels; demands arise from a finite customer population, each with an identical stochastic demand function. Customers are sequentially released to the market and select the firm with the lowest price among those whose inventory has not been exhausted by prior customers. The author shows that no Nash equilibrium exists unless the retailers can be partitioned into two groups, with the second group of market “entrants” announcing their pricing and stocking decisions *after* the decisions of the first group are revealed, and unless the number of entrants is sufficiently large. Another such model is Deneckere et al. (1997), addressing a market with a continuum of identical retailers offering a completely homogeneous product. (See also Deneckere et al. 1996 and Deneckere and Peck 1995 for related models.)

Most directly related to our paper are Birge et al. (1998), Carr et al. (1999), and Van Mieghem and Dada (1999). The former consider the special case of our model with *two* competing retailers, confining themselves to a characterization of the equilibrium behavior in the retailer game under a given pair of constant wholesale prices. Carr et al. (1999) characterize the price equilibrium in a single-echelon setting where each retailer satisfies demand up to an exogenously given, albeit random, capacity level. Van Mieghem and Dada (1999), as part of a larger study of the value of various types of postponement, analyze a similar model, except that the retailer’s capacities are chosen endogenously by the retailers in a first-stage game, and except that the retailers, offering a

completely homogeneous product, choose sales quantities rather than retail prices (the retailers thus face Cournot, as opposed to Bertrand, competition).

The remainder of this paper is organized as follows: Section 2 addresses the model with noncompeting retailers, while in §3 we analyze the case of general competing retailers. All proofs are relegated to the appendix.

## 2. Noncompeting Retailers

In this section, we analyze the (single-period) model in which the supplier sells to  $N$  independent retailers. Each retailer  $i = 1, \dots, N$  faces a random demand volume, the distribution of which depends on his own price  $p_i$  only. Let

$D_i(p)$  = retailer  $i$ ’s (random) demand when charging a retail price  $p$ ,  $i = 1, \dots, N$ .

The  $D_i(p)$ -variables have a general, but known, continuous cdf  $G_i(\cdot | p)$ , which is differentiable with respect to  $p$  and with inverse cumulative distribution function (cdf)  $G_i^{-1}(\cdot | p)$ .  $D_i(p)$  is (strictly) stochastically decreasing in  $p$ , i.e.,  $G_i(x | p) > G_i(x | p')$  for all  $x$  and  $p < p'$ . Moreover,  $\lim_{p \rightarrow \infty} pE[D_i(p)] = 0$ , i.e., expected revenues decrease to zero as the price increases to infinity.

At the start of the period, each retailer  $i$  chooses his retail price  $p_i$  and  $y_i$ , the quantity to order from the supplier. For each retailer, the same uniform price applies to all units sold. This means that the retailers do not apply price discrimination by segmenting their markets. Moreover, as, e.g., for catalog retailers, the sales process does not allow for price adjustments on the basis of intermediate demand and inventory observations during the single period. Settings with price adjustments need to be analyzed with multi-period models as in Bernstein and Federgruen (2004). Any unmet demand is lost, while any excess inventory at retailer  $i$  can be salvaged to an outside firm at a given per-unit salvage rate  $-\infty < v_i < \infty$ . The supplier has ample capacity to satisfy all retailer orders and does so with a constant per-unit procurement cost  $c_i$  for retailer  $i$ .

Assume first that the supplier charges retailer  $i$  a constant per-unit wholesale price  $w_i$ , combined with a commitment to buy back unsold inventory at a per unit rate  $b_i$ . To avoid trivial settings, assume  $v_i < b_i < w_i$  and  $v_i < c_i$ ,  $i = 1, \dots, N$ . First, consider the case of a *single* retailer, omitting all subscripts. The integrated system has the expected profit function

$$\pi_i(y, p) = (p - c)y - (p - v)E[y - D(p)]^+. \quad (1)$$

Similarly, the retailer’s profit function under the *constant*  $(w, b)$ -contract is given by

$$\pi_R(y, p) = (p - w)y - (p - b)E[y - D(p)]^+. \quad (2)$$

$\pi_R$  is strictly concave in  $y$ . Thus, for any retail price  $p > w$ , the retailer's optimal order quantity is the *unique* root of  $\partial\pi_R/\partial y$ , giving rise to the well-known newsvendor solution (note that  $\lim_{y \rightarrow \infty} \partial\pi_R(y, p)/\partial y < 0 < \partial\pi_R(0, p)/\partial y$ )

$$y_R(p) = G^{-1}\left(\frac{p-w}{p-b} \mid p\right). \quad (3)$$

Substituting  $y_R(p)$  in (2), the retailer's optimal profits can be expressed as a continuous function of his retail price  $p$  only. Note that profits are nonpositive when  $p = 0$  or when  $p \rightarrow \infty$  (as  $\pi_R \leq (p-w) \cdot (y - E[y - D(p)]^+) \leq (p-w)E[D(p)]$ , and using  $\lim_{p \rightarrow \infty} pE[D(p)] = 0$ ). Because the profit function is continuous in  $p$ , there exists an optimal price/order quantity combination  $(p_R^*, y_R(p_R^*))$ . (We ignore the trivial situation where the retailer is unable to ensure himself of a positive profit.) By the same arguments, the integrated system has an optimal solution  $(p^I, y^I = y_I(p^I))$ , where

$$y_I(p) = G^{-1}\left(\frac{p-c}{p-v} \mid p\right).$$

In general, it is hard to identify conditions under which the optimal solution  $(p^I, y^I)$  is *unique*. On the other hand, if the random variable  $D(p)$  is of the multiplicative form  $D(p) = d(p)\epsilon$  for some random variable  $\epsilon$ , it follows from the proof of Theorem 3 below that  $\tilde{\pi}(p) = \pi(p, y_I(p))$  is *log concave*—hence unimodal—under the broadly satisfied conditions of this theorem. In the next section, we characterize in a more general setting the impact of various parameters on the optimal retailer choices.

The decentralized system can be coordinated by choosing  $w = c$  and  $b = v$  because in this case the profit functions  $\pi_R$  and  $\pi_I$  coincide. As mentioned in §1, this contract is hardly satisfying because it leaves the supplier with zero profits. We now verify that no other  $(w, b)$ -contract is capable of coordinating the system. Thus, if an optimal solution  $(p^I, y^I)$  for the integrated system is to be adopted by the retailer, it must optimize his expected profit function  $\pi_R$ , and hence it must satisfy the first-order conditions

$$\frac{\partial\pi_R}{\partial y}(p^I, y^I) = (p^I - w) - (p^I - b)G(y^I \mid p^I) = 0, \quad (4)$$

$$\begin{aligned} \frac{\partial\pi_R}{\partial p}(p^I, y^I) &= y^I - E[y^I - D(p^I)]^+ \\ &\quad - (p^I - b) \int_0^{y^I} \frac{\partial G}{\partial p}(u \mid p^I) du = 0. \end{aligned} \quad (5)$$

Equations (4) and (5) are a system of *two linear* equations in *two* unknowns  $(w, b)$  with a *unique* solution. The equation in (5) involves only the variable  $b$ , whose coefficient

$$\int_0^{y^I} \frac{\partial G}{\partial p}(u \mid p^I) du < 0,$$

because  $(\partial G/\partial p)(u \mid p^I) < 0$  by the strict stochastic monotonicity of  $D(p)$ . Thus, (5) has a unique solution in  $b$  which, when substituted into (4), results in a unique solution for  $w$ . Because the pair  $(c, v)$  satisfies (4) and (5), no other contract with constant wholesale price and buy-back rate coordinates the system when the retail price  $p$  is *endogenously* determined. This is in sharp contrast to the case of an exogenously specified retail price  $p$ , where Pasternack (1985) exhibited a *continuum* of coordinating  $(w, b)$ -contracts.

We now show that coordination can be achieved with a *linear* price-discount sharing (PDS) scheme. Here, the wholesale price is a linear function of the retail price, i.e.,  $w = w(p)$ , and is specified as

$$\begin{aligned} w &= w^0 - \alpha\Delta p \quad \text{or} \quad \Delta w = \alpha\Delta p, \\ \text{where } \Delta p &= p^0 - p \quad \text{and} \quad \Delta w = w^0 - w. \end{aligned} \quad (6)$$

$w^0$  is a constant *base* or *gross* wholesale price,  $p^0$  an arbitrary reference value—e.g., the “list price”—and  $\alpha > 0$  a *constant*. The PDS scheme is to be combined with a traditional buy-back arrangement, at a buy-back rate set at a given constant  $\delta$  below the (net) wholesale price, i.e.,

$$b = w - \delta. \quad (7)$$

Under the scheme (6)–(7), the supplier compensates the retailer for every sold unit at the rate of  $\alpha\delta$  for every dollar the retailer discounts from the reference value  $p^0$ . Lal et al. (1996) consider such a shared price discount scheme—however, restricting the retailer to one possible discount size only; they also discuss a cooperative merchandising agreement in the consumer packaged-goods industry, which embodies this type of PDS scheme. In an article in the *Sloan Management Review*, Ailawadi et al. (1999) report on an increasing trend towards trade promotions<sup>2</sup> and explain that the most effective discount schemes tie the supplier's price directly to the retailer price according to a given PDS scheme. For the special case where  $D(p) = 4,000p^{-2}$  with probability one, they assert that no *constant* wholesale price that leaves the supplier with a positive margin will result in optimal supply-chain-wide performance, nor will any scheme that offers a *single* discounted wholesale price value if the retailer is willing to adopt a retail price at (or below) the chainwide optimal price level  $p^I$ . On the other hand, they propose a specific nonlinear PDS scheme under which perfect coordination is

<sup>2</sup>In 1996, the percentage of retail sales made ‘on deal’ across 40 packaged goods categories included in the Market Fact Book averaged about 37%, up almost 5% from 1991” (p. 83).

achieved.<sup>3</sup> Our results below show that the simpler, linear PDS scheme (6) can be used for a given, constant sharing fraction  $\alpha$  across the entire price range; moreover, the fraction  $\alpha$  can be chosen arbitrarily between 0 and 1, giving rise to a *continuum* of perfect coordination schemes. See, however, §3, where a *nonlinear* PDS scheme is required under retailer competition. The authors also report that in a series of interviews, several retail executives in charge of trade promotions indicate that virtually 100% of their promotions involve PDS schemes instead of traditional quantity-discount schemes.

The pricing scheme also bears close resemblance to the traditional “billback” or “count-recount” schemes; see Blattberg and Neslin (1990, Chapter 11). The only difference is that in the latter, the supplier adjusts the base wholesale price only if the retailer *reduces* his price. On the other hand, if the reference value  $p^0 > p^l$  and the scheme induces the retailer to adopt  $p^l$ , this difference is immaterial. For example, it can be shown that under a price-only contract with a constant wholesale price  $w > c$ , the retailer adopts a price  $p_R^* > p^l$ , an inefficiency resulting from *double marginalization*. If this is the status quo ante, it makes sense to choose  $p^0 = p_R^*$ . We will see, however, that coordination can be achieved under (6)–(7) for an arbitrary choice of the reference value  $p^0$  and a continuum of  $\alpha$ -values in  $(0, 1)$ . Let

$$w^0 = \alpha p^0 + (1 - \alpha)c \quad \text{and} \quad \delta = (1 - \alpha)(c - v). \quad (8)$$

**THEOREM 1.** Let  $N = 1$ . (a) For any pricing scheme with a constant per-unit wholesale price  $w$  and constant buy-back rate  $b < w$ , the retailer has an optimal response  $(p_R^*, y_R^* = y_R(p_R^*))$ . However, except for the “trivial” scheme with  $w = c$  and  $b = v$ , the decentralized chain generates expected systemwide profits that are strictly below those achievable under centralization.

(b) The linear PDS scheme (6), combined with (7), with base wholesale price  $w^0$  and buy-back discount  $\delta$  as in (8), results in perfect coordination for any reference value  $p^0$  and any  $0 < \alpha < 1$ .

Equation (A.1) in the proof of Theorem 1 shows that under the scheme (6)–(7), the fraction of the supply chain profits earned by the supplier is given by  $\alpha$ , and that of the retailer by  $(1 - \alpha)$ . Participation constraints for the supplier and the retailer, e.g., ensuring that their expected profits are in excess of those achieved prior to coordination, result in a lower bound  $\underline{\alpha}$  and

<sup>3</sup> The retailer is *fully* compensated for any price discounts in the range from \$14 to \$12 ( $\Delta w = \Delta p$ ), but only 75% of retail price discounts in the range from \$12 to \$8 are shared by the supplier ( $\Delta w = 0.75\Delta p$ ) and an increasingly lower percentage for retail price discounts beyond the \$8 value ( $\Delta w = 0.35\Delta p$ ) for price discounts below the \$5 value.

upper bound  $\bar{\alpha}$ , respectively. The exact choice of  $\alpha \in [\underline{\alpha}, \bar{\alpha}]$  depends on the chain members’ bargaining powers, but all such choices result in perfect coordination. Substituting (8) into (6)–(7), one verifies that  $w = c + \alpha(p - c)$  and  $b = v + \alpha(p - v)$ . A larger value of  $\alpha$  thus results in a larger wholesale price and buy-back rate, as well as higher profits for the supplier, confirming empirical observations that suppliers often prefer more generous buy-back commitments, even when combined with less than equal increases in the wholesale price (see, e.g., Padmanabhan and Png 1995). (That is,  $\alpha_1 < \alpha_2 \Leftrightarrow w(\alpha_2) - w(\alpha_1) < b(\alpha_2) - b(\alpha_1)$  for any retail price  $p$ , because  $c > v$ .) Clearly, the higher wholesale price applies to *all* units purchased by the retailer, but the increased buy-back rate applies only to the unsold ones. Padmanabhan and Png (1995) report examples of suppliers (e.g., McKesson, a major national distributor of hospital supplies) offering a *menu* of alternative returns policies, in which options with more generous return privileges are linked to higher wholesale prices.

The proof of Theorem 1(b) reveals that, under (6)–(7),  $\pi_R(p, y) = (1 - \alpha)\pi_I(p, y) = (1 - \alpha)R(p, y) - (1 - \alpha)cy$ , with  $R(p, y)$  the expected revenues (from direct sales *and* salvage) earned by the retailer in the absence of a buy-back agreement. This shows that an alternative coordination scheme arises when charging the retailer a wholesale price  $w = (1 - \alpha)c$ , but requiring him to share a fraction  $\alpha$  of his revenues with the supplier. This alternative scheme is equivalent to the PDS scheme in the sense that it generates identical profits for both firms for *any* demand realization.<sup>4</sup> Recall from §1 that the introduction of such revenue-sharing schemes in the video rental industry has drastically improved the performance of the supply chains, even though it requires an outside party to monitor and report the revenues of the retailers to the studios.<sup>5</sup> In contrast, implementation of the PDS scheme (6) only requires monitoring of the retail price. This is particularly easy when the retailer publishes his (uniform) price via publicly available media such as catalogs, newspapers, magazine ads, or the Internet.<sup>6</sup> In other words, the discount-

<sup>4</sup> A third alternative coordination scheme is based on sharing the revenues from *direct* sales only, where now the percentage paid to the supplier is specified as a (nonlinear) function of the retail price; see Cachon and Lariviere (2001).

<sup>5</sup> Shapiro (1998, p. B1) reports: “Blockbuster recently signed a multiyear pact with the company [Rentrak], whose proprietary information system records each and every rental at its clients’ stores and acts as the go-between for them with the studios. In exchange, Rentrak gets 10% of rental revenues, with the stores and studios getting 45% each.”

<sup>6</sup> Even in settings outside the one addressed by our model, in which the retailers use nonuniform prices, the scheme still incents the retailers to report all sales to which a discount off the reference value  $p^0$  is applied, provided  $p^0$  is set sufficiently high.

sharing scheme avoids the moral hazards associated with alternatives such as revenue-sharing schemes.<sup>7</sup> Finally, the compliance of PDS schemes with state and federal trade regulations has been tested and affirmed; see, e.g., South Carolina Revenue Ruling #94-8 (South Carolina Department of Revenue 1994).<sup>8</sup> It is unknown whether, in rental industries, perfect coordination can be achieved with PDS schemes. In any case, for rental revenues, where the same physical unit is rented many times for variable durations, as opposed to sales of consumable products, PDS schemes lose their implementation advantages over revenue-sharing schemes, because even under uniform pricing transactions need to be monitored under both types of schemes. This may explain why revenue-sharing schemes have been implemented for rental services, in particular the video rental industry (which typically uses uniform pricing throughout the season).

REMARK 1. In general, scheme (6) cannot be implemented as a traditional quantity-discount scheme, i.e., a scheme in which the wholesale price is specified as a function of the order size or the annual sales volume. While for any given retail price  $p$  the optimal corresponding order quantity is uniquely specified by

$$y_R(p) = y_i(p) = G^{-1}\left(\frac{p-c}{p-v} \mid p\right),$$

the opposite may fail to be true: A given order quantity may, optimally, correspond with several retail prices. As an example, take  $D(p) = (a - \beta p)\epsilon$  with  $\epsilon$  normally distributed with mean 1 and standard deviation  $s$ . Then,

$$y(p) = (a - \beta p) + s(a - \beta p)\Phi^{-1}\left(\frac{p-c}{p-v}\right)$$

with  $\Phi^{-1}(\cdot)$  the inverse cdf of a *standard* normal. Observe that a lower retail price induces a *larger* mean and standard deviation of the demand, but it also *lowers* the need for safety stocks. That is, it lowers the number  $\Phi^{-1}((p-c)/(p-v))$  of required standard deviations. Indeed,  $y(p)$  fails to be monotone, i.e., it fails to have an inverse when, e.g.,  $\alpha = 20$ ,  $\beta = 1$ , and  $s = 0.5$ . This implies that the retail price cannot be expressed

<sup>7</sup> Indeed, Ailawadi et al. (1999) argue that “Linking manufacturer price to retailer price rather than the quantity bought (or the amount sold) by the retailer is a simple solution to the coordination problem. It is easily understood, administered, and monitored; it also alleviates many problems and costs involved in other trade promotions” (p. 91).

<sup>8</sup> The ruling states: “A brewery may offer promotions involving a beer price reduction to wholesalers contingent upon the wholesalers passing price savings on to retailers, and a wholesaler may offer promotions involving a beer price reduction to retailers contingent upon the retailers passing price savings on to consumers.”

as a function of the order quantity, so that the PDS scheme cannot be specified as a traditional discount scheme.

Turning to the *general*  $N$ -retailer model, Theorem 1 carries over because the integrated profit function  $\pi_i$  is separable in all  $(p_i, y_i)$ -pairs. In particular, no scheme with *constant* wholesale prices and buy-back rates  $\{(w_i, b_i)\}$  induces coordination except for the trivial scheme  $w_i = c, b_i = v$ , but the following generalization of (6)–(7) does. For any given reference price  $p^0$ , let  $\Delta p_i = p_i - p^0$ . For any  $0 < \alpha_i < 1, i = 1, \dots, N$ , let

$$\begin{aligned} w_i &= \alpha_i p^0 + (1 - \alpha_i)c + \alpha_i \Delta p_i, \\ b_i &= w_i - (1 - \alpha_i)(c - v_i). \end{aligned} \quad (9)$$

### 3. General Competing Retailers

In this section, we consider the general case of  $N$  *competing* retailers. Here, the demand faced by any retailer  $i$ —i.e., the cdf of  $D_i$ —depends on the entire price vector  $p$ :  $\tilde{G}_i(x \mid p_1, \dots, p_N)$  for all  $i = 1, \dots, N$ . The analysis of the decentralized supply chain is considerably more complex in this case because the decisions made by any retailer  $i$  impact not just his own expected profits, but those of all other retailers as well. In the case of noncompeting retailers (§2), the recommended retailer strategies are *dominant*, i.e., each retailer’s strategy is optimal irrespective of the strategies chosen by the other retailers. The best that can be hoped for in the case of competing retailers is the existence of a Nash equilibrium.

We restrict ourselves to the case where the random variables  $D_i(p), i = 1, \dots, N$ , are of the multiplicative form  $D_i(p) = d_i(p)\epsilon_i$ , with  $\epsilon_i$  a general random variable whose distribution is independent of the price vector  $p$ . Equivalently,

$$\tilde{G}_i(x \mid p) = G_i\left(\frac{x}{d_i(p)}\right), \quad (10)$$

with  $G_i(\cdot)$  the cdf of  $\epsilon_i$  and  $g_i(\cdot)$  its pdf. (Most of the results in this section carry over to the case of *additive* demand shocks, i.e.,  $D_i(p) = d_i(p) + \epsilon_i$ , again with  $\epsilon_i$  a random variable whose distribution is independent of the price vector  $p$ ; see Remark 2 below.) The multiplicative model implies that the coefficients of variation are independent of the price vector  $p$ . Without loss of generality, we normalize  $E\epsilon_i = 1, i = 1, \dots, N$ , so that  $ED_i(p) = d_i(p)$ . Because the products offered by the retailers are substitutes, we make the standard assumption that the demand functions  $d_i(p)$  are differentiable with

$$\frac{\partial d_i(p)}{\partial p_i} \leq 0 \quad \text{and} \quad \frac{\partial d_i(p)}{\partial p_j} \geq 0, \quad i \neq j. \quad (11)$$

In addition, we will assume

CONDITION (A). For each  $i = 1, \dots, N$ , the function  $\log d_i(p)$  has increasing differences<sup>9</sup> in  $(p_i, p_j)$  for all  $j \neq i$ .

Milgrom and Roberts (1990) identified the Linear, Logit, Cobb-Douglas, and CES demand functions as satisfying Condition (A). (It is easily verified that Condition (A) is essentially *weaker* than the requirement that the expected demand functions  $d_i(p)$  have increasing differences in  $(p_i, p_j)$ ,  $j \neq i$  themselves.) Finally, we assume that each retailer  $i$  chooses his price  $p_i$  from a closed interval  $[p_i^{\min}, p_i^{\max}]$ .

We first characterize the equilibrium (i.e., the set of retailer orders and prices) when wholesale prices and buy-back rates are constant. Let  $\pi_i(p, y)$  denote the expected profits for retailer  $i$ :

$$\begin{aligned} \pi_i(p, y) &= (p_i - w_i)y_i - (p_i - b_i)E[y_i - D_i(p)]^+ \\ &= (p_i - w_i)y_i - (p_i - b_i)E[y_i - d_i(p)\epsilon_i]^+. \end{aligned} \quad (12)$$

While a retailer's price  $p$  impacts on the profits of *all* retailers, his order quantity  $y$  affects his *own* profits only. It thus follows from (10) and (12) that for any price  $p_i$ , retailer  $i$ 's optimal corresponding order is given by the equation  $\bar{G}_i(y_i | p) = G_i(y_i/d_i(p)) = (p_i - w_i)/(p_i - b_i)$ , or

$$y_i(p) = d_i(p)G_i^{-1}\left(\frac{p_i - w_i}{p_i - b_i}\right). \quad (13)$$

This observation permits us to reduce the noncooperative game in the  $(p, y)$ -space to one in which retailers compete with a single instrument ( $p$ ) only. We refer to this game as the *reduced retailer game*, as opposed to the *original retailer game*, in which each retailer competes with his price variable and order quantity. Indeed, substituting (13) into (12), we get the retailers' profits as a function of  $p$  only:

$$\begin{aligned} \tilde{\pi}_i(p) &= d_i(p) \left[ (p_i - w_i)G_i^{-1}\left(\frac{p_i - w_i}{p_i - b_i}\right) \right. \\ &\quad \left. - (p_i - b_i)E\left[G_i^{-1}\left(\frac{p_i - w_i}{p_i - b_i}\right) - \epsilon_i\right]^+ \right] \\ &= \pi_i^{\det}(p | w_i)L_i(f_i(p_i)), \quad \text{where} \end{aligned} \quad (14)$$

$\pi_i^{\det}(p | w_i) = (p_i - w_i)d_i(p)$  = retailer  $i$ 's profits under the price vector  $p$ , in the *deterministic* system where no uncertainty prevails, i.e., demand for retailer  $i$  equals  $d_i(p)$  with probability one so that  $y_i(p) = d_i(p)$ ,

$f_i(p_i) = (p_i - w_i)/(p_i - b_i)$  = the critical fractile for retailer  $i$ , and

<sup>9</sup> A function  $f(x_1, \dots, x_N)$  has increasing differences in  $(x_i, x_j)$  if  $f(x_1, \dots, x_i^1, \dots, x_N) - f(x_1, \dots, x_i^2, \dots, x_N)$  is increasing in  $x_j$  for all  $x_i^2 < x_i^1$ . If the function  $f$  is twice differentiable, the property is equivalent to  $\partial^2 f / \partial x_i \partial x_j \geq 0$ .

$L_i(f) = G_i^{-1}(f) - f^{-1}E[G_i^{-1}(f) - \epsilon_i]^+ = G_i^{-1}(f) - f^{-1} \int_{-\infty}^{G_i^{-1}(f)} G_i(u) du = (\int_{-\infty}^{G_i^{-1}(f)} u g_i(u) du) / f$  = the factor by which retailer  $i$ 's profits are reduced because of the prevailing demand uncertainty.

(The second equality in the definition of  $L_i(f)$  follows by integration by parts.)

In other words, under a given price vector  $p$  in the market, each retailer  $i$ 's expected profits equal the profit value in the *deterministic* system multiplied with a *loss factor* that depends only on the retailer's chosen critical fractile  $f_i$  and the shape of the cdf of  $\epsilon_i$ . The following lemma establishes an important property of this loss factor.

LEMMA 1. For all  $i = 1, \dots, N$ , the loss factor  $L_i(f)$  defined on  $(0, 1)$ , increases from  $G_i^{-1}(0)$  to one as the critical fractile  $f$  increases from zero to one.

In other words, retailer  $i$ 's profit is always *less* than what would be achieved in the deterministic system, but the extent of the profit reduction decreases with the size of the retailer's critical fractile  $f_i$ , i.e., as the cost of understockage  $(p_i - w_i)$  dominates relative to the cost of overstockage  $(w_i - b_i)$ .

THEOREM 2. Assume Condition (A) applies. Fix the vectors  $b < w$ .

(a) There exists a Nash equilibrium  $p^*$  for the reduced retailer game that arises under the  $(w, b)$ -payment scheme, and  $(p^*, y(p^*))$  is a Nash equilibrium in the original game.

(b) If the reduced retailer game has multiple Nash equilibria, these equilibria constitute a sublattice of  $\mathfrak{R}^N$ . In particular, there exists a smallest and largest equilibrium  $\underline{p}$  and  $\bar{p}$ , respectively.

(c) The equilibrium  $\bar{p}$  is preferred by all  $N$  retailers among all Nash equilibria.

REMARK 2. Theorem 2 continues to apply in the model with additive demand shocks. Following the analysis above, we verify that, in this model,  $\tilde{\pi}_i(p) = \pi_i^{\det}(p | w_i) + (p_i - w_i)L_i(f_i(p_i))$ . The proof in this case is thus analogous to the proof given in the appendix.

Because under Condition (A) the retailer game is "log-supermodular"—see Topkis (1998)—it is well known that both  $\underline{p}$  and  $\bar{p}$  can be computed easily by a so-called *tatōnnement* or round-robin scheme: Starting with an arbitrary price vector  $p^0 \leq \underline{p}$  or  $p^0 \geq \bar{p}$ , e.g.,  $p^0 = p^{\min} = (p_1^{\min}, \dots, p_N^{\min})$  or  $p^0 = p^{\max} = (p_1^{\max}, \dots, p_N^{\max})$ , in the  $k$ th iteration  $p^k$  is obtained from  $p^{k-1}$  by determining  $p_i^k = \arg \max_{p_i} \tilde{\pi}_i(p_i, p_{-i}^{k-1})$ ; the sequence  $\{p^k\}$  converges to  $\underline{p}$  and  $\bar{p}$ , respectively. A *unique* equilibrium thus exists if and only if the algorithm converges to the same point when starting at  $p^{\min}$  and at  $p^{\max}$ .

We have not encountered any instances with multiple equilibria. Moreover, let

$$\hat{p}_i^{\min} = \max\{p_i^{\min}, 2w_i - b_i\}. \quad (15)$$

The following theorem guarantees that a *unique* Nash equilibrium exists in the (possibly restricted) price space  $\prod_{i=1}^N [\hat{p}_i^{\min}, p_i^{\max}]$ , provided *two* conditions are satisfied. (The restriction  $p_i \geq \hat{p}_i^{\min}$  is equivalent to the cost of *understockage*  $p_i - w_i \geq w_i - b_i$ , the cost of *overstockage*. This assumption applies, at least in equilibrium, to almost every industry: Virtually without exception, retailers restrict their price choices to values under which the cost of understockage exceeds that of overstockage—i.e., under which they are incited to provide a fill rate of at least 50%.) The first condition is the most generally known sufficient condition for a unique Nash equilibrium to exist in the *deterministic* retailer game; see, e.g., Milgrom and Roberts (1990) and Vives (1990):

$$(D) \quad -\frac{\partial^2 \log \pi_i^{\det}(p | w_i = b_i)}{\partial p_i^2} \geq \sum_{j \neq i} \frac{\partial^2 \log \pi_i^{\det}(p | w_i = b_i)}{\partial p_i \partial p_j},$$

$i = 1, \dots, N.$

Condition (D) is satisfied for all of the above stated classes of demand functions  $\{d_i(p)\}$  with minor parameter restrictions; see Bernstein et al. (1999).

The second condition provides a restriction for the distributions of the *random* variables  $\{\epsilon_i\}$ :

$$(S) \quad \psi_i(x) \stackrel{\text{def}}{=} \left[ -2x + \frac{\bar{G}_i(x)}{g_i(x)} \right] \int_{-\infty}^x u g_i(u) du - \bar{G}_i(x) x^2 \leq 0,$$

for all  $x \geq m_i, i = 1, \dots, N,$

where  $m_i$  = the median of the distribution  $G_i$ . This condition is, e.g., satisfied for all distributions whose hazard rate  $h(x) \stackrel{\text{def}}{=} g(x)/\bar{G}(x) \geq 1/2x$  for all  $x \geq m_i$ . This class includes most of the commonly used distributions, e.g., the exponentials and all normals with mean one and standard deviation  $s \leq 1$ . Condition (S) is also satisfied for all power distribution, i.e., when  $G_i(x)$  is of the form  $G_i(x) = ((k_i + 1)/k_i)^{-k_i} x_i^k$  for any constant  $k_i \geq 0$  and  $0 \leq x \leq (k_i + 1)/k_i$ .

**THEOREM 3.** Assume Conditions (A), (D), and (S) apply. The retailer game that arises under any given  $(w, b)$ -payment scheme has  $\bar{p}$  as the unique Nash equilibrium in the (possibly restricted) price space  $\prod_{i=1}^N [\hat{p}_i^{\min}, p_i^{\max}]$ .

Recall from Theorem 2(c) that, even in the presence of multiple equilibria,  $\bar{p}$  is the equilibrium that is preferred by all  $N$  retailers. Moreover, as explained above, the restriction  $p_i \geq \hat{p}_i^{\min}$  applies, at least in equilibrium, to almost every industry. Thus, markets tend to converge to  $\bar{p}$ , the unique equilibrium in the (possibly restricted) price space and the one that is preferred by all retailers.

We now investigate what impact the wholesale prices and buy-back rates have on the equilibrium. In the deterministic model, it is well known (see, e.g., Milgrom and Roberts 1990, Topkis 1998, and Vives

1990) that an increase in *one* of the wholesale prices results in an increase of *all* equilibrium retailer prices, provided Conditions (A) and (D) are satisfied. We show that this result continues to apply in the general *stochastic* model, under Conditions (A) and (D) and a strengthening of Condition (S) as follows:

$$(S_w) \quad \psi_i(x) \stackrel{\text{def}}{=} \left[ -x + \frac{\bar{G}_i(x)}{g_i(x)} \right] \int_{-\infty}^x u g_i(u) du - \bar{G}_i(x) x^2 \leq 0$$

for all  $x \geq m_i, i = 1, \dots, N.$

(Recall that we identified Condition (S) as a general sufficient condition to guarantee the existence of a unique Nash equilibrium.) Similarly, one would expect that an increase in *one* of the buyback rates, say  $b_i$ , will result in a *decrease* of *all* retailers' equilibrium prices: The increase in  $b_i$  results in an increase of retailer  $i$ 's profit margin under the given retailer price; the increased profit margin allows retailer  $i$  to reduce his price, which induces all competitors offering substitutable products to reduce their price as well. Indeed, we prove that this conjecture holds, again under Conditions (A), (D), and a further strengthening of (S), as follows:

$$(S_b) \quad \text{For all } i = 1, \dots, N \text{ and all } x \geq m_i,$$

$$\psi_i(x) \stackrel{\text{def}}{=} \left[ -\left(2 - \frac{1}{G_i(x)}\right)x + \frac{\bar{G}_i(x)}{g_i(x)} \right] \int_{-\infty}^x u g_i(u) du$$

$$- \bar{G}_i(x) x^2 + \frac{\int_{-\infty}^x u g_i(u) du}{G_i(x)} \leq 0.$$

Clearly,  $(S_b) \Rightarrow (S_w) \Rightarrow (S)$  because  $G_i(x) \leq 1$ . Condition  $(S_w)$  continues to hold for all exponentials, normals with mean one, and standard deviation  $s \leq 1$ , as well as the power distributions. Likewise, Condition  $(S_b)$  holds for all exponentials and power distributions. For normals with mean one and standard deviation  $s \leq 1$ , it holds for all  $x \geq 1.87$ . (For example, when  $s = 1$ , the condition is satisfied as long as each retailer adopts a safety stock of at least 1.8 standard deviations.)

**THEOREM 4.** Assume Conditions (A) and (D) apply.

(a) Under Condition  $(S_w)$ , the retailer game has a unique Nash equilibrium in which all retailer prices in  $p^*$  increase when one or more of the wholesale prices in  $w$  increase.

(b) Under Condition  $(S_b)$ , the retailer game has a unique Nash equilibrium in which all retailer prices in  $p^*$  decrease when one or more of the buy-back rates in  $b$  increase.

In the absence of retailer competition, we showed that the chain can be coordinated with a simple linear PDS scheme. We now show that with price competition, perfect coordination can still be induced with a PDS scheme, albeit that this scheme now requires a



nonlinear component. As before, let  $\tilde{\pi}_i(p)$  denote the expected profits in the integrated system under the price vector  $p$  and optimal corresponding order quantities  $\{y_j^l(p), j = 1, \dots, N\}$ . Analogous to (13), one verifies that

$$y_j^l(p) = d_j(p)G_j^{-1}\left(\frac{p_j - c_j}{p_j - v_j}\right), \quad j = 1, \dots, N.$$

Analogous to the derivation of (14), it is now easily verified that

$$\tilde{\pi}_i(p) = \sum_{j=1}^N \pi_{i,j}^{\det}(p) L_j \left(\frac{p_j - c_j}{p_j - v_j}\right), \quad (16)$$

where, for  $j = 1, \dots, N$ ,  $\pi_{i,j}^{\det}(p) \stackrel{\text{def}}{=} (p_j - c_j)d_j(p)$  denotes the profits obtained in the *deterministic* integrated system for sales at retailer  $j$ , and where  $L_j$  is the factor by which the profits arising from sales at retailer  $j$  are reduced due to the demand uncertainty. Recall from (13) that under a PDS scheme  $\{w_j(p_j), b_j(p_j)\}$ ,

$$y_j(p) = d_j(p)G_j^{-1}\left(\frac{p_j - w_j(p_j)}{p_j - b_j(p_j)}\right), \quad j = 1, \dots, N.$$

Thus, to induce perfect coordination, it is desirable to ensure that under any prevailing price vector  $p$ , each retailer  $j$  adopts an order quantity  $y_j(p) = y_j^l(p)$ , which is equivalent to the equation

$$G_j^{-1}\left(\frac{p_j - c_j}{p_j - v_j}\right) = G_j^{-1}\left(\frac{p_j - w_j(p_j)}{p_j - b_j(p_j)}\right),$$

and by the monotonicity of the inverse cdf,

$$\frac{p_j - w_j(p_j)}{p_j - b_j(p_j)} = \frac{p_j - c_j}{p_j - v_j}, \quad j = 1, \dots, N. \quad (17)$$

In other words, the pricing scheme must ensure that the retailers face the same critical fractile in the decentralized system as they do in the integrated system. Under (17), it follows from (14) that for each retailer  $i = 1, \dots, N$ ,

$$\tilde{\pi}_i(p) = (p_i - w_i)d_i(p)L_i\left(\frac{p_i - c_i}{p_i - v_i}\right). \quad (18)$$

Let  $p^l$  denote an optimal price vector for the integrated system. We now design the pricing scheme as an application of Groves' (1973) mechanism. This means that we ensure for each retailer  $i$  that when all other retailers' prices are chosen as in  $p^l$ , his profit function in the decentralized system is a monotone transformation of the *reduced* integrated profit function, i.e.,  $\tilde{\pi}_i(p_i, p_{-i}^l) = \Gamma_i(\tilde{\pi}_i(p_i, p_{-i}^l))$  for some monotone transformation  $\Gamma_i: \mathfrak{R} \rightarrow \mathfrak{R}$ . More specifically, we choose  $\Gamma_i$  as an affine transformation, as follows:

$$\tilde{\pi}_i(p_i, p_{-i}^l) = (1 - \alpha_i)[\tilde{\pi}_i(p_i, p_{-i}^l) - C_i], \quad (19)$$

where  $0 < \alpha_i < 1$ , and

$$C_i = \sum_{j \neq i} (p_j^l - c_j) d_j(p^l) L_j \left(\frac{p_j^l - c_j}{p_j^l - v_j}\right)$$

represents the optimal expected profits in the integrated system derived from all sales *excluding those at retailer  $i$* . Substituting (18) and (16) into (19) and rearranging the terms, we obtain

$$w_i = \alpha_i p_i + (1 - \alpha_i) c_i + (1 - \alpha_i) \cdot \sum_{j \neq i} (p_j^l - c_j) \frac{[d_j(p^l) - d_j(p_i, p_{-i}^l)] L_j \left(\frac{p_j^l - c_j}{p_j^l - v_j}\right)}{d_i(p_i, p_{-i}^l) L_i \left(\frac{p_i - c_i}{p_i - v_i}\right)}.$$

Choosing an arbitrary reference price value  $p_i^0$  (e.g.,  $p_i^0 = p_i^l$ , as the list price or recommended retailer price), the wholesale-price scheme can be rewritten as

$$w_i = w_i^0 + \alpha_i \Delta p_i - (1 - \alpha_i) \cdot \sum_{j \neq i} (p_j^l - c_j) \frac{[d_j(p_i, p_{-i}^l) - d_j(p^l)] L_j \left(\frac{p_j^l - c_j}{p_j^l - v_j}\right)}{d_i(p_i, p_{-i}^l) L_i \left(\frac{p_i - c_i}{p_i - v_i}\right)}, \quad (20)$$

where  $w_i^0 = \alpha_i p_i^0 + (1 - \alpha_i) c_i$  and  $\Delta p_i = p_i - p_i^0$ . The scheme thus generalizes the *linear* PDS scheme for noncompeting retailers. The competitive dynamics between the retailers, in particular the impact any retailer's price decision has on the sales of all other retailers, induces the need for a *nonlinear* correction term, the magnitude of which is directly related to the magnitude of the cross effects  $\partial d_j / \partial p_i$ ,  $j \neq i$ , in the demand functions, because  $d_j(p_i, p_{-i}^l) - d_j(p^l) = \int_{p_i^l}^{p_i} \partial d_j / \partial p_i(u, p_{-i}^l) du$ . Note that when retailer  $i$  chooses his price  $p_i > p_i^l$ , the correction term is negative, i.e., the wholesale price is adjusted *downwards* compared to its value in the noncompetitive model. On the other hand, when the retailer sets his price too low, i.e.,  $p_i < p_i^l$ , the wholesale price is adjusted *upwards* compared to its value in the absence of competition. In equilibrium, when  $p_i = p_i^l$ , the correction term vanishes and the wholesale prices (buy-back rates) are given by the same weighed averages of the retailer price and per-unit procurement cost (salvage value) as in the noncompetitive model. We conclude:

**THEOREM 5.** For any vector of price discount shares  $\alpha$ ,  $0 < \alpha < 1$ , the pair of vectors  $(p^l, y^l)$  is a Nash equilibrium in the retailer game induced by the wholesale PDS scheme (20) (and corresponding buy-back rates determined via (17)). In particular, the scheme induces perfect coordination, with equilibrium wholesale prices and buy-back rates given by  $c_i < w_i^* = \alpha_i p_i^l + (1 - \alpha_i) c_i < p_i^l$  and  $v_i < b_i^* = \alpha_i p_i^l + (1 - \alpha_i) v_i < w_i^*$ .

The question remains whether perfect coordination can be achieved with a set of *constant* wholesale prices  $w^*$  and buy-back rates  $b^*$ . Theorem 1 shows that in the absence of competition, the only such scheme has  $w_i^* = c_i$  and  $b_i^* = v_i$  for all  $i = 1, \dots, N$ , and this scheme is hardly appealing because it eliminates the supplier's profits entirely. In the presence of competition, the following theorem affirms the existence of a coordinating scheme with constant wholesale prices, provided conditions apply that guarantee the existence of a unique price equilibrium that increases in the vector of wholesale prices. (Recall that Theorem 4(a) identifies a broad class of demand distributions under which this is guaranteed.) Moreover, the coordinating vector of wholesale prices  $w^*$  satisfies the desired inequalities  $c_i < w_i^* < p_i^l$  for all  $i = 1, \dots, N$ , thus invoking *double marginalization* as a mechanism to *induce*, rather than *prohibit*, perfect coordination.

**THEOREM 6.** *Assume Conditions (A), (D), and (S<sub>w</sub>) apply, and  $p^l$  is an interior point of the feasible domain.*

(a) *There is a constant vector of wholesale prices  $w^*$ , with  $c \leq w^* < p^l$ , which in combination with the buy-back rates  $b^* = v$  induces perfect coordination.*

(b) *If, in addition,  $\partial d_i / \partial p_j > 0$  for  $j \neq i$ ,  $c_i < w_i^* < p_i^l$  for all  $i = 1, \dots, N$ .*

**REMARK 3.** If the inequality in (S<sub>w</sub>) is strict, one easily verifies from the above proof that the coordinating vector of wholesale prices  $w^*$  is *unique*.

We have thus identified two possible coordination schemes. The constant wholesale-pricing scheme is clearly simpler than the PDS scheme. For the latter, a wholesale price table needs to be offered to the retailers as opposed to a single cost value. Such wholesale price tables directly tying the wholesale price to different retail price ranges are commonly used; see the abovementioned *Sloan Management Review* article by Ailawadi et al. (1999) for a discussion and concrete examples. A table represents a step-function approximation for the general nonlinear scheme (20). As with all such approximations, a proper balance needs to be found between *accuracy* and *complexity*, i.e., the number of price ranges in the table. The constant wholesale-pricing scheme has the additional advantage of ensuring that the pair of vectors  $(p^l, y^l)$  arises as a *unique* retailer equilibrium while the PDS scheme may give rise to alternative Nash equilibria. On the other hand, the constant wholesale-pricing scheme can only be applied for the broad, though somewhat restrictive, class of demand functions and distributions implied by Conditions (A), (D), and (S<sub>w</sub>), while the PDS scheme is always applicable. More importantly, the constant-pricing scheme allows for only a *single* vector of wholesale prices  $w^*$  (see Remark 3) which, as shown in the proof of Theorem 6, may result

in very small or zero margins for the supplier when the competitive cross effects in the demand functions ( $\partial d_i / \partial p_j$ ,  $j \neq i$ ) are small or zero, respectively. The PDS scheme allows for a *continuous menu* of pricing schemes by varying the share parameters  $\alpha_i$  from 0 to 1. The retailers may prefer a contract with a value of  $\alpha$  close to one, resulting in a low gross margin but a high buy-back rate and correspondingly small purchasing risk.

## Appendix. Proofs

**PROOF OF THEOREM 1.** Part (a) is shown in §2. (b) Replace in (2),  $w$  by (6) and  $b$  by (7), with  $w^0$  and  $\delta$  specified as in (8):

$$\begin{aligned} \pi_R(p, y) &= (p - w)y - (p - w + \delta)E[y - D(p)]^+ \\ &= (p - w^0 - \alpha(p - p^0))y - [p - w^0 - \alpha(p - p^0) \\ &\quad + (1 - \alpha)(c - v)]E[y - D(p)]^+ \\ &= (1 - \alpha)(p - c)y - [(1 - \alpha)(p - c) \\ &\quad + (1 - \alpha)(c - v)]E[y - D(p)]^+ \\ &= (1 - \alpha)\pi_i(p, y). \end{aligned} \tag{A.1}$$

Thus, any optimal solution  $(p^l, y^l)$  of  $\pi_i(\cdot, \cdot)$  optimizes  $\pi_R$  as well.  $\square$

**PROOF OF LEMMA 1.** As a function of  $x_i = G_i^{-1}(f_i)$ , an increasing function of  $f_i$ , the loss factor can be written as  $L_i(x_i) = (\int_{-\infty}^{x_i} u g_i(u) du) / G_i(x_i)$ , with

$$\begin{aligned} L'_i(x_i) &= (x_i g_i(x_i) G_i(x_i) - g_i(x) \int_{-\infty}^{x_i} u g_i(u) du) / G_i^2(x_i) \\ &= g_i(x_i) E[x_i - \epsilon_i]^+ / G_i^2(x_i) > 0. \end{aligned}$$

Moreover,

$$\begin{aligned} \lim_{f \searrow 0} L_i(f) &= \lim_{f \searrow 0} \frac{d(\int_{-\infty}^{G_i^{-1}(f)} u g_i(u) du)}{df} \\ &= \lim_{f \searrow 0} G_i^{-1}(f) g_i(G_i^{-1}(f)) \frac{1}{g_i(G_i^{-1}(f))} = G_i^{-1}(0), \end{aligned}$$

while  $\lim_{f \nearrow 1} L_i(f) = \int_{-\infty}^{\infty} u g_i(u) du = 1$  because  $E\epsilon_i = 1$ .  $\square$

**PROOF OF THEOREM 2.** To prove parts (a) and (b), we show that the reduced retailer game is log-supermodular; see Milgrom and Roberts (1990). Because each of the retailers competes with a single instrument chosen from a compact set, it suffices to show that for all  $i = 1, \dots, N$ ,  $\log \tilde{\pi}_i(p)$  has increasing differences in  $(p_i, p_j)$  for all  $j \neq i$ . By (14),

$$\log \tilde{\pi}_i(p) = \log d_i(p) + \log(p_i - w_i) + \log L_i(f_i(p_i)). \tag{A.2}$$

The required property follows from Condition (A) and the fact that the second and third terms in (A.2) only depend on the variable  $p_i$ . To prove part (c), we show that retailer  $i$  prefers the price vector  $\bar{p}$  over all  $p \leq \bar{p}$  in general, and hence, by part (b), over all other Nash equilibria in particular. Note from (11) and (14) that  $\tilde{\pi}_i(p)$  is increasing in  $p_j$  for all  $j \neq i$ . Thus, for all  $p \leq \bar{p}$ ,  $\tilde{\pi}_i(p_i, p_{-i}) \leq \tilde{\pi}_i(p_i, \bar{p}_{-i}) \leq \tilde{\pi}_i(\bar{p}_i, \bar{p}_{-i})$ , where the second inequality follows from the fact that  $\bar{p}$  is a Nash equilibrium.  $\square$

PROOF OF THEOREM 3. Because (A) applies, the retailer game has a Nash equilibrium, by Theorem 2. Uniqueness follows from Milgrom and Roberts (1990) if

$$-\frac{\partial^2 \log \tilde{\pi}_i(p)}{\partial p_i^2} \geq \sum_{j \neq i} \frac{\partial^2 \log \tilde{\pi}_i(p)}{\partial p_i \partial p_j}.$$

(Note that the condition is analogous to (D).) Let  $\hat{L}_i(p_i) \stackrel{\text{def}}{=} \int_{-\infty}^{G_i^{-1}(f_i(p_i))} u g_i(u) du$ . By (14),

$$\begin{aligned} \log \tilde{\pi}_i(p) &= \log(p_i - w_i) d_i(p) - \log f_i(p_i) + \log \hat{L}_i(p_i) \\ &= \log(p_i - w_i) d_i(p) + \log \frac{p_i - b_i}{p_i - w_i} + \log \hat{L}_i(p_i) \\ &= \log(p_i - b_i) d_i(p) + \log \hat{L}_i(p_i) \\ &= \log \pi_i^{\text{det}}(p \mid w_i = b_i) + \log \hat{L}_i(p_i). \end{aligned} \quad (\text{A.3})$$

In view of (D) and the fact that the second term to the right of (A.3) depends only on retailer  $i$ 's own price  $p_i$ , it suffices to show that  $(\partial^2 \log \hat{L}_i(p_i)) / \partial p_i^2 \leq 0$ . One easily verifies that

$$\frac{\partial \log \hat{L}_i(p_i)}{\partial p_i} = \frac{(w_i - b_i)(p_i - b_i)^{-2} G_i^{-1}(f_i(p_i))}{\hat{L}_i(p_i)}, \quad (\text{A.4})$$

$$\begin{aligned} \frac{\partial^2 \log \hat{L}_i(p_i)}{\partial p_i^2} &= \frac{(w_i - b_i)}{\hat{L}_i^2(p_i)} \left[ \left\{ -2 \frac{G_i^{-1}(f_i(p_i))}{(p_i - b_i)^3} + \frac{(w_i - b_i)}{(p_i - b_i)^4 g_i(G_i^{-1}(f_i(p_i)))} \right\} \right. \\ &\quad \left. \cdot \hat{L}_i(p_i) - \frac{(w_i - b_i)}{(p_i - b_i)^4} [G_i^{-1}(f_i(p_i))]^2 \right] \\ &= \frac{(w_i - b_i)(p_i - b_i)^{-3}}{\hat{L}_i^2} \left[ \left( -2x_i + \frac{\bar{G}_i(x_i)}{g(x_i)} \right) \int_{-\infty}^{x_i} u g_i(u) du - x_i^2 \bar{G}_i(x_i) \right], \end{aligned}$$

substituting  $x_i = G_i^{-1}(f_i(p_i))$ . By Condition (S), we conclude that  $(\partial^2 \log \hat{L}_i(p_i)) / \partial p_i^2 \leq 0$ , because  $x_i \geq m_i$  as  $f_i \geq 1/2$  by (15).  $\square$

PROOF OF THEOREM 4. (a) Because  $(S_w) \Rightarrow (S)$ , a unique Nash equilibrium  $p^*$  exists by Theorem 3. By Milgrom and Roberts (1990), it suffices to show that  $\partial^2 \log \tilde{\pi}_i(p) / \partial p_i \partial w_i \geq 0$  for all  $i = 1, \dots, N$ . By (A.3), this is equivalent to  $\partial^2 \log \hat{L}_i(p) / \partial p_i \partial w_i \geq 0$  because the first term in (A.3) does not depend on  $w$ . Using (A.4), we obtain

$$\begin{aligned} \frac{\partial^2 \log \hat{L}_i(p_i)}{\partial p_i \partial w_i} &= \frac{(p_i - b_i)^{-2}}{\hat{L}_i^2(p_i)} \left[ \left( G_i^{-1}(f_i(p_i)) - \frac{w_i - b_i}{p_i - b_i} \frac{1}{g_i(G_i^{-1}(f_i(p_i)))} \right) \hat{L}_i(p_i) \right. \\ &\quad \left. + \frac{w_i - b_i}{p_i - b_i} (G_i^{-1}(f_i(p_i)))^2 \right] \\ &= \frac{(p_i - b_i)^{-2}}{\hat{L}_i^2(p_i)} \left[ \left( x - \frac{\bar{G}_i(x)}{g_i(x)} \right) \int_{-\infty}^x u g_i(u) du + x^2 \bar{G}_i(x) \right] \geq 0, \end{aligned}$$

again substituting  $x = G_i^{-1}(f_i) \geq 1$  because  $f_i > 1/2$ , by (15).

(b) Because  $(S_b) \Rightarrow (S)$ , the proof is analogous to that of part (a), and it suffices to show that

$$\begin{aligned} \frac{\partial^2 \log \tilde{\pi}(p)}{\partial p_i \partial b_i} &= \frac{\partial^2 \pi_i^{\text{det}}(p \mid b_i)}{\partial p_i \partial b_i} + \frac{\partial^2 \log \hat{L}_i(p_i)}{\partial p_i \partial b_i} \\ &= \frac{1}{(p_i - b_i)^2} + \frac{\partial^2 \log \hat{L}_i(p_i)}{\partial p_i \partial b_i} \leq 0. \end{aligned}$$

Using (A.4), we obtain

$$\begin{aligned} \frac{\partial^2 \log \hat{L}_i(p_i)}{\partial p_i \partial b_i} &= \frac{(p_i - b_i)^{-2}}{\hat{L}_i^2(p_i)} \left[ \left( -G_i^{-1}(f_i(p_i)) + \frac{(w_i - b_i)}{p_i - b_i} \right) \frac{(p_i - w_i)}{p_i - b_i} \right. \\ &\quad \left. \cdot \frac{1}{g_i(G_i^{-1}(f_i(p_i)))} \right] \hat{L}_i(p_i) \\ &\quad + 2 \left( \frac{w_i - b_i}{p_i - b_i} \right) G_i^{-1}(f_i(p_i)) \hat{L}_i(p_i) \\ &\quad - \left( \frac{w_i - b_i}{p_i - b_i} \right) \left( \frac{p_i - w_i}{p_i - b_i} \right) (G_i^{-1}(f_i(p_i)))^2 \Big] \\ &= \frac{G(x)(p_i - b_i)^{-2}}{\hat{L}_i^2(p_i)} \left[ \left( - \left( 2 - \frac{1}{G_i(x)} \right) x + \frac{\bar{G}_i(x)}{g_i(x)} \right) \right. \\ &\quad \left. \cdot \int_{-\infty}^x u g_i(u) du - x^2 \bar{G}_i(x) \right. \\ &\quad \left. + \frac{\int_{-\infty}^x u g_i(u) du}{G(x)} \right] \geq 0, \end{aligned}$$

again substituting  $x = G_i^{-1}(f_i)$ .  $\square$

PROOF OF THEOREM 5. Fix  $i = 1, \dots, N$ . Because  $p^l$  maximizes  $\tilde{\pi}_i$ ,  $p_i^l$  maximizes  $\tilde{\pi}_i(\cdot, p_{-i}^l)$  as well as  $\tilde{\pi}_i(\cdot, p_{-i}^l)$ , its monotone transformation (19). This implies that  $p^l$  is a Nash equilibrium in the retailer game induced by the scheme (20).  $\square$

PROOF OF THEOREM 6. (a) Fix  $b^* = v$ . Theorem 2 shows that the functions  $\log \tilde{\pi}_i(\cdot, p_{-i}^l \mid w)$  are unimodal for all choices of  $w$  (recall that  $(S_w)$  implies  $(S)$ ). Thus, if for some vector  $w^*$ ,  $\partial \log \tilde{\pi}_i(p_i^l, p_{-i}^l \mid w^*) / \partial p_i = 0$  for all  $i = 1, \dots, N$ ,  $p^l$  is a Nash equilibrium in the retailer game induced by the  $(w^*, b^* = v)$ -scheme. By Theorem 4, we know that  $(\partial \log \tilde{\pi}_i) / \partial p_i$  is an increasing function of  $w_i$ . It therefore suffices to show that

$$\frac{\partial \log \tilde{\pi}_i(p_i^l, p_{-i}^l \mid w_i = c_i)}{\partial p_i} \leq 0 \quad \text{and} \quad (\text{A.5})$$

$$\lim_{w_i \searrow p_i^l} \frac{\partial \log \tilde{\pi}_i(p_i^l, p_{-i}^l \mid w_i)}{\partial p_i} > 0. \quad (\text{A.6})$$

Recall that

$$\tilde{\pi}_i(p) = (p_i - w_i) d_i(p) L_i \left( \frac{p_i - w_i}{p_i - v_i} \right),$$

while

$$\tilde{\pi}_i(p) = (p_i - c_i) d_i(p) L_i \left( \frac{p_i - c_i}{p_i - v_i} \right) + \sum_{j \neq i} (p_j - c_j) d_j(p) L_j \left( \frac{p_j - c_j}{p_j - v_j} \right).$$

Therefore,

$$\frac{\partial \tilde{\pi}_i(p_i^l, p_{-i}^l \mid w_i = c_i)}{\partial p_i} \leq \frac{\partial \tilde{\pi}_i(p^l)}{\partial p_i} = 0, \quad (\text{A.7})$$

where the inequality follows from  $\partial d_j / \partial p_i \geq 0$  for all  $j \neq i$ , and the equality from the fact that  $p^l$  is an interior maximum of the function  $\tilde{\pi}_i$ . (A.5) thus follows from the

chain rule of differentiation, while by Lemma 1,

$$\begin{aligned} \frac{\partial \log \tilde{\pi}_i(p_i^l, p_{-i}^l | w_i)}{\partial p_i} &= \frac{\partial d_i(p^l)/\partial p_i}{d_i(p^l)} + \frac{1}{p_i^l - w_i} \\ &\quad + \frac{L_i \left( \frac{p_i^l - w_i}{p_i^l - v_i} \right)}{L_i \left( \frac{p_i^l - w_i}{p_i^l - v_i} \right)} \frac{w_i - b_i}{(p_i^l - b_i)^2} \\ &\geq \frac{\partial d_i(p^l)/\partial p_i}{d_i(p^l)} + \frac{1}{p_i^l - w_i}, \end{aligned}$$

thus verifying (A.6).

(b) If  $\partial d_i/\partial p_i > 0$ , the inequality in (A.7), and hence that in (A.5), is strict, implying that  $w_i^* > c_i$ .  $\square$

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