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## DECIDABILITY OF PERIODICITY FOR INFINITE WORDS (\*)

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Abstract. — *We show that it is decidable whether an infinite word generated by iterated morphism is ultimately periodic or not.*

Résumé. — *Nous montrons qu'on peut décider si un mot infini engendré par morphisme itéré est ultimement périodique.*

### 1. INTRODUCTION

Let  $X$  be a finite alphabet and  $g$  a morphism of the free monoid  $X^*$ , prolongable in  $u_0 \in X^+$ , i. e. such that  $g(u_0) = u_0 u$ ,  $u \in X^+$ . Then:

$$g^i(u_0) = g^{i-1}(u_0)g^{i-1}(u)$$

and  $g$  defines a unique word, in general infinite, denoted by:

$$g^\omega(u_0) = u_0 u g(u) \dots g^i(u) \dots$$

An infinite word  $\mathcal{M}$  is (*ultimately*) *periodic* if  $\mathcal{M} = vw^\omega = vwvw\dots$  for finite words  $v$  and  $w$ . The question of deciding whether  $g^\omega(u_0)$  is periodic or not has been raised recently, in connection with the  $\omega$ -sequence equivalence problem for DOL systems [1, 2], the adherence equivalence problem for DOL languages [3], and with the subword complexity of infinite words [4].

We give a simple proof of decidability for this question, using the notion of elementary morphism (*see* [5]). After some preliminaries, we give an algorithm for elementary morphisms in section 2 and for arbitrary morphisms in section 3.

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A subword  $u$  of an infinite word  $\mathcal{M}$  is *biprolongable* if and only if there exist distinct letters  $x$  and  $y$  such that  $ux$  and  $uy$  are subwords of  $\mathcal{M}$ . Let  $c(n)$  be the number of distinct subwords of  $\mathcal{M}$  of length  $n$ . Then  $c(n) \leq c(n+1)$  and  $\mathcal{M}$  is periodic if and only if  $c(n)$  is bounded. But if  $u$  is a biprolongable subword of  $\mathcal{M}$ ,  $|u|=n$ , then  $c(n+1) \geq c(n)+1$ . Hence the following property.

LEMMA 1: *An infinite word  $\mathcal{M}$  is ultimately periodic if and only if the length of its biprolongable subwords is bounded.*

Let  $g: X^* \rightarrow X^*$  be a morphism. It is *simplifiable* if there exist an alphabet  $Y$ ,  $|Y| < |X|$  and two morphisms  $f: X^* \rightarrow Y^*$ ,  $h: Y^* \rightarrow X^*$  such that  $g = h \circ f$ . A morphism  $g$  is *elementary* if and only if it is not simplifiable. In this case  $g$  is injective and the set  $\{g(x), x \in X\}$  is a code with bounded delay from left to right ([5], p. 131). In particular if  $g(xu)$  is a prefix of  $g(yv)$ ,  $x \neq y$  then  $g(xu)$  has a bounded length.

Finally a letter  $x \in X$  is *growing* (for  $g$ ) if  $|g^n(x)|$ ,  $n \geq 0$  is unbounded. We denote  $C \subset X$  the set of growing letters and  $B = X \setminus C$  the set of *bounded* letters.

## 2. THE CASE OF ELEMENTARY MORPHISMS

LEMMA 2: *The infinite word  $\mathcal{M} = g^\omega(u_0)$ , with  $g$  elementary, is ultimately periodic if and only if  $\mathcal{M}$  has no biprolongable subword of the form  $xu$ ,  $x \in C$ ,  $u \in B^*$ .*

*Proof:* Assume  $xu_1$  is biprolongable. There exist infinite suffixes  $xu_1y_1v_1$  and  $xu_1z_1w_1$  of  $\mathcal{M}$  for distinct letters  $y_1$  and  $z_1$ . But since  $\mathcal{M} = g(\mathcal{M})$ ,  $g(y_1v_1)$  and  $g(z_1w_1)$  are also suffixes of  $\mathcal{M}$ . Because  $g$  is elementary, their greatest common prefix,  $u_2$ , is finite, and  $g(xu_1)u_2$  is biprolongable. Similarly there exists  $u_3$  such that  $g(g(xu_1)u_2)u_3$  is biprolongable, and so on. Thus we can construct an infinite sequence of biprolongable words, with unbounded length since  $x$  is growing. Hence  $\mathcal{M}$  is not periodic by lemma 1.

Conversely, assume that there is no biprolongable factor of the form  $xu$ . We consider two cases.

*First case:*  $\mathcal{M}$  contains only a finite number of occurrences of growing letters. Then there is only one such occurrence, and  $g(u_0) = u_0u$  with  $u \in B^+$ . Moreover  $|g^i(u)|$ ,  $i \geq 0$ , is bounded, and there is a smallest  $n$  such that  $g^{n+1}(u) = g^i(u)$ ,  $i \leq n$ . But then:

$$\mathcal{M} = u_0 u g(u) \dots g^{i-1}(u) [g^i(u) \dots g^n(u)]^\omega$$

is ultimately periodic.

*Second case:*  $\mathcal{M}$  contains an infinite number of occurrences of growing letters,  $\mathcal{M} = \alpha_0 x_1 \alpha_1 x_2 \alpha_2 \dots$ ,  $x_i \in C$ ,  $\alpha_i \in B^*$ . Let  $n$  be the smallest integer such that  $x_{n+1} = x_i$ ,  $i \leq n$ . Since there is no biprolongable word of the form  $xu$ , we have  $\mathcal{M} = \alpha_0 x_1 \alpha_1 \dots x_{i-1} \alpha_{i-1} [x_i \alpha_i \dots x_n \alpha_n]^\omega$  and  $\mathcal{M}$  is ultimately periodic. ■

**COROLLARY 3:** *If  $\mathcal{M} = g^\omega(u_0)$ , with  $g$  elementary, we can decide if  $\mathcal{M}$  is ultimately periodic.*

*Proof:* Consider the following procedure:

Compute the subset of growing letters,  $C$ .

If  $g(u_0)$  contains only one occurrence of letter from  $C$  then  $\mathcal{M}$  is ultimately periodic.

If  $g(u_0)$  contains several occurrences of letters from  $C$  then:

– compute the shortest prefix  $p$  of  $\mathcal{M}$  containing two occurrences of the same growing letter  $x_i$ :

$$p = \alpha_0 x_1 \alpha_1 \dots x_i \alpha_i x_{i+1} \dots x_n \alpha_n x_i;$$

- for all  $xu$  prefix of some  $x_j \alpha_j$ ,  $i \leq j \leq n$ , check if  $xu$  is biprolongable;
- $\mathcal{M}$  is ultimately periodic if and only if no  $xu$  is biprolongable.

This procedure gives the right answer by lemma 2. Moreover each step is effectively computable: one can determine if a letter is growing, and one can determine if a given word  $xu$  is biprolongable (this comes from the fact that for a given  $n$  one can compute all subwords of length  $n$  of  $\mathcal{M}$ , see [5], p. 210-212). ■

### 3. THE CASE OF ARBITRARY MORPHISMS

**THEOREM 4:** *It is decidable whether  $\mathcal{M} = g^\omega(u_0)$  is ultimately periodic or not for an arbitrary morphism  $g$ .*

*Proof:* By induction on the size of the alphabet,  $|X|$ .

If  $|X| = 1$  then  $\mathcal{M}$  is always periodic.

Assume the theorem is true for alphabets of size  $< |X|$  and let  $g: X^* \rightarrow X^*$  be an arbitrary morphism. If  $g$  is elementary then we decide if  $\mathcal{M}$  is periodic by corollary 3. If  $g$  is not elementary we compute  $Y$ ,  $f: X^* \rightarrow Y^*$  and  $h: Y^* \rightarrow X^*$  such that  $g = h \circ f$  and  $|Y| < |X|$ . Let  $g' = f \circ h$ , and  $u'_0 = f(u_0)$ . Then:

$$g'(u'_0) = g'(f(u_0)) = f(g(u_0)) = f(u_0 u) = u'_0 f(u),$$

where  $g(u_0) = u_0 u$ . So  $g'(u'_0)$  starts with  $u'_0$  and defines an infinite word  $\mathcal{M}' = g'^{\omega}(u'_0)$ . Moreover  $\mathcal{M} = h(\mathcal{M}')$  and  $\mathcal{M}' = f(\mathcal{M})$ , and  $\mathcal{M}$  is ultimately periodic if and only if  $\mathcal{M}'$  is. Therefore, by induction hypothesis we can decide if  $\mathcal{M}'$  is periodic or not. Since the construction of  $g'$  from  $g$  is effective (see [5], p. 17), we can decide whether  $\mathcal{M}$  is periodic or not. This proves the inductive step. ■

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