DECIDABILITY OF SECOND-ORDER THEORIES AND AUTOMATA ON INFINITE TREES[1]

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1. **Introduction.** In this note we announce the solvability of the decision problem of the (monadic) second-order theory of two successor functions (S2S). This answers a question raised by Büchi [1].

The above decidability result turns out to be very powerful in that many difficult, often seemingly unrelated, decision problems are reducible to it. Thus we are able to deduce: the decidability of the first-order theory of the lattice of closed subsets of the real line (in answer to Grzegorczyk [6]); the decidability of the second-order theory of countable linearly ordered sets; decidability of theory of countable Boolean algebras with quantification permitted over ideals; and many other results. All the decidability procedures obtained here are elementary recursive in the sense of Kalmar. Due to the fact that we use reductions to a second-order theory, our decidability proofs are very direct. Through appropriate interpretations, the set variables of S2S allow us to talk about all structures in a certain class.

The method of solution involves the development of a theory of finite automata operating on infinite trees. Complete details will be published elsewhere.

1. **Theory of n successor functions.** Let $T=\{0, 1\}^*$ be the set of all finite words on $\{0, 1\}$. The functions $r_0(x)=x0$, $r_1(x)=x1$, $x\in T$, are called the *successor functions*. On T define the relation $x\leqq y \equiv \exists z[y=xz]$; and the lexicographic total ordering $x\preceq y\equiv x\leqq y \vee \exists z\, \exists u\, \exists v[x=z0u\wedge y=z1v]$.

Let Λ denote the empty sequence. A *path* π of T is a subset $\pi\subset T$ such that (1) $\Lambda\in\pi$; (2) for each $x\in\pi$, either $x0\in\pi$ or $x1\in\pi$; (3) for each $\Lambda\neq x\in\pi$, the predecessor node y of x is in π.

For $\mathfrak{M}$ a structure and L a formal language, $\mathrm{Th}(\mathfrak{M}, L)$ will denote the theory of $\mathfrak{M}$ in the language L. If $\mathcal{K}$ is a class of similar structures, then $\mathrm{Th}(\mathcal{K}, L)=\bigcap_{\mathfrak{M}\in\mathcal{K}} \mathrm{Th}(\mathfrak{M}, L)$. If L is (monadic) second-order, then we denote $\mathrm{Th}(\mathfrak{M}, L)$ by $\mathrm{Th}_2(\mathfrak{M})$. If L' is second-order and the set variables are restricted to range over finite subsets of the domain, then $\mathrm{Th}(\mathfrak{M}, L')$ is called the *weak second-order theory* of $\mathfrak{M}$.

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$\mathrm{Th}_2(\mathfrak{N}_2)$, where $\mathfrak{N}_2=\langle T, r_0, r_1, \leqq, \preceq\rangle$, is denoted by S2S and called the *second-order theory of two successor functions*. In a similar way we define SnS—the second-order theory of n successor functions, for any $1\leqq n\leqq\omega$.

THEOREM 1.1. *The second-order theory of two successor functions* (S2S) *is decidable.*

By direct interpretations we get

COROLLARY 1.2. SnS *is decidable for every* $1\leqq n\leqq\omega$.

The proof of the decidability of S2S employs automata on infinite trees in a manner to be explained in §3.

2. **Applications.** Let $\mathcal{K}_{\leqq}^{\omega}$ be the class of all linearly ordered sets $\langle\overline{A}, \leqq\rangle$ such that $c(\overline{A})\leqq\omega$.

THEOREM 2.1. $\mathrm{Th}_2(\mathcal{K}_{\leqq}^{\omega})$, *the second-order theory of countable linearly ordered sets, is decidable.*

PROOF. It is readily seen that for every $\langle\overline{A}, \leqq\rangle\in\mathcal{K}_{\leqq}^{\omega}$ there exists a set $A\subseteq T$ so that $\langle\overline{A}, \leqq\rangle\simeq\langle A, \sqsubseteq|A\rangle$. This directly implies decidability of $\mathrm{Th}_2(\mathcal{K}_{\leqq}^{\omega})$.

The notion of a subset $A\subseteq T$ being finite is definable in S2S by a formula $\mathrm{F}n(\boldsymbol{A})$. It follows that S2S remains decidable upon inclusion of set variables ranging over finite sets. We get as a corollary the following result of Laüchli [7] which strengthens Ehrenfeucht's result [4]. In contrast with the treatment in [4], [7], we get here elementary recursive decision procedures.

COROLLARY 2.2. *The weak second-order theory of linearly ordered sets is decidable.*

The following result is related to Büchi's Theorem 1′ of [2].

COROLLARY 2.3. *The second-order theory of countable well-ordered sets is decidable.*

Let $\mathcal{K}_f$ be the class of all structures $\langle A, f\rangle$, where $f: A\to A$; and $\mathcal{K}_f^{\omega}$ be the class of all $\langle A, f\rangle\in\mathcal{K}_f$ with $c(A)\leqq\omega$.

THEOREM 2.4. $\mathrm{Th}_2(\mathcal{K}_f^{\omega})$, *the second-order theory of a unary function with a countable domain, is decidable.*

The proof is accomplished by reproducing in $\mathfrak{N}_2$, through appropriate definitions, the general structure $\langle A, f\rangle\in\mathcal{K}_f^{\omega}$.

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COROLLARY 2.5. *The weak second-order theory of a unary function is decidable.*

This is a strengthened version of Ehrenfeucht's result [3], where he announced the decidability of the first-order theory of a unary function.

Let $CD=\{0, 1\}^\omega$ with product topology. Each path $\pi\subset T$ is the set of all finite initials of a unique element $\phi: \omega\to\{0, 1\}$ of CD. Thus, we shall view the paths as elements of CD, and sets of paths as subsets of CD.

THEOREM 2.6. *Let* $\mathrm{Cl}(\boldsymbol{B}, \boldsymbol{A})$ *be* $[\boldsymbol{B}\subseteq\boldsymbol{A}]\wedge\mathrm{Path}(\boldsymbol{B})$, *and* $F_\sigma(\boldsymbol{B}, \boldsymbol{A})$ *be* $\mathrm{Fn}(\boldsymbol{A}\cap\boldsymbol{B})\wedge\mathrm{Path}(\boldsymbol{B})$. $\{\pi\mid\mathfrak{N}_2\models\mathrm{Cl}(\pi, A)\}$ *ranges, with* $A\subseteq T$, *over all closed subsets of* CD, *and* $\{\pi\mid\mathfrak{N}_2\models F_\sigma(\pi, A)\}$ *ranges over all* F_σ *subsets of* CD.

THEOREM 2.7. *Let* $\mathfrak{C}=\langle CD, \leqq\rangle$ *be Cantor's discontinuum with the usual ordering. Let* L *be a language appropriate to* $\mathfrak{C}$ *which has (besides the individual variables) set variables,* $\boldsymbol{C}_1, \boldsymbol{C}_2, \cdots$, *ranging over closed subsets of* CD, *and set variables* $\boldsymbol{D}_1, \boldsymbol{D}_2, \cdots$, *ranging over* F_σ *subsets of* CD. $\mathrm{Th}(\mathfrak{C}, L)$ *is decidable.*

The above result carries over from CD to the segment $[0, 1]$ with the usual topology and order. This implies an affirmative answer to Grzegorczyk's question [6] whether the first-order theory of the lattice of all closed subsets of the real line is decidable.

Denote the class of all Boolean algebras by $\mathcal{K}_B$, and the class of countable Boolean algebras by $\mathcal{K}_B^\omega$. Let L_I be the language appropriate for $\mathcal{K}_B$, which has set variables ranging over *ideals* of the Boolean algebras.

THEOREM 2.8. $\mathrm{Th}(\mathcal{K}_B^\omega, L_I)$, *the theory of countable Boolean algebras with quantification over ideals, is decidable.*

This follows from Theorem 2.7 and the fact that CD is the Stone space of the free Boolean algebra with a denumerable number of generators.

As a corollary we get the following improvement of Tarski's result [8]; and of Ershov's result [5, Theorem 9] to the effect that the first-order theory of Boolean algebras with a distinguished *maximal* ideal is decidable.

THEOREM 2.9. *The first-order theory of Boolean algebras with a sequence of distinguished ideals is decidable.*

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3. **Automata on infinite trees.** For a mapping $\phi: A \to B$, define $In(\phi) = \{b \mid b \in B,\ c(\phi^{-1}(b)) \geqq \omega\}$. In the following, Σ denotes a finite set called the *alphabet.*

DEFINITION. A Σ-*(valued) tree* is a pair (v, T) such that $v: T \to \Sigma$. The set of all Σ-trees will be denoted by V_Σ.

DEFINITION. A Σ-*automaton* is a system $\mathfrak{A} = \langle S, M, S_0, F\rangle$ where S is a finite set; $M: S \times \Sigma \to P(S \times S)$ ($P(A)$ denotes the set of all subsets of A); $S_0 \subseteq S$; and $F \subseteq P(S)$.

DEFINITION. A *run* of $\mathfrak{A}$ on the Σ-tree $t = (v, T)$ is a mapping $r: T \to S$ such that for $y \in T$, $(r(y0), r(y1)) \in M(r(y), v(y))$.

The automaton $\mathfrak{A}$ *accepts* t if there exists an $\mathfrak{A}$-run r on t such that $r(\Lambda) \in S_0$, and for every path π of T, $In(r \mid \pi) \in F$. The set $T(\mathfrak{A})$ of Σ-trees *defined* by $\mathfrak{A}$ is $T(\mathfrak{A}) = \{t \mid t \in V_\Sigma,\ t \text{ accepted by } \mathfrak{A}\}$. A set $A \subseteq V_\Sigma$ is *automaton definable* if for some $\mathfrak{A}$, $A = T(\mathfrak{A})$.

Let $t = (v, T)$ be a $\Sigma \times \Sigma_1$-tree and let $p(x, y) = x$. The *projection* $p(t)$, by definition, is the Σ-tree (pv, T).

THEOREM 3.1. *If A, $B \subseteq V_\Sigma$ and $C \subseteq V_{\Sigma \times \Sigma_1}$ are automaton definable, then so are $A \cup B$, $V_\Sigma - A$, and $p(C)$. Automata defining the latter sets can be effectively obtained from automata defining A, B and C.*

THEOREM 3.2. *There exists an effective (even elementary-recursive) procedure for deciding for every automaton $\mathfrak{A}$ whether $T(\mathfrak{A}) = \varnothing$.*

For a set $A \subseteq T$, let $\chi_A: T \to \{0, 1\}$ be the characteristic function of A. Denote $\{0, 1\}^n$ by Σ^n, $n < \omega$. With $\vec{A} = (A_1, \cdots, A_n)$, associate the Σ^n-tree $(v_{\vec{A}}, T)$ defined by $v_{\vec{A}}(x) = (\chi_{A_1}(x), \cdots, \chi_{A_n}(x))$, $x \in T$. The mapping $\tau: \vec{A} \to (v_{\vec{A}}, T)$ sets up a one-to-one correspondence between $P(T)^n$ and V_{Σ^n}.

THEOREM 3.3. *There exists an (elementary recursive) effective procedure for assigning to every formula $F(\boldsymbol{A}_1, \cdots, \boldsymbol{A}_n)$ of* S2S *a Σ^n-automaton $\mathfrak{A}_F$ so that*

$$T(\mathfrak{A}_F) = \tau(\{(A_1, \cdots, A_n) \mid \mathfrak{N}_2 \models F(A_1, \cdots, A_n)\}).$$

The combination of Theorems 3.2 and 3.3 at once implies the decidability of S2S. In fact, Theorem 3.3 gives us a complete picture of the relations definable in S2S. Through the interpretations used, we also get information about definability in all the theories proved decidable in §2.

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