Decidability Results in First–Order Hybrid Petri Nets*

Fabio Balduzzi (*), Angela Di Febbraro (*),
Alessandro Giua (**), Carla Seatzu (**)

(*) Dip. di Automatica ed Informatica, Politecnico di Torino, Italy

(**) Dip. di Ingegneria Elettrica ed Elettronica, Università di Cagliari, Italy
{giua,seatzu}@diee.unica.it

Abstract

In this paper we tackle the decidability of marking reachability for a hybrid formalism based on Petri nets. The model we consider is the untimed version of First–Order Hybrid Petri Nets: it combines a discrete Petri net and a continuous Petri net, the latter being a fluid version of a usual discrete Petri net.

It is suggested that the decidability results should be pursued exploiting a hierarchy of models as it has been done in the framework of Hybrid Automata. In this paper we define the class of Single–Rate Hybrid Petri Nets: the continuous dynamics of these nets is such that the vector of the marking derivatives of the continuous places is constant but for a scalar factor. This class of nets can be seen as the counterpart of timed automata with skewed clocks. We prove that the reachability problem for this class can be reduced to the reachability problem of an equivalent discrete net and thus it is decidable.

1 Introduction

Modeling, analysis, and control of hybrid systems are topics presently attracting the attention of ever more researchers. As a result, several models for hybrid systems, i.e., systems presenting both time-driven and event-driven dynamics, have been conceived. A rich related literature already exists, where one can find not only survey papers about the different hybrid models like [9], but also special publications embracing any aspect relevant to hybrid systems [14, 10, 3, 4, 5].

In this framework, a well known hybrid modeling methodology is based on *Hybrid Automata* (HA), which can be considered to be a generalization of the *Timed Automata* (TA) originally presented by [1]. A HA consists of a classical automaton provided with a continuous state that may continuously evolve in time with different dynamics or have discontinuous jumps at the occurrence of a discrete event belonging to a predefined set of feasible events. Significant results about the decidability and the complexity of special sub–classes of this model have been presented by different authors [16, 19].

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Petri nets (PN) were first introduced, and are still successfully used, to describe and analyze discrete event systems [17]. Recently, much effort has been devoted to apply these models to hybrid systems as well. A recent survey of the relevant literature can be found in [12].

The model considered in this paper was first presented by [8] and is inspired from the approach of David and Alla [11]. In this model, called hereafter *First–Order Hybrid Petri Nets* (FOHPN), places and transitions can be either continuous or discrete: continuous places hold fluid, discrete places contain a non–negative integer number of tokens. Note that here, differently from [8], it is assumed that no timing structure is associated to the firing of discrete transitions. This is consistent with the definition of HA, where the variable "time" is only associated to the continuous evolution.

The first part of the paper is devoted to a comparison between FOHPN and HA, highlighting relationships which are worth commenting on. A major difference consists in the fact that the discrete state space of a FOHPN, i.e., the set of discrete markings of the net, may be infinite, while the discrete state space of a HA, i.e., the set of locations, must be finite. In all other respects, however, a FOHPN is a special case of a HA, and any FOHPN with bounded discrete places may also be modeled by a HA.

The recent results on HA have shown that a trade-off between modeling power and analytical tractability is necessary. To this end, several special classes of HA have been studied: timed automata, timed automata with skewed clocks, multirate and rectangular automata (initialized or not) [19].

Since FOHPN have only recently been introduced, very little is known about their decidability properties. This motivates the authors in defining and exploring a hierarchy of models of increasing complexity.

In a preliminary work [6] the class of Unitary–Rate Hybrid Petri Nets (URHPN) was presented. A URHPN consists of a FOHPN with a single continuous transition whose firing increases the marking of all continuous places with the same rate. For this class, that can be seen as the FOHPN counterpart of a timed automaton, it was proven that the marking reachability problem is decidable. This result is not surprising, because the reachability problem is known to be decidable for TA [1].

In this paper we extend this result, introducing a new class of FOHPN named *Single–Rate Hybrid Petri Nets* (SRHPN) that can be seen as the FOHPN counterpart of a timed automaton with skewed clocks. This class, that strictly includes the class of URHPN, consists of a FOHPN with a single continuous transition whose firing speed v is constant, and whose firing may increase the marking of different continuous places with different rates. Thus, the continuous dynamic is such that the marking of each continuous place increases with a single constant rate that depends on the place. Note however that all the results presented in this paper also hold if the firing speed of the continuous transition is not constant but may take value in a real interval, i.e., $v \in [V', V]$. In this case one has a single–rate evolution but for a scalar factor that may vary in time.

We have already remarked that in general FOHPN are a less powerful model than HA. However, when we pose structural restrictions to the two formalisms and compare corresponding subclasses, the situation changes. As an example, when comparing SRHPN and TA with skewed clocks, it is clear that the two models are significantly different and neither one can be seen as a special case of the other one. TA can model "reset" of the continuous state, while SRHPN can model "jumps of constant magnitude" of the continuous state and may also have an infinite discrete state space. It will be proved that the reachability problem for SRHPN can be reduced to the reachability problem of an equivalent discrete PN and thus it is decidable. Note that the complexity of this problem is usually high [20] unless the net considered belong to special classes. Nevertheless, this result is interesting, because the reachability problem for a TA with skewed clocks is known to be undecidable [15].

The paper is structured as follows. Section 2 presents the formal definition of hybrid Petri nets and the rules governing their evolution. In Section 3 the definition of hybrid automata is given. In Section 4 the relations between hybrid automata and hybrid Petri nets are explored. In Section 5 the considered special class of hybrid Petri nets, called SRHPN, is defined and it is proved that for this net the reachability problem is decidable.

2 First–Order Hybrid Petri Nets

The Petri net formalism used in this paper can be seen as the "untimed" version of the model presented in [8], in the sense that no timing structure is associated to the firing of discrete transitions. For a more comprehensive introduction to place/transition Petri nets see [17].

An (untimed) First–Order Hybrid Petri Net (FOHPN) is a structure $N = (P, T, Pre, Post, \mathcal{C})$.

The set of *places* $P = P_d \cup P_c$ is partitioned into a set of *discrete* places P_d (represented as circles) and a set of *continuous* places P_c (represented as double circles). The cardinality of P, P_d and P_c is denoted n, n_d and n_c.

The set of *transitions* $T = T_d \cup T_c$ is partitioned into a set of discrete transitions T_d and a set of continuous transitions T_c (represented as double boxes). The cardinality of T, T_d and T_c is denoted q, q_d and q_c.

The pre- and post-incidence functions that specify the arcs are (here $\mathbb{R}_0^+ = \mathbb{R}^+ \cup \{0\}$): $Pre, Post : P_d \times T \to \mathbb{N},\ P_c \times T \to \mathbb{R}_0^+$.. It is required (well-formed nets) that for all $t \in T_c$ and for all $p \in P_d$, $Pre(p, t) = Post(p, t)$.

The function $\mathcal{C} : T_c \to \mathbb{R}_0^+ \times \mathbb{R}_\infty^+$ specifies the firing speeds associated to continuous transitions (here $\mathbb{R}_\infty^+ = \mathbb{R}^+ \cup \{\infty\}$). For any continuous transition $t_j \in T_c$ let $\mathcal{C}(t_j) = (V'_j, V_j)$, with $V'_j \leq V_j$. Here V'_j represents the minimum firing speed (mfs) and V_j represents the maximum firing speed (MFS).

The preset (postset) of transition t is denoted as ${}^\bullet t$ ($t^\bullet$) and its restriction to continuous or discrete places as ${}^{(d)}t = {}^\bullet t \cap P_d$ or ${}^{(c)}t = {}^\bullet t \cap P_c$. Similar notation may be used for presets and postsets of places. The incidence matrix of the net is defined as $\mathbf{C}(p, t) = Post(p, t) - Pre(p, t)$. The restriction of $\mathbf{C}$ to P_X and T_Y ($X, Y \in \{c, d\}$) is denoted $\mathbf{C}_{XY}$. Note that by the well-formedness hypothesis $\mathbf{C}_{dc} = 0$: this ensures that the firing of a continuous transition cannot modify the discrete marking of the net.

A marking $\mathbf{m} : P_d \to \mathbb{N},\ P_c \to \mathbb{R}_0^+$ is a function that assigns to each discrete place a non-negative number of tokens, represented by black dots and assigns to each continuous place a fluid volume; m_p denotes the marking of place p. The value of a marking at time τ is denoted $\mathbf{m}(\tau)$. The restriction of $\mathbf{m}$ to P_d and P_c are denoted with $\mathbf{m}^d$ and $\mathbf{m}^c$, respectively. An FOHPN system

$(N, \mathbf{m}(\tau_0))$ is an FOHPN N with an initial marking $\mathbf{m}(\tau_0)$.

The enabling of a discrete transition depends on the marking of all its input places, both discrete and continuous.

Definition 1. *Let $(N, \mathbf{m})$ be an FOHPN system. A discrete transition t is enabled at $\mathbf{m}$ if for all $p_i \in {}^\bullet t$, $m_i \geq Pre(p_i, t)$.* ■

A continuous transition is enabled only by the marking of its input discrete places. The marking of its input continuous places, however, is used to distinguish between strongly and weakly enabling.

Definition 2. *Let $(N, \mathbf{m})$ be an FOHPN system. A continuous transition t is enabled at $\mathbf{m}$ if for all $p_i \in {}^{(d)}t$, $m_i \geq Pre(p_i, t)$.*

An enabled transition $t \in T_c$ is:

- *strongly enabled at $\mathbf{m}$ if for all places $p_i \in {}^{(c)}t$, $m_i > 0$;*
- *weakly enabled at $\mathbf{m}$ if for some $p_i \in {}^{(c)}t$, $m_i = 0$.* ■

In the following the hybrid dynamics of an FOHPN is described. The time–driven behavior associated to the firing of continuous transitions is considered first, and then the event–driven behavior associated to the firing of discrete transitions.

The instantaneous firing speed (IFS) at time τ of a transition $t_j \in T_c$ is denoted $v_j(\tau)$. The equation which governs the evolution in time of the marking of a place $p_i \in P_c$ can be written as

$$\dot{m}_i(\tau) = \sum_{t_j \in T_c} C(p_i, t_j) v_j(\tau) \tag{1}$$

where $\mathbf{v}(\tau) = [v_1(\tau), \ldots, v_{n_c}(\tau)]^T$ is the IFS vector at time τ. Indeed Equation 1 holds assuming that at time τ no discrete transition is fired and that all speeds $v_j(\tau)$ are continuous in τ.

The enabling state of a continuous transition t_j defines its admissible IFS v_j.

- If t_j is not enabled then $v_j = 0$.
- If t_j is strongly enabled, then it may fire with any firing speed $v_j \in [V'_j, V_j]$.
- If t_j is weakly enabled, then it may fire with any firing speed $v_j \in [V'_j, \overline{V}_j]$, where $\overline{V}_j \leq V_j$. The value of $\overline{V}_j$ depends on the flow entering the empty input continuous places of t_j, since t_j cannot remove more fluid from any empty place than the quantity that enters due to the firing of other transitions.

It is possible now to characterize the set of all admissible IFS vectors.

Definition 3. (admissible IFS vectors)
Let $(N, \mathbf{m})$ be an FOHPN system. Let $T_{\mathcal{E}}(\mathbf{m}) \subset T_c$ ($T_{\mathcal{N}}(\mathbf{m}) \subset T_c$) be the subset of continuous transitions enabled (not enabled) at $\mathbf{m}$, and $P_{\mathcal{E}} = \{p_i \in P_c \mid m_i = 0\}$ be the subset of empty continuous places. Any admissible IFS vector $\mathbf{v}$ at $\mathbf{m}$ is a feasible solution of the following linear

set:

$$\begin{cases} (a) & V_j - v_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\mathbf{m}) \\ (b) & v_j - V'_j \geq 0 & \forall t_j \in T_{\mathcal{E}}(\mathbf{m}) \\ (c) & v_j = 0 & \forall t_j \in T_{\mathcal{N}}(\mathbf{m}) \\ (d) & \sum_{t_j \in T_{\mathcal{E}}} C(p, t_j) v_j \geq 0 & \forall p \in P_{\mathcal{E}}(\mathbf{m}) \end{cases} \tag{2}$$

The set of all feasible solutions is denoted $\mathcal{S}(N, \mathbf{m})$. ■

Constraints of the form (2.a), (2.b), and (2.c) follow from the firing rules of continuous transitions. Constraints of the form (2.d) follow from (1), because if a continuous place is empty then its fluid content cannot decrease.

Note that the set $\mathcal{S}$ is a function of the marking of the net. Thus as $\mathbf{m}$ changes it may vary as well. In particular it changes at the occurrence of the following macro–events: (a) a discrete transition fires, thus changing the discrete marking and enabling/disabling a continuous transition; (b) a continuous place becomes empty, thus changing the enabling state of a continuous transition from strong to weak.

Let τ_k and τ_{k+1} be the occurrence times of two consecutive macro–events of this kind; it is assumed that within the interval of time $[\tau_k, \tau_{k+1})$ the IFS vector is constant and denoted as $\mathbf{v}(\tau_k)$. Then the continuous behavior of an FOHPN for $\tau \in [\tau_k, \tau_{k+1})$ is described by

$$\begin{cases} \mathbf{m}^c(\tau) & = \mathbf{m}^c(\tau_k) + \mathbf{C}_{cc}\mathbf{v}(\tau_k)(\tau - \tau_k) \\ \mathbf{m}^d(\tau) & = \mathbf{m}^d(\tau_k). \end{cases} \tag{3}$$

The firing of a discrete transition t_j at $\mathbf{m}(\tau)$ yields the marking

$$\begin{cases} \mathbf{m}^c(\tau) & = \mathbf{m}^c(\tau^-) + \mathbf{C}_{cd}\boldsymbol{\sigma}(\tau) \\ \mathbf{m}^d(\tau) & = \mathbf{m}^d(\tau^-) + \mathbf{C}_{dd}\boldsymbol{\sigma}(\tau) \end{cases} \tag{4}$$

where $\boldsymbol{\sigma}(\tau)$ is the firing count vector associated to the firing of transition t_j.

2.1 Firing sequence and reachability

Now, some definitions that will be useful in the following are provided.
Definition 4. *(Event Step) Let us consider a FOHPN system $(N, \mathbf{m})$. If $t \in T_d$ is enabled at $\mathbf{m}$, t may fire. The firing of t determines a new marking $\tilde{\mathbf{m}} = \mathbf{m} + Post(\cdot, t) - Pre(\cdot, t)$ and we write $\mathbf{m}[t\rangle\tilde{\mathbf{m}}$.* ■

We can use a similar notation for the marking variation due to the firing of continuous transitions.
Definition 5. *(Time Step) Let $(N, \mathbf{m})$ be a FOHPN system and let $\mathbf{v} \in \mathcal{S}(N, \mathbf{m})$ be the IFS vector constant within a time step of length $\overline{\tau} \in \mathbb{R}^+$. The marking $\tilde{\mathbf{m}}$ reached at the end of the step is*

$$\begin{cases} \tilde{\mathbf{m}}^d = \mathbf{m}^d \\ \tilde{\mathbf{m}}^c = \displaystyle\int_0^{\overline{\tau}} \mathbf{C}_{cc}\mathbf{v}(\tau)d\tau + \mathbf{m}^c \geq \mathbf{0} \end{cases}$$

and we write $\mathbf{m}[\overline{\tau}\rangle\tilde{\mathbf{m}}$. ■
Definition 6. *Let $(N, \mathbf{m})$ be a FOHPN system. A firing sequence $\sigma = \alpha_1 \cdots \alpha_k \in (T_d \cup \mathbb{R}^+)^*$ is enabled from a marking $\mathbf{m}$ if $\mathbf{m}[\alpha_1\rangle\mathbf{m}_1[\alpha_2\rangle\ \mathbf{m}_2 \cdots [\alpha_k\rangle\tilde{\mathbf{m}}$ holds. To denote that the firing of σ from $\mathbf{m}$ determines the marking $\tilde{\mathbf{m}}$ we write $\mathbf{m}[\sigma\rangle\tilde{\mathbf{m}}$.* ■

3 Hybrid Automata

A hybrid automaton [18, 19] is a structure $H = (L, act, inv, E)$ defined as follows.

- L is a finite set of locations.
- $act : L \to Inclusions$ is a function that associates to each location $l \in L$ a differential inclusion of the form $\dot{\mathbf{x}} \in act_l(\mathbf{x}) \subseteq \mathbb{R}^n$ where $act_l(\mathbf{x})$ is a set-valued map; if $act_l(\mathbf{x})$ is a singleton then it is a differential equation.

  A solution of a differential inclusion with initial condition $\mathbf{x}_0 \in \mathbb{R}^n$ is any differentiable function $\boldsymbol{\phi}(\tau)$, where $\boldsymbol{\phi} : \mathbb{R} \to \mathbb{R}^n$ such that $\boldsymbol{\phi}(0) = \mathbf{x}_0$ and $\dot{\boldsymbol{\phi}}(\tau) \in act_l(\boldsymbol{\phi}(\tau))$.
- $inv : L \to Invariants$ is a function that associates to each location $l \in L$ an invariant $inv_l \subset \mathbb{R}^n$.

  An invariant function is $\mathbf{x} \in inv_l$. The invariant function constrains the behaviour of the automaton state during time steps within a given subset of $\mathbb{R}^n$.
- $E \subset L \times Guards \times Jump \times L$ is the set of edges. An edge $e = (l, g, j, l') \subset E$ is an edge from location l to l' with guard g and jump relation j.

  A guard is $g \subset \mathbb{R}^n$. An edge is enabled when the state $\mathbf{x} \in g$.

  A jump relation is $j \subset \mathbb{R}^n \times \mathbb{R}^n$. During the jump, $\mathbf{x}$ is set to $\mathbf{x}'$ provided $(\mathbf{x}, \mathbf{x}') \in j$. When j is the identity relation, the continuous state does not change.

The state of the hybrid automaton is the pair $(l, \mathbf{x})$ where $l \in L$ is the discrete location, and $\mathbf{x} \in \mathbb{R}^n$ is the continuous state. The hybrid automaton starts from some initial state $(l_0, \mathbf{x}_0)$. The trajectory evolves with the location remaining constant and the continuous state $\mathbf{x}$ evolving within the invariant function at that location, and its first derivative remains within the differential inclusion at that location. When the continuous state satisfies the guard of an edge from location l to location l', a jump can be made to location l'. During the jump, the continuous state may get initialized to a new value $\mathbf{x}'$. The new state is the pair $(l', \mathbf{x}')$. The continuous state $\mathbf{x}'$ now moves within the invariant function with the new differential inclusion, followed some time later by another jump, and so on.

In this paper the interest is focused on two special classes of Hybrid Automata, the *Timed Automata* and the *Timed Automata with Skewed Clocks*. Let us recall the definition of a *rectangle*, before defining them.

Definition 7. *An n-dimensional rectangle is a set of the form* $r = [l_1, u_1] \times \cdots \times [l_n, u_n]$ *with* $l_i, u_i \in \mathbb{Z}_{\pm\infty}$. *The i-th component of r is* $r_i = [l_i, u_i]$. *The set of all n-dimensional rectangles is* $Rect_n$. ■

Definition 8. *An n-dimensional timed automaton* $R = (L, act, E)$ *is an hybrid automaton in which the set Inclusions contains the single element* $\mathbf{1} \in \mathbb{R}^n$, *i.e.,* $\dot{x}_i = 1$ *for each i at every control location; Guard =* $Rect_n$; *Jump =* $\{j \mid j = j_1 \times \cdots \times j_n\}$ *where* $j_i = [l_i, u_i]$ *or* $j_i = id$. *The relation* $[l_i, u_i] = \{(\mathbf{x}, \mathbf{x}') \mid \mathbf{x}' \in [l_i, u_i]\}$ *and id is the identity relation.* ■

Since the differential equation is fixed at each location in the timed automaton, we denote the timed automaton by $T = (L, E)$.

Definition 9. *An n-dimensional* timed automaton with skewed clocks $R = (L, act, E)$ *is an hybrid automaton in which the set Inclusions contains the single element* $\mathbf{v} \in (\mathbb{R}^+)^n$*, i.e.,* $\dot{x}_i = v_i$ *for each i at every location; Guard* $= Rect_n$*; Jump* $= \{j \mid j = j_1 \times \cdots \times j_n$ *where* $j_i = [l_i, u_i]$ *or* $j_i = id\}$*. Here we consider the relation* $[l_i, u_i] = \{(\mathbf{x}, \mathbf{x}') \mid \mathbf{x}' \in [l_i, u_i]\}$ *and id is the identity relation.* ■

Note that, following [1, 19], we are assuming that the behaviour of such classes of HA is not constrained by any invariant function.

4 Hybrid Petri nets and hybrid automata

In this section the relations between hybrid automata and hybrid Petri nets are explored. The table below summarizes the differences existing between the two hybrid models. The first three rows deal with the state definition; the 4-th and the 5-th rows with the continuous dynamics, and the last two rows with the discrete evolution associated to the occurrence of events in hybrid automata and to the firing of discrete transitions in hybrid Petri nets.

	FOHPN	HA
discrete state	$\mathbf{m}^d \in \mathbb{N}^{n_d}$	$l \in L$
continuous state	$\mathbf{m}^c \in \mathbb{R}^{n_c}$	$\mathbf{x} \in \mathbb{R}^n$
state	$(\mathbf{m}^d, \mathbf{m}^c) \in \mathbb{N}^{n_d} \times \mathbb{R}^{n_c}$	$(l, \mathbf{x}) \in L \times \mathbb{R}^n$
activity	$\dot{\mathbf{m}}^c \in \{\mathbf{C}_{cc}\mathbf{v} \mid \mathbf{v} \in \mathcal{S}(N, \mathbf{m})\ \}$	$\dot{\mathbf{x}} \in act_l(\mathbf{x})$
invariant	$\mathbf{m}^c \in (\mathbb{R}_0^+)^{n_c}$	$\mathbf{x} \in inv_l$
guard	$\{\mathbf{m} \mid \mathbf{m} \geq Pre(\cdot, t)\}$	$g \subset \mathbb{R}^n$
jump	$\{(\mathbf{m}^c, \tilde{\mathbf{m}}^c) \mid \tilde{\mathbf{m}}^c = \mathbf{m}^c + \mathbf{C}_{cd}(\cdot, t)\}$	$j \subset \mathbb{R}^n \times \mathbb{R}^n$

In both models the state consists of a discrete part (location, discrete marking) and a continuous part (continuous state, continuous marking). In the hybrid automaton only the location $l \in L$ is represented in the transition structure, while the continuous state $\mathbf{x} \in \mathbb{R}^n$ is given using an algebraic formalism. In the FOHPN the net marking $\mathbf{m} = (\mathbf{m}^d, \mathbf{m}^c) \in \mathbb{N}^{n_d} \times (\mathbb{R}_0^+)^{n_c}$ represents with a single formalism both discrete and continuous state. Another important difference is the fact that a Petri net may have an infinite number of discrete markings (i.e., the discrete state space may be infinite), while the locations of an automaton may only take values within a finite set L. This is the hybrid equivalent of the difference between finite state automata and place/transition Petri nets. We remark that it may be possible to define a HA with a set of locations $L = \mathbb{N}$, but in the general case this model cannot be finitely described, lacking any structure: thus L is usually defined as a finite set.

The activity function which constrains $\dot{\mathbf{x}} \in act_l(\mathbf{x})$ finds its counterpart in hybrid Petri nets. In fact, the continuous marking of a FOHPN varies with $\dot{\mathbf{m}}^c = \mathbf{C}_{cc}\mathbf{v}$, where $\mathbf{v} \in \mathcal{S}(N, \mathbf{m})$ is the IFS at $\mathbf{m}$. Clearly, in the case of FOHPN this set has a special structure: given the characterization of eq. (2) one can see that it is a linear convex set and thus a FOHPN is similar to the special sub–class *linear* HA [2]).

Similarly, while in a hybrid automaton the continuous state at each location l may be constrained by an arbitrary invariant function inv_l, in a hybrid Petri net the only constraint to continuous marking is that it must be non–negative and it is the same for all discrete markings. Note, however, that this invariant function is not explicitly given in the model, i.e., in the definition of

FOHPN we did not mention any constraint on the continuous markings. This constraint follows from the definition of initial marking (that must be non negative) and from the fact that the set of admissible IFS vectors is defined so that for each empty continuous place the output flow can never exceed the input flow. Finally, we also observe that while no upper bound of the continuous marking — this is an invariant function that may be useful in many cases — can be directly imposed, it may be possible to bound a continuous place adding to the net a new complementary continuous place, a technique also used in discrete nets.

The guard $g \subset \mathbb{R}^n$ associated to an edge of a HA corresponds to the enabling condition $\mathbf{m} \geq Pre(\cdot, t)$ associated to a discrete transition t of a FOHPN. Note first of all that while each edge of a HA represents a single event, in a FOHPN a single transition may represent different events. Thus, the fact that the transition enabling depends on the discrete marking $\mathbf{m}^d$ is used to specify that a given transition corresponds to an event that may be enabled only by a subset of locations. On the other hand, the fact that the transition enabling depends also on the continuous marking $\mathbf{m}^c$ is the FOHPN counterpart of the guard g of an hybrid automaton. Note also that the enabling of a FOHPN is a guard with special structure: it is a "right closed set", i.e., if $(\mathbf{m}^d, \mathbf{m}^c) \in g$, and $\tilde{\mathbf{m}}^c \geq \mathbf{m}^c \Longrightarrow (\mathbf{m}^d, \tilde{\mathbf{m}}^c) \in g$.

Finally, in hybrid automata the jump relation $j \subset \mathbb{R}^n \times \mathbb{R}^n$ defines for each edge the updated value that the continuous state assumes when the location varies, i.e., when an event occurs. The updated value may always be the same each time the event occurs, may depend on the value that the continuous state has before the event occurs, or may also be non deterministic, in the sense that the relation j may be one–to–many. In a FOHPN, on the other hand, the firing of a discrete transition t produces a *constant variation* on the continuous marking, i.e., if the continuous marking *before* the transition firing is any vector $\mathbf{m}^c$ the updated marking $\tilde{\mathbf{m}}^c = \mathbf{m}^c + \mathbf{C}_{cd}(\cdot, t)$ will differ from it by an additive quantity $\mathbf{C}_{cd}(\cdot, t)$. Thus, while the jump relation of a HA may be used to associate to an event firing a variable variation of the continuous state, a FOHPN can only produce constant discrete marking variations. In particular, in FOHPN the reset of the continuous marking is not possible. Furthermore, we remark that in FOHPN each transition firing updates not only the continuous marking but the discrete marking as well. The discrete marking updating is used to specify the updated discrete state (location) reached after the occurrence of the event from a given discrete marking (location).

To summarize, FOHPN can be seen as a restriction of HA with the only exception that a FOHPN may have an infinite number of locations. Note, however, that the generality of HA has as a consequence the fact that most properties are undecidable unless very strong restrictions are added to the basic model. When we pose these restrictions on HA to obtain special sub–classes, and compare these classes with the corresponding sub–classes of FOHPN, the two formalisms are rather different and neither one can be seen as special case of the other one.

The main feature that any FOHPN lacks with respect to an HA is the fact that the content of a continuous place cannot be reset to zero. We remark, however, that the possibility of resetting the contents of continuous places using *flush–out* arcs is currently being investigated for particular hybrid Petri net models [13].

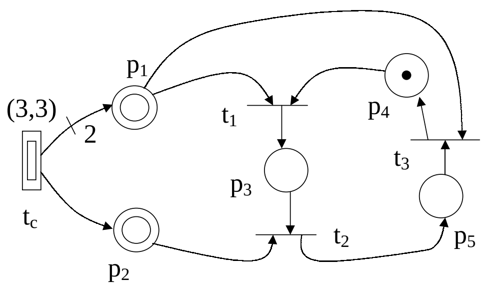

Figure 1: A Single–Rate Hybrid Petri Net.

5 Single–Rate Hybrid Petri Nets

In this section we define the special class of *Single–Rate* FOHPN that can be seen as the net counterpart of timed automata with skewed clocks. It consists of a FOHPN where the continuous dynamics is such that the marking of each continuous place constantly increases with an integer slope.

Definition 10. *A Single–Rate Hybrid Petri Net (SRHPN) is a FOHPN where:*

- $T_c = \{t_c\}$,
- ${}^\bullet t_c = \emptyset$,
- $\mathcal{C}(t_c) = (v, v)$ *where* $v \in \mathbb{N}^+$,
- $\forall i \mid p_i \in P_c \, : \, Post(p_i, t_c) = w_i \in \mathbb{N}^+$ *and* $\{w_i\}$ *is a prime set, i.e., the* w_i*'s do not have a factor common to all of them,*
- $Pre, Post \in \mathbb{N}^{n \times q}$. ■

Thus a Single–Rate Hybrid Petri Net has a *single* continuous transition t_c that is always enabled — because it has no input places — and whose firing speed is constant. The marking of all continuous places increases with constant rate during a time step. Discontinuous variations of continuous markings may only follow the firing of discrete transitions.

The special structure of this net is such that at each step $\mathcal{S}(N, \mathbf{m}) = \{v\}$ is a singleton set and this set is always the same regardless of $\mathbf{m}$. It is important to note that all results presented in this paper still hold if we consider $\mathcal{C}(t_c) = (V', V)$. In this case the set of admissible IFSs $\mathcal{S}(N, \mathbf{m})$ is a segment and the marking of all continuous places may increase with different rates during a time step but the rates associated to different places always have the same ratio.

We have assumed without loss of generality that the set of all w_i — the weights of the arcs from the continuous transition t_c to the continuous places p_i — is a prime set. In fact, if this is not the case, we can always consider an equivalent net N' with the same structure as the original one but with different values of both v and w_i. The new firing speed would be $v' = v \cdot GCD(w_1, \cdots, w_{n_c})$ and the new weights would be $w'_i = w_i / GCD(w_1, \cdots, w_{n_c})$, where GCD denotes the greatest common divisor.

Furthermore, we assume that all arcs have integer weights. Such an assumption has been introduced for simplicity. In fact, whenever $Pre, Post \in \mathbb{Q}^{n \times q}$ all the weights could be multiplied

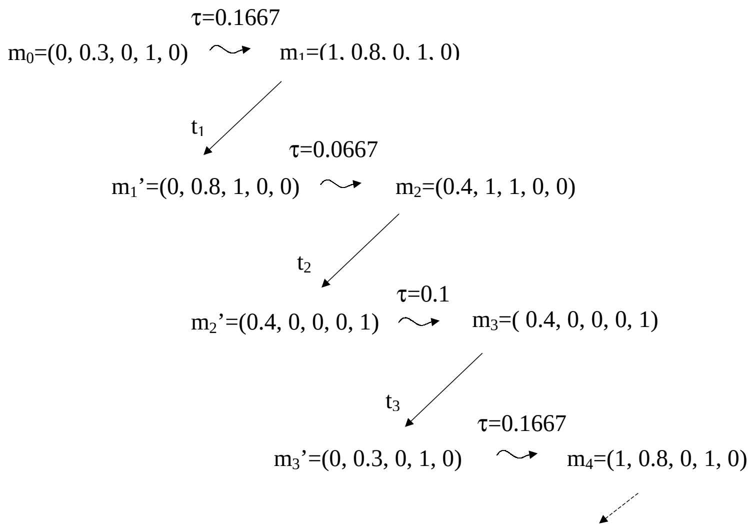

Figure 2: The reachability graph of the SRHPN in figure 1.

by the least common multiple of the denominators of all the constants appearing in $Pre, Post$ to get a new hybrid net that is isomorphic with a new one where $Pre, Post \in \mathbb{N}^{n\times q}$.

The evolution of SRHPN can be related to that of timed HA with skewed clocks. In fact, the continuous evolution (due to the firing of the transition t_c) is such that each continuous variable m_{p_i}, i.e., the marking of each continuous place p_i, has a constant derivative equal to vw_i during a time interval in which no discrete transition fires. Thus the derivative of each continuous variable is constant, but non necessarily equal to 1. Furthermore, different variables can have different derivatives.
Remark 11. Note that the class of Unitary–Rate Hybrid Petri Nets, formally defined by the authors in [6], are a particular case of SRHPN where the weights of the arcs from continuous transition to continuous places are all 1's.
Example 12. The FOHPN in figure 1 is a SRHPN. It represents a production system with two continuous flows of parts (type 1 and type 2) that are put into two buffers (places p_1 and p_2). The batch processing of parts, represented by the cycle of discrete transitions, requires first a unit of part type 1, then a unit of part type 2 and then again a unit of part type 1.

Its reachability graph is shown in figure 2 under the assumption that $\mathbf{m}_0 = (0,\ 0.3,\ 0,\ 1,\ 0)^T$. It has been drawn in accordance with the following rule. The firing of the continuous transition is represented only if it produces a variation on the enabling condition of the net. Note however that the continuous transition is always enabled and always fires with a constant rope equal to 3. Therefore, there exists a time–step enabled from all right-most markings of each row of the graph: such a time–step of length τ adds a marking quantity 2τ to m_{p_1} and a marking quantity τ to m_{p_1}. ■

Now, we prove that the reachability problem for SRHPN is decidable.

Let us first define an equivalence relation on $(\mathbb{R}_0^+)^m$.
Definition 13. *Given a vector* $\mathbf{w} = (w_1, \cdots, w_m)^T \in (\mathbb{N}^+)^m$ *where* $\{w_i\}$ *is a prime set, we say*

that a vector $\mathbf{x} \in (\mathbb{R}_0^+)^m$ *is w-consistent with* $\mathbf{y} \in (\mathbb{R}_0^+)^m$ *if:*

$$\exists\, b \in [0,1) \;:\; \forall i = 1, \cdots, m,\; \langle y_i \rangle = \langle x_i + w_i\, b \rangle$$

where $\langle \cdot \rangle$ *denotes the fractional part and we write* $\mathbf{x} \sim_w \mathbf{y}$*. The equivalence classes of this relation are denoted* $[\mathbf{x}]_w$*.* ■

Example 14. Let $\mathbf{x} = (0,\, 0.3)^T$ and $\mathbf{w} = (2,\, 1)^T$. In figure 3 the set of vectors w-consistent with $\mathbf{x}$ are represented in the plane (x_1, x_2) and lie on a family of parallel lines. All lines are equally spaced and are characterized by a constant slope equal to 2. ■

Definition 15. *Given a vector* $\mathbf{x} \in (\mathbb{R}_0^+)^m$ *and a vector* $\mathbf{y} \in [\mathbf{x}]_w$*, we define the vector* $\boldsymbol{\gamma}(\mathbf{x}, \mathbf{y}, \mathbf{w}) = \mathbf{x} + \hat{\tau}\mathbf{w}$ *where*

$$\hat{\tau} = \min\{\tau \geq 0 \mid \forall i = 1, \cdots, m,\; \langle x_i + \tau w_i \rangle = \langle y_i \rangle\}$$

and we call $\boldsymbol{\gamma}$ *the w–cover of* $\mathbf{x}$ *with the same fractional part of* $\mathbf{y}$*. Note that* $\hat{\tau} \in [0,1)$ *because* $\{w_i\}$ *is a prime set.* ■

In plain words, if we consider a point $\mathbf{x}$ and if we move along the direction corresponding to the vector $\mathbf{w}$ — i.e., we move along the unique line in $[\mathbf{x}]_w$ passing through $\mathbf{x}$ — the w-cover $\boldsymbol{\gamma}(\mathbf{x}, \mathbf{y}, \mathbf{w})$ is the first point we reach with the same fractional part of $\mathbf{y}$.

Now, let us provide a constructive algorithm to determine the numerical value of $\hat{\tau}$ and thus the vector $\boldsymbol{\gamma}$.

Algorithm 16. Observe that, since $\hat{\tau} \in [0,1)$, the integer part of each component of $\boldsymbol{\gamma}$ may differ from the integer part of the corresponding component of $\mathbf{x}$ by a quantity that belongs to the set:

$$\mathcal{I}_i = \begin{cases} \{0, 1, \cdots, w_i - 1\} & \text{if } \langle x_i \rangle \leq \langle y_i \rangle \\ \{1, 2, \cdots, w_i\} & \text{otherwise.} \end{cases}$$

Compute $g_i = \langle \gamma_i \rangle - \langle x_i \rangle \equiv \langle y_i \rangle - \langle x_i \rangle$ (this is a known quantity) and define $k_i = \lfloor \gamma_i \rfloor - \lfloor x_i \rfloor$. Thus we need to solve for $\tau \in [0,1)$ the following system of equations

$$\gamma_i = x_i + \tau w_i, \quad (i = 1, \cdots, m),$$

that can be rewritten as

$$k_i + g_i = \tau w_i, \quad (i = 1, \cdots, m),$$

where the unknown terms are $\tau \in [0,1)$ and $k_i \in \mathcal{I}_i$.

If there exists one value of $\hat{\tau} \in [0,1)$ such that $\langle \mathbf{x} + \hat{\tau}\mathbf{w} \rangle = \langle \mathbf{y} \rangle$, then this value is unique and can be computed as follows.

for all $i = 1, 2, \cdots, m$
  begin
    $\mathcal{J}_i := \emptyset$;
    for all $k_i \in \mathcal{I}_i$, $\mathcal{J}_i := \mathcal{J}_i \cup \{\dfrac{k_i + g_i}{w_i}\}$;
  end
$\hat{\tau} := \cap_{i=1}^m \mathcal{J}_i$;
$\boldsymbol{\gamma} := \mathbf{x} + \hat{\tau}\mathbf{w}$. □

Now, let us provide a necessary condition for a marking $\tilde{\mathbf{m}}$ to be reachable.

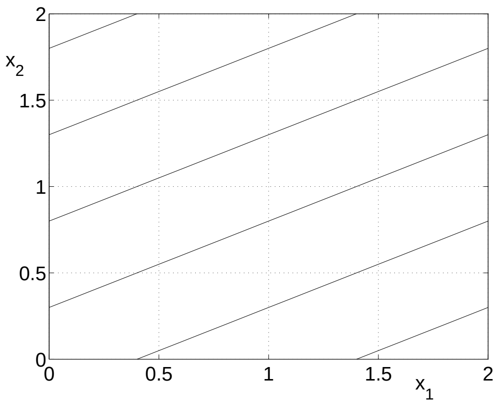

Figure 3: The equivalence class $[(0,\,0.3)]_w$, $\mathbf{w} = (2,\,1)^T$.

Lemma 17. *Let* $(N, \mathbf{m})$ *be a SRHPN system with* $\mathbf{w} = \mathbf{C}_{cc}$*, i.e., Post*$(p_i, t_c) = w_i$ *for* $i = 1, \cdots, n_c$*. If* $\tilde{\mathbf{m}} \in R(N, \mathbf{m})$ *then* $\tilde{\mathbf{m}}^c \in [\mathbf{m}^c]_w$*.*

Proof. If $\tilde{\mathbf{m}} \in R(N, \mathbf{m})$, then there exists a firing sequence $\sigma = \alpha_1, \alpha_2, \cdots, \alpha_k$ such that

$$\mathbf{m}[\alpha_1\rangle \mathbf{m}_1[\alpha_2\rangle \mathbf{m}_2 \cdots [\alpha_k\rangle \tilde{\mathbf{m}}.$$

It is enough to show that $\mathbf{m}_i^c \in [\mathbf{m}_{i-1}^c]_w$ and the result follows from the transitivity of the equivalence relation.

(Event step) Since all the arc weights are integers, the firing of a discrete transition produces no variation on the fractional parts of a continuous marking. Thus, if $\mathbf{m}_{i-1}[\alpha_i\rangle \mathbf{m}_i$ and $\alpha_i \in T_d$, then $\langle \mathbf{m}_{i-1} \rangle = \langle \mathbf{m}_i \rangle$ and $\mathbf{m}_i^c \in [\mathbf{m}_{i-1}^c]_w$.

(Time step) The firing of the continuous transition may produce a variation on the fractional parts of the continuous markings. Unless $\mathbf{w} = \mathbf{1}$, the variations of different marking components have different magnitude. However, their ratio is always the same since the arc weights are constant. Thus, if $\alpha_i = \overline{\tau} \in \mathbb{R}^+$, then $\mathbf{m}_i^c = \int_0^{\overline{\tau}} \mathbf{C}_{cc} v(\tau) d\tau + \mathbf{m}_{i-1}^c$. However, $v(\tau)$ is constant and equal to v and $\mathbf{C}_{cc} = (w_1, \cdots, w_{n_c})^T$ by hypothesis, hence

$$\begin{cases} m_{i,p_1} = w_1\, v\, \overline{\tau} + m_{i-1,p_1} \\ \vdots \\ m_{i,p_{n_c}} = w_{n_c}\, v\, \overline{\tau} + m_{i-1,p_{n_c}}. \end{cases}$$

Now, let $b = \langle v\overline{\tau} \rangle$, then $\forall p \in P_c, \langle m_{i,p} \rangle = \langle m_{i-1,p} + w_i\, b \rangle$. Thus, $\mathbf{m}_i^c \in [\mathbf{m}_{i-1}^c]_w$. This completes the proof. □

Example 18. Let us consider the SRHPN system $(N, \mathbf{m}_0)$ in example 12 with initial marking $\mathbf{m}_0 = (0,\, 0.3,\, 0,\, 1,\, 0)^T$. In figure 4 the set of all continuous markings reachable from $\mathbf{m}_0$ is represented. Obviously, this is a subset of $[\mathbf{m}_0^c]_w$.

Lines have been distinguished as continuous, dash and dash–dot lines. Continuous line (1) corresponds to the set of continuous markings reachable when the discrete marking is equal to $\mathbf{m}^d = (1,\, 0,\, 0)^T$, dash line (2) corresponds to the set of continuous markings reachable when the discrete marking is equal to $\mathbf{m}^d = (0,\, 1,\, 0)^T$, and dash-dot line (3) corresponds to the set

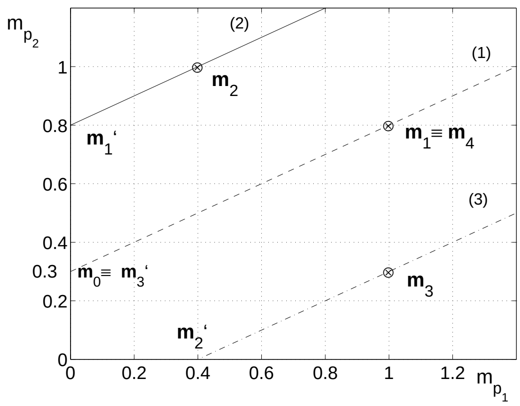

Figure 4: The set of continuous markings for the SRHP in example 18.

of continuous markings reachable when the discrete marking is equal to $\mathbf{m}^d = (0,\ 0,\ 1)^T$. The discrete marking changes every times one of the discrete transitions fires and discrete transitions can only fire alternatively.

Let us examine all possible evolutions of the net when the initial marking is $\mathbf{m}_0$. During the first 0.1667 time instants, no discrete transition is enabled and t_c fires until the marking moving along line (1) reaches $\mathbf{m}_1 = (1,\ 0.8,\ 0,\ 1,\ 0)^T$. Now t_1 becomes enabled and may fire changing the marking to $\mathbf{m}'_1$. Note however that t_1 does not necessarily fire as soon as $\mathbf{m}_1$ is reached; it may fire from any other point on line (1) greater than $\mathbf{m}_1$ thus reaching a corresponding point on line (2). For all markings on line (2) smaller than $\mathbf{m}_2$ no discrete transition is enabled and only the continuous transition fires until $\mathbf{m}_2$ is reached. Now t_2 becomes enabled and may fire changing the marking to $\mathbf{m}'_2$. Note however that t_2 is not required to fire as soon as $\mathbf{m}_2$ is reached; it may fire from any point on line (2) greater than $\mathbf{m}_2$ thus reaching a corresponding point on line (3). No discrete transition is enabled until $\mathbf{m}_3$ is reached, when t_3 may fire thus reaching $\mathbf{m}'_3 \equiv \mathbf{m}_0$ on line (1), i.e., the system cames back to the initial marking.

We also observe that the markings $\mathbf{m}_0, \mathbf{m}_1, \mathbf{m}'_1$, etc. that characterize the net evolution correspond to the markings in the reachability graph in figure 2. ■

Now, let us define a transformation on a hybrid Petri net.

Definition 19. *Let $N = (P, T, Pre, Post, \mathcal{C})$ be a FOHPN. We define* discretized PN associated *to N the P/T net $\lfloor N \rfloor = (P', T', Pre', Post')$ with:*

- *$P' = P$, i.e., $\lfloor N \rfloor$ has as many places as N, but they are all discrete,*
- *$T' = T$, i.e., $\lfloor N \rfloor$ has as many transitions as N, but they are all discrete,*
- *$Pre'(p, t) = \lfloor Pre(p, t) \rfloor$,*
- *$Post'(p, t) = \lfloor Post(p, t) \rfloor$,*

where $\lfloor \cdot \rfloor$ denotes the integer part. ■

Example 20. In figure 5 the discretized PN corresponding to the FOHPN in figure 1 is shown. ■

The following proposition shows that the discretized net can be used to determine if a marking $\mathbf{m}$ is reachable from $\mathbf{m}_0$ if the two makings have the same fractional part.

Proposition 21. *Let $(N, \mathbf{m}_0)$ be a SRHPN system and consider the discrete PN system $(\lfloor N \rfloor, \lfloor \mathbf{m}_0 \rfloor)$ associated to N. Given any marking $\mathbf{m}$ with $\langle \mathbf{m} \rangle = \langle \mathbf{m}_0 \rangle$, it holds $\mathbf{m} \in R(N, \mathbf{m}_0)$ iff $\lfloor \mathbf{m} \rfloor \in R(\lfloor N \rfloor, \lfloor \mathbf{m}_0 \rfloor)$.*

Proof. Let us denote $\bar{t}$ the discrete transition of $\lfloor N \rfloor$ corresponding to the continuous transition t_c in N and let v be the constant firing speed associated to t_c.

First, let us observe that $\mathbf{m} \in R(N, \mathbf{m}_0)$ iff $\exists\, \sigma$ such that $\mathbf{m}_0[\sigma\rangle \mathbf{m}$. Since the continuous transition in $(N, \mathbf{m}_0)$ is always enabled, this implies that $\exists\, \tilde{\sigma} = \sigma_\tau \sigma_T$ such that $\mathbf{m}_0[\tilde{\sigma}\rangle \mathbf{m}$, where $\sigma_\tau \in \mathbb{R}_0^+$ and $\sigma_T \in T_d^*$, i.e., if $\mathbf{m}$ is reachable, then it may also be reached by a "normalized sequence" where a single time step occurs first, and all the event steps occur only at the end.

Since $\langle \mathbf{m} \rangle = \langle \mathbf{m}_0 \rangle$, then $\mathbf{m}$ is reached from $\mathbf{m}_0$ by firing t_c for a time interval whose length is a multiple of $1/v$, i.e., $\sigma_\tau = k/v$.

Finally, the result follows from the fact that the firing of each discrete transition in N finds its counterpart in $\lfloor N \rfloor$ and the firing of t_c for a time interval of length $1/v$ corresponds to the firing of $\bar{t}$ in $\lfloor N \rfloor$. □

Now, we provide a necessary and sufficient condition for a marking $\mathbf{m}$ in a SRHPN to be reachable.

Theorem 22. *Let $(N, \mathbf{m}_0)$ be a SRHPN system with $Post(p_i, t_c) = w_i$, for $i = 1, \cdots, n_c$. Then, $\mathbf{m} \in R(N, \mathbf{m}_0)$ iff $\mathbf{m}^c \in [\mathbf{m}_0^c]_w$ and $\lfloor \mathbf{m} \rfloor \in R(\lfloor N \rfloor, \tilde{\mathbf{m}})$ where*

$$\begin{cases} \tilde{\mathbf{m}}^c = \lfloor \boldsymbol{\gamma}(\mathbf{m}_0^c, \mathbf{m}^c, \mathbf{w}) \rfloor \\ \tilde{\mathbf{m}}^d = \mathbf{m}_0^d, \end{cases}$$

and $\lfloor N \rfloor$ is the discretized net associated to N.

Proof. As in the proof of the previous proposition, we observe that $\mathbf{m} \in R(N, \mathbf{m}_0)$ iff there exists a normalized sequence $\sigma = \sigma_\tau \sigma_T$ such that $\mathbf{m}_0[\sigma\rangle \mathbf{m}$.

The firing sequence σ_τ can be written as $\sigma_\tau = \sigma'_\tau \sigma''_\tau$, where $\sigma'_\tau \in [0, 1/v)$, and $\sigma''_\tau = k/v$, with $k \in \mathbb{N}_0^+$. Therefore, $\mathbf{m}_0[\sigma'_\tau\rangle \mathbf{m}'_0[\sigma''_\tau\rangle \mathbf{m}'[\sigma_T\rangle \mathbf{m}$. Obviously, $\langle \mathbf{m}'_0 \rangle = \langle \mathbf{m}' \rangle = \langle \mathbf{m} \rangle$.

We further observe that the difference in the fractional part between $\mathbf{m}_0$ and $\mathbf{m}$ is due to the time step σ'_τ, that has a length less than $1/v$ and whose firing yields $\mathbf{m}'_0$ from $\mathbf{m}_0$. Moreover $\mathbf{m}'_0 = \int_0^{\sigma'_\tau} v \mathbf{C}_{cc} d\tau + \mathbf{m}_0 = v\sigma'_\tau \mathbf{C}_{cc} + \mathbf{m}_0 = b\mathbf{w} + \mathbf{m}_0$, where $b = v\sigma'_\tau \in [0, 1)$ and $\mathbf{w} = \mathbf{C}_{cc}$. Therefore, $\mathbf{m}_0'^c$ is exactly the w-cover of $\mathbf{m}_0^c$ with the same fractional part of $\mathbf{m}^c$, i.e., $\mathbf{m}_0'^c = \boldsymbol{\gamma}(\mathbf{m}_0^c, \mathbf{m}^c, \mathbf{w})$ and the integer part of $\mathbf{m}'_0$ is exactly the marking $\tilde{\mathbf{m}}$ defined in the theorem statement.

Finally, by virtue of proposition 21, since $\mathbf{m}'_0$ and $\mathbf{m}$ have the same fractional part, then $\mathbf{m} \in R(N, \mathbf{m}'_0)$ if and only if $\lfloor \mathbf{m} \rfloor \in R(\lfloor N \rfloor, \lfloor \mathbf{m}'_0 \rfloor)$. □

Example 23. Let us consider the SRHPN system (N, m_0) in example 12 with initial marking $\mathbf{m}_0 = (1.3,\ 0.5,\ 0,\ 1,\ 0)^T$. Here $\mathbf{w} = (2,\ 1)^T$. We want to determine whether $\mathbf{m} = (5.1,\ 1.9,\ 0,\ 0,\ 1)^T \in R(N, \mathbf{m}_0)$ by applying theorem 22.

Clearly $\mathbf{m}^c \in [\mathbf{m}_0^c]_w$ because if we take $b = 0.4$, then $\forall p_i \in P_c$, $\langle m_{p_i} \rangle = \langle m_{0,p_i} + bw_i \rangle$.

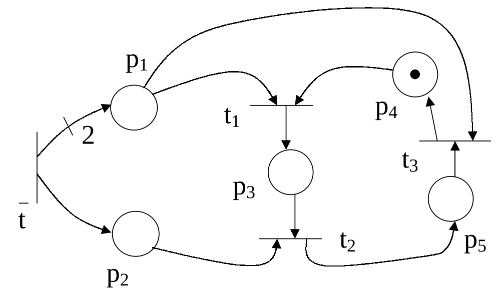

Figure 5: The discretized PN corresponding to the SRHPN in figure 1.

By applying algorithm 16, we compute the marking $\boldsymbol{\gamma}(\mathbf{m}_0^c, \mathbf{m}^c, \mathbf{w}) = (2.1,\ 0.9)^T$ and we define $\tilde{\mathbf{m}} = \lfloor (2.1,\ 0.9,\ 0,\ 1,\ 0)^T \rfloor$ with $\tilde{\mathbf{m}}^c = \lfloor \boldsymbol{\gamma}(\mathbf{m}_0^c,\ \mathbf{m}^c, \mathbf{w}) \rfloor$ and $\tilde{\mathbf{m}}^d = \mathbf{m}_0^d$. If we consider the discretized PN in figure 5 we see that $\lfloor \mathbf{m} \rfloor = (5,\ 1,\ 0,\ 0,\ 1)^T$ is reachable from $\tilde{\mathbf{m}} = (2,\ 0,\ 0,\ 1,\ 0)^T$. In fact, the firing sequence, say, $\overline{\sigma} = t_1\ \overline{t}\ \overline{t}\ t_2$ is such that $\tilde{\mathbf{m}}\ [\overline{\sigma}\rangle\ \lfloor \mathbf{m} \rfloor$. Therefore, we can conclude that $\mathbf{m} \in R(N, \mathbf{m}_0)$. ■

By virtue of the above theorem 22, the results on the reachability of discrete Petri nets can be extended to SRHPN, thus proving the validity of the following corollary.
Corollary 24. *The reachability problem is decidable for SRHPN.*

Proof. Follows from theorem 22 and from the fact that the reachability problem is decidable for discrete PN [20]. □

6 Conclusions

Although Hybrid Automata can be seen under most aspects as a generalization of First–Order Hybrid Petri Nets, restricted classes of HA and FOHPN are different models that describe different classes of hybrid systems.

As an example, we have studied in this paper *Single–Rate Hybrid Petri Nets*, a model that can be seen as the Petri net counterpart of a Timed Automaton with skewed clocks. The reachability problem for a hybrid net in this class has been reduced to the reachability problem of a corresponding discrete Petri net, and thus it is decidable.

To study this class of nets, in one of the examples we have informally used the reachability graph analysis that has been developed for discrete nets. It may be interesting to find out if a technique based on the reachability/coverability graph may always be applied to this hybrid model and which properties can be studied with it.

So far, no results on the decidability properties of the general FOHPN model are known. Towards the goal of proving/disproving this general result, we feel it is worth defining and exploring sub–classes of FOHPN of increasing complexity. These structures may extend the classes of models for which important properties can be shown to be decidable and can be studied with standard tools of discrete Petri nets.

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