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DECIDING UNAMBIGUITY AND SEQUENTIALITY OF POLYNOMIALLY AMBIGUOUS MIN-PLUS AUTOMATA

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Abstract. This paper solves the unambiguity and the sequentiality problem for polynomially ambiguous min-plus automata. This result is proved through a decidable algebraic characterization involving so-called metatransitions and an application of results from the structure theory of finite semigroups. It is noteworthy that the equivalence problem is known to be undecidable for polynomially ambiguous automata.

1. Introduction

Min-plus and max-plus automata are studied under various names in the literature, e.g. distance, finance, or cost automata. They have also appeared in various contexts: logical problems in formal language theory (star height, finite power property, star problem for traces) [6, 12, 13, 23, 20], study of dynamics of some discrete event systems (DES) [1, 2], automatic speech recognition [21], and database theory [3].

The sequentiality/unambiguity problem is one of the most intriguing open problems for min-plus automata: decide (constructively) whether some given min-plus automaton admits a sequential/unambiguous equivalent. This problem is wide open despite the fact it was studied by several researchers, e.g. [15, 19, 21].

In 2004, Klimann, Lombardy, Mairesse, and Prieur showed that this problem is decidable for finitely ambiguous min-plus automata [15]. For the sequentiality problem, Mohri presented an imperfect algorithm (which is not a decision algorithm) in 1997 [21].

In the present paper, we show a new partial solution to the sequentiality/unambiguity problem: we show that this problem is decidable provided that the input automaton is polynomially ambiguous. Polynomially ambiguous min-plus automata are much more involved objects than finitely ambiguous ones, e.g. the equivalence problem is undecidable for polynomially, but decidable for finitely ambiguous min-plus automata [16, 8]. In fact, all the key ideas in [15] for finitely ambiguous min-plus automata (namely the decomposition technique and the pumping arguments) do not carry over to polynomially ambiguous min-plus automata and we have to develop advanced proof techniques. We develop a theory of

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so-called metatransitions and establish a decidable algebraic characterization of the polynomially ambiguous min-plus automata which admit an unambiguous equivalent. To prove the characterization, we utilize some techniques from the limitedness problem for distance and desert automata [18, 24, 12, 13], results from the structure theory of finite semigroups as the factorization forest theorem along with various new ideas. The proof for the sufficiency of the construction leads to an intriguing combination of two Burnside problems.

2. Preliminaries

2.1. Notations

Let Σ be a finite alphabet. The notion of a $\sharp$-expression is due to [7]. Every $a \in \Sigma$ is a *$\sharp$-expression*. For $\sharp$-expressions r and s, the expressions rs and $r^\sharp$ are *$\sharp$-expressions*. For a $\sharp$-expression r and $k \geq 0$, let $r(k)$ be the word obtained by replacing every $\sharp$ by k.

Let $\mathbb{N} = \{0, 1, \dots\}$. Let $\mathbb{Z}_\omega = (\mathbb{Z} \cup \{\omega, \infty\}, \mathsf{min}, +, \infty, 0)$ be the semiring whereas min is the minimum for the ordering $\cdots \leq -1 \leq 0 \leq 1 \cdots \leq \omega \leq \infty$ and $m + n$ is defined as usual if $m, n \in \mathbb{Z}$ but as maximum of m and n if $m \in \{\omega, \infty\}$ or $n \in \{\omega, \infty\}$. The *tropical semiring* $\mathbb{Z}_\infty$ is the restriction of $\mathbb{Z}_\omega$ to $\mathbb{Z} \cup \{\infty\}$.

Let Q be a finite set. For $k \geq 1$, matrices $M_1, \dots, M_k, M \in \mathbb{Z}_\omega^{Q \times Q}$, and $p_0, \dots, p_k \in Q$, we denote $M_1[p_0, p_1] + \cdots + M_k[p_{k-1}, p_k]$ by $(M_1, \dots, M_k)[p_0, \dots, p_k]$, and we denote $M[p_0, p_1] + \cdots + M[p_{k-1}, p_k]$ by $M[p_0, \dots, p_k]$.

Let $M \in \mathbb{Z}_\omega^{Q \times Q}$. We set $\mathsf{mind}(M) = \mathsf{min}\{M[p,p] \,|\, p \in Q\}$. If some entry of M belongs to $\mathbb{Z}$, then $\mathsf{min}(M)$ (resp. $\mathsf{max}(M)$) is the minimum (resp. maximum) of the set $\{M[p,q] \,|\, p, q \in Q, M[p,q] \in \mathbb{Z}\}$, and $\mathsf{span}(M) = \mathsf{max}(M) - \mathsf{min}(M)$. Otherwise, $\mathsf{span}(M) = 0$.

The boolean semiring is $\mathbb{B} = (\{0,1\}, +, \cdot, 0, 1)$, and we denote by $\alpha : \mathbb{Z}_\omega \to \mathbb{B}$ the morphism defined by $\alpha(\infty) = 0$ and $\alpha(z) = 1$ for $z \neq \infty$.

Given $P \subseteq Q$ and $M \in \mathbb{B}^{Q \times Q}$, we let $P \cdot M = \{q \in Q \,|$ there is some $p \in P$ such that $M[p,q] = 1\}$ and $M \cdot P = \{q \in Q \,|$ there is some $p \in P$ such that $M[q,p] = 1\}$.

We generalize all these notions (except mind) to matrices which are not quadratic.

Let T be a set and $\cdot : T \times T \dashrightarrow T$ be partial mapping. We assume that $\cdot$ is associative, i.e., if for $p, q, r \in T$, either both products $(pq)r$ and $p(qr)$ are undefined or both products are defined and $(pq)r = p(qr)$. Let $T_0 = T \uplus \{0\}$. We extend $\cdot$ to T_0 by setting $pq = 0$ for $p, q \in T$ for which pq is undefined in T. Clearly, T_0 is a semigroup with zero 0.

2.2. Min-Plus Automata

A *min-plus automaton* is a tuple $\mathcal{A} = [Q, \mu, \lambda, \varrho]$ whereas Q is a nonempty, finite set of *states*, $\mu : \Sigma^* \to \mathbb{Z}_\infty^{Q \times Q}$ is a homomorphism, and $\lambda, \varrho \in \mathbb{Z}_\infty^Q$. A min-plus automaton $\mathcal{A}$ computes a mapping $|\mathcal{A}| : \Sigma^* \to \mathbb{Z}_\infty$ by $|\mathcal{A}|(w) = \lambda\mu(w)\varrho$ for $w \in \Sigma^*$.

Two min-plus automata are *equivalent* if and only if they compute the same mapping. We call a state $q \in Q$ *accessible* (resp. *co-accessible*) if there is a $v \in \Sigma^*$ such that $(\lambda\mu(v))[q] \in \mathbb{Z}$ (resp. $(\mu(v)\varrho)[q] \in \mathbb{Z}$). If every state is accessible and co-accessible, then we call $\mathcal{A}$ *trim*.

Let $I = \{\, q \in Q \,|\, \lambda[q] \in \mathbb{Z} \,\}$ and $F = \{\, q \in Q \,|\, \varrho[q] \in \mathbb{Z} \,\}$. If $|I| = 1$, and for every $a \in \Sigma$, $p \in Q$, there exists at most one $q \in Q$ satisfying $\mu(a)[p,q] \in \mathbb{Z}$, then we call $\mathcal{A}$ *sequential*.

Let $w = a_1 \cdots a_{|w|} \in \Sigma^*$. A sequence $p_0, \dots, p_{|w|}$ is a *path (in $\mathcal{A}$) from p_0 to $p_{|w|}$ for w* if $(\mu(a_1), \dots, \mu(a_{|w|}))[p_0, \dots, p_{|w|}] \in \mathbb{Z}$. We call $p_0, \dots, p_{|w|}$ *accepting* if $p_0 \in I$, $p_{|w|} \in F$.

If there exists some polynomial $P : \mathbb{N} \to \mathbb{N}$ such that for every $w \in \Sigma^*$, there are at most $P(|w|)$ accepting paths for w, then $\mathcal{A}$ is called *polynomially ambiguous*. If the same condition is satisfied for a constant $n \in \mathbb{N}$, then $\mathcal{A}$ is called *finitely ambiguous*. If there is at most one path for each word, then $\mathcal{A}$ is called *unambiguous*. The mapping $f : \{a,b\}^* \to \mathbb{Z}_\infty$ defined as $f(w) = \mathsf{min}\{k \mid ba^kb \text{ is a factor of } w\}$ can be computed by a polynomially ambiguous min-plus automaton, but not by a finitely ambiguous min-plus automaton [14].

The following characterization is used implicitly in [10, 11, 22] (cf. Proof of Theorem 3.1 in [11] or Lemma 4.3 in [10]).

Theorem 2.1. *A trim min-plus automaton $\mathcal{A}$ is polynomially ambiguous if and only if for every state q and every $w \in \Sigma^*$, there is at most one path for w from q to q.*

We need the following characterization.

Lemma 2.2. *Let $\mathcal{A} = [Q, \mu, \lambda, \varrho]$ be a trim, unambiguous min-plus automaton. Let $w \in \Sigma^*$, $k \geq 1$, and $q_0, \ldots, q_k \in Q$ such that there is path for w from q_{i-1} to q_i for every $1 \leq i \leq k$.*

There are $\pi_1, \pi_2, \pi_3 \in Q^*$ such that $|\pi_1\pi_3| \leq |Q|$, $|\pi_2| \leq |Q|$, and $q_0 \ldots q_k \in \pi_1\pi_2^*\pi_3$.

3. Overview

3.1. Metatransitions

The combination of a forward and backward parsing was one of the key ideas by Hashiguchi in various papers on the finite power property and distance automata (e.g. in [4, 5]). Metatransitions formalize this idea in an algebraic fashion. Metatransitions form a semigroup, and the homomorphism $\alpha : \mathbb{Z}_\omega \to \mathbb{B}$ extends in a natural way to a homomorphism between the semigroups of metatransitions. Henceforth, we can utilize semigroup theoretic approaches by Simon, Leung, and Kirsten (e.g. [18, 23, 24, 12, 13]) on metatransitions. Consequently, the concept of a metatransition compromises the combinatorial approach by Hashiguchi and the algebraic approach by Simon and Leung in the research on min-plus automata. Several results in this section were already shown in [9].

Let Q be a finite set. A *metatransition over $\mathbb{Z}_\omega$ and Q* is a tuple $\left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ M\ \begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\right)$, whereas

MT1.: $P_0, P_1, R_0, R_1 \subseteq Q$,

MT2.: $M \in \mathbb{Z}_\omega^{(P_0 \cap R_0) \times (P_1 \cap R_1)}$,

MT3.: $(P_0 \cap R_0) \cdot \alpha(M) = (P_1 \cap R_1)$ and $(P_0 \cap R_0) = \alpha(M) \cdot (P_1 \cap R_1)$.

Two metatransitions $\left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ M\ \begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} P_0' \\ R_0' \end{smallmatrix}\ M'\ \begin{smallmatrix} P_1' \\ R_1' \end{smallmatrix}\right)$ are called *concatenable* if and only if $P_1 = P_0'$ and $R_1 = R_0'$. In this case, their product yields $\left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ MM'\ \begin{smallmatrix} P_1' \\ R_1' \end{smallmatrix}\right)$.

Let $\mathsf{MT}(\mathbb{Z}_\omega, Q)$ be the set consisting of all metatransitions over Q. Then, $\mathsf{MT}(\mathbb{Z}_\omega, Q)_0$ is a semigroup with a zero.

We define metatransitions over $\mathbb{B}$ and Q in the same way.[1] We extend the homomorphism $\alpha : \mathbb{Z}_\omega \to \mathbb{B}$ to $\alpha : \mathsf{MT}(\mathbb{Z}_\omega, Q)_0 \to \mathsf{MT}(\mathbb{B}, Q)_0$ by setting

$$\alpha\left(\left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ M_1\ \begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\right)\right) = \left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ \alpha(M_1)\ \begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\right) \quad \text{and} \quad \alpha(0) = 0.$$

[1] In ($\mathsf{MT3}$), $\alpha(M)$ is replaced by M.

Let $M' \in \mathbb{Z}_\omega^{Q\times Q}$ and $P_0, R_1 \subseteq Q$. Let $P_1 = P_0 \cdot \alpha(M')$, $R_0 = \alpha(M') \cdot R_1$, and let M be the restriction of M' to $(P_0 \cap R_0) \times (P_1 \cap R_1)$. We denote $\left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ M\ \begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\right)$ by $[\![P_0, M', R_1]\!]$ and call it the *metatransition induced by* P_0, M', R_1. We also say that $[\![P_0, M', R_1]\!]$ is induced by M'.

Lemma 3.1. *Let* $t_1 = \left(\begin{smallmatrix} P_0 \\ R_0 \end{smallmatrix}\ M_1\ \begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\right)$, $t_2 = \left(\begin{smallmatrix} P_1 \\ R_1 \end{smallmatrix}\ M_2\ \begin{smallmatrix} P_2 \\ R_2 \end{smallmatrix}\right) \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$ *and* $M_1', M_2' \in \mathbb{Z}_\omega^{Q\times Q}$. *If* $t_1 = [\![P_0, M_1', R_1]\!]$ *and* $t_2 = [\![P_1, M_2', R_2]\!]$, *then* $t_1 t_2 = [\![P_0,\, M_1' M_2',\, R_2]\!]$.

Let $k \geq 1$ and let $M_1', \ldots, M_k' \in \mathbb{Z}_\omega^{Q\times Q}$. Let $P_0, R_k \subseteq Q$. As above, the matrices $M_1', \ldots, M_k'$ induce with P_0, R_k a sequence of concatenable metatransitions: For $0 < i \leq k$, let $P_i = P_{i-1} \cdot \alpha(M_i')$ and $R_{i-1} = \alpha(M_i') \cdot R_i$. Finally let M_i be the restriction of M_i' to $(P_{i-1} \cap R_{i-1}) \times (P_i \cap R_i)$ and $t_i = \left(\begin{smallmatrix} P_{i-1} \\ R_{i-1} \end{smallmatrix}\ M_i\ \begin{smallmatrix} P_i \\ R_i \end{smallmatrix}\right)$ for $1 \leq i \leq k$.

Clearly, $t_i = [\![P_{i-1}, M_i', R_i]\!]$. Moreover, $P_i \cap R_i \neq \emptyset$ for some $0 \leq i \leq k$ if and only if $P_i \cap R_i \neq \emptyset$ for every $0 \leq i \leq k$. By Lemma 3.1, we obtain $t_1 \cdots t_k = [\![P_0,\, M_1 \cdots M_k,\, R_k]\!]$.

3.2. The Semigroup of Metatransitions of an Automaton

Let $\mathcal{A} = [Q, \mu, \lambda, \varrho]$ be a min-plus automaton, $I = \{\, q \in Q \mid \lambda[q] \in \mathbb{Z} \,\}$ and $F = \{\, q \in Q \mid \varrho[q] \in \mathbb{Z} \,\}$.

Let $n \geq 1$ and $w_1, \ldots, w_n \in \Sigma^*$ be a sequence of words. As above, the matrices $\mu(w_1), \ldots, \mu(w_n)$ induce with $P_0 = I$ and $R_n = F$ a sequence of concatenable metatransitions $t_1, \ldots, t_n$.

Let $q_0, \ldots, q_n \in Q$. If $\lambda[q_0] + \big(\mu(w_1), \ldots, \mu(w_n)\big)[q_0, \cdots, q_n] + \varrho[q_n] \in \mathbb{Z}$, then we have $q_i \in P_i \cap R_i$ for every $0 \leq i \leq k$. Conversely, for every $1 \leq i \leq k$ and every $q \in P_i \cap R_i$, $\mathcal{A}$ can read $w_1 \cdots w_i$ from an initial state to q, and it can read $w_{i+1} \cdots w_n$ from q to an accepting state. In this sense, the metatransitions $t_1, \ldots, t_n$ represent exactly the accepting paths for w in $\mathcal{A}$. The matrices inside $t_1, \ldots, t_n$ are the matrices $\mu(w_1), \ldots, \mu(w_n)$ restricted to the entries which occur in accepting paths for $w_1 \ldots w_n$.

We have $P_i \cap R_i \neq \emptyset$ for some $0 \leq i \leq k$ if and only if $P_i \cap R_i \neq \emptyset$ for every $0 \leq i \leq k$ if and only if $\mathcal{A}$ accepts $w_1 \ldots w_n$.

The most beautiful property is the following: let $0 \leq i < j \leq n$ and assume $P_i = P_j$ and $R_i = R_j$. We consider the sequence of words $w' = w_1, \ldots, w_i, (w_{i+1}, \ldots, w_j,)^k w_{j+1}, \ldots, w_n$ for some $k \geq 0$. By applying μ to each word in w', we obtain a sequence of matrices. As above, these matrices induce with $P_0 = I$ and $R_n = F$ a sequence of metatransitions. Clearly, we obtain the sequence $t_1, \ldots, t_i, (t_{i+1}, \ldots, t_j,)^k t_{j+1}, \ldots, t_{|w|}$.

Although this property looks quite obvious, it is of crucial importance since it enables us to apply pumping- and Burnside-techniques.

We associate to $\mathcal{A}$ a subsemigroup of $\mathsf{MT}(\mathbb{Z}_\omega, Q)_0$. We call some set $S \subseteq Q$ a *P-clone of* $\mathcal{A}$ (resp. an *R-clone of* $\mathcal{A}$) if there exists some word $v \in \Sigma^*$ such that $S = I \cdot \alpha(\mu(v))$ (resp. $S = \alpha(\mu(v)) \cdot F$). Let $\mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) =$

$$\Big\{ [\![P_0, \mu(a), R_1]\!] \,\Big|\, a \in \Sigma, P_0 \text{ is a P-clone}, R_1 \text{ is a R-clone},\ P_0 \cdot \mu(a) \cdot R_1 \neq \infty \Big\}.$$

The condition $P_0 \cdot \mu(a) \cdot R_1 \neq \infty$ ensures that $\mu(a)$ does not restrict to a $\emptyset \times \emptyset$-matrix.

Let $\langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0$ be the subsemigroup of $\mathsf{MT}(\mathbb{Z}_\omega, Q)_0$ generated by $\mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \cup \{0\}$. By Lemma 3.1, we can show that $\langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0$ consists of 0 and metatransitions of the form $[\![P_0, \mu(w), R_1]\!]$ for P-clones P_0, R-clones R_1, and words $w \in \Sigma^+$ satisfying $P_0 \cdot \mu(w) \cdot R_1 \neq \infty$.

For every metatransition $t_2 \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0$, there are $t_1, t_3 \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0$ such that $t_1t_2t_3 \neq 0$ and $t_1t_2t_3 = [\![I, \mu(w), F]\!]$ for some word $w \in \Sigma^*$.

By removing the weights from $\mathcal{A}$, we can define in the same way a set $\mathsf{MT}(\mathbb{B}, \mathcal{A})$ and the subsemigroup $\langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$ of $\mathsf{MT}(\mathbb{B}, \mathcal{A})$.

3.3. On Metatransitions with an Idempotent Structure

Let $\mathcal{A}$ be a polynomially ambiguous min-plus automaton and let $e = \begin{pmatrix} {}^P_R & M & {}^P_R \end{pmatrix} \in \langle \mathsf{MT}(\mathbb{Z}_\omega, Q)\rangle_0$ be a metatransition with an *idempotent structure*, i.e., assume $\alpha(ee) = \alpha(e)$. We define a relation $\leq_e$ on $P \cap R$ by setting $p \leq_e q$ iff $M[p, q] \neq \infty$. This relation is "almost a partial order" in that it satisfies the three following properties.

i) Clearly, $\leq_e$ is transitive.

ii) The relation $\leq_e$ is antisymmetric. Let $p \neq q \in P \cap R$ such that $p \leq_e q \leq_e p$. Clearly, e is induced by $\mu(v)$ for some $v \in \Sigma^*$. Then, $\mathcal{A}$ can read v^2 from p to p in two paths. One path stays at p, the other path goes from p to q and back. This contradicts Theorem 2.1.

iii) For every $q \in P \cap R$, there exist $p, r \in P \cap R$, $p \leq_e q \leq_e r$ such that $p \leq_e p$ and $r \leq_e r$. By (MT3), there are $q_1, q_2, \ldots \in P \cap R$ such that $q \leq_e q_1 \leq_e q_2 \leq_e \cdots$ By transitivity and finiteness of $P \cap R$, there is some r among $q_1, q_2, \ldots$ such that $q \leq_e r \leq_e r$. The proof for p is similar.

Lemma 3.2. *Let $e = \begin{pmatrix} {}^P_R & M & {}^P_R \end{pmatrix} \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0$ with an idempotent structure. Let $k \geq 1$ and $p, q \in P \cap R$ such that $M^k[p, q] \neq \infty$.*

There are $p = p_0, \ldots, p_k = q$ in $P \cap R$ such that $M^k[p, q] = M[p_0, \ldots, p_k]$. Moreover, if $k > |P \cap R|$, then we can choose $p_0, \ldots, p_k$ such that there are $0 \leq i < j \leq k$ such that $p_i = p_{i+1} = \cdots = p_j$ and $p_0, \ldots, p_i, p_{j+1}, \ldots, p_k$ does not contain a cycle.

The last claim of Lemma 3.2 just says that for large k, one can choose the sequence $p_0, \ldots, p_k$ to be almost constant up to a short cycle-free prefix and suffix. The total length of the prefix and the suffix is at most $|P \cap R|$.

It is important that for the antisymmetry of $\leq_e$ and for Lemma 3.2, we do not need to assume $e \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0$, it suffices that $e \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$ and $\alpha(e) = \alpha(ee) \in \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$.

3.4. Stabilization

Let $\mathcal{A}$ be a polynomially ambiguous min-plus automaton. Let $e = \begin{pmatrix} {}^P_R & M & {}^P_R \end{pmatrix} \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$ such that $\alpha(e) = \alpha(ee) \in \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$. Assume $\mathsf{mind}(M) = 0$.

We define $M^\sharp$, the *stabilization of* M. The idea of $M^\sharp$ is to understand the sequence $(M^k)_{k\geq 1}$. Let $p, q \in P \cap R$.

If $M^k[p, q] = \infty$, for some $k \geq 1$, then $M^k[p, q] = \infty$ for every $k \geq 1$. In this case, we define $M^\sharp[p, q] = \infty$.

Assume $M[p, q] \neq \infty$. Lemma 3.2 is crucial to understand the sequence $(M^k[p, q])_{k\geq 1}$. From $\mathsf{mind}(M) = 0$, we can easily deduce a lower bound on $(M^k[p, q])_{k\geq 1}$.

We say that some sequence $p_0, \ldots, p_k \in P \cap R$ satisfies (S1), if $p_0 = p$, $p_k = q$, and $M[p_0, \ldots, p_k] \in \mathbb{Z}$. If $p_0, \ldots, p_k$ satisfies (S1) and there exists some $0 \leq i \leq k$ such that $M[p_i, p_i] = 0$, then we say that $p_0, \ldots, p_k$ satisfies (S2).

Assume there exists a sequence which satisfies (S2). Then, there exists a sequence $p_0, \ldots, p_k$ for some $k < |P \cap R|$ which satisfies (S2) such that $m = M[p_0, \ldots, p_k]$ is minimal among all sequences which satisfy (S2). In this case, $(M^k[p,q])_{k\geq 1}$ is ultimately constant m and we define $M^\sharp[p,q] = m$.

Assume that there does not exist a sequence which satisfies (S2) although $M[p,q] \neq \infty$. We can conclude that the sequence $(M^k[p,q])_{k\geq 1}$ is either ultimately ω, or it tends to infinity, since (S1)-sequences cannot utilize the zeros on the main diagonal of M. In this case, we set $M^\sharp[p,q] = \omega$.

Consequently, $M^\sharp[p,q]$ describes the behaviour of $(M^k[p,q])_{k\geq 1}$.

For $p \in P \cap R$ satisfying $M[p,p] = 0$, we have $M^\sharp[p,p] = 0$.

We generalize the definition of $M^\sharp$ by weakening the assumption $\mathsf{mind}(M) = 0$ to $\mathsf{mind}(M) \in \mathbb{Z}$. We still assume $\alpha(e) = \alpha(ee) \in \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$.

We normalize M. Let $m = \mathsf{mind}(M)$ and define $\bar{M}$ by $\bar{M}[p,q] = M[p,q] - m$ for[2] $p,q \in P \cap R$. Clearly, $\alpha(\bar{M}) = \alpha(M)$ and $\mathsf{mind}(\bar{M}) = 0$. We define $M^\sharp = \bar{M}^\sharp$.

For $k \geq 1$, we have $M^k[p,q] = km + \bar{M}^k[p,q]$. Let $p,q,p',q' \in P \cap R$. For $k \geq 1$, we have $M^k[p,q] - M^k[p',q'] = \bar{M}^k[p,q] - \bar{M}^k[p',q']$, provided that $M^k[p,q] \in \mathbb{Z}$.

If $M^\sharp[p,q]$ and $M^\sharp[p',q']$ are integers, then the entries $[p,q]$ and $[p',q']$ are ultimately constant in $(\bar{M}^k)_{k\geq 1}$, i.e., the entries $[p,q]$ and $[p',q']$ grow or sink synchronized in the sequence $(M^k)_{k\geq 1}$ and for every k beyond some bound, we have $M^k[p,q] - M^k[p',q'] = M^\sharp[p,q] - M^\sharp[p',q']$.

However, if $M^\sharp[p,q] = \omega$ and $M^\sharp[p',q'] \in \mathbb{Z}$, then either $(M^k[p,q])_{k\geq 1}$ is ultimately ω, or the difference $(M^k[p,q] - M^k[p',q'])_{k\geq 1}$ tends to infinity.

Given $e = \begin{pmatrix} {}^P_R & M & {}^P_R \end{pmatrix} \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$ satisfying $\alpha(e) = \alpha(ee) \in \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$ and $\mathsf{mind}(M) \in \mathbb{Z}$, we define its *stabilization* $e^\sharp = \begin{pmatrix} {}^P_R & M^\sharp & {}^P_R \end{pmatrix} \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$. We have $\alpha(e^\sharp) = \alpha(e)$ and $e^\sharp \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$.

Finally, let $t \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$ and let M be the matrix in t. We define $\mathsf{span}(t) = \mathsf{span}(M)$ and generalize the notions min, max, and mind in the same way.

Lemma 3.3. (1) *For concatenable* $t_1, t_2 \in \mathsf{MT}(\mathbb{Z}, Q)$, $\mathsf{span}(t_1 t_2) \leq \mathsf{span}(t_1) + \mathsf{span}(t_2)$.
(2) *For* $e \in \mathsf{MT}(\mathbb{Z}_\omega, Q)$ *for which* $e^\sharp$ *is defined, we have* $\mathsf{span}(e^\sharp) \leq |Q|\mathsf{span}(e)$.

3.5. Main Results, Conclusions, and Open Questions

Let $\mathcal{A}$ be a polynomially ambiguous min-plus automaton and let $\mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})$ as in Section 3.2. Let $\langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp$ be the least semigroup which

(1) contains $\mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})$ and the zero of $\mathsf{MT}(\mathbb{Z}_\omega, Q)$,
(2) is closed under the product of metatransitions, and
(3) is closed under stabilization, i.e., for every $e \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp$, we have $e^\sharp \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp$, provided that $e^\sharp$ is defined.

We state our main characterization:

Theorem 3.4. *Let* $\mathcal{A}$ *be a polynomially ambiguous min-plus automaton.*
The following assertions are equivalent:

(1) *There exists some metatransition* $t \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp$ *such that every entry in* t *belongs to* $\{\omega, \infty\}$.
(2) *The min-plus automaton* $\mathcal{A}$ *has no unambiguous equivalent.*

[2] Whereas $\infty - m = \infty$ and $\omega - m = \omega$.

(3) *Every unambiguous min-plus automaton $\tilde{\mathcal{A}}$ which accepts the same language as $\mathcal{A}$ satisfies one of the following conditions:*
   (3a) *There are $u, v, w \in \Sigma^*$ such that $uv^k w$ is accepted by $\mathcal{A}$ and $\tilde{\mathcal{A}}$ for $k \geq 1$, and for growing k, the sequence $\left(|\tilde{\mathcal{A}}|(uv^k w) - |\mathcal{A}|(uv^k w)\right)_{k \geq 1}$ tends to infinity.*
   (3b) *There is a $\sharp$-expression r such that $r(k)$ is accepted by $\mathcal{A}$ and $\tilde{\mathcal{A}}$ for $k \geq 1$, and for growing k, the sequence $\left(|\mathcal{A}|(r(k)) - |\tilde{\mathcal{A}}|(r(k))\right)_{k \geq 1}$ tends to infinity.*

The reader might complain that (as seen in Section 3.4) one entry in the main diagonal of a stabilization $e^\sharp$ is 0, and hence, some matrix t as in Theorem 3.4(1) cannot exist. However, by applying both stabilization and multiplication, metatransitions in which every entry is either ω or ∞ may arise.

For illustration, let us consider Theorem 3.4 for the particular case that $\mathcal{A}$ is unambiguous. Let $e = \left(\begin{smallmatrix} P \\ R \end{smallmatrix}\ M\ \begin{smallmatrix} P \\ R \end{smallmatrix}\right) \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0^\sharp$ be with an idempotent structure. Since P (resp. R) is a P- (resp. R-clone), there are $u, v \in \Sigma^*$ such that $P = I \cdot \alpha(\mu(u))$ and $R = \alpha(\mu(v)) \cdot F$. If $|P \cap R| > 1$, then we can construct two different accepting paths for uv. Hence, $P \cap R = 1$ and M is a (1×1)-matrix. By $(\mathsf{MT}3)$, the entry of M cannot be ∞. If the entry of M is an integer, then $\mathsf{mind}(M)$ yields the only entry of M, and thus, the entry of the normalization $\bar{M}$ is 0, i.e., the entry of $M^\sharp = \bar{M}^\sharp$ is 0. Consequently, ω's cannot arise in the closure $\langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0^\sharp$, and in particular, (1) in Theorem 3.4 is not satisfied.

Note that (3) $\Rightarrow$ (2) in Theorem 3.4 is obvious. We will prove (1) $\Rightarrow$ (3) in Section 4. We assume some t as in (1) and assume some $\tilde{\mathcal{A}}$ as in (3) which does not satisfy (3a). Then, we show (3b): as t is constructed from metatransitions in $\mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})$ by using multiplication and stabilization, r is constructed from letters by using concatenation and $\sharp$-powers.

We will prove (2) $\Rightarrow$ (1) in Section 5. It leads to an intriguing combination of two Burnside problems over metatransitions which are remotely related to problems considered by Simon and Leung, e.g. [18, 23, 24].

Theorem 3.5. *Given a polynomially ambiguous min-plus automaton $\mathcal{A}$, we can decide whether $\mathcal{A}$ has an unambiguous equivalent, or whether it has a sequential equivalent.*

Proof. To decide the existence of an unambiguous equivalent, one process searches for some $t \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0^\sharp$ as in Theorem 3.4(1). A simultaneous process enlists all unambiguous min-plus automata, and checks (using an algorithm in [17]) whether one of them is equivalent to $\mathcal{A}$. By Theorem 3.4, exactly one of the processes terminates. To decide the existence of a sequential equivalent, the algorithm decides at first whether there exists an unambiguous equivalent $\mathcal{A}'$. If so, it applies an algorithm in [15, 21] to $\mathcal{A}'$. ∎

It is interesting to have by Theorem 3.5 a decidability result for a class of min-plus automata for which the equivalence problem is undecidable [16]. Many interesting questions arise from our approach and from the introduced proof techniques. The central question is of course whether or how our approach can be generalized to arbitrary min-plus automata. Another question is whether we can achieve complexity results or a practical algorithm.

Further questions are: can we characterize the existence of a sequential equivalent in terms of the stabilization closure $\langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0^\sharp$? Is the existence of a finitely ambiguous (resp. finitely sequential) equivalent decidable? Is the membership problem of $\langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A}) \rangle_0^\sharp$ decidable? Are our techniques helpful to decide the open equivalence problem between a polynomially and a finitely ambiguous min-plus automaton?

4. Necessity

We prove (1) $\Rightarrow$ (3) in Theorem 3.4. We assume some polynomially ambiguous min-plus automaton $\mathcal{A} = [Q, \mu, \lambda, \varrho]$ which satisfies (1). We assume an unambiguous automaton $\tilde{\mathcal{A}} = [\tilde{Q}, \tilde{\mu}, \tilde{\lambda}, \tilde{\varrho}]$ which accepts the same language and show (3).

Since $\mathcal{A}$ satisfies Theorem 3.4(1), there exists some $s \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp$ such that every entry in s is ω or ∞. We can assume that s is of the form $s = \begin{pmatrix} I & & I_s \\ F_s & M & F \end{pmatrix}$ for some $F_s, I_s \subseteq Q$ and some M. Since $\alpha(s) \in \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$, we have $I \cap F_s \neq \emptyset$ and $I_s \cap F \neq \emptyset$.

To explain the idea, let us assume that s is of the form $s = t_1 e_2^\sharp t_3 e_4^\sharp t_5$ for some metatransitions $t_1, e_2, t_3, e_4, t_5 \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0$, i.e., there is no ω in t_1, e_2, t_3, e_4, t_5. Let $u_1, \ldots, u_5 \in \Sigma^*$ such that t_1, e_2, t_3, e_4, t_5 are induced by $\mu(u_1), \ldots, \mu(u_5)$ with I and F. We denote by $M_1, \ldots, M_5$ the matrices inside t_1, e_2, t_3, e_4, t_5.

Let ℓ be some extremely large multiple of $|\tilde{Q}|!$. We show that (3a) or (3b) is satisfied.

At first, we consider the output of $\mathcal{A}$ on words $u_1 u_2^{\ell k} u_3 u_4^{\ell} u_5$ for large, growing k. The output of $\mathcal{A}$ on such words for large k is mainly determined by $u_2^{\ell k}$. It should be clear that for large growing k, the output of $\mathcal{A}$ grows by $\ell \cdot \mathsf{mind}(M_2)$ per k, i.e., the growth rate is $\ell \cdot \mathsf{mind}(M_2)$ per k.

Similarly, the output of $\mathcal{A}$ on $u_1 u_2^{\ell} u_3 u_4^{\ell k} u_5$ for large, growing k has a growth rate of $\ell \cdot \mathsf{mind}(M_4)$ per k.

However, what happens for words $u_1 u_2^{\ell k} u_3 u_4^{\ell k} u_5$ for large, growing k. Assume the growth rate of the output of $\mathcal{A}$ on this sequence is $\ell \cdot (\mathsf{mind}(M_2) + \mathsf{mind}(M_4))$. Assume some extremely large k and consider some accepting path π for the word $u_1 u_2^{\ell k} u_3 u_4^{\ell k} u_5$. Assume that the weight of π yields $|\mathcal{A}|(u_1 u_2^{\ell k} u_3 u_4^{\ell k} u_5)$. We decompose π into $\pi_1, \ldots, \pi_5$ which correspond to $u_1, u_2^{\ell k}, u_3, u_4^{\ell k}, u_5$, and denote the first and last states of $\pi_1, \ldots, \pi_5$ by $i_0, \ldots, i_5$. For example π_2 starts in i_1, ends in i_2 and reads $u_2^{\ell k}$.

To achieve the growth rate of $\ell \cdot (\mathsf{mind}(M_2) + \mathsf{mind}(M_4))$ per k, $\mathcal{A}$ has to read almost every u_2 with a weight of $\mathsf{mind}(M_2)$, and has read almost every u_4 with a weight of $\mathsf{mind}(M_4)$. Hence, the paths π_2 and π_4 have to utilize the least entries on the main diagonal on M_2 and M_4, respectively.

We can factorize π_2 into ℓk factors such that each factor reads u_2. Let us denote by $r_0, \ldots, r_{\ell k}$ the first and last states of these factors, in particular, $i_1 = r_0$ and $r_{\ell k} = i_2$. Since, π_2 utilizes a least entry on the main diagonal of M_2, $r_0, \ldots, r_{\ell k}$ utilize a 0 on the main diagonal of the normalization $\bar{M}_2$, i.e., $r_0, \ldots, r_{\ell k}$ satisfy (S2). Hence, $M_2^\sharp[i_1, i_2] = \bar{M}_2^\sharp[i_1, i_2] \in \mathbb{Z}$. By the same argument, we obtain $M_4^\sharp[i_3, i_4] \in \mathbb{Z}$. Consequently, $s[i_0, i_5] \leq (M_1, M_2^\sharp, M_3, M_4^\sharp, M_5)[i_0, \ldots, i_5] \in \mathbb{Z}$, i.e., $s[i_0, i_5] \in \mathbb{Z}$ which contradicts the choice of s.

Consequently, the growth rate of the output of $\mathcal{A}$ on words $u_1 u_2^{\ell k} u_3 u_4^{\ell k} u_5$ for large, growing k is strictly larger than $\ell \cdot (\mathsf{mind}(M_2) + \mathsf{mind}(M_4))$.

Next, we analyze how $\tilde{\mathcal{A}}$ reads $u_1 u_2^{\ell} u_3 u_4^{\ell} u_5$. Let π be the unique accepting path of $u_1 u_2^{\ell} u_3 u_4^{\ell} u_5$ in $\tilde{\mathcal{A}}$. As above, we decompose π into $\pi_1, \ldots, \pi_5$ which correspond to $u_1, u_2^{\ell}, u_3, u_4^{\ell}, u_5$, and denote the first and last states of $\pi_1, \ldots, \pi_5$ by $i_0, \ldots, i_5$.

For the structure of π_2 and π_4, Lemma 2.2 is very helpful. Since ℓ is extremely larger than $|\tilde{Q}|$, π_2 consists mainly of a short cycle π_2' which is looped many times. Let n_2 be the number of u_2's which are read in this cycle. Let m_2 the weight of π_2' divided by n_2. The value m_2 can be understood as the relative cycle weight of π_2.

Now, we consider the output of $\tilde{\mathcal{A}}$ on words $u_1 u_2^{\ell k} u_3 u_4^{\ell} u_5$ for large, growing k. Since the factors u_2 are read in many looped π_2' cycles, the growth rate of the output of $\tilde{\mathcal{A}}$ is $\ell k m_2$ per k.

By applying the same argument on π_4 we obtain some m_4, and the growth rate of the output of $\tilde{\mathcal{A}}$ on words $u_1 u_2^{\ell} u_3 u_4^{\ell k} u_5$ for large, growing k is $\ell k m_4$ per k.

Since $\tilde{\mathcal{A}}$ is unambiguous, the growth rate of the output of $\tilde{\mathcal{A}}$ on words $u_1 u_2^{\ell k} u_3 u_4^{\ell k} u_5$ for large, growing k is $\ell k(m_2 + m_4)$ per k.

Now, at least one of the following three cases occurs:

- $km_2 > \mathsf{mind}(M_2)$ Then, on words $u_1 u_2^{\ell k} u_3 u_4^{\ell} u_5$ for growing k, the output of $\tilde{\mathcal{A}}$ grows faster than the output of $\mathcal{A}$. Hence, we have (3a) by using u_1, u_2^{ℓ}, $u_3 u_4^{\ell} u_5$ as u, v, w.
- $km_4 > \mathsf{mind}(M_4)$ Like the previous case.
- $km_2 \leq \mathsf{mind}(M_2)$ and $km_4 \leq \mathsf{mind}(M_4)$ We consider words $u_1 u_2^{\ell k} u_3 u_4^{\ell k} u_5$ for growing k. The growth rate of $\mathcal{A}$ on these words is strictly larger than $\ell \cdot (\mathsf{mind}(M_2) + \mathsf{mind}(M_4))$ per k, whereas the growth rate of $\tilde{\mathcal{A}}$ is less than $\ell k(m_2 + m_4)$ per k. Hence, we have (3b) by using $u_1 (u_2^{\ell})^{\sharp} u_3 (u_4^{\ell})^{\sharp} u_5$ as r.

Thus, we have shown (3) in the particular case that s is of the form $t_1 e_2^{\sharp} t_3 e_4^{\sharp} t_5$. It is straightforward to generalize this argument for s which are of the form $t_1 e_2^{\sharp} t_3 \ldots e_{n-1}^{\sharp} t_n$ for some n. However, this generalization is not sufficient. The real technical challenge is to prove (3) for some s which is generated by nesting stabilizations, e.g., if s is of the form $t_1 \big(e_2^{\sharp} t_3 e_4^{\sharp}\big)^{\sharp} t_5$ or if s is generated by arbitrarily many nested stabilizations.

To deal with these cases, we have to develop the same argumentation as above in a tree-like fashion. As above, we assume by Theorem 3.4(1) some $s \in \langle \mathsf{MT}(\mathbb{Z}_{\omega}, \mathcal{A}) \rangle_0^{\sharp}$ which is of the form $s = \begin{pmatrix} I & & I_s \\ F_s & M & F \end{pmatrix}$ whereas every entry in M belongs to $\{\omega, \infty\}$.

We define the notion of a $\sharp$-tree. Its nodes are labeled with triples (w, t, t') whereas $w \in \Sigma^*$, $t \in \langle \mathsf{MT}(\mathbb{Z}, \mathcal{A}) \rangle_0^{\sharp}$, and $t' \in \langle \mathsf{MT}(\mathbb{Z}, \mathcal{A}) \rangle_0$, satisfying $\alpha(t) = \alpha(t')$.

Let $k \geq 1$. We define now $\sharp$-trees of rank k. For every $a \in \Sigma$, every P-clone P and every R-clone R, there is a *$\sharp$-tree* which consists of a single node labeled with $\big(a, t, t\big)$, whereas $t = [\![P, \mu(a), R]\!]$.

Let T_1, T_2 be $\sharp$-trees and assume that their roots are labeled with (w_1, t_1, t_1') and (w_2, t_2, t_2'), respectively. If t_1 and t_2 are concatenable, then we construct a *$\sharp$-tree* as follows: its root is labeled with $(w_1 w_2,\ t_1 t_2,\ t_1' t_2')$. Its successors are T_1 and T_2.

Let T_1 be a $\sharp$-tree and assume that its root is labeled with (w_1, t_1, t_1'). If $t_1^{\sharp}$ is defined, then we construct another *$\sharp$-tree*: its root is labeled with $(w_1^k, t_1^{\sharp}, t_1'^k)$ and has k copies of T_1 as successors.

For every $t \in \langle \mathsf{MT}(\mathbb{Z}, \mathcal{A}) \rangle_0^{\sharp}$, there are some $w \in \Sigma^*$, $t' \in \langle \mathsf{MT}(\mathbb{Z}, \mathcal{A}) \rangle_0$, and a $\sharp$-tree whose root is labeled with (w, t, t').

Consequently, there are $w \in \Sigma^*$, $s' \in \langle \mathsf{MT}(\mathbb{Z}, \mathcal{A}) \rangle_0$, and a $\sharp$-tree whose root is labeled with (w, s, s'). We can naturally associate a $\sharp$-expression r to this $\sharp$-tree in a bottom-up-manner, and we have $r(k) = w$.

We can then prove various conditions in a bottom-up induction over the nodes of the $\sharp$-tree. The key argumentation is as follows: we assume that w does not admit a factorization into three words which prove (3a). Under this assumption, we can show that $|\mathcal{A}|(w)$ is much larger than $|\tilde{\mathcal{A}}|(w)$, and we can show in particular that r can be used to prove (3b).

5. Sufficiency

We show (2) $\Rightarrow$ (1) in Theorem 3.4 by contraposition. We assume a polynomially ambiguous min-plus automaton $\mathcal{A} = [Q, \mu, \lambda, \varrho]$ which does not satisfy (1), and we construct an equivalent unambiguous automaton. We assume that the entries of λ and ϱ are 0 or ∞.

The construction of an unambiguous equivalent relies on the following proposition:

Proposition 5.1. *Let $\mathcal{A}$ be a polynomially ambiguous min-plus automaton, and assume that $\mathcal{A}$ does not satisfy (1) in Theorem 3.4.*

There is some $Y \geq 0$ such that the following assertion is true:

For every $t = \begin{pmatrix} {}^P_R & M & {}^{P'}_{R'} \end{pmatrix} \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0$, there is some $t' = \begin{pmatrix} {}^P_R & M' & {}^{P'}_{R'} \end{pmatrix} \in \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp$ satisfying:

(A1): $\alpha(t) = \alpha(t')$

(A2): *For every $p \in P \cap R$, $q \in P' \cap R'$, satisfying $M[p,q] \neq \infty$ and*

$$M[p,q] \geq \mathsf{min}(M) + Y, \quad \text{we have} \quad M'[p,q] = \omega.$$

The proof of Proposition 5.1 leads us to an intriguing combination of two Burnside problems for metatransitions. The main proof of Proposition 5.1 utilizes an inductive argument via the factorization forest theorem for the homomorphism $\alpha : \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp \to \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$. The induction step for metatransitions with an idempotent structure leads us to another Burnside problem itself. To solve this inner Burnside problem, we consider subsemigroups $T_e = \langle \mathsf{MT}(\mathbb{Z}_\omega, \mathcal{A})\rangle_0^\sharp \cap \alpha^{-1}(e)$ for idempotents $e \in \langle \mathsf{MT}(\mathbb{B}, \mathcal{A})\rangle_0$. This inner Burnside problem is then shown by methods which are remotely related to techniques by Simon and Leung for the limitedness problem of distance automata [18, 23, 24].

To prove Proposition 5.1 by an induction via the factorization forest theorem, we have to add two more technical conditions to get a stronger inductive hypothesis.

One can deduce Y from the proof of Proposition 5.1. It is elementary but superexponential. Knowing Y is not required to show the decidability in Theorem 3.5.

We construct now an unambiguous equivalent $\mathcal{A}'$ of $\mathcal{A}$.

For every R-clone R satisfying $R \cap I \neq \emptyset$, we add an initial state (I, δ, R) to $\mathcal{A}'$, whereas δ is a $(0, \ldots, 0)$ tuple of dimension $I \cap R$.

Next, we construct for every state of $\mathcal{A}'$ the outgoing transitions and the follow state.

Let (P, δ, R) be some already constructed state of $\mathcal{A}'$. For every $a \in \Sigma$ and every R-clone R' satisfying $R = \alpha(\mu(a)) \cdot R'$, we add a transition and a state to $\mathcal{A}$ as follows:

(1) Let $t = \begin{pmatrix} {}^P_R & M & {}^{P'}_{R'} \end{pmatrix}$ be the metatransition induced by $\mu(a)$ with P and R'.
(2) Let $\hat{\delta} = \delta \cdot M$. Hence, $\hat{\delta}$ is a tuple of dimension $(P' \cap R')$.
(3) We normalize $\hat{\delta}$. For every $q \in P' \cap R'$, we set $\delta'[q] = \hat{\delta}[q] - \mathsf{min}(\hat{\delta})$.
(4) We introduce a transition from (P, δ, R) to (P', δ', R').
(5) The label and the weight of this transition are a and $\mathsf{min}(\hat{\delta})$, respectively.

In this way, we can construct the entire min-plus automaton $\mathcal{A}'$. At this point of the construction, the set of states might become infinite.

Some state (P, δ, R) is an accepting state if $R = F$. The accepting weight is 0.

Consider some word $w = a_1 \ldots a_n \in \Sigma^*$ which is accepted by $\mathcal{A}'$. Denote the states of the accepting path for w in $\mathcal{A}'$ by (P_i, δ_i, R_i) for $i \in \{0, \ldots, n\}$. In particular, $P_0 = I$ and $R_n = F$. For $i \in \{1, \ldots, n\}$, denote by m_i the transition weight of the i-th transition of π.

Let $1 \leq i \leq n$. By an induction on i, we can show that for every $q \in P_i \cap R_i$, the sum $m_1 + \cdots + m_i + \delta_i[q]$ is exactly $(I \cdot \mu(a_1 \ldots a_i))[q]$. The sum $m_1 + \cdots + m_n$ is then the minimum of $(I \cdot \mu(w))[q]$ for $q \in F$, i.e., the sum $m_1 + \cdots + m_n$ is $\lambda\mu(w)\varrho = \mathcal{A}(w)$.

Conversely, consider some word $w = a_1 \ldots a_n \in \Sigma^*$ which is accepted by $\mathcal{A}$. We can construct an accepting path for w in $\mathcal{A}'$ as follows. Let $t_1, \ldots, t_n$ be the metatransitions induced by $\mu(a_1), \ldots, \mu(a_n)$ with I and F. Denote $t_i = \left({}^{P_{i-1}}_{R_{i-1}} \; M_i \; {}^{P_i}_{R_i} \right)$. The state (P_0, δ, R_0) (whereas δ is the $(0, \ldots, 0)$ tuple of dimension $P_0 \cap R_0$) is the first state of the constructed path. Then, we proceed along the above steps (1) to (5) for each a_i and each R_i for $i \in \{1, \ldots, n\}$ and obtain an accepting path for w in $\mathcal{A}'$. We can apply the above argumentation to show that the sum of the transition weights is exactly $\lambda\mu(w)\varrho = \mathcal{A}(w)$.

Consequently, $|\mathcal{A}|$ and $|\mathcal{A}'|$ are equivalent, and it is easy to verify that $\mathcal{A}'$ is unambiguous. However, a major problem remained: we cannot show that $\mathcal{A}'$ has finitely many states. We overcome this problem by changing step (3) in the construction above as follows:

(3') We normalize $\hat{\delta}$. For every $q \in P' \cap R'$, we set $\delta''[q] = \hat{\delta}[q] - \mathsf{min}(\hat{\delta})$. Then, we construct δ' by replacing in δ'' every non-∞ entry which is larger than $2Y$ by ω.

By using (3') instead of (3), the set of states of $\mathcal{A}'$ will be finite. We have to show that the construction of $\mathcal{A}'$ is still correct, that is that every entry that becomes too large can be replaced by ω.

Let $u_1, u_2 \in \Sigma^*$ and assume that $\mathcal{A}$ accepts u_1u_2. Let $I = \{\, q \in Q \mid \lambda[q] \in \mathbb{Z} \,\}$ and $F = \{\, q \in Q \mid \varrho[q] \in \mathbb{Z} \,\}$. We denote $t_1 = [\![I, \mu(u_1), \alpha(\mu(u_2)) \cdot F]\!] = \left({}^{I}_{R_0} \; M_1 \; {}^{P_1}_{R_1} \right)$ and $t_2 = [\![I \cdot \alpha(\mu(u_1)), \mu(u_2), F]\!] = \left({}^{P_1}_{R_1} \; M_2 \; {}^{P_2}_{F} \right)$. Then, $t_1t_2 = \left({}^{I}_{R_0} \; M_1M_2 \; {}^{P_2}_{F} \right)$, and moreover, $|\mathcal{A}|(u_1u_2)$ is the least entry in M_1M_2, i.e., $|\mathcal{A}|(u_1u_2) = \mathsf{min}(M_1M_2)$.

Let $p_0 \in I \cap R_0$, let $p_1 \in P_1 \cap R_1$, and let $p_2 \in P_2 \cap F$. Assume $(M_1, M_2)[p_0, p_1, p_2] \in \mathbb{Z}$.

Moreover, assume that $M_1[p_0, p_1] \geq \mathsf{min}(M_1) + 2Y$, (the Y from Proposition 5.1) but nevertheless $(M_1, M_2)[p_0, p_1, p_2] = \mathsf{min}(M_1M_2)$. Intuitively, the path along p_0, p_1, p_2 has after reading u_1 from p_0 to p_1 a very large weight (in comparison to the path which has a weight of $\mathsf{min}(M_1)$), but nevertheless, by reading u_2 from p_1 to p_2 the weight of the path becomes smaller and smaller and finally the path has a weight of $\mathsf{min}(M_1M_2)$, i.e., it is the path with the least weight.

Let $q_0 \in I \cap R_0$, let $q_1 \in P_1 \cap R_1$, and let $q_2 \in P_2 \cap F$. Assume $(M_1, M_2)[q_0, q_1, q_2] \in \mathbb{Z}$.

We have $(M_1, M_2)[q_0, q_1, q_2] \geq \mathsf{min}(M_1M_2) = (M_1, M_2)[p_0, p_1, p_2]$. Hence, we have $M_1[q_0, q_1] \geq M_1[p_0, p_1] - Y$ or $M_2[q_1, q_2] \geq M_2[p_1, p_2] + Y \geq \mathsf{min}(M_2) + Y$. However, $M_1[q_0, q_1] \geq M_1[p_0, p_1] - Y$ implies $M_1[q_0, q_1] \geq \mathsf{min}(M_1) + Y$ (by the above assumption on $M_1[p_0, p_1]$). Consequently, we have $M_1[q_0, q_1] \geq \mathsf{min}(M_1) + Y$ or $M_2[q_1, q_2] \geq \mathsf{min}(M_2) + Y$.

Now, let t_1' and t_2' be the matrices which exist by Proposition 5.1. By (A2), we have $M_1'[q_0, q_1] = \omega$ or $M_2'[q_1, q_2] = \omega$ whereas M_1' resp. M_2' are the matrices in t_1' resp. t_2'.

Since this argumentation holds for every q_0, q_1, q_2 (in particular for p_0, p_1, p_2) every entry of $t_1't_2'$ is ω or ∞, i.e., $t_1't_2'$ shows that (1) in Theorem 3.4 is satisfied, which is a contradiction.

Consequently, the above assumed p_0, p_1, p_2 cannot exist.

Let $w \in \Sigma^*$, and let π be an accepting path. Assume the weight of π is $|\mathcal{A}|(w)$. By the above observation, π can intermediately not have a much larger (i.e. $2Y$ larger) weight than another accepting path.

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