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Decision Aiding

The evidential reasoning approach for MADA under both probabilistic and fuzzy uncertainties

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Abstract

Many multiple attribute decision analysis (MADA) problems are characterised by both quantitative and qualitative attributes with various types of uncertainties. Incompleteness (or ignorance) and vagueness (or fuzziness) are among the most common uncertainties in decision analysis. The evidential reasoning (ER) approach has been developed in the 1990s and in the recent years to support the solution of MADA problems with ignorance, a kind of probabilistic uncertainty. In this paper, the ER approach is further developed to deal with MADA problems with both probabilistic and fuzzy uncertainties.

In this newly developed ER approach, precise data, ignorance and fuzziness are all modelled under the unified framework of a distributed fuzzy belief structure, leading to a fuzzy belief decision matrix. A utility-based grade match method is proposed to transform both numerical data and qualitative (fuzzy) assessment information of various formats into the fuzzy belief structure. A new fuzzy ER algorithm is developed to aggregate multiple attributes using the information contained in the fuzzy belief matrix, resulting in an aggregated fuzzy distributed assessment for each alternative. Different from the existing ER algorithm that is of a recursive nature, the new fuzzy ER algorithm provides an analytical means for combining all attributes without iteration, thus providing scope and flexibility for sensitivity analysis and optimisation. A numerical example is provided to illustrate the detailed implementation process of the new ER approach and its validity and wide applicability.

Keywords: Multiple attribute decision analysis; Uncertainty modelling; The evidential reasoning approach; Utility; Fuzzy sets; Fuzzy ranking; Product selection

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1. Introduction

Many real world multiple attribute decision analysis (MADA) problems are characterised with both quantitative and qualitative attributes. For instance, the design evaluation of an engineering product may require the simultaneous consideration of several attributes such as cost, quality, safety, reliability, maintainability and environmental impact; in selection of its suppliers, an organisation needs to take account of such attributes as quality, technical capability, supply chain management, financial soundness, environmental, ethical, health and safety standards, and general factors. Most of such attributes are qualitative and could only be properly assessed using human judgments, which are subjective in nature and are inevitably associated with uncertainties caused due to the human being's inability to provide complete judgments, or the lack of information, or the vagueness of the meanings about attributes and their assessments. Such uncertainties can be classified into two main classes: ignorance (incompleteness) and fuzziness (vagueness). The Dempster–Shafer (D–S) theory of evidence (Dempster, 1967; Shafer, 1976) provides an appropriate framework to model ignorance whilst fuzziness can be well treated using fuzzy set theory (Zadeh, 1975, 1978).

Although the D–S theory was not originally proposed in relation to artificial intelligence (AI), it has found wide applications in AI and expert systems over the past two decades (Yager et al., 1994; Wallery, 1996; Anand et al., 1996; Benferhat et al., 2000; Denoeux, 1997, 2000a; Guan and Bell, 1997; Bryson and Mobolurin, 1999; Yager, 1999; Hullermeier, 2001; He et al., 2001; Davis and Hall, 2003). In decision analysis under uncertainty, the D–S theory has been mostly used as an alternative approach to Bayes decision theory (Yager et al., 1994). It was not until 1994 that the D–S theory was first combined with a distributed modelling framework to develop the Evidential Reasoning (ER) approach for dealing with MADA problems with probabilistic uncertainty (Yang and Singh, 1994; Yang and Sen, 1994a,b). In recent years, there have been several other attempts to use the D–S theory of evidence for MADA from other perspectives (Chen, 1997; Bauer, 1997; Beynon et al., 2000, 2001).

Different from most conventional MADA methods, the ER approach describes each attribute at an alternative by a distributed assessment using a belief structure. The main advantage of doing so is that both precise data and subjective judgments with uncertainty can be consistently modelled under the unified framework. The ER approach provides a novel procedure for aggregating multiple attributes based on the distributed assessment framework and the evidence combination rule of the D–S theory. It has since been applied to MADA problems in engineering design assessment (Yang et al., 1994, 1997, 1998, 2001a,b), system safety analysis and synthesis (Wang et al., 1995, 1996), software safety analysis (Wang, 1997; Wang and Yang, 2001), organisational self-assessment (Yang et al., 2001a,b; Siow et al., 2001), contractor selection (Sonmez et al., 2001, 2002), and so on.

Extensive research dedicated to the ER approach has been conducted in recent years. Firstly, the rule and utility-based information transformation techniques were proposed within the ER modelling framework (Yang, 2001). This work enables the ER approach to deal with a wide range of MADA problems having precise data, random numbers and subjective judgments with probabilistic uncertainty in a way that is rational, transparent, reliable, systematic and consistent. Then, the in-depth research into the ER algorithm has been conducted by treating the unassigned belief degree in two parts, one caused due to the incompleteness and the other caused due to the fact that each attribute plays only one part in the whole assessment process because of its relative weight (Yang and Xu, 2002a). This work leads to a rigorous yet pragmatic ER algorithm that satisfies several common sense rules governing any approximate reasoning based aggregation procedures. The ER approach has thus been equipped with the desirable capability of generating the upper and lower bounds of the degree of belief for any incomplete assessment, which are crucial to measure the degree of ignorance. Thirdly, the decision analysis process of the ER approach was fully investigated, which reveals the nonlinear features of the ER aggregation process (Yang and Xu, 2002b). This work provides guidance on conducting sensitivity analysis using the ER approach. Last but not least, a window-based and graphically designed intelligent decision system (IDS) has been developed to implement the ER approach, which provides a flexible and easy to use interface for modelling, analysis and reporting. The ER approach and the IDS software have already been successfully applied to deal with a wide range of MADA problems as mentioned earlier.

The current ER approach does not take account of vagueness or fuzzy uncertainty. In many decision problems with qualitative attributes, however, it may be difficult to define assessment grades as independent crisp sets. It would be more natural to define assessment grades using subjective and vague linguistic terms, which may overlap in their meanings. For example, the assessment grades "*good*" and "*very good*" are difficult to be expressed as clearly distinctive crisp sets, but quite natural to be defined as two dependent fuzzy sets. In other words, the intersection of the two fuzzy sets may not be empty. Fuzzy assessment approaches have been widely researched and developed for decision analysis under fuzzy uncertainty (Carlsson and Fuller, 1996; Ribeiro, 1996; Liang, 1999; Yeh et al., 2000; Chen, 2001). Nevertheless, pure fuzzy MADA approaches are unable to handle probabilistic uncertainties such as ignorance as modelled in the belief structure. As such, there is a clear need to combine the D–S theory and the fuzzy set theory for handling both types of uncertainties. Indeed, there have been several attempts to generalize the D–S theory of evidence to fuzzy sets (Ishizuka et al., 1982; Yager, 1982, 1996, 2002; Yen, 1990; Lucas and Araabi, 1999; Denoeux, 1999, 2000b). However, these efforts were mainly focused on the normalization of fuzzy belief structures and the other theoretical issues related to the combination of evidence under fuzzy environments. In fact, none of these efforts was directed to deal with MADA problems with both probabilistic and fuzzy uncertainties. This research has been conducted to fill the gap.

In this paper, the ER approach will be further developed to take into account fuzzy assessment grades, resulting in a new ER approach for MADA under both probabilistic and fuzzy uncertainties. In particular, a distributed fuzzy belief structure will be constructed to model precise data, ignorance and fuzziness under the unified framework, leading to a fuzzy belief decision matrix. A utility-based grade match method will be proposed to transform both numerical data and qualitative (fuzzy) assessment information of various formats into the fuzzy belief structure. A new fuzzy ER algorithm will be developed to aggregate multiple attributes using the information contained in the matrix, resulting in an aggregated fuzzy distributed assessment for each alternative. Different from the existing ER algorithm that is of a recursive nature, the new fuzzy ER algorithm provides an analytical means for combining all attributes in one go without iteration, thus providing scope and flexibility for sensitivity analysis and optimisation. A numerical example will be examined to demonstrate the detailed implementation process of the new ER approach and its validity and wide applicability.

The paper is organized as follows. Section 2 provides a brief description of the D–S theory and the existing ER approach. In Section 3, the new ER approach for MADA under both probabilistic and fuzzy uncertainties will be fully investigated, including the development of the fuzzy ER algorithm, the grade match method for fuzzy information transformation, and the computational formula for generating fuzzy expected utilities. Section 4 presents the investigation of a car selection problem using fuzzy and complete assessment information to show the detailed implementation process of the new ER approach and its validity and wide applicability. The paper is concluded in Section 5. The derivation of the fuzzy ER algorithm and the other formulae is provided in Appendices A and B.

2. The ER approach for MADA under probabilistic uncertainty

The ER approach is characterised by a distributed modelling framework capable of modelling both precise data and ignorance, by an evidential reasoning algorithm for aggregating both complete and incomplete information and by the interval utility for characterising incomplete assessments and for ranking alternatives. The current version of the ER approach can be used to deal with MADA problems with probabilistic uncertainty, as briefly introduced in this section. Before the introduction, some basic concepts of the Dempster–Shafer theory of evidence are discussed.

2.1. Basics of the evidence theory

The evidence theory was first developed by Dempster (1967) in the 1960s and later extended and refined by Shafer (1976) in the 1970s. The evidence theory is related to Bayesian probability theory in the sense that they both can update subjective beliefs given new evidence. The major difference between the two theories is that the evidence theory is capable of combining evidence and dealing with ignorance in the evidence combination process. The basic concepts and definitions of the evidence theory relevant to this paper are briefly described as follows.

Let $\Theta = \{H_1, \ldots, H_N\}$ be a collectively exhaustive and mutually exclusive set of hypotheses, called the frame of discernment. A basic probability assignment (bpa) is a function $m: 2^\Theta \to [0,1]$, called a mass function and satisfying

$$m(\Phi) = 0 \quad \text{and} \quad \sum_{A \subseteq \Theta} m(A) = 1,$$

where Φ is an empty set, A is any subset of Θ, and 2^Θ is the power set of Θ, which consists of all the subsets of Θ, i.e. $2^\Theta = \{\Phi, \{H_1\}, \ldots, \{H_N\}, \{H_1 \cup H_2\}, \ldots, \{H_1 \cup H_N\}, \ldots, \Theta\}$. The assigned probability (also called probability mass) $m(A)$ measures the belief exactly assigned to A and represents how strongly the evidence supports A. All assigned probabilities sum to unity and there is no belief in the empty set Φ. The probability assigned to Θ, i.e. $m(\Theta)$, is called the degree of ignorance. Each subset $A \subseteq \Theta$ such that $m(A) > 0$ is called a focal element of m. All the related focal elements are collectively called the body of evidence.

Associated with each bpa are a belief measure (*Bel*) and a plausibility measure (*Pl*) which are both functions: $2^\Theta \to [0, 1]$, defined by the following equations, respectively:

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad \text{and} \quad Pl(A) = \sum_{A \cap B \neq \Phi} m(B),$$

where A and B are subsets of Θ. $Bel(A)$ represents the exact support to A, i.e. the belief of the hypothesis A being true; $Pl(A)$ represents the possible support to A, i.e. the total amount of belief that could be potentially placed in A. $[Bel(A), Pl(A)]$ constitutes the interval of support to A and can be seen as the lower and upper bounds of the probability to which A is supported. The two functions can be connected by the equation

$$Pl(A) = 1 - Bel(\overline{A}),$$

where $\overline{A}$ denotes the complement of A. The difference between the belief and the plausibility of a set A describes the ignorance of the assessment for the set A (Shafer, 1976).

Since $m(A)$, $Bel(A)$ and $Pl(A)$ are in one-to-one correspondence, they can be seen as three facets of the same piece of information. There are several other functions such as commonality function, doubt function, and so on, which can also be used to represent evidence. They all represent the same information and provide flexibility in a variety of reasoning applications.

The kernel of the evidence theory is the Dempster's rule of combination by which the evidence from different sources is combined. The rule assumes that the information sources are independent and use the orthogonal sum to combine multiple belief structures $m = m_1 \oplus m_2 \oplus \cdots \oplus m_K$, where $\oplus$ represents the operator of combination. With two belief structures m_1 and m_2, the Dempster's rule of combination is defined as follows:

$$[m_1 \oplus m_2](C) = \begin{cases} 0, & C = \Phi, \\ \frac{\sum_{A\cap B=C} m_1(A)m_2(B)}{1-\sum_{A\cap B=\Phi} m_1(A)m_2(B)}, & C \neq \Phi, \end{cases}$$

where A and B are both focal elements and $[m_1 \oplus m_2](C)$ itself is a bpa. The denominator, $1 - \sum_{A\cap B=\Phi} m_1(A)m_2(B)$ is called the normalization factor, $\sum_{A\cap B=\Phi} m_1(A)m_2(B)$ is called the degree of conflict, which measures the conflict between the pieces of evidence. Several researchers have investigated the combination of the evidence theory and fuzzy sets. In this paper, the principle investigated by Yen (1990) is used to develop a new algorithm as discussed in Section 3.

Note that the crude application of the D–S theory and the combination rule can lead to irrational conclusions in the aggregation of multiple pieces of evidence in conflict (Murphy, 2000). This issue is addressed in the ER approach introduced in the next section by generating basic probability assignment through the combination of belief degrees and normalised weights and by normalising the combined probability masses.

2.2. The ER distributed modelling framework—the belief structure

The evidence theory was first introduced to deal with MADA problem under uncertainty in the early 1990s (Yang and Singh, 1994; Yang and Sen, 1994a,b) by designing a novel belief decision matrix to model a MADA problem and creating a unique attribute aggregation process based on Dempster's evidence combination rule. Suppose a MADA problem has M alternatives a_l ($l = 1, \ldots, M$), one upper level attribute, referred to as a *general* attribute, and L lower level attributes e_i ($i = 1, \ldots, L$), called *basic* attributes. Suppose the relative weights of the L basic attributes are given and denoted by $w = (w_1, \ldots, w_L)$, which are normalised to satisfy the following condition:

$$0 \leqslant w_i \leqslant 1 \quad \text{and} \quad \sum_{i=1}^{L} w_i = 1. \tag{1}$$

In most conventional MADA methods, a score (value or utility) is used to assess an alternative on an attribute and a MADA problem is modelled by a decision matrix. It is argued that a single score can only represent the average performance but not the true diverse nature of a subjective assessment. In many decision situations, however, a human judgement may need to be used but may not be modelled using a precise number without pre-aggregating various types of information, which often is a complicated process and can hardly be consistent, reliable or systematic. MADA should be aimed at not only generating average scores for ranking alternatives but also identifying the strengths and weaknesses of alternatives. A belief structure as described in this section is therefore designed to capture the performance distribution of a qualitative assessment. The kernel of the belief structure is the use of focal values or evaluation grades to which an alternative can be assessed.

Suppose the M alternatives are assessed at the L attributes on the basis of N common crisp evaluation grades H_n ($n = 1, \ldots, N$), which are required to be mutually exclusive and collectively exhaustive. If alternative a_l is assessed to a grade H_n on an attribute e_i with a degree of belief of $\beta_{n,i}(a_l)$, we denote this assessment by $S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)), n = 1, \ldots, N\}$, which is a distributed assessment and referred to as a belief structure, where $\beta_{n,i}(a_l) \geqslant 0$ and $\sum_{n=1}^{N} \beta_{n,i}(a_l) \leqslant 1$. If $\sum_{n=1}^{N} \beta_{n,i}(a_l) = 1$, the assessment is said to be complete; otherwise, it is incomplete. Note that $\sum_{n=1}^{N} \beta_{n,i}(a_l) = 0$ denotes total ignorance about the assessment of a_l on e_i. For example, if the quality of an engine is assessed to "*good*" with a belief degree of 70% and to "*excellent*" with a degree of belief of 30%, this assessment will be complete; if it is assessed to 50% "*good*" and to 30% "*excellent*", then the assessment will be incomplete with a degree of incompleteness of 20%. The individual assessments of all alternatives on each basic attribute are represented by a belief decision matrix, defined as follows:

$$D_g = (S(e_i(a_l)))_{L\times M}. \tag{2}$$

2.3. The recursive ER algorithm

One simple approach for attribute aggregation is to transform a belief structure into a single score and then aggregate attributes on the basis of the scores using traditional methods such as the additive utility function approach. However, such transformation would hide performance diversity shown in a distribution assessment, leading to the possible failure of identifying strengths and weaknesses of an alternative on higher level attributes. In the ER approach, attribute aggregation is based on evidential reasoning rather than directly manipulating (e.g. adding) scores. In other words, the given assessments of alternatives on the individual basic attributes are treated as evidence and which evaluation grades the general attribute should be assessed to is treated as hypotheses (Yang and Singh, 1994; Yang and Sen, 1994a,b). Dempster's evidence combination rule is then employed and revised to create a novel process for such attribute aggregation (Yang, 2001; Yang and Xu, 2002a). The revision of the rule is necessary due to the need to handle conflicting evidence and follow common sense rules for attribute aggregation in MADA. The detailed analysis and the rationale on the development of the attribute aggregation process can be found in the references (Yang, 2001; Yang and Xu, 2002a). The process is briefly described as the following steps.

First of all, a degree of belief given to an assessment grade H_n for an alternative a_l on an attribute e_i is transformed into a basic probability mass by multiplying the given degree of belief by the relative weight of the attribute using the following equations:

$$m_{n,i} = m_i(H_n) = w_i\beta_{n,i}(a_l), \quad n = 1,\dots,N;\ i = 1,\dots,L, \tag{3}$$

$$m_{H,i} = m_i(H) = 1 - \sum_{n=1}^{N} m_{n,i} = 1 - w_i\sum_{n=1}^{N}\beta_{n,i}(a_l), \quad i = 1,\dots,L, \tag{4}$$

$$\bar{m}_{H,i} = \bar{m}_i(H) = 1 - w_i, \quad i = 1,\dots,L, \tag{5}$$

$$\tilde{m}_{H,i} = \tilde{m}_i(H) = w_i\Big(1 - \sum_{n=1}^{N}\beta_{n,i}(a_l)\Big), \quad i = 1,\dots,L, \tag{6}$$

with $m_{H,i} = \bar{m}_{H,i} + \tilde{m}_{H,i}$ and $\sum_{i=1}^{L} w_i = 1$.

Note that the probability mass assigned to the whole set $H, m_{H,i}$ which is currently unassigned to any individual grades, is split into two parts: $\bar{m}_{H,i}$ and $\tilde{m}_{H,i}$, where $\bar{m}_{H,i}$ is caused by the relative importance of the attribute e_i and $\tilde{m}_{H,i}$ by the incompleteness of the assessment on e_i for a_l.

The second step is to aggregate the attributes by combining the basic probability masses generated above, or reasoning based on the given evidence (Yang and Singh, 1994). Due to the assumptions that the evaluation grades are mutually exclusive and collectively exhaustive and that assessments on a basic attribute are independent of assessments on other attributes, or utility independence among attributes (Keeney and Raiffa, 1993), the Dempster's combination rule can be directly applied to combine the basic probability masses in a recursive fashion. In the belief decision matrix framework, the combination process can be developed into the following recursive ER algorithm (Yang, 2001; Yang and Xu, 2002a):

$$\{H_n\}: m_{n,I(i+1)} = K_{I(i+1)}[m_{n,I(i)}m_{n,i+1} + m_{n,I(i)}m_{H,i+1} + m_{H,I(i)}m_{n,i+1}], \tag{7}$$

$$m_{H,I(i)} = \bar{m}_{H,I(i)} + \tilde{m}_{H,I(i)}, \quad n = 1, \ldots, N,$$

$$\{H\}: \tilde{m}_{H,I(i+1)} = K_{I(i+1)}[\tilde{m}_{H,I(i)}\tilde{m}_{H,i+1} + \tilde{m}_{H,I(i)}\bar{m}_{H,i+1} + \bar{m}_{H,I(i)}\tilde{m}_{H,i+1}], \tag{8}$$

$$\{H\}: \bar{m}_{H,I(i+1)} = K_{I(i+1)}[\bar{m}_{H,I(i)}\bar{m}_{H,i+1}], \tag{9}$$

$$K_{I(i+1)} = \left[1 - \sum_{n=1}^{N}\sum_{\substack{t=1\\t\neq n}}^{N} m_{n,I(i)}m_{t,i+1}\right]^{-1}, \quad i = 1, \ldots, L-1, \tag{10}$$

$$\{H_n\}: \beta_n = \frac{m_{n,I(L)}}{1 - \bar{m}_{H,I(L)}}, \quad n = 1, \ldots, N, \tag{11}$$

$$\{H\}: \beta_H = \frac{\tilde{m}_{H,I(L)}}{1 - \bar{m}_{H,I(L)}}. \tag{12}$$

In the above equations, $m_{n,I(i)}$ denotes the combined probability mass generated by aggregating i attributes; $m_{n,I(i)}m_{n,i+1}$ measures the relative support to the hypothesis that the general attribute should be assessed to the grade H_n by both the first i attributes and the $(i+1)$th attribute; $m_{n,I(i)}m_{H,i+1}$ measures the relative support to the hypothesis by the first i attributes only; $m_{H,I(i)}m_{n,i+1}$ measures the relative support to the hypothesis by the $(i+1)$th attribute only. It is assumed in the above equations that $m_{n,I(1)} = m_{n,1}$ $(n = 1,\ldots, N)$, $m_{H,I(1)} = m_{H,1}$, $\bar{m}_{H,I(1)} = \bar{m}_{H,1}$ and $\tilde{m}_{H,I(1)} = \tilde{m}_{H,1}$. Note that the aggregation process does not depend on the order in which attributes are combined.

β_n and β_H represent the belief degrees of the aggregated assessment, to which the general attribute is assessed to the grades H_n and H, respectively. The combined assessment can be denoted by $S(y(a_l)) = \{(H_n, \beta_n(a_l)), n = 1, \ldots, N\}$. It has been proved that $\sum_{n=1}^{N}\beta_n + \beta_H = 1$ (Yang and Xu, 2002a). Yang and Xu also put forward four axioms and proved the rationality and validity of the above recursive ER algorithm. The nonlinear features of the above aggregation process were also investigated in detail (Yang and Xu, 2002b). In the above ER algorithm, Eqs. (7)–(10) are the direct implementation of the Dempster's evidence combination rule within the belief decision matrix; the weigh normalisation process shown in Eq. (1), the assignment of the basic probability masses shown in Eqs. (3)–(6) and the normalisation of the combined probability masses shown in Eqs. (11) and (12) are developed to ensure that the ER algorithm can process conflicting evidence rationally and satisfy common sense rules for attribute aggregation in MADA (Yang and Xu, 2002a).

2.4. The utility interval based ER ranking method

The above ER algorithm allows each attribute to have its own set of evaluation grades. Before the aggregation, however, all different sets of evaluation grades have to be transformed into a unified set of assessment grades using either the rule or utility-based equivalence transformation techniques. The interested reader may refer to the paper by Yang (2001).

In order to compare M alternatives at the presence of incomplete assessments, maximum, minimum and average utilities are introduced and used to rank them. Suppose the utility of an evaluation grade H_n is $u(H_n)$, then the expected utility of the aggregated assessment $S(y(a_l))$ is defined as follows:

$$u(S(y(a_l))) = \sum_{n=1}^{N}\beta_n(a_l)u(H_n). \tag{13}$$

The belief degree $\beta_n(a_l)$ stands for the lower bound of the likelihood that a_l is assessed to H_n, whilst the corresponding upper bound of the likelihood is given by $(\beta_n(a_l) + \beta_H(a_l))$ (Yang, 2001; Yang and Xu, 2002a), which leads to the establishment of a utility interval if the assessment is incomplete. Without loss of generality, suppose the least preferred assessment grade having the lowest utility is H_1 and the most preferred assessment grade having the highest utility is H_N. Then the maximum, minimum and average utilities of a_l can be calculated by

$$u_{\max}(a_l) = \sum_{n=1}^{N-1} \beta_n(a_l)u(H_n) + (\beta_N(a_l) + \beta_H(a_l))u(H_N), \tag{14}$$

$$u_{\min}(a_l) = (\beta_1(a_l) + \beta_H(a_l))u(H_1) + \sum_{n=2}^{N} \beta_n(a_l)u(H_n), \tag{15}$$

$$u_{\text{avg}}(a_l) = \frac{u_{\max}(a_l) + u_{\min}(a_l)}{2}. \tag{16}$$

It is obvious that if $u(H_1) = 0$ then $u(S(y(a_l))) = u_{\min}(a_l)$; if original assessments $S(e_i(a_l))$ in the belief decision matrix are all complete, then $\beta_H(a_l) = 0$ and $u(S(y(a_l))) = u_{\min}(a_l) = u_{\max}(a_l) = u_{\text{avg}}(a_l)$. It has to be made clear that the above utilities are only used for characterizing a distributed assessment but not for attribute aggregation. According to the maximum, minimum utilities and the corresponding utility interval, the ranking of two alternatives can be made as follows. If $u_{\min}(a_l) \geqslant u_{\max}(a_k)$, a_l is said to be preferred to a_k; if $u_{\min}(a_l) = u_{\min}(a_k)$ and $u_{\max}(a_l) = u_{\max}(a_k)$, a_l is said to be indifferent to a_k. In other cases, average utility may be used to generate an average ranking, but this kind of ranking may be inconclusive and unreliable. To produce a reliable ranking, the quality of original assessments must be improved by reducing imprecision or incompleteness present in the original information associated with a_l and a_k.

3. The ER approach for MADA under both probabilistic and fuzzy uncertainties

In the ER approach introduced above, the assessment grades are assumed to be crisp and independent of each other. In many situations, however, an assessment grade may represent a vague or fuzzy concept or standard and there may be no clear cut between the meanings of two adjacent grades. In this paper, we will drop the above assumption and allow the grades to be fuzzy and dependent. To simplify the discussion and without loss of generality, fuzzy sets will be used to characterise such assessment grades and it is assumed that only two adjacent fuzzy grades have the overlap of meanings. This represents the most common features of fuzzy uncertainty in decision analysis. Note that the principle of the following method can be extended to more general cases.

3.1. The new ER distributed modelling framework the fuzzy belief structure

Suppose a general set of fuzzy assessment grades $\{H_n\}$ $(n = 1, \ldots, N)$ are dependent on each other, which may be either triangular or trapezoidal fuzzy sets or their combinations. Assuming that only two adjacent fuzzy assessment grades may intersect, we denote by $H_{n,n+1}$ the fuzzy intersection subset of the two adjacent fuzzy assessment grades H_n and H_{n+1} (see Fig. 1).

Since fuzzy assessment grades and belief degrees are used, then $S(e_i(a_l))$ as defined in Section 2.1 contains both fuzzy sets (grades) and belief degrees. The former can model fuzziness or vagueness and the latter incompleteness or ignorance. As such, $S(e_i(a_l))$ is referred to as fuzzy belief structure in this paper.

3.2. The new ER algorithm for MADA under both probabilistic and fuzzy uncertainties

In the derivation of Eqs. (7)–(10), it was assumed that the evaluation grades are independent of each other. Due to the dependency of the adjacent fuzzy assessment grades on each other as shown in Fig. 1, Eqs. (7)–(10) can no longer be employed without modification to aggregate attributes assessed using such fuzzy grades. However, the evidence theory provides scope to deal such fuzzy assessments. The ideas similar to those used to develop the non-fuzzy evidential reasoning algorithm (Yang and Singh, 1994; Yang and Sen, 1994a,b) are used to deduce the following fuzzy evidential reasoning algorithm. The new challenge is that the intersection of two adjacent evaluation grades H_n and H_{n+1} is $H_{n,n+1}$, which is not empty as shown in Fig. 1. Another difference is that the normalisation has to be conducted after all pieces of evidence have been combined in order to preserve the property that the generated belief and plausibility functions still represent the lower and upper bounds of the combined degrees of belief (Yen, 1990).

In this subsection, a fuzzy ER algorithm will be developed using the similar technique used in Yang and Singh (1994). Following the assumptions on the fuzzy assessment grades made in the previous subsection, based on the belief decision matrix as shown in Eq. (2) and the basic probability masses generated using Eqs. (3)–(6), it is proven in Appendix A that the following analytical (non-recursive) fuzzy ER algorithm can be used to aggregate the L basic attributes for alternative a_l $(l = 1, \ldots, M)$:

$$\{H_n\}: \quad m(H_n) = k\left\{\prod_{i=1}^{L}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{L} m_i(H)\right\}, \quad n = 1, \ldots, N, \tag{17}$$

$$\begin{aligned}\{\overline{H}_{n,n+1}\}: \quad m(\overline{H}_{n,n+1}) = k\mu_{H_{n,n+1}}^{\max}\Bigg\{&\prod_{i=1}^{L}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_n) + m_i(H)] \\ &- \prod_{i=1}^{L}[m_i(H_{n+1}) + m_i(H)] + \prod_{i=1}^{L} m_i(H)\Bigg\}, \quad n = 1, \ldots, N-1,\end{aligned} \tag{18}$$

$$\{H\}: \quad \tilde{m}(H) = k\left\{\prod_{i=1}^{L} m_i(H) - \prod_{i=1}^{L} \bar{m}_i(H)\right\}, \tag{19}$$

$$\{H\}: \quad \bar{m}(H) = k\left[\prod_{i=1}^{L} \bar{m}_i(H)\right],$$

$$\begin{aligned}k = \Bigg\{&\sum_{n=1}^{N-1}\left(1 - \mu_{H_{n,n+1}}^{\max}\right)\left(\prod_{i=1}^{L}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{L} m_i(H)\right) \\ &+ \sum_{n=1}^{N-1}\mu_{H_{n,n+1}}^{\max}\left(\prod_{i=1}^{L}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_{n+1}) + m_i(H)]\right) \\ &+ \prod_{i=1}^{L}[m_i(H_N) + m_i(H)]\Bigg\}^{-1},\end{aligned} \tag{20}$$

$$\{H_n\}: \quad \beta_n = \frac{m(H_n)}{1 - \bar{m}(H)}, \quad n = 1, \ldots, N, \tag{21}$$

$$\{\overline{H}_{n,n+1}\}: \quad \beta_{n,n+1} = \frac{m(\overline{H}_{n,n+1})}{1 - \bar{m}(H)}, \quad n = 1, \ldots, N-1, \tag{22}$$

$$\{H\}: \quad \beta_H = \frac{\tilde{m}(H)}{1 - \bar{m}(H)}, \tag{23}$$

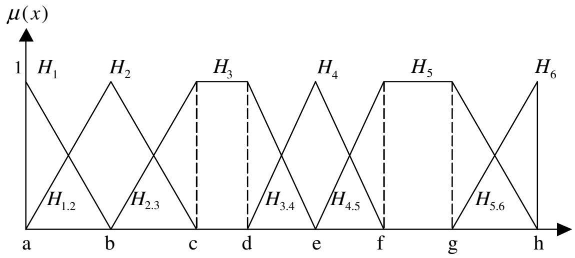

Fig. 1. The mutual relationships between fuzzy assessment grades.

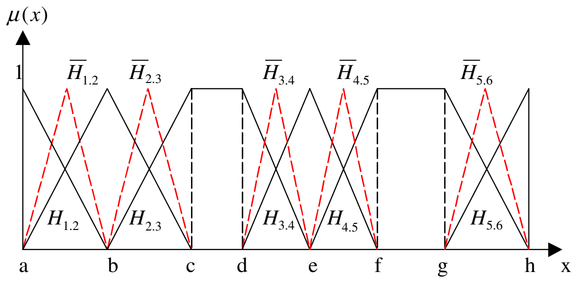

Fig. 2. Normalized fuzzy intersection subsets.

where $\overline{H}_{n,n+1}$ is a normalized fuzzy subset for the fuzzy intersection subset $H_{n,n+1}$ whose maximum degree of membership is represented by $\mu^{\max}_{H_{n,n+1}}$ and is usually less than one, as shown in Fig. 2.

$H_{n,n+1}$ is normalised to $\overline{H}_{n,n+1}$ as shown in Eq. (18) so that $H_{n,n+1}$ can be measured as a formal fuzzy set with the maximum membership degree being one, therefore assessed in the same scale as the other defined fuzzy evaluation grades such as H_n (Fig. 1). The normalisation of $H_{n,n+1}$ seems logical because the probability mass $m(\overline{H}_{n,n+1})$ assigned to the fuzzy intersection subset is directly related to the height of $H_{n,n+1}$. In other words, how H_n and H_{n+1} are interrelated is thus taken into account in the calculation of the belief assigned to their intersection. Without the normalisation, $m(\overline{H}_{n,n+1})$ would remain constant as long as H_n and H_{n+1} intersect, however small or large the intersection might be. Since $\overline{H}_{n,n+1}$ (or $H_{n,n+1}$) is not an originally defined fuzzy evaluation grade, however, its degree of belief (or $\beta_{n,n+1}$) should eventually be reassigned to H_n and H_{n+1}. The detailed assignment approach will be discussed in the late part of Section 3.4.

3.3. Fuzzy grade utility

Utility is one of the most important concepts in decision analysis. It reflects a decision maker (DM)'s preferences for various values of a variable and measures the relative strength of desirability that the DM has for those values. A function that reflects the DM's preferences is referred to as a utility function. For different decision problems, the same DM may have different preferences and utilities as well as utility functions. For the same decision problem, different DMs may have different preferences, utilities and utility functions in different circumstances.

In crisp MADA, utilities corresponding to crisp assessment grades can be represented by singleton numerical values. In fuzzy MADA, however, utilities corresponding to fuzzy assessment grades can no longer be represented by singleton numerical values because the evaluation grades are fuzzy sets. In this paper, we define utilities corresponding to fuzzy assessment grades by fuzzy grade utilities or fuzzy utilities

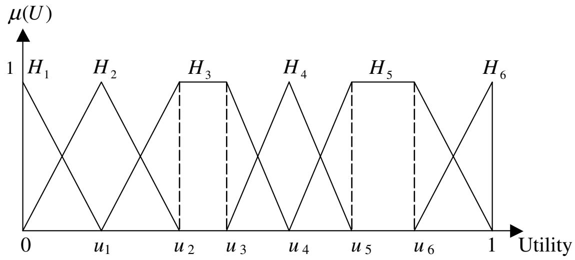

Fig. 3. Fuzzy grade utilities corresponding to fuzzy assessment grades.

for short. If a fuzzy assessment grade is a triangular fuzzy number, its corresponding fuzzy grade utility should also be a triangular fuzzy number. If a fuzzy assessment grade is a trapezoidal fuzzy number, its corresponding fuzzy utility will also be a trapezoidal fuzzy number. If fuzzy assessment grades are the combinations of triangular and trapezoidal fuzzy numbers, their corresponding grade utilities will be trapezoidal fuzzy numbers. In other words, a fuzzy grade utility should have the same form as its corresponding fuzzy assessment grade. The fuzzy grade utilities corresponding to the fuzzy assessment grades in Fig. 1 are shown in Fig. 3. Fuzzy grade utilities will be used as a basis for fuzzy grade transformation as discussed later. Note that the definition of fuzzy sets and the estimation of utilities are problem specific (Liu et al., 2004).

3.4. The grade match method for transforming fuzzy assessment grades

The fuzzy ER algorithm proposed in Section 3.2 is based on the assumption that the L basic attributes employ the same set of fuzzy assessment grades as the general attribute. In practice, however, to facilitate data collection and assessment, each basic attribute may be assessed using a specific set of fuzzy assessment grades different from those for other attributes. In such cases, before the above fuzzy ER algorithm can be applied, it is necessary to transform the belief decision matrix described in forms of different sets of fuzzy assessment grades into one expressed in the same set of fuzzy assessment grades. This mirrors the normalisation process in traditional MADA methods to transform attributes to the same space to facilitate trade-off analysis among attributes. In the development of the non-fuzzy evidential reasoning approach, an information transformation process was proposed (Yang, 2001). In this section, a similar information transformation technique is proposed to transform various sets of fuzzy evaluation grades into a unified set of grades on the basis of the fuzzy grade utilities.

Suppose the basic attribute e_i employs its own set of fuzzy assessment grades $\widetilde{H}_{n_i}$ $(n_i = 1,\dots,N_i)$, based on which the following distributed assessments are provided: $\widetilde{S}(e_i(a_l)) = \{(\widetilde{H}_{n_i}, \tilde{\beta}_{n_i,i}(a_l)),\ n_i = 1,\dots,N_i\}$ with $\tilde{\beta}_{n_i,i}(a_l) \geqslant 0$ and $\sum_{n_i=1}^{N_i}\tilde{\beta}_{n_i,i}(a_l) \leqslant 1$. The question is how to find the relations between the two different sets of fuzzy assessment grades H_n $(n = 1,\dots,N)$ and $\widetilde{H}_{n_i}$ $(n_i = 1,\dots,N_i)$, so that the latter can be equivalently represented by the former in some sense. One of such transformation techniques is to compare their fuzzy grade utilities and relative position relations. Fig. 4 shows the typical relative position relations between one basic fuzzy assessment grade $\widetilde{H}_{n_i}$ and two general assessment grades H_n and H_{n+1}.

From Fig. 4 one can see that $\widetilde{H}_{n_i}$ lies completely between H_n and H_{n+1}, and has no intersection with any other general fuzzy assessment grades. Therefore, it is sufficient to use only H_n and H_{n+1} to represent $\widetilde{H}_{n_i}$. Suppose $\widetilde{H}_{n_i}$ intersects H_n with an area of $(S_n + S_{n,n+1})$ and H_{n+1} with an area of $(S_{n,n+1} + S_{n+1})$, where $S_{n,n+1}$ is the common area of $\widetilde{H}_{n_i}$ intersecting both H_n and H_{n+1}. The minimum distance between the peaks of $\widetilde{H}_{n_i}$ and H_n is denoted by d_n and that between the peaks of $\widetilde{H}_{n_i}$ and H_{n+1} by d_{n+1}. The representation of $\widetilde{H}_{n_i}$ by H_n and H_{n+1} must at least follow the following three axioms.

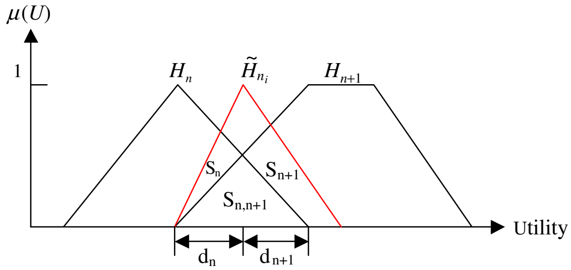

Fig. 4. The relations between different fuzzy assessment grades.

Axiom 1. If $\widetilde{H}_{n_i}$ is entirely included within a fuzzy assessment grade H_n, then it should completely belong to H_n.

Axiom 2. If $\widetilde{H}_{n_i}$ only intersects a fuzzy assessment grade H_n and has no non-empty intersection subset with any other grade, then it should also completely belong to H_n, no matter whether it is entirely included in H_n or not.

Axiom 3. If $\widetilde{H}_{n_i}$ intersects several fuzzy assessment grades at the same time but it is not entirely included in any one of them, then it should belong to each of them to certain extents.

It seems logical that the belief degree of $\widetilde{H}_{n_i}$ belonging to an assessment grade, say H_n, is related to $S_n, S_{n+1}, S_{n,n+1}, d_n$ and d_{n+1}. Intuitively, a large S_n and a small S_{n+1} together with a small d_n and a large d_{n+1} should imply a high degree of belief to which $\widetilde{H}_{n_i}$ belongs to H_n. Special care must be taken in handling $S_{n,n+1}$. Axiom 1 requires that the allocation of $S_{n,n+1}$ be related to the distances d_n and d_{n+1} as well as the areas S_n and S_{n+1}. In particular, if d_n and S_{n+1} are both zero, then $S_{n,n+1}$ should be completely allocated to H_n; if d_{n+1} and S_n are both zero, then $S_{n,n+1}$ should be completely allocated to H_{n+1}. As such, the following two allocation factors AF_n and AF_{n+1} are introduced:

$$AF_n = \frac{1}{2}\left[\left(1 - \frac{d_n}{d_n + d_{n+1}}\right) + \frac{S_n}{S_n + S_{n+1}}\right], \tag{24}$$

$$AF_{n+1} = \frac{1}{2}\left[\left(1 - \frac{d_{n+1}}{d_n + d_{n+1}}\right) + \frac{S_{n+1}}{S_n + S_{n+1}}\right]. \tag{25}$$

It is obvious that $AF_n + AF_{n+1} = 1$. Also, if d_n and S_{n+1} are both zero, then $AF_n = 1$ and $AF_{n+1} = 0$; if d_{n+1} and S_n are both zero, then $AF_n = 0$ and $AF_{n+1} = 1$. The allocation factors are used to assign the belief degrees to which $\widetilde{H}_{n_i}$ is allocated to H_n and H_{n+1} as follows:

$$Bel(\widetilde{H}_{n_i} \subset H_n) = \frac{S_n + AF_n \cdot S_{n,n+1}}{S_n + S_{n,n+1} + S_{n+1}}, \tag{26}$$

$$Bel(\widetilde{H}_{n_i} \subset H_{n+1}) = \frac{S_{n+1} + AF_{n+1} \cdot S_{n,n+1}}{S_n + S_{n,n+1} + S_{n+1}}. \tag{27}$$

It can be shown that the above assignments of the belief degrees satisfy the three axioms. The concept of assigning belief degrees through introducing allocation factors can be further extended to general cases where a basic fuzzy assessment grade $\widetilde{H}_{n_i}$ intersects more than two general fuzzy assessment grades. Fig. 5 shows the case of $\widetilde{H}_{n_i}$ intersecting H_n, H_{n+1} and H_{n+2}.

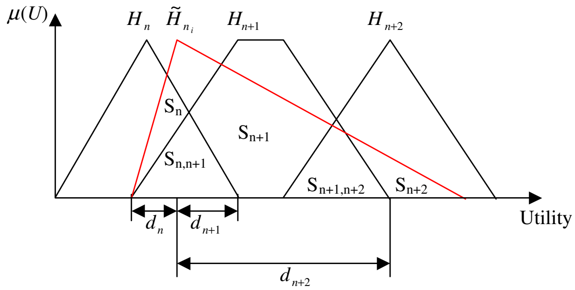

Fig. 5. The relationships between the fuzzy assessment grades.

Based on similar considerations as for defining AF_n and AF_{n+1} and from Fig. 5, the allocation factors for distributing $S_{n,n+1}$ between the fuzzy assessment grades H_n and H_{n+1} and for distributing $S_{n+1,n+2}$ between the fuzzy assessment grades H_{n+1} and H_{n+2} are given as follows:

$$AF_n(S_{n,n+1}) = \frac{1}{2}\left[\left(1-\frac{d_n}{d_n+d_{n+1}}\right)+\frac{S_n}{S_n+S_{n+1}+S_{n+1,n+2}}\right], \tag{28}$$

$$AF_{n+1}(S_{n,n+1}) = \frac{1}{2}\left[\left(1-\frac{d_{n+1}}{d_n+d_{n+1}}\right)+\frac{S_{n+1}+S_{n+1,n+2}}{S_n+S_{n+1}+S_{n+1,n+2}}\right], \tag{29}$$

$$AF_{n+1}(S_{n+1,n+2}) = \frac{1}{2}\left[\left(1-\frac{d_{n+1}}{d_{n+1}+d_{n+2}}\right)+\frac{S_{n+1}+S_{n,n+1}}{S_{n+1}+S_{n,n+1}+S_{n+2}}\right], \tag{30}$$

$$AF_{n+2}(S_{n+1,n+2}) = \frac{1}{2}\left[\left(1-\frac{d_{n+2}}{d_{n+1}+d_{n+2}}\right)+\frac{S_{n+2}}{S_{n+1}+S_{n,n+1}+S_{n+2}}\right]. \tag{31}$$

Based on the above allocation factors, the belief degrees to which $\widetilde{H}_{n_i}$ is assigned to H_n, H_{n+1} and H_{n+2} are calculated as follows:

$$Bel(\widetilde{H}_{n_i} \subset H_n) = \frac{S_n + AF_n(S_{n,n+1})\cdot S_{n,n+1}}{S_n+S_{n,n+1}+S_{n+1}+S_{n+1,n+2}+S_{n+2}}, \tag{32}$$

$$Bel(\widetilde{H}_{n_i} \subset H_{n+1}) = \frac{S_{n+1} + AF_{n+1}(S_{n,n+1})\cdot S_{n,n+1} + AF_{n+1}(S_{n+1,n+2})\cdot S_{n+1,n+2}}{S_n+S_{n,n+1}+S_{n+1}+S_{n+1,n+2}+S_{n+2}}, \tag{33}$$

$$Bel(\widetilde{H}_{n_i} \subset H_{n+2}) = \frac{S_{n+2} + AF_{n+2}(S_{n+1,n+2})\cdot S_{n+1,n+2}}{S_n+S_{n,n+1}+S_{n+1}+S_{n+1,n+2}+S_{n+2}} \tag{34}$$

with $\sum_{n=1}^{N} Bel(\widetilde{H}_{n_i} \subset H_n) = 1$. It can be shown that the above assignments of the belief degrees also satisfy the three axioms. This information transformation technique is referred to as the grade match technique based on the fuzzy utilities or the grade match technique for short. Using this technique, $\widetilde{H}_{n_i}$ can be represented as $\{(H_n, Bel(\widetilde{H}_{n_i} \subset H_n)),\ n = 1,\ldots,N\}$. Thus, the general set of fuzzy assessment grades H_n $(n = 1,\ldots,N)$ can be used to represent the basic set of fuzzy assessment grades $\widetilde{H}_{n_i}$ $(n_i = 1,\ldots,N_i)$. A distributed assessment based on the basic set of fuzzy assessment grades $\widetilde{S}(e_i(a_l)) = \{(\widetilde{H}_{n_i}, \tilde{\beta}_{n_i,i}(a_l)),\ n_i = 1,\ldots,N_i\}$ can then be expressed as $S(e_i(a_l)) = \{(H_n, \beta_{n,i}(a_l)),\ n = 1,\ldots,N\}$ using the general set of fuzzy assessment grades, where $\beta_{n,i}(a_l) = \sum_{n_i=1}^{N_i} \tilde{\beta}_{n_i,i}(a_l) Bel(\widetilde{H}_{n_i} \subset H_n)$.

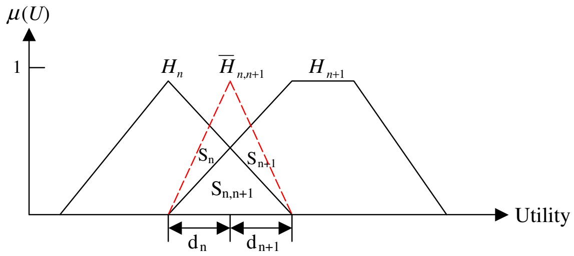

Fig. 6. The reassignment of the belief degree of fuzzy intersection subset.

Now we can solve the problem of reassigning the belief degree of the fuzzy intersection subset $\overline{H}_{n,n+1}$. Since $\overline{H}_{n,n+1}$ intersects H_n and H_{n+1} only (Fig. 6), the degree of belief assigned to $\overline{H}_{n,n+1}$ should only be reassigned to the two fuzzy grades H_n and H_{n+1}. From Fig. 6, we have

$$AF_n = \frac{1}{2}\left[\left(1 - \frac{d_n}{d_n + d_{n+1}}\right) + \frac{S_n}{S_n + S_{n+1}}\right], \tag{35}$$

$$AF_{n+1} = \frac{1}{2}\left[\left(1 - \frac{d_{n+1}}{d_n + d_{n+1}}\right) + \frac{S_{n+1}}{S_n + S_{n+1}}\right], \tag{36}$$

$$Bel(\overline{H}_{n,n+1} \subset H_n) = \frac{S_n + AF_n \cdot S_{n,n+1}}{S_n + S_{n,n+1} + S_{n+1}}, \tag{37}$$

$$Bel(\overline{H}_{n,n+1} \subset H_{n+1}) = \frac{S_{n+1} + AF_{n+1} \cdot S_{n,n+1}}{S_n + S_{n,n+1} + S_{n+1}}. \tag{38}$$

Thus, the belief degree $\beta_{n,n+1}$ can be divided into two parts: $\beta_{n,n+1}Bel(\overline{H}_{n,n+1} \subset H_n)$ and $\beta_{n,n+1}Bel(\overline{H}_{n,n+1} \subset H_{n+1})$. The former should be added to β_n and the latter to β_{n+1}. It is easy to prove that if $\mu_{U(H_n)} + \mu_{U(H_{n+1})} \equiv 1$ at any utility value, where $\mu_{U(H_n)}$ is the membership degree of a utility value belonging to the fuzzy set H_n and $\mu_{U(H_{n+1})}$ is the membership degree of the same utility value belonging to the fuzzy set H_{n+1}, then $Bel(\overline{H}_{n,n+1} \subset H_n) = Bel(\overline{H}_{n,n+1} \subset H_{n+1}) \equiv 0.5$. Therefore, the final belief degree that supports the fuzzy assessment grade H_n should include three parts: $\beta_n + \beta_{n,n+1}Bel(\overline{H}_{n,n+1} \subset H_n) + \beta_{n-1,n}Bel(\overline{H}_{n-1,n} \subset H_n)$ for $n = 2,\ldots,N-1$. The belief degree for H_1 is given by $\beta_1 + \beta_{1,2}Bel(\overline{H}_{1,2} \subset H_1)$ and $\beta_N + \beta_{N-1,N}Bel(\overline{H}_{N-1,N} \subset H_N)$ for H_N. The belief degree which supports the whole set $H = \{H_1,\ldots,H_N\}$ is still β_H. For convenience, we denote the above final belief degrees by $\beta_{1F}, \beta_{2F},\ldots,\beta_{NF}$ and β_H. Therefore, the final aggregated assessment can be expressed as $S(y(a_l)) = \{(H_n, \beta_{nF}(a_l)),\ n = 1,\ldots,N\}$.

3.5. The expression of exact data using fuzzy assessment grades

Quantitative attributes are normally measured by numerical data. To use the ER approach to conduct decision analysis using quantitative attributes together with qualitative attributes having fuzzy assessment grades, all numerical data need be transformed into distributed assessments. This is logical as the assessment of a quantitative attribute can also be properly characterised by fuzzy assessment grades. Take a price attribute for example. We may say that any prices between P_1 and P_2 are "*good*" which is a vague or fuzzy concept, but their degrees of being "*good*" are different. Thus, we may use a fuzzy number to describe an assessment grade "*good*" for the price attribute.

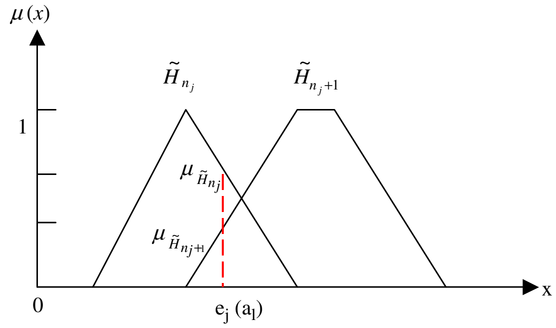

Fig. 7. Distribution assessments of quantitative basic attributes.

Suppose a basic attribute e_j is a quantitative attribute, which has its own set of fuzzy assessment grades $\widetilde{H}_{n_j}$ $(n_j = 1, \ldots, N_j)$. For an exact numerical value $e_j(a_l)$, we define its degrees of belief to the assessment grades by normalising its membership degrees to these grades. It can be seen from Fig. 7 that a numerical value $e_j(a_l)$ belongs to two different fuzzy assessment grades $\widetilde{H}_{n_j}$ and $\widetilde{H}_{n_j+1}$ with membership degrees of $\mu_{\tilde{H}_{n_j}}$ and $\mu_{\tilde{H}_{n_j+1}}$, respectively.

To obtain the distributed assessment of an alternative a_l on the basic attribute e_j, these two membership degrees are normalised to generate its degrees of belief as follows:

$$\tilde{\beta}_{n_j,j}(a_l) = \frac{\mu_{\tilde{H}_{n_j}}}{\mu_{\tilde{H}_{n_j}} + \mu_{\tilde{H}_{n_j+1}}}, \tag{39}$$

$$\tilde{\beta}_{n_j+1,j}(a_l) = \frac{\mu_{\tilde{H}_{n_j+1}}}{\mu_{\tilde{H}_{n_j}} + \mu_{\tilde{H}_{n_j+1}}}. \tag{40}$$

Thus, we have the distributed assessment $\widetilde{S}(e_j(a_l)) = \{(\widetilde{H}_{n_j}, \tilde{\beta}_{n_j,j}(a_l)),\ (\widetilde{H}_{n_j+1}, \tilde{\beta}_{n_{j+1},j}(a_l))\}$. After all numerical data are transformed into distributed assessments, if their evaluation grades are different from the general set of fuzzy assessment grades, the grade match technique can be used to transform the former into the latter, as discussed in Section 3.3.

3.6. Fuzzy expected utilities for characterising alternatives

Different from the fuzzy grade utilities, fuzzy expected utilities are calculated for alternatives. They are employed to compare and rank alternatives. The fuzzy expected utility of an aggregated assessment $S(y(a_l))$ for alternative a_l is defined as follows:

$$u(S(y(a_l))) = \sum_{n=1}^{N} \beta_{nF}(a_l) u(H_n), \tag{41}$$

where $u(H_n)$ is the fuzzy grade utility of the assessment grade H_n. Accordingly, the fuzzy expected utility $u(S(y(a_l)))$ is also a fuzzy number. However, the existence of the upper and lower bounds of the degrees of belief may lead to the maximum and minimum fuzzy expected utilities. Without loss of generality, suppose H_1 is the least preferred fuzzy assessment grade, which has the lowest fuzzy grade utility, and H_N is the most preferred fuzzy assessment grade, which has the highest fuzzy grade utility. Then the maximum and the minimum fuzzy expected utilities of alternative a_l are calculated by

$$u_{\max}(a_l) = \sum_{n=1}^{N-1} \beta_{nF}(a_l)u(H_n) + (\beta_{NF}(a_l) + \beta_H(a_l))u(H_N), \tag{42}$$

$$u_{\min}(a_l) = (\beta_{1F}(a_l) + \beta_H(a_l))u(H_1) + \sum_{n=2}^{N} \beta_{nF}(a_l)u(H_n), \tag{43}$$

which are both fuzzy numbers. They will be different if there exist incomplete assessments. In the case of all original assessments $S(e_i(a_l))$ being complete, then there is $\beta_H(a_l) = 0$ and therefore we have $u(S(y(a_l))) = u_{\min}(a_l) = u_{\max}(a_l)$. Based on the concept of fuzzy expected utilities, the comparison of alternatives reduces to the comparison of the maximum and the minimum fuzzy expected utilities of alternatives.

4. Use fuzzy grades and complete information to assess cars

In this section, a numerical example is examined to demonstrate the implementation process of the new ER approach to deal with MADA problems with both probabilistic and fuzzy uncertainties.

4.1. Problem description of a car evaluation problem with fuzzy assessment grades

Consider a performance assessment problem of executive cars on seven basic performance attributes, four of which are quantitative with the others being qualitative. The quantitative attributes are *Acceleration* (seconds from 0 mph to 60 mph), *Braking* (feet from 60 mph to 0 mph), *Horsepower* (hp) and *Fuel economy* (mpg), while the qualitative attributes are *Handling*, *Ride quality* and *Powertrain*. For convenience, we denote them by $e_1, e_2, e_4, e_7, e_3, e_5, e_6$, respectively, of which e_1 and e_2 are for minimisation and the others for maximisation. The data are shown in Table 1 (Yang, 2001). The relative weight of the ith attribute is denoted by w_i ($i = 1, \ldots, 7$).

Suppose the performance of a car is classified into six categories (grades), which are *Top* (T), *Excellent* (E), *Good* (G), *Average* (A), *Poor* (P) and *Worst* (W). They consist of the general set of assessment grades:

$$H = \{H_j, j = 1, \ldots, 6\} = \{Worst, Poor, Average, Good, Excellent, Top\}.$$

In this paper, different from Yang (2001) it is assumed that all the six assessment grades are either triangular or trapzoidal fuzzy numbers, as shown in Fig. 8 and also defined in Table 2. The corresponding fuzzy grade utilities are also assumed as shown in Fig. 9 and also defined in Table 2. Note that these assumptions are made to demonstrate the implementation process of the new ER approach, though other types of fuzzy numbers can also be used, depending upon specific circumstances. Note that in practice the fuzzy sets and their utilities need to be assigned or estimated by analysts and decision makers concerned and in general sensitivity analysis can be conducted to examine the impact of defined utilities on the final assessments.

Table 1
Original performance assessment of executive cars

Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Acceleration (e_1)	8.8	8.0	7.7	8.4	8.0	7.9
Braking (e_2)	128	124	127	134	135	126
Handling (e_3)	*B*	*A*	*B*	*B*–	*B*+	*A*
Horsepower (e_4)	196	152	182	183	138	171
Ride quality (e_5)	*A*–	*B*–	*B*	*B*+	*B*+	*A*–
Powertrain (e_6)	*B*	*B*+	*A*	*B*	*A*–	*A*
Fuel economy (e_7)	20	20	21	20	19	20

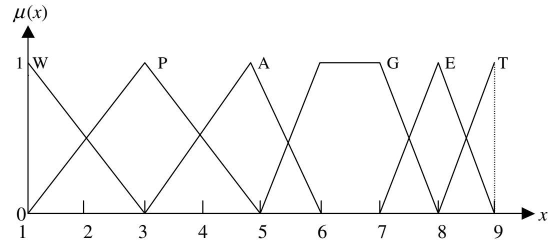

Fig. 8. The general set of fuzzy assessment grades for car evaluation.

Table 2
Membership functions of the fuzzy assessment grades and their fuzzy utilities

Linguistic term	*Worst* (*W*)	*Poor* (*P*)	*Average* (*A*)	*Good* (*G*)	*Excellent* (*E*)	*Top* (*T*)
Membership functions of fuzzy assessment grades	(1, 1, 3)	(1, 3, 5)	(3, 5, 6)	(5, 6, 7, 8)	(7, 8, 9)	(8, 9, 9)
Membership functions of fuzzy grade utilities	(0, 0, 0.2)	(0, 0.2, 0.4)	(0.2, 0.4, 0.6)	(0.4, 0.6, 0.7, 0.85)	(0.7, 0.85, 1)	(0.85, 1, 1)

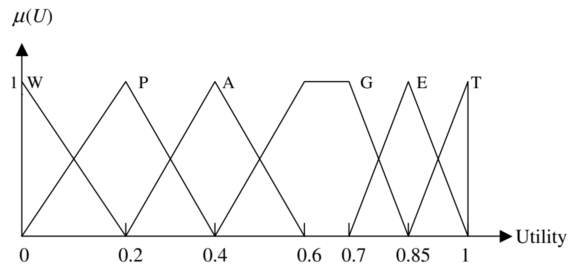

Fig. 9. Fuzzy utilities of the general set of fuzzy assessment grades.

For simplicity, suppose all qualitative attributes are assessed with reference to this general set of fuzzy assessment grades. The assessment results are presented in Table 1, where

$$C^- \Longleftrightarrow \{(Worst, 1.0)\},\ C \Longleftrightarrow \{(Worst, 0.6), (Poor, 0.4)\},\ C^+ \Longleftrightarrow \{(Poor, 0.6), (Average, 0.4)\},$$
$$B^- \Longleftrightarrow \{(Average, 1.0)\},\ B \Longleftrightarrow \{(Average, 0.4), (Good, 0.6)\},\ B^+ \Longleftrightarrow \{(Good, 1.0)\},$$
$$A^- \Longleftrightarrow \{(Good, 0.6), (Excellent, 0.4)\},\ A \Longleftrightarrow \{(Excellent, 0.6), (Top, 0.4)\},\ A^+ \Longleftrightarrow \{(Top, 1.0)\}.$$

The symbol “ $\Longleftrightarrow$ ” means “is equivalent to”.

4.2. Characterising quantitative data using fuzzy assessment grades

The four quantitative attributes may use either different sets of fuzzy assessment grades or the same set of grades as the general attribute, depending on the need of real assessments. For illustration purpose, a different set of fuzzy assessment grades is employed in this paper, similar to the general set of grades in words but not the same in meanings. The set of fuzzy assessment grades for the quantitative attributes is also called *Top* ($\widetilde{T}$), *Excellent* ($\widetilde{E}$), *Good* ($\widetilde{G}$), *Average* ($\widetilde{A}$), *Poor* ($\widetilde{P}$) and *Worst* ($\widetilde{W}$). They form the basic set of fuzzy assessment grades for quantitative attributes:

$$\widetilde{H} = \{\widetilde{H}_j, j = 1, \ldots, 6\} = \{Worst\ (\widetilde{W}), Poor\ (\widetilde{P}), Average\ (\widetilde{A}), Good\ (\widetilde{G}), Excellent\ (\widetilde{E}), Top\ (\widetilde{T})\}.$$

For different quantitative attributes, although the words of the assessment grades are the same, the definitions of their membership functions are different, which are shown in Figs. 10–13 and also defined in Table 3. The corresponding fuzzy grade utilities are shown in Fig. 14 and defined in Table 4.

Given the fuzzy assessment grades and the corresponding membership functions, the data shown in Table 1 for quantitative attributes can easily be transformed into distributed assessments using formulas (39) and (40). Take the *acceleration time* of car 1 for example. The *acceleration time* of car 1 is 8.8 seconds, which is between the fuzzy assessment grades *Poor* ($\widetilde{P}$) and *Average* ($\widetilde{A}$) (see Fig. 10). As far as *acceleration time* is concerned, the membership degrees of car 1 belonging to the assessment grades *Poor* ($\widetilde{P}$) and *Average* ($\widetilde{A}$) can be calculated as follows:

$$\mu_{\widetilde{P}} = \frac{8.8 - 8.7}{9.2 - 8.7} = 0.2, \quad \mu_{\widetilde{A}} = \frac{9.2 - 8.8}{9.2 - 8.7} = 0.8.$$

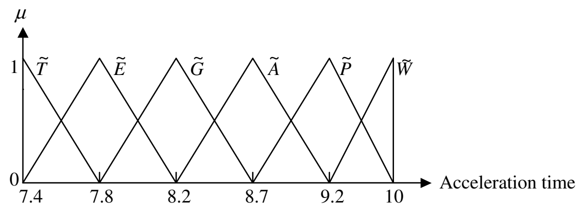

Fig. 10. Membership functions of fuzzy assessment grades for acceleration time.

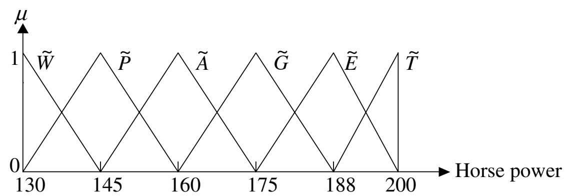

Fig. 11. Membership functions of fuzzy assessment grades for horsepower.

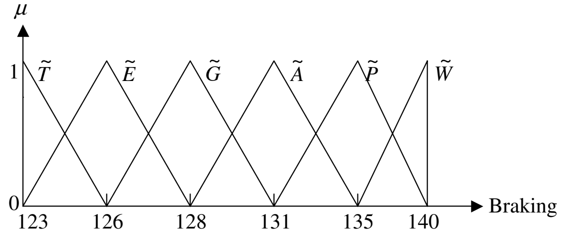

Fig. 12. Membership functions of fuzzy assessment grades for braking.

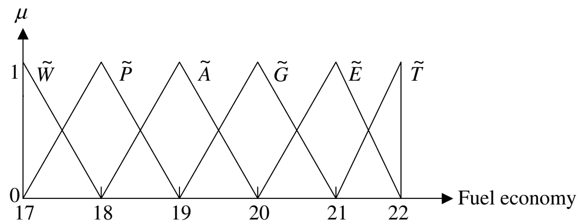

Fig. 13. Membership functions of fuzzy assessment grades for fuel economy.

Table 3
Membership functions of fuzzy assessment grades for quantitative attributes

Linguistic term	Worst ($\widetilde{W}$)	Poor ($\widetilde{P}$)	Average ($\widetilde{A}$)	Good ($\widetilde{G}$)	Excellent ($\widetilde{E}$)	Top ($\widetilde{T}$)
Acceleration time	(9.2, 10, 10)	(8.7, 9.2, 10)	(8.2, 8.7, 9.2)	(7.8, 8.2, 8.7)	(7.4, 7.8, 8.2)	(7.4, 7.4, 7.8)
Horsepower	(130, 130, 145)	(130, 145, 160)	(145,160,175)	(160,175,188)	(175,188,200)	(188,200,200)
Braking	(135,140,140)	(131,135,140)	(128, 131, 135)	(126, 128, 131)	(123, 126, 128)	(123, 123, 126)
Fuel economy	(17, 17, 18)	(17, 18, 19)	(18, 19, 20)	(19, 20, 21)	(20, 21, 22)	(21, 22, 22)

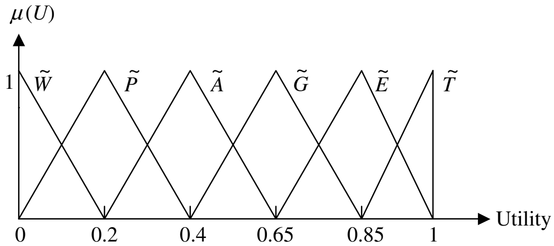

Fig. 14. Fuzzy utilities of the basic set of fuzzy assessment grades.

Table 4
Membership functions of fuzzy grade utilities

Linguistic term	Worst	Poor	Average	Good	Excellent	Top
Membership function	(0, 0, 0.2)	(0, 0.2, 0.4)	(0.2, 0.4, 0.65)	(0.4, 0.65, 0.85)	(0.65, 0.85, 1)	(0.85, 1, 1)

The belief degrees of car 1 belonging to *Poor* ($\widetilde{P}$) and *Average* ($\widetilde{A}$) are therefore determined by

$$\tilde{\beta}_{\tilde{P}} = \frac{0.2}{0.8 + 0.2} = 0.2, \quad \tilde{\beta}_{\tilde{A}} = \frac{0.8}{0.8 + 0.2} = 0.8.$$

Accordingly, we get the distributed assessment $\widetilde{S}(e_1(Car\ 1)) = \{(\widetilde{P}, 0.2), (\widetilde{A}, 0.8)\}$. All the other quantitative data can be transformed in the same way, leading to the distributed assessments as shown in Table 5.

4.3. Transformation of original information using the grade match method

Because the basic set of fuzzy assessment grades $\widetilde{H}$ is different from the general set of fuzzy assessment grades H, in order to use the fuzzy ER algorithm to aggregate attributes, it is necessary to transform the former into the latter. The transformation must be based on fuzzy grade utilities. Take the transformation of the fuzzy assessment grade *Average* ($\widetilde{A}$) for example. The relationships between $\widetilde{A}$ and the fuzzy assessment grades P, A and G are shown in Fig. 15. Because the area of $\widetilde{A}$ intersecting P only is zero and the distance between the peaks of $\widetilde{A}$ and A is also zero, the area S_{PA} should not be assigned to the fuzzy assessment

Table 5
Distributed assessments of executive cars with seven attributes

Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Acceleration	$\{(\widetilde{P},0.2),\ (\widetilde{A},0.8)\}$	$\{(\widetilde{G},0.5),\ (\widetilde{E},0.5)\}$	$\{(\widetilde{E},0.75),\ (\widetilde{T},0.25)\}$	$\{(\widetilde{A},0.4),\ (\widetilde{G},0.6)\}$	$\{(\widetilde{G},0.5),\ (\widetilde{E},0.5)\}$	$\{(\widetilde{G},0.25),\ (\widetilde{E},0.75)\}$
Braking	$\{(\widetilde{G},1.0)\}$	$\{(\widetilde{E},0.3333),\ (\widetilde{T},0.6667)\}$	$\{(\widetilde{G},0.5),\ (\widetilde{E},0.5)\}$	$\{(\widetilde{P},0.75),\ (\widetilde{A},0.25)\}$	$\{(\widetilde{P},1.0)\}$	$\{(\widetilde{E},1.0)\}$
Handling	$\{(A,0.4),\ (G,0.6)\}$	$\{(E,0.6),\ (T,0.4)\}$	$\{(A,0.4),\ (G,0.6)\}$	$((A,1.0)\}$	$\{(G,1.0)\}$	$\{(E,0.6),\ (T,0.4)\}$
Horsepower	$\{(\widetilde{E},0.3333),\ (\widetilde{T},0.6667)\}$	$\{(\widetilde{P},0.5333),\ (\widetilde{A},0.4667)\}$	$\{(\widetilde{G},0.4615),\ (\widetilde{E},0.5385)\}$	$\{(\widetilde{G},0.3846),\ (\widetilde{E},0.6154)\}$	$\{(\widetilde{W},0.4667),\ (\widetilde{P},0.5333)\}$	$\{(\widetilde{A},0.2667),\ (\widetilde{G},0.7333)\}$
Ride quality	$\{(G,0.6),\ (E,0.4)\}$	$\{(A,1.0)\}$	$\{(A,0.4),\ (G,0.6)\}$	$\{(G,1.0)\}$	$\{(G,1.0)\}$	$\{(G,0.6),\ (E,0.4)\}$
Powertrain	$\{(A,0.4),\ (G,0.6)\}$	$\{(G,1.0)\}$	$\{(E,0.6),\ (T,0.4)\}$	$\{(A,0.4),\ (G,0.6)\}$	$\{(G,0.6),\ (E,0.4)\}$	$\{(E,0.6),\ (T,0.4)\}$
Fuel economy	$\{(\widetilde{G},1.0)\}$	$\{(\widetilde{G},1.0)\}$	$\{(\widetilde{E},1.0)\}$	$\{(\widetilde{G},1.0)\}$	$\{(\widetilde{A},1.0)\}$	$\{(\widetilde{G},1.0)\}$

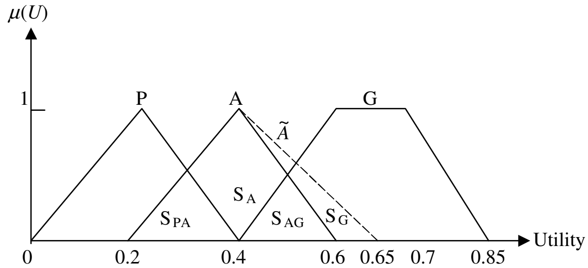

Fig. 15. The relationships between the fuzzy assessment grades.

grade P. Therefore, S_{PA} is entirely allocated to the fuzzy assessment grade A. The area S_{AG}, however, should be assigned between the fuzzy assessment grades A and G because there is a separate area between $\widetilde{A}$ and either A or G. The computational process of the areas is given below (see Appendix B for formulae):

$$S_{PA} = \frac{1}{2} \times (0.4 - 0.2) \times \frac{0.4 - 0.2}{(0.4 + 0.4) - (0.2 + 0.2)} = \frac{1}{2} \times 0.2 \times \frac{1}{2} = 0.05,$$

$$S_{AG} = \frac{1}{2} \times (0.6 - 0.4) \times \frac{0.6 - 0.4}{(0.6 + 0.6) - (0.4 + 0.4)} = \frac{1}{2} \times 0.2 \times \frac{1}{2} = 0.05,$$

$$S_A = \frac{1}{2} \times (0.6 - 0.2) \times 1 - S_{PA} - S_{AG} = 0.2 - 0.05 - 0.05 = 0.1,$$

$$S_G = \frac{1}{2} \times (0.65 - 0.4) \times \frac{0.65 - 0.4}{(0.65 + 0.6) - (0.4 + 0.4)} - S_{AG}$$
$$= \frac{1}{2} \times 0.25 \times \frac{0.25}{0.45} - 0.05 = 0.01944.$$

The distances between the peaks of $\widetilde{A}$ and P, A and G are determined by $d_{\tilde{A}P} = 0.4 - 0.2 = 0.2, d_{\tilde{A}A} = 0,$ and $d_{\tilde{A}G} = 0.6 - 0.4 = 0.2$. The allocation factors for the common areas are calculated by using formulas (28)–(31) as follows:

$$AF_P(S_{PA}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{A}P}}{d_{\tilde{A}P} + d_{\tilde{A}A}}\right) + \frac{0}{0 + S_A + S_{AG}}\right] = 0,$$

$$AF_A(S_{PA}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{A}A}}{d_{\tilde{A}P} + d_{\tilde{A}A}}\right) + \frac{S_A + S_{AG}}{0 + S_A + S_{AG}}\right] = 1,$$

$$AF_A(S_{AG}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{A}A}}{d_{\tilde{A}A} + d_{\tilde{A}G}}\right) + \frac{S_{PA} + S_A}{S_{PA} + S_A + S_G}\right]$$

$$= \frac{1}{2}\left[1 + \frac{0.05 + 0.1}{0.05 + 0.1 + 0.01944}\right] = 0.9426,$$

$$AF_G(S_{AG}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{A}G}}{d_{\tilde{A}A} + d_{\tilde{A}G}}\right) + \frac{S_G}{S_{PA} + S_A + S_G}\right]$$

$$= \frac{1}{2}\left[0 + \frac{0.01944}{0.05 + 0.1 + 0.01944}\right] = 0.0574.$$

Having obtained the above allocation factors, we have the following assignments of the belief degrees:

$$Bel_P = \frac{AF_P(S_{PA}) \cdot S_{PA}}{S_{PA} + S_A + S_{AG} + S_G} = 0,$$

$$Bel_A = \frac{AF_A(S_{PA}) \cdot S_{PA} + S_A + AF_A(S_{AG}) \cdot S_{AG}}{S_{PA} + S_A + S_{AG} + S_G}$$

$$= \frac{1 \times 0.05 + 0.1 + 0.9426 \times 0.05}{0.05 + 0.1 + 0.05 + 0.01944} = 0.8983,$$

$$Bel_G = \frac{AF_G(S_{AG}) \cdot S_{AG} + S_G}{S_{PA} + S_A + S_{AG} + S_G} = \frac{0.0574 \times 0.05 + 0.01944}{0.05 + 0.1 + 0.05 + 0.01944} = 0.1017.$$

Therefore, we get the rule of grade match between $\widetilde{A}$ and P, A and G based on fuzzy utilities

$$\widetilde{A} \Longleftrightarrow \{(A, 0.8983), (G, 0.1017).\}$$

Similarly, the relations between the fuzzy assessment grades "*Excellent* ($\widetilde{E}$)" and "*Good* (G)", "*Excellent* (E)" and "*Top* (T)" are shown in Fig. 16. Because there is no separate area between $\widetilde{E}$ and T except for the common area S_{ET} and the peaks of $\widetilde{E}$ and E overlap, the common area S_{ET} should not be allocated to the fuzzy assessment grade T. However, the common area S_{GE} should be allocated between the fuzzy assessment grades G and E. Accordingly, we have the following results:

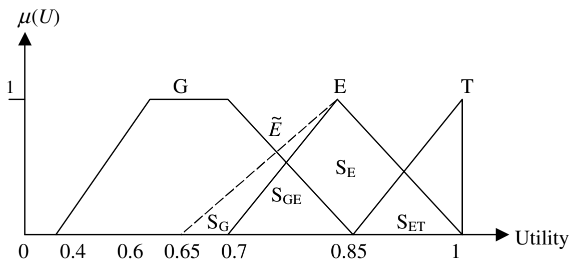

Fig. 16. The relations between the fuzzy assessment grades.

$$S_{GE} = \frac{1}{2} \times (0.85 - 0.7) \times \frac{0.85 - 0.7}{(0.85 + 0.85) - (0.7 + 0.7)} = \frac{1}{2} \times 0.15 \times \frac{1}{2} = 0.0375,$$

$$S_{ET} = \frac{1}{2} \times (1 - 0.85) \times \frac{1 - 0.85}{(1 + 1) - (0.85 + 0.85)} = \frac{1}{2} \times 0.15 \times \frac{1}{2} = 0.0375,$$

$$S_G = \frac{1}{2} \times (0.85 - 0.65) \times \frac{0.85 - 0.65}{(0.85 + 0.85) - (0.7 + 0.65)} - S_{GE}$$
$$= \frac{1}{2} \times 0.2 \times \frac{0.2}{0.35} - 0.0375 = 0.01964,$$

$$S_E = \frac{1}{2} \times (1 - 0.7) \times 1 - S_{GE} - S_{ET} = 0.15 - 0.0375 - 0.0375 = 0.075,$$

$$d_{\tilde{E}G} = 0.85 - 0.7 = 0.15, d_{\tilde{E}E} = 0, \quad \text{and} \quad d_{\tilde{E}T} = 1 - 0.85 = 0.15,$$

$$AF_G(S_{GE}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{E}G}}{d_{\tilde{E}G} + d_{\tilde{E}E}}\right) + \frac{S_G}{S_G + S_E + S_{ET}}\right]$$
$$= \frac{1}{2}\left[0 + \frac{0.01964}{0.01964 + 0.075 + 0.0375}\right] = 0.0743,$$

$$AF_E(S_{GE}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{E}E}}{d_{\tilde{E}G} + d_{\tilde{E}E}}\right) + \frac{S_E + S_{ET}}{S_G + S_E + S_{ET}}\right]$$
$$= \frac{1}{2}\left[1 + \frac{0.075 + 0.375}{0.01964 + 0.075 + 0.0375}\right] = 0.9257$$

$$AF_E(S_{ET}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{E}E}}{d_{\tilde{E}E} + d_{\tilde{E}T}}\right) + \frac{S_{GE} + S_E}{S_{GE} + S_E + 0}\right] = 1,$$

$$AF_T(S_{ET}) = \frac{1}{2}\left[\left(1 - \frac{d_{\tilde{E}T}}{d_{\tilde{E}E} + d_{\tilde{E}T}}\right) + \frac{0}{S_{GE} + S_E + 0}\right] = 0,$$

$$Bel_G = \frac{AF_G(S_{GE}) \cdot S_{GE} + S_G}{S_G + S_{GE} + S_E + S_{ET}} = \frac{0.0743 \times 0.0375 + 0.01964}{0.01964 + 0.0375 + 0.075 + 0.0375} = 0.1322,$$

$$Bel_E = \frac{AF_E(S_{GE}) \cdot S_{GE} + S_E + AF_E(S_{ET}) \cdot S_{ET}}{S_G + S_{GE} + S_E + S_{ET}}$$
$$= \frac{0.9257 \times 0.0375 + 0.075 + 1 \times 0.0375}{0.01964 + 0.0375 + 0.075 + 0.0375} = 0.8678,$$

$$Bel_T = \frac{AF_T(S_{ET}) \cdot S_{ET}}{S_G + S_{GE} + S_E + S_{ET}} = 0.$$

The rule of grade match between $\widetilde{E}$ and G, E and T based on fuzzy utilities is given by

$$\widetilde{E} \Longleftrightarrow \{(G, 0.1322), (E, 0.8678)\}.$$

Since the relations between $\widetilde{W}$ and W, $\widetilde{P}$ and P, and $\widetilde{G}$ and G are all inclusion relations, that is, the former is completely included in the latter, we have the following rules of grade match based on fuzzy utilities: $\widetilde{W} \Longleftrightarrow W$, $\widetilde{P} \Longleftrightarrow P$, $\widetilde{G} \Longleftrightarrow G$, and $\widetilde{T} \Longleftrightarrow T$. Based on these acquired rules of grade match, Table 5 with different sets of fuzzy assessment grades can be transformed into Table 6, which has only one unified set of fuzzy assessment grades.

Table 6
Transformed distributed assessment of executive cars

Performance	Car 1	Car 2	Car 3	Car 4	Car 5	Car 6
Acceleration	$\{(P,0.2), (A,0.71864), (G,0.08136)\}$	$\{(G,0.5661), (E,0.4339)\}$	$\{(G,0.09915), (E,0.65085), (T,0.25)\}$	$\{(A,0.35932), (G,0.64068)\}$	$\{(G,0.5661), (E,0.4339)\}$	$\{(G,0.34915), (E,0.65085)\}$
Braking	$\{(G,1.0)\}$	$\{(G,0.04406), (E,0.28924), (T,0.6667)\}$	$\{(G,0.5661), (E,0.4339)\}$	$\{(P,0.75), (A,0.224575), (G,0.025425)\}$	$\{(P,1.0)\}$	$\{(G,0.1322), (E,0.8678)\}$
Handling	$\{(A,0.4), (G,0.6)\}$	$\{(E,0.6), (T,0.4)\}$	$\{(A,0.4), (G,0.6)\}$	$\{(A,1.0)\}$	$\{(G,1.0)\}$	$\{(E,0.6), (T,0.4)\}$
Horsepower	$\{(G,0.04406), (E,0.28924), (T,0.6667)\}$	$\{(P,0.5333), (A,0.419237), (G,0.047463)\}$	$\{(G,0.53269), (E,0.46731)\}$	$\{(G,0.465956), (E,0.534044)\}$	$\{(W,0.4667), (P,0.5333)\}$	$\{(A,0.239577), (G,0.760423)\}$
Ride quality	$\{(G,0.6), (E,0.4)\}$	$\{(A,1.0)\}$	$\{(A,0.4), (G,0.6)\}$	$\{(G,1.0)\}$	$\{(G,1.0)\}$	$\{(G,0.6), (E,0.4)\}$
Powertrain	$\{(A,0.4), (G,0.6)\}$	$\{(G,1.0)\}$	$\{(E,0.6), (T,0.4)\}$	$\{(A,0.4), (G,0.6)\}$	$\{(G,0.6), (E,0.4)\}$	$\{(E,0.6), (T,0.4)\}$
Fuel economy	$\{(G,1.0)\}$	$\{(G,1.0)\}$	$\{(G,0.1322), (E,0.8678)\}$	$\{(G,1.0)\}$	$\{(A,0.8983), (G,0.1017)\}$	$\{(G,1.0)\}$

4.4. Aggregation of attributes using the fuzzy ER algorithm

The assessment information shown in Table 6 is represented in the same format. Therefore, they can be aggregated using the fuzzy ER algorithm. Suppose the seven attributes are of equal importance, that is, $w_i = 1/7$ ($i = 1, \ldots, 7$). Before using the ER algorithm to aggregate attributes, we still need to determine the maximum degree of membership for each fuzzy intersection subset. Table 7 shows the maximum degree of membership for each fuzzy intersection subset. The assessments of a car on all the seven attributes are aggregated into an overall assessment using the fuzzy ER algorithm described in Section 3.1, as shown in Table 8.

Since fuzzy intersection subsets $\overline{WP}$, $\overline{PA}$, $\overline{AG}$, $\overline{GE}$ and $\overline{ET}$ are not the defined fuzzy assessment grades, the belief degrees assigned to them need to be reassigned to the defined fuzzy assessment grades. The

Table 7
Maximum membership degrees of each fuzzy intersection

Fuzzy intersection subset	WP	PA	AG	GE	ET
Maximum degree of membership ($\mu^{\max}_{H_{n,n+1}}$)	0.5	0.5	0.5	0.5	0.5

Table 8
Overall distribution assessments of executive cars

	W	$\overline{WP}$	P	$\overline{PA}$	A	$\overline{AG}$	G	$\overline{GE}$	E	$\overline{ET}$	T	H
Car 1	0	0	0.0217	0.0015	0.1782	0.066	0.5526	0.0286	0.0768	0.0024	0.0723	0
Car 2	0	0	0.0630	0.0053	0.1759	0.0447	0.3632	0.0389	0.1676	0.0102	0.1313	0
Car 3	0	0	0	0	0.0893	0.0160	0.3226	0.0864	0.3973	0.0165	0.072	0
Car 4	0	0	0.0803	0.0128	0.2365	0.0849	0.5094	0.0190	0.0571	0	0	0
Car 5	0.0548	0.0046	0.1905	0.0143	0.1055	0.0337	0.4694	0.0261	0.1013	0	0	0
Car 6	0	0	0	0	0.0252	0.0049	0.3563	0.1025	0.4029	0.0211	0.0871	0

way to transform them is similar to transforming the basic fuzzy assessment grades to the general grades. Take the transformation of $\overline{WP}$ for example. Fig. 17 describes the relationships between $\overline{WP}$ and W and P. The calculations for the reassignments are as follows:

$$S_{WP} = \frac{1}{2} \times (0.2 - 0) \times \frac{0.2 - 0}{(0.2 + 0.2) - (0 + 0)} = \frac{1}{2} \times 0.2 \times \frac{1}{2} = 0.05,$$

$$S_W = \frac{1}{2} \times (0.2 - 0) \times \frac{0.2 - 0}{(0.2 + 0.1) - (0 + 0)} - S_{WP} = \frac{1}{2} \times 0.2 \times \frac{0.2}{0.3} - 0.05 = 0.0167,$$

$$S_P = \frac{1}{2} \times (0.2 - 0) \times \frac{0.2 - 0}{(0.2 + 0.2) - (0 + 0.1)} - S_{WP} = \frac{1}{2} \times 0.2 \times \frac{0.2}{0.3} - 0.05 = 0.0167,$$

$$d_{\overline{WP}W} = 0.1 - 0 = 0.1 \quad \text{and} \quad d_{\overline{WP}P} = 0.2 - 0.1 = 0.1,$$

$$AF_W(S_{WP}) = \frac{1}{2}\left[\left(1 - \frac{d_{\overline{WP}W}}{d_{\overline{WP}W} + d_{\overline{WP}P}}\right) + \frac{S_W}{S_W + S_P}\right] = \frac{1}{2}\left[\left(1 - \frac{0.1}{0.1 + 0.1}\right) + \frac{0.0167}{0.0167 + 0.0167}\right] = 0.5,$$

$$AF_P(S_{WP}) = \frac{1}{2}\left[\left(1 - \frac{d_{\overline{WP}P}}{d_{\overline{WP}W} + d_{\overline{WP}P}}\right) + \frac{S_P}{S_W + S_P}\right] = \frac{1}{2}\left[\left(1 - \frac{0.1}{0.1 + 0.1}\right) + \frac{0.0167}{0.0167 + 0.0167}\right] = 0.5,$$

$$Bel_W = \frac{AF_W(S_{WP}) \cdot S_{WP} + S_W}{S_W + S_{WP} + S_P} = \frac{0.5 \times 0.05 + 0.0167}{0.0167 + 0.05 + 0.0167} = 0.5,$$

$$Bel_P = \frac{AF_P(S_{WP}) \cdot S_{WP} + S_P}{S_W + S_{WP} + S_P} = \frac{0.5 \times 0.05 + 0.0167}{0.0167 + 0.05 + 0.0167} = 0.5.$$

Thus, we get the rule of grade match for $\overline{WP}$: $\overline{WP} \Longleftrightarrow \{(W, 0.5), (P, 0.5)\}$. Similarly, we get all the other rules of grade match for the other four fuzzy subsets: $\overline{PA} \Longleftrightarrow \{(P, 0.5), (A, 0.5)\}$, $\overline{AG} \Longleftrightarrow \{(A, 0.5), (G, 0.5)\}$, $\overline{GE} \Longleftrightarrow \{(G, 0.5), (E, 0.5)\}$, and $\overline{ET} \Longleftrightarrow \{(E, 0.5), (T, 0.5)\}$. These rules of grade match result in the reassignment of the belief degrees for all the fuzzy intersection subsets back to the defined general fuzzy assessment grades. The transformed overall distributed assessments are presented in Table 9.

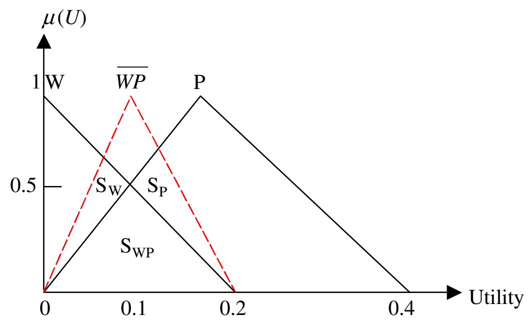

Fig. 17. The transformation of fuzzy intersection subset.

Table 9
Transformed overall distributed assessments of executive cars

	W	P	A	G	E	T	H
Car 1	0	0.02242	0.21195	0.59986	0.09229	0.07347	0
Car 2	0	0.06563	0.20092	0.40497	0.19211	0.13637	0
Car 3	0	0	0.09725	0.37375	0.44875	0.08025	0
Car 4	0	0.08666	0.28537	0.56134	0.066664	0	0
Car 5	0.05708	0.19989	0.12944	0.49925	0.11434	0	0
Car 6	0	0	0.02768	0.41	0.46471	0.09761	0

4.5. Ranking the cars

It can be seen from Table 9 that among six executive cars only car 5 is assessed to "*worst* (W)" to a certain degree and also it is to a large extent assessed to the grades not better than *good*. Therefore, we have sufficient reason to believe that car 5 should be the worst in performance. Both car 6 and car 3 are assessed to at least "*Average* (A)", and therefore they should be the best candidates. Furthermore, the performance of car 6 should be better than car 3 because the degrees of belief for car 6 being assessed to "*Good* (G)", "*Excellent* (E)" and "*Top* (T)" are all greater than those for car 3. So, car 6 should be ranked the best and car 3 the second best in performance. Since car 4 is not assessed to "*Top* (T)" and at most to "*Excellent* (E)", its performance should be worse than the performances of the other cars except for car 5. Comparing the numerical values of the belief degrees of car 1 and car 2, we can find that the degrees of belief for car 2 being assessed to "*Excellent* (E)" and "*Top* (T)" are both greater than those for car 1. So, we may conclude that the performance of car 2 is better than the performance of car 1. Based on the above observations we can get the following intuitive ranking order: $Car\ 6 \succ Car\ 3 \succ Car\ 2 \succ Car\ 1 \succ Car\ 4 \succ Car\ 5$.

More precise ranking order can be obtained by calculating and analysing the fuzzy expected utilities. Since there is no incomplete information or any degree of ignorance in this example, the maximum and the minimum fuzzy expected utilities are the same for each executive car. Table 10 shows the fuzzy expected utilities of the six executive cars, which are all trapezoidal fuzzy numbers. The reason for the fuzzy expected utilities to be trapezoidal fuzzy numbers is because the utility of the fuzzy assessment grade "*Good* (G)" is a trapezoidal fuzzy number. As long as there is one assessment grade that is a trapezoidal fuzzy number, the fuzzy expected utilities including the maximum and the minimum fuzzy expected utilities are trapezoidal fuzzy numbers. If the assessment grades are all triangular fuzzy numbers, then the fuzzy expected utilities will be triangular fuzzy numbers.

Table 10
Fuzzy expected utilities and ranking order of executive cars

	Fuzzy expected utility				Centroid and distance			Rank
	Lower bound	Most possible value		Upper bound	$\bar{u}_{\max}$	$\bar{\mu}(u_{\max})$	$D = \sqrt{\bar{u}_{\max}^2 + \bar{\mu}^2(u_{\max})}$	
Car 1	0.40939	0.60110	0.66109	0.81178	0.618309	0.3766	0.7240	4
Car 2	0.45256	0.63614	0.67664	0.81951	0.643494	0.3665	0.7405	3
Car 3	0.55129	0.72484	0.76221	0.90504	0.733772	0.3652	0.8196	2
Car 4	0.32825	0.52492	0.58106	0.74966	0.544184	0.3725	0.6595	5
Car 5	0.30563	0.48849	0.53842	0.70774	0.509188	0.3701	0.6295	6
Car 6	0.57780	0.74969	0.79069	0.92743	0.759086	0.3683	0.8437	1

The centroid method is used to rank the executive cars. To do so, the centroid points of their fuzzy expected utilities are first calculated and then used to rank the cars according to the distances from the centroid points to the origin. The greater the distance, the better the performance. The results are shown in the last four columns of Table 10. It can be seen that $Car\ 6 \succ Car\ 3 \succ Car\ 2 \succ Car\ 1 \succ Car\ 4 \succ Car\ 5$, which is identical to the ranking order obtained previously through intuitive analysis. Therefore, there are sufficient reasons to believe that the ranking order $Car\ 6 \succ Car\ 3 \succ Car\ 2 \succ Car\ 1 \succ Car\ 4 \succ Car\ 5$ is reliable.

5. Concluding remarks

Most real world multiple attribute decision analysis (MADA) problems involve various types of uncertainties, which significantly increase the complexity and difficulty in decision analysis. The solution of such problems requires powerful methods that are capable of dealing with both quantitative and qualitative attributes with various types of uncertainties in a way that is rational, systematic, reliable, flexible and transparent. The evidential reasoning (ER) approach developed in this paper provides a novel, flexible and systematic way to support MADA under both probabilistic and fuzzy uncertainties.

In particular, the proposed ER modelling framework can consistently accommodate numerical data and subjective judgments with both probabilistic and fuzzy uncertainties using the fuzzy belief structure, which allows one to describe incomplete assessment information with fuzziness in an explicit and hybrid manner. The new fuzzy ER algorithm provides a systematic yet strict procedure for aggregating both probabilistic and fuzzy information in an analytical fashion. The grade match method provides a novel and pragmatic way for transforming fuzzy evaluation grades from one form to another. This makes it possible for different attributes to use their own assessment grades and thus greatly increases the flexibility in decision analysis. A numerical example demonstrated the implementation process of the ER approach in handling MADA problems with both probabilistic and fuzzy uncertainties. It can be concluded that the new ER approach could be used to deal with a wide range of MADA problems with various types of uncertainties.

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Appendix A. The derivation of the fuzzy ER algorithm

The fuzzy evidential reasoning algorithm is developed based on the non-fuzzy evidential reasoning algorithm (Yang and Singh, 1994; Yang and Sen, 1994a,b; Yang, 2001; Yang and Xu, 2002a). The main difficulties in devising the fuzzy ER algorithm include how to treat the dependency of adjacent evaluation grades and how to conduct normalisation. The first difficulty is overcome by introducing intermediate evaluation grades $H_{n,n+1}$ as shown in this appendix and in Sections 3.1, 3.2 and 3.4. Since the normalization rule in the Dempster–Shafer (D–S) theory of evidence can be applied towards the end of the evidence combination process without changing the combination result (Yen, 1990), we first take no account of the normalization when combining all attributes and then apply the normalization at the end. Such an evidence combination process, as Yen (1990) claimed for combining fuzzy evidence, can preserve the unique feature of the D–S theory that the belief and plausibility measures provide the lower and the upper bounds of the degrees of belief.

First of all, let us combine two attributes. Table 11 shows the combination process of two attributes without normalization. The combined probability masses generated by aggregating the two attributes are given as follows. Note that $m_{1-l}(H_n)$ and $m_{1-l}(H)$ denote the belief degree assigned to H_n and H generated by combining the first l attributes.

$$\begin{aligned} m_{1-2}(H_n) &= m_1(H_n)m_2(H_n) + m_1(H_n)[\tilde{m}_2(H) + \bar{m}_2(H)] + [\tilde{m}_1(H) + \bar{m}_1(H)]m_2(H_n) \\ &= m_1(H_n)m_2(H_n) + m_1(H_n)m_2(H) + m_1(H)m_2(H_n) \\ &= m_1(H_n)[m_2(H_n) + m_2(H)] + m_1(H)[m_2(H_n) + m_2(H)] - m_1(H)m_2(H) \\ &= [m_1(H_n) + m_1(H)][m_2(H_n) + m_2(H)] - m_1(H)m_2(H) \\ &= \prod_{i=1}^{2}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{2} m_i(H), n = 1, \ldots, N, \end{aligned}$$

$$\begin{aligned} \tilde{m}_{1-2}(H) &= \tilde{m}_1(H)\tilde{m}_2(H) + \tilde{m}_1(H)\bar{m}_2(H) + \bar{m}_1(H)\tilde{m}_2(H) \\ &= \tilde{m}_1(H)[\tilde{m}_2(H) + \bar{m}_2(H)] + \bar{m}_1(H)[\tilde{m}_2(H) + \bar{m}_2(H)] - \bar{m}_1(H)\bar{m}_2(H) \\ &= [\tilde{m}_1(H) + \bar{m}_1(H)][\tilde{m}_2(H) + \bar{m}_2(H)] - \bar{m}_1(H)\bar{m}_2(H) = \prod_{i=1}^{2} m_i(H) - \prod_{i=1}^{2} \bar{m}_i(H), \end{aligned}$$

$$\bar{m}_{1-2}(H) = \bar{m}_1(H)\bar{m}_2(H) = \prod_{i=1}^{2} \bar{m}_i(H),$$

$$\begin{aligned} m_{1-2}(H_{n,n+1}) &= m_1(H_n)m_2(H_{n+1}) + m_1(H_{n+1})m_2(H_n) \\ &= [m_1(H_n) + m_1(H_{n+1})][m_2(H_n) + m_2(H_{n+1})] - m_1(H_n)m_2(H_n) - m_1(H_{n+1})m_2(H_{n+1}) \\ &= [m_1(H_n) + m_1(H_{n+1}) + m_1(H)][m_2(H_n) + m_2(H_{n+1}) + m_2(H)] - [m_1(H_n) + m_1(H)] \\ &\quad \times [m_2(H_n) + m_2(H)] - [m_1(H_{n+1}) + m_1(H)][m_2(H_{n+1}) + m_2(H)] + m_1(H)m_2(H) \\ &= \prod_{i=1}^{2}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{2}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{2}[m_i(H_{n+1}) + m_i(H)] \\ &\quad + \prod_{i=1}^{2} m_i(H)], \quad n = 1, \ldots, N-1. \end{aligned}$$

Table 11
The combination of two pieces of evidence

$m_1 \oplus m_2$			m_1							
			$m_1(H_1)$	$m_1(H_2)$	...	$m_1(H_n)$	...	$m_1(H_N)$	$m_1(H)$	
									$\tilde{m}_1(H)$	$\bar{m}_1(H)$
m_2	$m_2(H_1)$		$m(H_1) = m_1(H_1)m_2(H_1)$	$m(H_{1,2}) = m_1(H_2)m_2(H_1)$	...	$m(\Phi) = m_1(H_n)m_2(H_1)$	...	$m(\Phi) = m_1(H_N)m_2(H_1)$	$m(H_1) = \tilde{m}_1(H)m_2(H_1)$	$m(H_1) = \bar{m}_1(H)m_2(H_1)$
	$m_2(H_2)$		$m(H_{1,2}) = m_1(H_1)m_2(H_2)$	$m(H_2) = m_1(H_2)m_2(H_2)$	...	$m(\Phi) = m_1(H_n)m_2(H_2)$	...	$m(\Phi) = m_1(H_N)m_2(H_2)$	$m(H_2) = \tilde{m}_1(H)m_2(H_2)$	$m(H_2) = \bar{m}_1(H)m_2(H_2)$
	...		...	...	...		...	...	...	...
	$m_2(H_n)$		$m(\Phi) = m_1(H_1)m_2(H_n)$	$m(\Phi) = m_1(H_2)m_2(H_n)$	...	$m(H_n) = m_1(H_n)m_2(H_n)$	...	$m(\Phi) = m_1(H_N)m_2(H_n)$	$m(H_n) = \tilde{m}_1(H)m_2(H_n)$	$m(H_n) = \bar{m}_1(H)m_2(H_n)$
	...		...	...	...		...	...	...	...
	$m_2(H_N)$		$m(\Phi) = m_1(H_1)m_2(H_N)$	$m(\Phi) = m_1(H_2)m_2(H_N)$	...	$m(\Phi) = m_1(H_n)m_2(H_N)$	...	$m(H_N) = m_1(H_N)m_2(H_N)$	$m(H_N) = \tilde{m}_1(H)m_2(H_N)$	$m(H_N) = \bar{m}_1(H)m_2(H_N)$
	$m_2(H)$	$\tilde{m}_2(H)$	$m(H_1) = m_1(H_1)\tilde{m}_2(H)$	$m(H_2) = m_1(H_2)\tilde{m}_2(H)$	...	$m(H_n) = m_1(H_n)\tilde{m}_2(H)$	...	$m(H_N) = m_1(H_N)\tilde{m}_2(H)$	$m(H) = \tilde{m}_1(H)\tilde{m}_2(H)$	$m(H) = \bar{m}_1(H)\tilde{m}_2(H)$
		$\bar{m}_2(H)$	$m(H_1) = m_1(H_1)\bar{m}_2(H)$	$m(H_2) = m_1(H_2)\bar{m}_2(H)$	...	$m(H_n) = m_1(H_n)\bar{m}_2(H)$	...	$m(H_N) = m_1(H_N)\bar{m}_2(H)$	$m(H) = \tilde{m}_1(H)\bar{m}_2(H)$	$m(H) = \bar{m}_1(H)\bar{m}_2(H)$

Suppose the following equations are true for combining the first $(l-1)$ attributes, where $l_1\ l-1$:

$$m_{1-l_1}(H_n) = \prod_{i=1}^{l-1}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{l-1} m_i(H), \quad n = 1, \ldots, N,$$

$$\tilde{m}_{1-l_1}(H) = \prod_{i=1}^{l-1} m_i(H) - \prod_{i=1}^{l-1} \bar{m}_i(H),$$

$$\bar{m}_{1-l_1}(H) = \prod_{i=1}^{l-1} \bar{m}_i(H),$$

$$\begin{aligned} m_{1-l_1}(H_{n,n+1}) = {} & \prod_{i=1}^{l-1}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{l-1}[m_i(H_n) + m_i(H)] \\ & - \prod\nolimits_{i=1}^{l-1}[m_i(H_{n+1}) + m_i(H)] + \prod_{i=1}^{l-1} m_i(H), \quad n = 1, \ldots, N-1. \end{aligned}$$

The above combined probability masses are further aggregated with the lth attribute. The results are generated using Table 12. The combined probability masses are then given below.

$$\begin{aligned} m_{1-l}(H_n) &= m_{1-l_1}(H_n)m_l(H_n) + [\tilde{m}_{1-l_1}(H) + \bar{m}_{1-l_1}(H)]m_l(H_n) + m_{1-l_1}(H_n)[\tilde{m}_l(H) + \bar{m}_l(H)] \\ &= m_{1-l_1}(H_n)m_l(H_n) + m_{1-l_1}(H)m_l(H_n) + m_{1-l_1}(H_n)m_l(H) \\ &= [m_{1-l_1}(H_n) + m_{1-l_1}(H)]m_l(H_n) + [m_{1-l_1}(H_n) + m_{1-l_1}(H)]m_l(H) - m_{1-l_1}(H)m_l(H) \\ &= [m_{1-l_1}(H_n) + m_{1-l_1}(H)][m_l(H_n) + m_l(H)] - m_{1-l_1}(H)m_l(H) \\ &= \prod_{i=1}^{l-1}[m_i(H_n) + m_i(H)][m_l(H_n) + m_l(H)] - \prod_{i=1}^{l-1} m_i(H)m_l(H) \\ &= \prod_{i=1}^{l}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{l} m_i(H), \quad n = 1, \ldots, N, \end{aligned}$$

$$\begin{aligned} \tilde{m}_{1-l}(H) &= \tilde{m}_{1-l_1}(H)\tilde{m}_l(H) + \tilde{m}_{1-l_1}(H)\bar{m}_l(H) + \bar{m}_{1-l_1}(H)\tilde{m}_l(H) \\ &= \tilde{m}_{1-l_1}(H)[\tilde{m}_l(H) + \bar{m}_l(H)] + \bar{m}_{1-l_1}(H)[\tilde{m}_l(H) + \bar{m}_l(H)] - \bar{m}_{1-l_1}(H)\bar{m}_l(H) \\ &= [\tilde{m}_{1-l_1}(H) + \bar{m}_{1-l_1}(H)][\tilde{m}_l(H) + \bar{m}_l(H)] - \bar{m}_{1-l_1}(H)\bar{m}_l(H) \\ &= \left(\prod_{i=1}^{l-1}[\tilde{m}_i(H) + \bar{m}_i(H)] - \prod_{i=1}^{l-1}\bar{m}_i(H) + \prod_{i=1}^{l-1}\bar{m}_i(H)\right)[\tilde{m}_l(H) + \bar{m}_l(H)] - \prod_{i=1}^{l-1}\bar{m}_i(H)\bar{m}_l(H) \\ &= \prod_{i=1}^{l}[\tilde{m}_i(H) + \bar{m}_i(H)] - \prod_{i=1}^{l}\bar{m}_i(H) = \prod_{i=1}^{l} m_i(H) - \prod_{i=1}^{l}\bar{m}_i(H), \end{aligned}$$

$$\bar{m}_{1-l}(H) = \bar{m}_{1-l_1}(H)\bar{m}_l(H) = \prod_{i=1}^{l-1}\bar{m}_i(H)\bar{m}_l(H) = \prod_{i=1}^{l}\bar{m}_i(H),$$

Table 12
The combination of four pieces of evidence

$m_{1-l_1} \oplus m_l$			m_{1-l_1}													
			$m_{1-l_1}(H_1)$	$m_{1-l_1}(H_2)$	…	$m_{1-l_1}(H_n)$	…	$m_{1-l_1}(H_N)$	$m_{1-l_1}(H)$		$m_{1-l_1}(H_{1,2})$	…	$m_{1-l_1}(H_{n,n+1})$	…	$m_{1-l_1}(H_{N-1,N})$
									$\tilde{m}_{1-l_1}(H)$	$\bar{m}_{1-l_1}(H)$					
m_l	$m_l(H_1)$		$m(H_1) = m_{1-l_1}(H_1)\, m_l(H_1)$	$m(H_{1,2}) = m_{1-l_1}(H_2)\, m_l(H_1)$	…	$m(\Phi) = m_{1-l_1}(H_n) m_l(H_1)$	…	$m(\Phi) = m_{1-l_1}(H_N)\, m_l(H_1)$	$m(H_1) = \tilde{m}_{1-l_1}(H)\, m_l(H_1)$	$m(H_1) = \bar{m}_{1-l_1}(H)\, m_l(H_1)$	$m(H_{1,2}) = m_{1-l_1}(H_{1,2})\, m_l(H_1)$	…	$m(\Phi) = m_{1-l_1}(H_{n,n+1})\, m_l(H_1)$	…	$m(\Phi) = m_{1-l_1}(H_{N-1,N})\, m_l(H_1)$
	$m_l(H_2)$		$m(H_{1,2}) = m_{1-l_1}(H_1)\, m_l(H_2)$	$m(H_2) = m_{1-l_1}(H_2)\, m_l(H_2)$	…	$m(\Phi) = m_{1-l_1}(H_n)\, m_l(H_2)$	…	$m(\Phi) = m_{1-l_1}(H_N)\, m_l(H_2)$	$m(H_2) = \tilde{m}_{1-l_1}(H)\, m_l(H_2)$	$m(H_2) = \bar{m}_{1-l_1}(H)\, m_l(H_2)$	$m(H_{1,2}) = m_{1-l_1}(H_{1,2})\, m_l(H_2)$	…	$m(\Phi) = m_{1-l_1}(H_{n,n+1})\, m_l(H_2)$	…	$m(\Phi) = m_{1-l_1}(H_{N-1,N})\, m_l(H_2)$
	…		…	…	…	…	…	…	…	…	…	…	…	…	…
	$m_l(H_n)$		$m(\Phi) = m_{1-l_1}(H_1)\, m_l(H_n)$	$m(\Phi) = m_{1-l_1}(H_2)\, m_l(H_n)$	…	$m(H_n) = m_{1-l_1}(H_n)\, m_l(H_n)$	…	$m(\Phi) = m_{1-l_1}(H_N)\, m_l(H_n)$	$m(H_n) = \tilde{m}_{1-l_1}(H)\, m_l(H_n)$	$m(H_n) = \bar{m}_{1-l_1}(H)\, m_l(H_n)$	$m(\Phi) = m_{1-l_1}(H_{1,2})\, m_l(H_n)$	…	$m(H_{n,n+1}) = m_{1-l_1}(H_{n,n+1})\, m_l(H_n)$	…	$m(\Phi) = m_{1-l_1}(H_{N-1,N})\, m_l(H_n)$
	…		…	…	…	…	…	…	…	…	…	…	…	…	…
	$m_l(H_N)$		$m(\Phi) = m_{1-l_1}(H_1)\, m_l(H_N)$	$m(\Phi) = m_{1-l_1}(H_2)\, m_l(H_N)$	…	$m(\Phi) = m_{1-l_1}(H_n)\, m_l(H_N)$	…	$m(H_N) = m_{1-l_1}(H_N)\, m_l(H_N)$	$m(H_N) = \tilde{m}_{1-l_1}(H)\, m_l(H_N)$	$m(H_N) = \bar{m}_{1-l_1}(H) m_l(H_N)$	$m(\Phi) = m_{1-l_1}(H_{1,2})\, m_l(H_N)$	…	$m(\Phi) = m_{1-l_1}(H_{n,n+1})\, m_l(H_N)$	…	$m(H_{N-1,N}) = m_{1-l_1}(H_{N-1,N})\, m_l(H_N)$
	$m_l(H)$	$\tilde{m}_l(H)$	$m(H_1) = m_{1-l_1}(H_1)\, \tilde{m}_l(H)$	$m(H_2) = m_{1-l_1}(H_2)\, \tilde{m}_l(H)$	…	$m(H_n) = m_{1-l_1}(H_n)\, \tilde{m}_l(H)$	…	$m(H_N) = m_{1-l_1}(H_N)\, \tilde{m}_l(H)$	$m(H) = \tilde{m}_{1-l_1}(H)\, \tilde{m}_l(H)$	$m(H) = \bar{m}_{1-l_1}(H)\, \tilde{m}_l(H)$	$m(H_{1,2}) = m_{1-l_1}(H_{1,2})\, \tilde{m}_l(H)$	…	$m(H_{n,n+1}) = m_{1-l_1}(H_{n,n+1})\, \tilde{m}_l(H)$	…	$m(H_{N-1,N}) = m_{1-l_1}(H_{N-1,N})\, \tilde{m}_l(H)$
		$\bar{m}_l(H)$	$m(H_1) = m_{1-l_1}(H_1)\, \bar{m}_l(H)$	$m(H_2) = m_{1-l_1}(H_2)\, \bar{m}_l(H)$	…	$m(H_n) = m_{1-l_1}(H_n)\, \bar{m}_l(H)$	…	$m(H_N) = m_{1-l_1}(H_N)\, \bar{m}_l(H)$	$m(H) = \tilde{m}_{1-l_1}(H)\, \bar{m}_l(H)$	$m(H) = \bar{m}_{1-l_1}(H)\, \bar{m}_l(H)$	$m(H_{1,2}) = m_{1-l_1}(H_{1,2})\, \bar{m}_l(H)$	…	$m(H_{n,n+1}) = m_{1-l_1}(H_{n,n+1})\, \bar{m}_l(H)$	…	$m(H_{N-1,N}) = m_{1-l_1}(H_{N-1,N})\, \bar{m}_l(H)$

$$
\begin{aligned}
m_{1-l}(H_{n,n+1}) &= m_{1-l_1}(H_n)m_l(H_{n+1}) + m_{1-l_1}(H_{n+1})m_l(H_n) + m_{1-l_1}(H_{n,n+1})[m_l(H_n)\\
&\quad + m_l(H_{n+1}) + \tilde{m}_l(H) + \bar{m}_l(H)]\\
&= m_{1-l_1}(H_n)m_l(H_{n+1}) + m_{1-l_1}(H_{n+1})m_l(H_n) + m_{1-l_1}(H_{n,n+1})[m_l(H_n) + m_l(H_{n+1}) + m_l(H)]\\
&= m_{1-l_1}(H_n)[m_l(H_n) + m_l(H_{n+1}) + m_l(H)] - m_{1-l_1}(H_n)[m_l(H_n) + m_l(H)] + m_{1-l_1}(H_{n+1})\\
&\quad \times [m_l(H_n) + m_l(H_{n+1}) + m_l(H)] - m_{1-l_1}(H_{n+1})[m_l(H_{n+1}) + m_l(H)] + m_{1-l_1}(H)[m_l(H_n)\\
&\quad + m_l(H_{n+1}) + m_l(H)] - m_{1-l_1}(H)[m_l(H_n) + m_l(H_{n+1}) + m_l(H)] + m_{1-l_1}(H_{n,n+1})[m_l(H_n)\\
&\quad + m_l(H_{n+1}) + m_l(H)] = [m_{1-l_1}(H_n) + m_{1-l_1}(H_{n+1}) + m_{1-l_1}(H) + m_{1-l_1}(H_{n,n+1})][m_l(H_n)\\
&\quad + m_l(H_{n+1}) + m_l(H)] - m_{1-l_1}(H_n)[m_l(H_n) + m_l(H)] - m_{1-l_1}(H_{n+1})[m_l(H_{n+1}) + m_l(H)]\\
&\quad - m_{1-l_1}(H)[m_l(H_n) + m_l(H_{n+1}) + m_l(H)]\\
&= \prod_{i=1}^{l-1}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)][m_l(H_n) + m_l(H_{n+1}) + m_l(H)] - [m_{1-l_1}(H_n)\\
&\quad + m_{1-l_1}(H)][m_l(H_n) + m_l(H)] + m_{1-l_1}(H)[m_l(H_n) + m_l(H)] - [m_{1-l_1}(H_{n+1})\\
&\quad + m_{1-l_1}(H)][m_l(H_{n+1}) + m_l(H)] + m_{1-l_1}(H)[m_l(H_{n+1}) + m_l(H)]\\
&\quad - m_{1-l_1}(H)[m_l(H_n) + m_l(H_{n+1}) + m_l(H)] = \textstyle\prod_{i=1}^{l}[m_i(H_n) + m_i(H_{n+1})\\
&\quad + m_i(H)] - [m_{1-l_1}(H_n) + m_{1-l_1}(H)][m_l(H_n) + m_l(H)] - [m_{1-l_1}(H_{n+1})\\
&\quad + m_{1-l_1}(H)][m_l(H_{n+1}) + m_l(H)] + m_{1-l_1}(H)m_l(H)\\
&= \textstyle\prod_{i=1}^{l}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \displaystyle\prod_{i=1}^{l}[m_i(H_n) + m_i(H)]\\
&\quad - \prod_{i=1}^{l}[m_i(H_{n+1}) + m_i(H)] + \prod_{i=1}^{l}m_i(H), \quad n = 1, \dots, N-1.
\end{aligned}
$$

Therefore, the above equations are true for any $l \in \{1, \dots, L\}$. For $l = L$, we get the following un-normalized combined probability assignments generated by aggregating the L attributes

$$
m_{1-L}(H_n) = \prod_{i=1}^{L}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{L}m_i(H), \quad n = 1, \dots, N,
$$

$$
\tilde{m}_{1-L}(H) = \prod_{i=1}^{L}m_i(H) - \prod_{i=1}^{L}\bar{m}_i(H), \quad \bar{m}_{1-L}(H) = \prod_{i=1}^{L}\bar{m}_i(H),
$$

$$
\begin{aligned}
m_{1-L}(H_{n,n+1}) &= \prod_{i=1}^{L}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_n) + m_i(H)]\\
&\quad - \prod_{i=1}^{L}[m_i(H_{n+1}) + m_i(H)] + \prod_{i=1}^{L}m_i(H), \quad n = 1, \dots, N-1.
\end{aligned}
$$

Since the fuzzy subset $H_{n,n+1}$ is the intersection of the two fuzzy assessment grades H_n and H_{n+1}, its maximum degree of membership is normally not equal to unity. In order to capture the exact probability mass assigned to $H_{n,n+1}$, its membership function needs to be normalized. If this were not done, then probability mass assigned to $H_{n,n+1}$ would have nothing to do with its shape or height. In other words, as long as the two fuzzy assessment grades H_n and H_{n+1} intersect, the probability mass assigned to $H_{n,n+1}$ would always be the same, no matter how large or small the intersection subset may be. So, it is necessary to normalise the membership function of $H_{n,n+1}$. After normalization, we have

$$m_{1-L}(\overline{H}_{n,n+1}) = \mu^{\max}_{H_{n,n+1}} \cdot \left\{ \prod_{i=1}^{L}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_{n+1}) + m_i(H)] + \prod_{i=1}^{L} m_i(H) \right\}, \quad n = 1, \dots, N-1,$$

where $\overline{H}_{n,n+1}$ stands for the normalized fuzzy intersection subset for $H_{n,n+1}$.

Define by k the normalization constant for the fuzzy evidential combination. Then, we have the normalized combined probability masses as follows:

$$m(H_n) = km_{1-L}(H_n) = k\left\{ \prod_{i=1}^{L}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{L} m_i(H)\} \right\}, \quad n = 1, \dots, N,$$

$$\tilde{m}(H) = k\tilde{m}_{1-L}(H) = k\left\{ \prod_{i=1}^{L} m_i(H) - \prod_{i=1}^{L} \bar{m}_i(H) \right\}, \quad \bar{m}(H) = k\bar{m}_{1-L}(H) = k\left[\prod_{i=1}^{L} \bar{m}_i(H) \right],$$

$$\begin{aligned} m(\overline{H}_{n,n+1}) &= km_{1-L}(\overline{H}_{n,n+1}) \\ &= k\mu^{\max}_{H_{n,n+1}} \left\{ \prod_{i=1}^{L}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_n) + m_i(H)] \right. \\ &\quad \left. - \prod_{i=1}^{L}[m_i(H_{n+1}) + m_i(H)] + \prod_{i=1}^{L} m_i(H) \right\}, \quad n = 1, \dots, N-1, \end{aligned}$$

where k can be determined using the following normalization constraint condition:

$$\sum_{n=1}^{N} m(H_n) + \sum_{n=1}^{N-1} m(\overline{H}_{n,n-1}) + \tilde{m}(H) + \bar{m}(H) = 1$$

from which we get

$$\begin{aligned} k = \left\{ \sum_{n=1}^{N-1} (1 - \mu^{\max}_{H_{n,n+1}}) \left(\prod_{i=1}^{L}[m_i(H_n) + m_i(H)] - \prod_{i=1}^{L} m_i(H) \right) \right. \\ \left. + \sum_{n=1}^{N-1} \mu^{\max}_{H_{n,n+1}} \left(\prod_{i=1}^{L}[m_i(H_n) + m_i(H_{n+1}) + m_i(H)] - \prod_{i=1}^{L}[m_i(H_{n+1}) + m_i(H)] \right) + \prod_{i=1}^{L}[m_i(H_N) + m_i(H)] \right\}^{-1}. \end{aligned}$$

Appendix B. The formula for the computation of intersection area

As shown in Fig. 18, at the point of the intersection of the two fuzzy assessment grades A and B with the maximum degree of membership, their degrees of membership are the same, that is $\frac{y-a}{b-a} = \frac{d-y}{d-c}$, from which we derive the coordinates of the point of the intersection as follows:

$$y = \frac{bd - ac}{(b+d) - (a+c)},$$

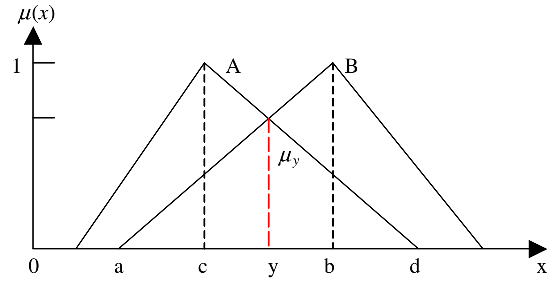

Fig. 18. The computation of area of fuzzy intersection subset.

$$\mu_y = \frac{d-a}{(b+d)-(a+c)}.$$

Accordingly, the area of the fuzzy intersection subset is obtained by

$$S_{\Delta a\mu_y d} = \frac{1}{2}(d-a)\cdot\mu_y = \frac{1}{2}\times(d-a)\times\frac{d-a}{(b+d)-(a+c)}$$

which was extensively used in Example 1.

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