DECISION METHODS IN THE THEORY OF ORDINALS[1]

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For an ordinal α, let $RS(\alpha)$, the restricted second order theory of $[\alpha, <]$, be the interpreted formalism containing the first order theory of $[\alpha, <]$ and quantification on monadic predicate variables, ranging over all subsets of α. For a cardinal γ, $RS(\alpha, \gamma)$ is like $RS(\alpha)$, except that the predicate variables are now restricted to range over subsets of α of cardinality less than γ. $\omega = \omega_0$ and ω_1 denote the first two infinite cardinals. In this note I will outline results concerning $RS(\alpha, \omega_0)$, which were obtained in the Spring of 1964 (detailed proofs will appear in [**8**]), and the corresponding stronger results about $RS(\alpha, \omega_1)$, which were obtained in the Fall of 1964.

The binary expansion of natural numbers can be extended to ordinals. If $x < 2^\alpha$, let ϕx be the finite subset $\{u_1, \cdots, u_n\}$ of α, given by $x = 2^{u_1} + \cdots + 2^{u_n}$, $u_n < \cdots < u_1$. ϕ is a one-to-one map of 2α onto all finite subsets of α. Let Exy stand for $(\exists u)[x = 2^u \wedge u \in \phi y]$, and note that the algorithm $i + j = s$, for addition in binary notation can be expressed in $RS(\alpha, \omega_0)$. It now is easy to see that the first order theory $FT[2^\alpha, +, E]$ is equivalent to $RS(\alpha, \omega_0)$, in the strong sense that the two theories merely differ in the choice of primitive notions; the binary expansion ϕ yields the translation. Similarly, $RS(\alpha, \gamma)$ can be reinterpreted as a first order theory. We will state our results in one of the two forms, and leave it to the reader to translate.

THEOREM 1. *For any α, there is a decision method for truth of sentences in $RS(\alpha, \omega_0)$. The same sentences are true in $RS(\alpha, \omega_0)$ and $RS(\beta, \omega_0)$, if and only if, $\alpha = \beta < \omega^\omega$ or else $\alpha, \beta \geqq \omega^\omega$ and have the same ω-tail.*

If $\alpha = z + \omega^y + \omega^n c_n + \cdots + \omega^0 c_0$, $y \geqq \omega$, then $z + \omega^y$ is called the *ω-head* of α, and $\omega^n c_n + \cdots + \omega^0 c_0$ is called the ω-tail of α.

THEOREM 2. *For any ordinals $\beta > \alpha > \omega^\omega$, $[2^\beta, +, E]$ is an elementary extension of $[2^\alpha, +, E]$, if and only if, α and β have the same ω-tail. The elementary embedding is then given by $h(2^{\alpha_0}x + y) = 2^{\beta_0}x + y$, whereby $x < 2^\tau$, $y < 2^{\alpha_0}$, τ is the common ω-tail of α and β, α_0 and β_0 are respectively the ω-heads of α and β.*

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Let $\alpha=\alpha_0+\tau\geqq\omega^\omega$, where α_0 is the ω-head and τ is the ω-tail of α. From Theorem 2 one easily shows: the ordinals definable in $\mathrm{FT}[2^\alpha, +, E]$ (in $\mathrm{FT}[2^\alpha, +]$) are those of form $2^{\alpha_0}x+y$, whereby $x<2^\tau$ and $y<2^{(\omega^\omega)}$. Actually, Theorems 1 and 2 are but samples of corollaries to Theorem 3, which completely describes the relations on ordinals definable in $\mathrm{FT}[2^\alpha, +, E]$.

The results on definability of individuals in $\mathrm{FT}[\omega^\alpha, +]$ have been obtained earlier by A. Ehrenfeucht [6]. His methods are quite different; a lucid presentation of this work occurs in [3]. In [3] and [4] it is stated that Ehrenfeucht also knew a decision method for $\mathrm{FT}[\omega^\alpha, +]$. However, it seems that nobody has checked out these ideas. The first published proof of the decidability of $\mathrm{FT}[\omega, +, E]$, i.e., of $\mathrm{RS}(\omega, \omega)$ occurs in [1], and a similar one in [7]. These are both based on my conjecture that $\mathrm{RS}(\omega, \omega)$ is just strong enough to express the behavior of finite automata.

The key to the understanding of $\mathrm{RS}(\alpha, \omega_0)$ is a natural extension of deterministic finite-state recursions to the transfinite. Let I (input states) and S (internal states) be finite sets. An automaton $\mathfrak{A}$ on I, S consists of an element $A\in S$ (initial state) a map $H: S\times I\to S$, a map $U: 2^S\to S$, and a subset $0\subseteq S$ (the output). Let $\sup_{t<x}(rt)$ stand for the set of all values which the function r takes on cofinal to x, i.e. $Y\in\sup_{t<x}(rt)\cdot\equiv\cdot(\forall z)_0^x(\exists t)_z^x[rt=Y]$. $[A, H, U]$ determines recursively an operator $s[o, \alpha]=\zeta i[o, \alpha)$ from I^α to $S^{\alpha+1}$, namely,

$$\begin{aligned} so &= A, \\ s(x+1) &= H[sx, ix], \\ sx &= U\Big[\sup_{t<x}(st)\Big], \quad x \text{ a limit.} \end{aligned}$$

An input sequence $i[o, \alpha)$ is said to be accepted by $\mathfrak{A}$, in case $s\alpha\in 0$. Extending the proofs given in [1], one now shows,

THEOREM 3. *Let $R(i_1, \cdots, i_n)$ be a relation on finite predicates on α. R is definable in $RS(\alpha, \omega_0)$ if and only if there is an automaton $\mathfrak{A}$ such that R consists of those finite $(i_1, \cdots, i_n)$ on α, for which the input signal $i[o, \alpha)$ is accepted by $\mathfrak{A}$.*

In fact there are effective methods, (1) for the construction of $\mathfrak{A}$ from a defining formula Σ of R (synthesis), and (2) for the construction of Σ from $\mathfrak{A}$ (analysis). Theorems 1 and 2 now follow by investigating the behavior of input-free automata.

Let us now consider $\mathrm{RS}(\alpha, \omega_1)$. The decidability of $\mathrm{RS}(\omega_0, \omega_1)$, i.e., $\mathrm{RS}(\omega_0)$ was proved in [2]. It is not difficult to extend the method used in [2], replacing ordinary automata recursions by transfinite automata. The result is,

THEOREM 1′. *For any countable ordinal* α, *$RS(\alpha)$ is decidable. For* $\alpha<\beta<\omega_1$, *$RS(\alpha)$ and $RS(\beta)$ are equivalent if and only if either* $\alpha=\beta<\omega^\omega$ *or* $\alpha, \beta\geqq\omega^\omega$ *and have the same ω-tail. Furthermore, $RS(\alpha, \omega_1)$ is decidable for any* α.

As in [2] we actually obtain a complete survey over definability in RS(α, ω_1). In particular, the analog to Theorem 2 holds.

Define the α-behavior of an automaton $\mathfrak{A}$ to be the set Bh($\mathfrak{A}$, α) consisting of all input-signals $i\,[{,}o\,\alpha)$ which are accepted by $\mathfrak{A}$. Thus, the ω-behaviors are the ordinary regular sets of finite automata theory.

THEOREM 4. *To any automaton $\mathfrak{A}$ with input (i, j) one can construct an automaton $\mathfrak{C}$ with input i, such that for any $\alpha\leqq\omega_1$ and any input-signal i of length* $<\alpha$, $i\in Bh(\mathfrak{C}, \alpha)\cdot\equiv\cdot(\exists j)(i, j)\in Beh(\mathfrak{A}, \alpha)$.

For $\alpha=\omega$ this is the well-known projection-lemma for behaviors of finite automata. The case $\alpha=\omega+1$ constitutes a significant improvement of the crucial Lemma 9 of [2], and has recently been obtained by R. McNaughton. His construction is very ingenious, and his $\mathfrak{C}$'s are by far the most intricate finite automata this writer has seen in action. The extension to $\alpha\leqq\omega_1$ is an exercise in handling ordinals. Using this improved form of Lemma 9, the definability result of [2] extends as follows,

THEOREM 3′. *To every RS-formula $\Sigma(i_1, \cdots, i_n)$ one can construct an automaton $\mathfrak{A}$, and to every automaton $\mathfrak{A}$ with 2^n-ary input $(i_1, \cdots, i_n)$ one can construct an RS-formula $\Sigma(i_1, \cdots, i_n)$, such that for any $\alpha<\omega_1$ the behavior $Bh(\mathfrak{A}, \alpha)$ is the relation defined by Σ in $RS(\alpha, \omega_1)$.*

The following problem remains unsolved: Is RS(ω_1) decidable?

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